RECOVERY OF A TIME-DEPENDENT HERMITIAN CONNECTION AND POTENTIAL APPEARING IN THE DYNAMIC SCHRÖDINGER EQUATION

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ABSTRACT. We consider, on a trivial vector bundle over a Riemannian manifold with boundary, the inverse problem of uniquely recovering time- and space-dependent coefficients of the dynamic, vector-valued Schrödinger equation from the knowledge of the Dirichlet-to-Neumann map. We show that the D-to-N map uniquely determines both the connection form and the potential appearing in the Schrödinger equation, under the assumption that the manifold is either a) two-dimensional and simple, or b) of higher dimension with strictly convex boundary and admits a smooth, strictly convex function.

1. INTRODUCTION

1.1. Statement of the Problem. Let T > 0 be fixed, and let (M, g) be a connected, compact, smooth Riemannian manifold of dimension $m \ge 2$ with boundary ∂M . In what follows, we shall additionally assume that (M, g) is non-trapping. Consider a trivial Hermitian vector bundle $E = M \times \mathbb{C}^n$ equipped with the Hermitian inner product $\langle \cdot, \cdot \rangle_E$.

We say that a connection $\nabla : C^{\infty}(M; E) \to C^{\infty}(M; E \otimes T^*M)$ is compatible with the Hermitian structure of E if for any sections $u, v \in C^{\infty}(M; E)$ it holds that

(1.1)
$$d\langle u, v \rangle_E = \langle \nabla u, v \rangle_E + \langle u, \nabla v \rangle_E,$$

where both sides of the above are regarded as sections of the cotangent bundle.

Such a connection has the form $\nabla = d + A$, where $A = A_i dx^i$ and each $A_i(x)$ is given by an $n \times n$ skew-Hermitian matrix. In what follows, we allow the connection form A to also depend smoothly on time, and write $\nabla^A : C^{\infty}((0,T) \times M; E) \to C^{\infty}((0,T) \times M; E \otimes T^*M)$ for the time-dependent connection corresponding to the connection form A. In other words, $\nabla^{A(t)}$ is a connection on $C^{\infty}(M; E)$ for each $t \in [0,T]$, and each $\nabla^{A(t)}$ is compatible with the Hermitian metric on E.

We can define a natural L^2 -inner product on $C^{\infty}((0,T) \times M; E)$ via

$$\langle u,v\rangle_{L^2((0,T)\times M;E)} = \int_0^T \int_M \langle u,v\rangle_E \, dV dt,$$

where dV denotes the usual Riemannian volume measure of (M, g). We can similarly define a natural L^2 -inner product on $C^{\infty}((0,T) \times M; E \otimes T^*M)$. For E-valued 1-forms $\alpha = \alpha_j dx^j$ and $\beta = \beta_j dx^j$, we set

$$\langle \alpha, \beta \rangle_{L^2((0,T) \times M; E \otimes T^*M)} = \int_0^T \int_M g^{ij} \langle \alpha_i, \beta_j \rangle_E \, dV dt$$

where q^{ij} denotes the inverse of the metric tensor.

We let $(\nabla^A)^*$ denote the adjoint of ∇^A with respect to the above inner products. We can then define the connection Laplacian $\Delta_A = -(\nabla^A)^* \nabla^A$, which corresponds to the connection form A.

We can compute local expressions for $(\nabla^A)^*$ and Δ_A . Consider a section $u \in C^{\infty}(M; E)$ and an *E*-valued 1-form $\beta = \beta_j dx^j$ supported on a local trivialisation. Since *A* is skew-Hermitian, it holds that

$$\langle Au,\beta\rangle_{L^2((0,T)\times M;E\otimes T^*M)} = \int_0^T \int_M g^{ij} \langle A_iu,\beta_j\rangle_E \, dVdt = -\int_0^T \int_M \langle u,g^{ij}A_i\beta_j\rangle_E \, dVdt$$

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Letting $(A, \beta)_g = g^{ij} A_i \beta_j$, we see that $(\nabla^A)^* = d^* - (A, \cdot)_g$. Therefore, we have

$$\Delta_A u = -d^* du - d^* (Au) + (A, du)_g + (A, Au)_g.$$

Recall that $d^*\alpha = -|g|^{-\frac{1}{2}} \partial_i (|g|^{\frac{1}{2}} g^{ij} \alpha_j)$, where |g| is the determinant of the metric tensor. Hence it holds that $d^*(Au) = (d^*A)u - (A, du)_g$. Thus, we conclude that

(1.2)
$$\Delta_A u = -d^* du + 2(A, du)_g - (d^* A)u + (A, Au)_g.$$

Lastly, we say that a section $V \in C^{\infty}((0,T) \times M; \mathbb{C}^{n \times n})$ is a potential if V is Hermitian or, equivalently, for any sections $u, v \in C^{\infty}((0,T) \times M; E)$ it holds that

$$\langle Vu, v \rangle_E = \langle u, Vv \rangle_E$$
.

Let Δ_A and V be as above and consider the following initial and boundary value problem for sections $u \in C^{\infty}((0,T) \times M; E)$.

(1.3)
$$i\partial_t u(t,x) + \Delta_{A(t)} u(t,x) + V(t,x)u(t,x) = 0 \text{ in } (0,T) \times M,$$
$$u(t,x) = f \text{ on } (0,T) \times \partial M,$$
$$u(0,x) = 0 \text{ in } M,$$

where the inhomogeneous Dirichlet data is given by $f \in C^{\infty}((0,T) \times \partial M; E)$ satisfying $f|_{t=0} = \partial_t f|_{t=0} = 0$. We can then define the associated Dirichlet-to-Neuman map via

$$\Lambda_{A,V}f = \nabla^A_\nu u \Big|_{(0,T) \times \partial M} ,$$

where ν denotes the outward pointing unit normal vector field on ∂M .

There is a natural gauge group associated with the equation above. Let $G: (0,T) \times M \to U(n)$ be a smooth map such that $G(t)|_{\partial M} = \mathrm{Id}$, and choose $A_2 = G^{-1}A_1G + G^{-1}dG$ and $V_2 = G^{-1}V_1G + iG^{-1}\partial_tG$. It then holds that $\nabla^{A_2} = G^{-1}\nabla^{A_1}G$, and hence that $\Delta_{A_2} = G^{-1}\Delta_{A_1}G$. We observe that if u solves (1.3) with $A = A_2$ and $V = V_2$, then Gu solves the equation with $A = A_1$ and $V = V_1$, since

$$(i\partial_t + \Delta_{A_1} + V_1)Gu = G(i\partial_t + \Delta_{A_2} + V_2)u = 0.$$

Furthermore, we observe that when the pairs (A_1, V_1) and (A_2, V_2) are as above, it holds that $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_2}$. Therefore, we can only hope to recover the pair (A, V) up to a gauge transform. The aim of the present work is to establish unique recovery of the connection form and potential from the knowledge of the Dirichlet-to-Neumann map, modulo gauge invariance.

1.2. **History of the Problem.** Literature dealing with the recovery of space- and time-dependent potentials of the dynamic Schrödinger equation is limited, even in the scalar case. For Euclidean domains, it was shown in [12] that the time-dependent electromagnetic potentials are uniquely determined by the Dirichletto-Neumann map. Logarithmic-stable determination was shown for the electric potential in [10], and this result was extended to the full electromagnetic potential in [8], provided that the time-independent part of the magnetic potential is sufficiently small. Indeed, it was only recently shown in [15] that time-dependent electromagnetic potentials in a Euclidean domain can be Hölder-stably recovered from the knowledge of the D-to-N map. We also mention here the recent work of [3], which establishes logarithmic and doublelogarithmic stability estimates for the same problem with partial data.

In the Riemannian setting, [4] and [5] establish, respectively, Hölder-stable recovery of a time-independent magnetic and electric potential of the dynamic Schrödinger equation on a simple manifold. These results were extended to simultaneous recovery of both electromagnetic potentials in [2]. In the case of time-dependent potentials in the Riemannian context, the only result is that of [16], establishing, on a simple manifold, the Hölder-stable recovery of both potentials from the knowledge of the Dirichlet-to-Neumann map.

In the case of the vector-valued dynamic Schrödinger equation, there are, to the best of the author's knowledge, no results establishing unique recovery even for time-independent coefficients. However, such results do exist for the related case of the stationary Schrödinger equation. In particular, for the stationary Schrödinger equation on a trivial vector bundle over a Euclidean domain, [13] establishes unique recovery

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of a connection form and potential from the knowledge of the Dirichlet-to-Neumann map. Additionally, for the stationary Schrödinger equation on a Hermitian vector bundle over a two-dimensional Riemann surface, [1] uniquely recovers the coefficients from Cauchy data at the boundary.

Let us also mention the paper [6], where it is conjectured that the Dirichlet-to-Neumann maps for two connection Laplacians coincide in the case of the stationary Schrödinger equation if and only if the associated connection forms are gauge equivalent. The present work solves this conjecture in the non-stationary case. More precisely, we show that the Dirichlet-to-Neumann map uniquely determines, up to gauge invariance, the space- and time-dependent connection form and potential appearing in the dynamic Schrödinger equation on a trivial vector bundle over a Riemannian manifold, provided that the manifold in question satisfies certain geometric conditions.

Finally, we note the works [7], [17], and [9], where various inverse problems for partial differential equations involving connections are considered. In particular, [7] establishes unique recovery of a time-independent unitary Yang-Mills connection on a Hermitian vector bundle, as well as recovering the bundle structure. Similarly, in the case of the wave equation, [17] uses techniques from the boundary control method to reconstruct a Riemannian manifold and Hermitian vector bundle with a time-independent compatible connection from the knowledge of the associated hyperbolic Dirichlet-to-Neumann map, and the work [9] considers the non-linear inverse problem of recovering a time-independent connection from a cubic wave equation on a Hermitian vector bundle over the Minkowski space \mathbb{R}^{1+3} .

Lastly, let us mention the recent work of [19], where the authors recover a time-dependent potential of the wave equation on a trivial vector bundle over a Euclidean domain from knowledge of the input-output operator on the partial boundary.

1.3. Geodesics and Parallel Transport. Let us assume that (M, g) is non-trapping, which is to say that every geodesic in M reaches the boundary in finite time. We now take a moment to recall certain key facts relating to the geodesics in M.

Given $x \in M$ and $\theta \in T_x M$, we denote by $\gamma_{x,\theta}$ the geodesic with initial point x and initial direction θ . We define the sphere bundle of M via

$$SM = \{ (x, \theta) \in TM : |\theta|_a = 1 \}.$$

Likewise, we define the submanifold of inner vectors $\partial_+ SM$ via

$$\partial_+ SM = \{ (x, \theta) \in SM : x \in \partial M, \ \langle \theta, \nu(x) \rangle_{g(x)} < 0 \},\$$

where $\nu(x)$ is the outward pointing unit normal vector at $x \in \partial M$, and we define the submanifold of outer vectors $\partial_{-}SM$ via

$$\partial_{-}SM = \{(x,\theta) \in SM : x \in \partial M, \ \langle \theta, \nu(x) \rangle_{g(x)} > 0\}.$$

Then, for $\gamma_{x,\theta}$ such that $(x,\theta) \in \partial_+ SM$, we can define the exit-time $\rho_+(x,\theta)$ of the geodesic in M by

$$\rho_+(x,\theta) = \min\{s > 0 : \gamma_{x,\theta}(s) \in \partial M\}.$$

Given the above, we further recall the parallel transport equations associated to a connection ∇^A . For any geodesic $\gamma_{x,\theta}$ with $(x,\theta) \in \partial_+ SM$, and any initial vector $w \in E_x$, we consider the parallel transport equation along $\gamma_{x,\theta}$, given by

(1.4)
$$\left[\partial_r + A(\gamma'_{x,\theta}(r))\right]W = 0$$
$$W(0) = w.$$

The transport of $w \in E_x$ along $\gamma_{x,\theta}$ is thus given by W(r).

It is frequently helpful to consider the fundamental matrix solution $U_A : [0, \rho_+(x, \theta)] \to U(n)$ of the parallel transport equation:

(1.5)
$$\begin{bmatrix} \partial_r + A(\gamma'_{x,\theta}(r)) \end{bmatrix} U_A = 0 \\ U_A(0) = \operatorname{Id}. \end{cases}$$

It is clear from the above that the transport of $w \in E_x$ along $\gamma_{x,\theta}$ is given by $W(r) = U_A(r) \cdot w$.

Given $(x, \theta) \in \partial_+ SM$, we define the scattering data for the connection as the map $C_A : \partial_+ SM \to U(n)$ given by

(1.6)
$$C_A(x,\theta) := U_A(\rho_+(x,\theta)).$$

Since elements of the gauge group must satisfy $G(t)|_{\partial M} = \text{Id}$, we note that the scattering data C_A is gauge invariant. We are now in a position to state the main results of the present work.

1.4. Main Results.

Theorem 1. Suppose that for j = 1, 2, we have connection forms $A_j \in C^{\infty}((0,T) \times M; \mathbb{C}^{n \times n} \otimes T^*M)$, and potentials $V_j \in C^{\infty}((0,T) \times M; \mathbb{C}^{n \times n})$. Then $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_2}$ implies that $C_{A_1} = C_{A_2}$.

Theorem 2. Assume the conditions of Theorem 1 hold, and assume further that M is either i) 2-dimensional and simple, or ii) of dimension $m \ge 3$ with strictly convex boundary, and admits a smooth strictly convex function. Then (A_1, V_1) is gauge equivalent to (A_2, V_2) .

These results are, as far as the author is aware, the first dealing with the recovery of coefficients appearing in the dynamic Schrödinger equation on a vector bundle. In fact, the above results are the first showing recovery of time-dependent coefficients of any linear second-order partial differential equation with variable coefficients of leading order, in the vector-valued case. The proof of these results relies on the construction of Gaussian beam solutions which allow recovery of the scattering data corresponding to the connection form. This data is then used to recover the connection form and potential of the Schrödinger equation via the inversion of attenuated ray-transforms with matrix-weights corresponding to the connection form and potential we wish to recover. This last step relies on the results of [20] and [21], which guarantee that the appropriate attenuated ray-transform is invertible when the base manifold is either i) two-dimensional and simple or ii) of higher dimension with strictly convex boundary and admits a smooth strictly convex function.

Here follows an outline of the present work. In section 2, we give some regularity results for the forward problem and for the Neumann trace. In section 3 we construct special Gaussian beam solutions for the Schrödinger equation. The proofs of Theorems 1 and 2 are given in sections 4 and 5 respectively.

2. The Forward Problem

Before we proceed, let us define for all $r, s \in (0, \infty)$ and for X = M or $X = \partial M$ the energy spaces $H^{r,s}((0,T) \times X; E) = H^r(0,T; L^2(X; E)) \cap L^2(0,T; H^s(X; E))$, together with the associated norm

$$\|u\|_{H^{r,s}((0,T)\times X;E)}^{2} = \|u\|_{H^{r}(0,T;L^{2}(X;E))}^{2} + \|u\|_{L^{2}(0,T;H^{s}(X;E))}^{2}$$

The main result of this section is the following well-posedness result for the Dirichlet-to-Neumann map:

Proposition 1. The IBVP (1.3) has a unique solution $u \in C^{\infty}((0,T) \times M; E)$. Further, there exists a constant C > 0 such that the associated Dirichlet-to-Neumann map satisfies the estimate

(2.1)
$$\|\Lambda_{A,V}f\|_{L^2((0,T)\times\partial M;E)} \le C \|f\|_{H^{\frac{9}{4},\frac{3}{2}}((0,T)\times\partial M;E)}.$$

The proof of the above result essentially reduces to proving a suitable energy estimate for the source problem for the Schrödinger equation. Thus, for $F \in C^{\infty}((0,T) \times M; E)$ we consider the solution of the source problem

(2.2)
$$(i\partial_t + \Delta_A + V)u = F(t, x) \text{ in } (0, T) \times M$$
$$u(t, x) = 0 \text{ on } (0, T) \times \partial M,$$
$$u(0, x) = 0 \text{ in } M.$$

Proposition 2. The source problem (2.2) satisfies the energy estimates

$$\begin{aligned} \|u\|_{L^{\infty}(0,T;L^{2}(M;E))} &\leq \|F\|_{L^{2}((0,T)\times M;E)} \\ \|u\|_{L^{\infty}(0,T;H^{1}(M;E))} &\leq \|F\|_{H^{1,0}((0,T)\times M;E)} \\ \|\partial_{t}u\|_{L^{\infty}(0,T;L^{2}(M;E))} &\leq \|F\|_{H^{1,0}((0,T)\times M;E)} \\ \|u\|_{H^{1,2}((0,T)\times M;E)} &\leq C \|F\|_{H^{1,0}((0,T)\times M;E)} . \end{aligned}$$

Proof. By taking the inner product of (2.2) with u and integrating by parts, we deduce that

(2.3)
$$i \int_{M} \langle \partial_{t} u, u \rangle_{E} \, dV_{g} - \int_{M} \left\langle \nabla^{A} u, \nabla^{A} u \right\rangle_{E} \, dV_{g} + \int_{M} \left\langle V u, u \right\rangle_{E} \, dV_{g} = \int_{M} \left\langle F, u \right\rangle_{E} \, dV_{g}$$

Taking the imaginary part of (2.3) yields

$$\frac{d}{dt} \Big(\|u(t)\|_{L^2(M;E)}^2 \Big) \le C \Big(\|F(t,\cdot)\|_{L^2(M;E)} \|u(t)\|_{L^2(M;E)} + \|u(t)\|_{L^2(M;E)}^2 \Big).$$

Then Grönwall's inequality tells us that

(2.4)
$$\|u\|_{L^{\infty}(0,T;L^{2}(M;E))} \leq C \|F\|_{L^{2}((0,T)\times M;E)}$$

On the other hand, taking the inner product of (2.2) with $\partial_t u$, we can integrate by parts to deduce that

(2.5)
$$i \int_{M} \langle \partial_{t} u, \partial_{t} u \rangle_{E} \, dV_{g} - \int_{M} \left\langle \nabla^{A} u, \nabla^{A} \partial_{t} u \right\rangle_{E} \, dV_{g} + \int_{M} \left\langle V u, \partial_{t} u \right\rangle_{E} \, dV_{g} = \int_{M} \left\langle F, \partial_{t} u \right\rangle_{E} \, dV_{g}$$
Then by setting

Then, by setting

$$\alpha(t; u, v) = \int_{M} \left\langle \nabla^{A} u, \nabla^{A} v \right\rangle_{E} dV_{g} - \int_{M} \left\langle V u, v \right\rangle_{E} dV_{g}$$

and

$$\alpha'(t; u, u) = \int_{M} \left\langle (\partial_{t} A)u, \nabla^{A} u \right\rangle_{E} dV_{g} + \int_{M} \left\langle \nabla^{A} u, (\partial_{t} A)u \right\rangle_{E} dV_{g} - \int_{M} \left\langle (\partial_{t} V)u, u \right\rangle_{E} dV_{g},$$

we can take the real part of (2.5) to conclude that

$$\frac{d}{ds}\alpha(s; u, u) = \alpha'(s; u, u) - 2 \operatorname{Re} \int_M \langle F(s, \cdot), \partial_t u(s) \rangle_E dV_g$$

By integrating, we can rewrite the above as

(2.6)
$$\alpha(t;u,u) = \int_0^t \alpha'(s;u,u)ds - 2 \operatorname{Re} \int_0^t \langle F(s,\cdot), \partial_t u(s) \rangle_{L^2(M;E)} ds$$

and since $\int_0^t \langle F(s,\cdot), \partial_t u(s) \rangle_{L^2(M;E)} ds = \langle F(t,\cdot), u(t) \rangle_{L^2(M;E)} - \int_0^t \langle \partial_t F(s,\cdot), u(s) \rangle_{L^2(M;E)} ds$, we can rewrite the identity (2.6) in the form (2.7)

$$\alpha(t; u, u) \le \int_0^t \alpha'(s; u, u) ds + 2 \, \|F(t)\|_{L^2(M; E)} \, \|u(t)\|_{L^2(M; E)} + 2 \int_0^t \|\partial_t F(s, \cdot)\|_{L^2(M; E)} \, \|u(s)\|_{L^2(M; E)} \, ds.$$

Further, by expanding the inner products appearing in $\alpha(t; u, u)$ and using Cauchy's inequality with $\varepsilon = \frac{1}{2}$, we can deduce that

$$\frac{1}{2} \|\nabla u(t)\|_{L^{2}(M; E \otimes T^{*}M)}^{2} \leq \alpha(t; u, u) + \|V(t)\|_{L^{\infty}(\mathbb{C}^{n \times n} \otimes T^{*}M)} \|u(t)\|_{L^{2}(M; E)}^{2} + \|A(t)\|_{L^{\infty}(\mathbb{C}^{n \times n} \otimes T^{*}M)}^{2} \|u(t)\|_{L^{2}(M; E)}^{2}$$

whence

$$\alpha(t; u, u) + \lambda \|u(t)\|_{L^{2}(M; E)}^{2} \ge \frac{1}{2} \|u(t)\|_{H^{1}(M; E)}^{2}$$

for $\lambda = \frac{1}{2} + \|V(t)\|_{L^{\infty}(M;\mathbb{C}^{n\times n}\otimes T^*M)} + \|A(t)\|_{L^{\infty}(M;\mathbb{C}^{n\times n}\otimes T^*M)}^2$. Then, combining the above with identity (2.7) and the definition of $\alpha'(t; u, v)$, we may deduce that $\|u(t)\|_{H^1(M;E)}^2$ is bounded above by

$$(2.8) \quad 2\lambda \|u(t)\|_{L^2(M;E)}^2 + 4 \|F(t)\|_{L^2(M;E)} \|u(t)\|_{L^2(M;E)} + C\Big(\int_0^t \|u(s)\|_{H^1(M;E)}^2 + \|\partial_t F(s,\cdot)\|_{L^2(M;E)}^2 \, ds\Big).$$

Let us briefly recall the inequality

$$\|F(t,\cdot)\|_{L^{2}(M;E)}^{2} = 2\operatorname{Re}\int_{0}^{t} \langle F(s,\cdot),\partial_{t}F(s,\cdot)\rangle_{L^{2}(M;E)} \, ds \leq \int_{0}^{t} \left(\|F(s,\cdot)\|_{L^{2}(M;E)}^{2} + \|\partial_{t}F(s,\cdot)\|_{L^{2}(M;E)}^{2}\right) \, ds.$$

Using the above in (2.8), together with Cauchy's inequality and the estimate (2.4), we deduce that

(2.9)
$$\|u(t)\|_{H^1(M;E)}^2 \le C \Big(\int_0^t \|u(s)\|_{H^1(M;E)}^2 \, ds + \|F\|_{H^{1,0}((0,T)\times M;E)}^2 \Big).$$

Then an application of Grönwall's inequality tells us that

(2.10)
$$\|u\|_{L^{\infty}(0,T;H^{1}(M;E))} \leq C \|F\|_{H^{1,0}((0,T)\times M;E)}.$$

For the next estimate, we begin by applying ∂_t to (2.2). Using the expression (1.2) for the connection Laplacian, we deduce that

$$(2.11) \qquad (i\partial_t + \Delta_A + V)\partial_t u = \partial_t F - 2(\partial_t A, du)_g + (\partial_t d^* A)u - (\partial_t A, Au)_g - (A, (\partial_t A)u)_g - (\partial_t V)u.$$

We now apply the estimate (2.4) to $\partial_t u$, replacing F appearing in (2.4) by the right-hand side of (2.11). We deduce, therefore, that

$$\begin{aligned} &\|\partial_{t}u\|_{L^{\infty}(0,T;L^{2}(M;E))} \\ \leq C \|\partial_{t}F - 2(\partial_{t}A, du)_{g} + (\partial_{t}d^{*}A)u - (\partial_{t}A, Au)_{g} - (A, (\partial_{t}A)u)_{g} - (\partial_{t}V)u\|_{L^{2}((0,T)\times M;E)}. \end{aligned}$$

Using the estimates (2.4) and (2.10), on the right-hand side of the above, we observe that

(2.12)
$$\|\partial_t u\|_{L^{\infty}(0,T;L^2(M;E))} \le C \|F\|_{H^{1,0}((0,T)\times M;E)}.$$

Lastly, we rearrange (2.2) to obtain

(2.13)
$$(\Delta_A + V)u = F - i\partial_t u \text{ in } (0, T) \times M$$
$$u = 0 \text{ on } (0, T) \times \partial M.$$

Then the bounds (2.4), (2.10) and (2.12) immediately imply the desired energy estimate

(2.14)
$$\|u\|_{H^{1,2}((0,T)\times M;E)} \le C \|F\|_{H^{1,0}((0,T)\times M;E)}.$$

We now turn to the proof of Proposition 1.

Proof of Proposition 1. The unique solvability of the IBVP (1.3) can be established in a similar manner to the scalar valued case, using the energy estimates of Proposition 2. Existence and uniqueness can then be proven using, for example, the Galerkin approach (see e.g. [15, Theorem 2.3] or [18, Section 3, Theorem 10.1]).

We now turn to establishing the bound (2.1) for the Dirichlet-to-Neumann map. Recall the initial and boundary value problem (1.3):

$$\begin{aligned} (i\partial_t + \Delta_A + V)u &= 0 \text{ in } (0,T) \times M, \\ u(t,x) &= f \text{ on } (0,T) \times \partial M, \\ u(0,x) &= 0 \text{ in } M, \end{aligned}$$

where the inhomogeneous Dirichlet data is given by $f \in C^{\infty}((0,T) \times \partial M; E)$ satisfying $f|_{t=0} = \partial_t f|_{t=0} = 0$.

Note that we can find $\Phi \in C^{\infty}((0,T) \times M; E)$ such that

$$\Phi(0, \cdot) = \partial_t \Phi(0, \cdot) = 0$$
 in M, $\Phi = f$ on ∂M ,

and

(2.15)
$$\|\Phi\|_{H^{3,2}((0,T)\times M;E)} \le C \|f\|_{H^{\frac{9}{4},\frac{3}{2}}((0,T)\times\partial M;E)}$$

for some C > 0, depending only on M and T. See [18, Chapter 4, Section 2] for a proof of this fact in the scalar case. The proof for vectors is analogous and, therefore, omitted. From the above, it holds that

(2.16)
$$F := -(i\partial_t + \Delta_A + V)\Phi$$

satisfies $F(0, \cdot) = 0$ in M. Then, letting v be the solution of (2.2) corresponding to the source term F defined in (2.16), we see that $u = \Phi + v$ is a solution to (1.3). Then, it follows by (2.14) that

$$\|u\|_{H^{1,2}((0,T)\times M;E)} \le C \|f\|_{H^{\frac{9}{4},\frac{3}{2}}((0,T)\times\partial M;E)}$$

and applying this estimate with f = 0 implies that such a solution u is unique. Finally, we observe that

$$\|\Lambda_{A,V}f\|_{L^{2}((0,T)\times\partial M;E)} \leq \|u\|_{H^{1,2}((0,T)\times M;E)} \leq C \|\Phi\|_{H^{3,2}((0,T)\times M;E)} \leq C \|f\|_{H^{\frac{9}{4},\frac{3}{2}}((0,T)\times\partial M;E)}.$$

3. Construction of Gaussian Beam Solutions

In this section, we shall construct Gaussian beam solutions to the Schrödinger equation which concentrate along geodesics in the high frequency limit.

Let (\widehat{M}, g) be a closed manifold. Recall that for a geodesic segment $\gamma: (a, b) \to \widehat{M}$ with no closed loops, there exist only finitely many values of $r \in (a, b)$ for which γ self-intersects at $\gamma(r)$. We begin by recording the following system of Fermi coordinates near a geodesic, which we shall later use to construct our Gaussian beam solutions.

Lemma 1. Let (\widehat{M},g) be a compact m-dimensional manifold without boundary, $m \geq 2$, and assume that $\gamma:(a,b)\to \widehat{M}$ is a unit-speed geodesic with no closed loops. Given a closed sub-interval $[a_0,b_0]$ of (a,b) such that $\gamma|_{[a_0,b_0]}$ self-intersects only at $\gamma(r_j)$ with $a_0 < r_1 < \cdots < r_K < b_0$, and setting $r_0 = a_0$, $r_{N+1} = b_0$, there exists an open cover $\{U_j, \phi_j\}_{j=0}^{K+1}$ of $\gamma([a_0, b_0])$ consisting of coordinate neighbourhoods with the following properties:

- $\phi_j(U_j) = I_j \times B$, where I_j are open intervals and $B = B(0, \delta')$ is an open ball in \mathbb{R}^{m-1} , where δ' can be taken arbitrarily small.
- $\phi_j(\gamma(r)) = (r, 0)$ for $r \in I_j$
- r_j only belongs to I_j and $\overline{I_j} \cap \overline{I_k} = \emptyset$, unless $|j-k| \le 1$ $\phi_j = \phi_k$ on $\phi_j^{-1}((I_j \cap I_k) \times B)$

Furthermore, the metric in these coordinates satisfies $g^{jk}|_{\gamma(r)} = \delta^{jk}$ and $\partial_i g^{jk}|_{\gamma(r)} = 0$.

Proof. See e.g. [11, Lemma 3.5] for details.

We now turn to the construction of the Gaussian beam solutions. We consider here a non-tangential unit-speed geodesic in M given by $\gamma: [0, L] \to M$. That is, $\gamma'(0)$ and $\gamma'(L)$ are both non-tangential to ∂M , and $\gamma(r) \in M^{int}$ for 0 < r < L. Note that this implies that the geodesic γ is not a closed loop in M.

We may then embed (M,g) in some closed manifold \widehat{M} , and extend γ to \widehat{M} as a unit-speed geodesic $\gamma: [-\varepsilon, L+\varepsilon] \to \widehat{M}$. Our aim is to construct a Gaussian beam solution near $\gamma([0, L])$. We fix a point x_0 on γ , and apply Lemma 1 on \widehat{M} with $a_0 < 0$ and $b_0 > L$ chosen so that $\gamma(a_0)$ and $\gamma(b_0)$ are in the interior of $\widehat{M} \setminus M$. This gives us a system of coordinates (r, y) around $x_0 = (r_0, 0)$, defined in a set $U = \{(r, y) : |r - r_0| < \delta, |y| < \delta'\}$ such that the geodesic near x_0 is given by $\Gamma = \{(r, 0) : |r - r_0| < \delta\}$.

The main aim of the present section is to establish the following result.

Proposition 3. Let (M, g) be non-trapping, let $\gamma : [0, L] \to M$ be a non-tangential geodesic, and let $s \gg 1$. There exists a function $v \in C^{\infty}((0,T) \times M; E)$, supported in a tubular neighbourhood of γ , which is an approximate solution of the Schrödinger equation in the sense that

(3.1)
$$\|(i\partial_t + \Delta_A + V)v\|_{L^2((0,T) \times M;E)} = O(s^{-1}), \qquad \|v\|_{L^2((0,T) \times M;E)} = O(1).$$

Away from the self-intersections of γ , this approximate solution is given by $v(t,r,y) = e^{is(\Psi(r,y)-st)}a(t,r,y)$, where the phase function Ψ has the form $\Psi(r, y) = r + \frac{1}{2}H(r)y \cdot y + O(|y|^3)$ and the amplitude satisfies

$$a(t,r,0) = c_0 \tilde{\chi}(t) e^{-\frac{1}{2} \int_{r_0}^r \operatorname{tr} H(\tilde{r}) d\tilde{r}} U_A w + O(s^{-1})$$

for some choices of arbitrary constant c_0 , compactly supported smooth function $\tilde{\chi}$ and initial vector w.

Proof. The first step is the local construction of such an approximate solution. That is, we wish to construct a solution v of the Schrödinger equation in U, with the form

$$v = e^{is(\Psi(r,y) - st)}a(s;t,r,y),$$

where $\Psi \in C^{\infty}(M; \mathbb{C})$, $a \in C^{\infty}((0, T) \times M; E)$ are given near Γ , with a supported in $\{|y| < \delta'/2\}$. For convenience, we shall supress the dependence on s of the amplitude function a.

In practice, we will determine the functions Ψ, a by solving certain Eikonal and transport equations up to Nth order on Γ , much as one would in the analogous construction for the stationary Schrödinger equation (see e.g. [6, Theorem 5.4]). As a result of using the phase $e^{is(\Psi-st)}$, we obtain the same Eikonal equation for Ψ as in [6], but derive easier transport equations for the amplitude a (compared to [6, Theorem 5.4], where a satisfies transport equations of $\overline{\partial}$ -type).

Therefore, let us begin by deriving these Eikonal and transport equations. Recalling the expression (1.2) for the connection Laplacian, we first compute the Schrödinger operator applied to v:

(3.2)
$$(i\partial_t + \Delta_A + V)v = e^{is(\Psi - st)}(i\partial_t + \Delta_A + V)a + s^2 e^{is(\Psi - st)} \left(1 - (d\Psi, d\Psi)_g\right)a + 2ise^{is(\Psi - st)} \left((d\Psi, \nabla^A a)_g + \frac{1}{2}(\Delta_g \Psi)a\right).$$

In light of the above, we seek Ψ satisfying the Eikonal equation

(3.3) $(d\Psi, d\Psi)_g - 1 = 0 \text{ to } N \text{th order on } \Gamma,$

exactly as in the case of the stationary Schrödinger equation. On the other hand, the amplitude a should satisfy, up to a small error, the transport equation

$$s\Big((d\Psi,\nabla^A a)_g + \frac{1}{2}(\Delta_g\Psi)a\Big) - \frac{i}{2}(i\partial_t + \Delta_A + V)a = 0 \quad \text{to } N\text{th order on } \Gamma.$$

Thus, we begin by seeking a solution Ψ of (3.3) having the form $\Psi = \sum_{j=0}^{N} \Psi_j$, where

$$\Psi_j(r,y) = \sum_{|\alpha|=j} \frac{\Psi_{j,\alpha}(r)}{\alpha!} y^{\alpha}$$

Let us also write the metric in the form $g^{jk} = \sum_{l=0}^{N} g_l^{jk} + r_{N+1}^{jk}$, where

$$g_l^{jk}(r,y) = \sum_{|\beta|=l} \frac{g_{l,\beta}^{j\kappa}(r)}{\beta!} y^{\beta}, \quad r_{N+1}^{jk} = O(|y|^{N+1}).$$

By the properties of the Fermi coordinates, we observe that $g_0^{jk} = \delta^{jk}$ and $g_1^{jk} = 0$. Thus, we can immediately choose $\Psi_0(r) = r$ and $\Psi_1(r, y) = 0$. Then, for j, k = 1...m and $\alpha, \beta = 2...m$, we have

$$g^{jk}\partial_{j}\Psi\partial_{k}\Psi - 1 = (1 + g_{2}^{11} + \cdots)(1 + \partial_{r}\Psi_{2} + \cdots)(1 + \partial_{r}\Psi_{2} + \cdots) + 2(g_{2}^{1\alpha} + \cdots)(1 + \partial_{r}\Psi_{2} + \cdots)(\partial_{y^{\alpha}}\Psi_{2} + \cdots) + (\delta^{\alpha\beta} + g_{2}^{\alpha\beta} + \cdots)(\partial_{y^{\alpha}}\Psi_{2} + \partial_{y^{\alpha}}\Psi_{3} + \cdots)(\partial_{y^{\beta}}\Psi_{2} + \partial_{y^{\beta}}\Psi_{3} + \cdots) - 1 = [2\partial_{r}\Psi_{2} + \nabla_{y}\Psi_{2} \cdot \nabla_{y}\Psi_{2} + g_{2}^{11}] + \sum_{p=3}^{N} \left[2\partial_{r}\Psi_{p} + 2\nabla_{y}\Psi_{2} \cdot \nabla_{y}\Psi_{p} + \sum_{l=0}^{p} g_{l}^{11} \sum_{\substack{j+k=p-l\\0\leq j,k$$

In the last equality, we have chosen to collect the terms into homogeneous polynomials in y (so that the first term is the second degree part of the right-hand side, and the rest are the parts of degree $p = 3, \ldots, N$). We first choose Ψ_2 such that the second-degree term $[2\partial_r\Psi_2 + \nabla_y\Psi_2 \cdot \nabla_y\Psi_2 + g_2^{11}]$ vanishes.

To this end, we choose $\Psi_2(r, y) = \frac{1}{2}H(r)y \cdot y$, where H is a smooth, symmetric, complex matrix solving the matrix Riccati equation

(3.5)
$$H'(r) + H^2(r) = F(r),$$

and F(r) is the symmetric matrix such that $g_2^{11}(r, y) = -F(r)y \cdot y$. If we impose some initial condition $H(r_0) = H_0$ on this equation, where H_0 is chosen to be a complex symmetric matrix with $\text{Im}(H_0)$ positive definite, then [14, Lemma 2.56] implies that the matrix Riccati equation above has a unique smooth symmetric solution H(r), for which Im(H(r)) is positive definite.

We now choose Ψ_3 so that the term corresponding to p = 3 in the right-hand side of (3.4) vanishes. We obtain the equation

$$2\partial_r \Psi_3 + 2\nabla_y \Psi_2 \cdot \nabla_y \Psi_3 = F(r, y),$$

where F is a third-order polynomial in y which only depends on Ψ_2 and g. This gives us a linear system of first-order ODEs for the Taylor coefficients $\Psi_{3,\alpha}(r)$, which can be solved uniquely if we prescribe some initial conditions at r_0 . We may, then, repeat this argument in order to obtain Ψ_4, \ldots, Ψ_N by solving ODEs on Γ , given initial conditions at r_0 .

Thus, we have $\Psi(r, y) = r + \frac{1}{2}H(r)y \cdot y + \tilde{\Psi}$, where $\tilde{\Psi} = O(|y|^3)$. We now turn to finding the amplitude *a* such that, up to a small error, we have

$$s\Big((d\Psi,\nabla^A a)_g + \frac{1}{2}(\Delta_g\Psi)a\Big) - \frac{i}{2}(i\partial_t + \Delta_A + V)a = 0 \quad \text{to } N\text{th order on }\Gamma.$$

We choose a of the form

$$a = s^{\frac{m-1}{4}} (a_0 + s^{-1}a_1 + \dots + s^{-N}a_N) \chi(y/\delta')$$

where χ is a smooth function such that $\chi = 1$ for $|y| \leq 1/4$ and $\chi = 0$ for $|y| \geq 1/2$. Letting $\eta = \Delta_g \Psi$, it is enough to find a_j such that

$$(d\Psi, \nabla^A a_0)_g + \frac{1}{2}\eta a_0 = 0 \quad \text{to } N \text{th order on } \Gamma$$

$$(d\Psi, \nabla^A a_1)_g + \frac{1}{2}\eta a_1 - \frac{i}{2}(i\partial_t + \Delta_A + V)a_0 = 0 \quad \text{to } N \text{th order on } \Gamma$$

$$(3.6)$$

$$(d\Psi, \nabla^A a_N)_g + \frac{1}{2}\eta a_N - \frac{i}{2}(i\partial_t + \Delta_A + V)a_{N-1} = 0 \quad \text{to } N\text{th order on } \Gamma.$$

We can write $\eta = \sum_{l=0}^{N} \eta_l + r_{N+1}$ and $a_0 = a_{00} + \cdots + a_{0N}$, where each η_l , a_{0l} is a homogeneous polynomial of order j in y, and the remainder r_{N+1} is $O(|y|^{N+1})$. Thus, writing $A = A(\gamma')dr + A(\partial_{y^{\alpha}})dy^{\alpha}$, we can rewrite the transport equation for a_0 in the system (3.6) above as

$$(1+g_{2}^{11}+\cdots)\left(1+\partial_{r}\Psi_{2}+\cdots\right)\left(\partial_{r}+A(\gamma')\right)\left(a_{00}+a_{01}+\cdots\right) + \left(g_{2}^{1\alpha}+\cdots\right)\left(1+\partial_{r}\Psi_{2}+\cdots\right)\left(\partial_{y^{\alpha}}+A(\partial_{y^{\alpha}})\right)\left(a_{00}+a_{01}+\cdots\right) + \left(g_{2}^{\alpha 1}+\cdots\right)\left(1+\partial_{y^{\alpha}}\Psi_{2}+\cdots\right)\left(\partial_{r}+A(\gamma')\right)\left(a_{00}+a_{01}+\cdots\right) + \left(\delta^{\alpha\beta}+g_{2}^{\alpha\beta}+\cdots\right)\left(\partial_{y^{\alpha}}\Psi_{2}+\partial_{y^{\alpha}}\Psi_{3}+\cdots\right)\left(\partial_{y^{\beta}}+A(\partial_{y^{\beta}})\right)\left(a_{00}+a_{01}+\cdots\right) + \frac{1}{2}(\eta_{0}+\eta_{1}+\cdots)(a_{00}+a_{01}+\cdots) = 0.$$

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We can then write $A(\theta) = \sum_{l=0}^{N} A_l(\theta) + R_{N+1}(\theta)$, where the entries of each $A_l(\theta)$ are homogeneous polynomials in y of order l, and the remainder $R_{N+1}(\theta)$ is $O(|y|^{N+1})$. As a result, we deduce that the left-hand side of (3.7) becomes

$$\partial_r a_{00} + A_0(\gamma')a_{00} + \frac{1}{2}\eta_0 a_{00} + \partial_r a_{01} + A_0(\gamma')a_{01} + A_1(\gamma')a_{00} + \nabla_y \Psi_2 \cdot \nabla_y a_{01} + \nabla_y \cdot A_0(\gamma')a_{00} + \frac{1}{2}\eta_0 a_{01} + \frac{1}{2}\eta_1 a_{00} + \cdots$$

We wish to find a_{00} such that the first line of the above expression vanishes. To this end, we note that $\eta_0(r) = \Delta_g \Psi(r, 0) = g^{\alpha\beta} \partial_{y^{\alpha}} [H_{\alpha\beta} y^{\alpha}] = \operatorname{tr} H(r)$. We therefore choose a_{00} such that

$$\partial_r a_{00} + A_0 (\gamma'(r)) a_{00} + \frac{1}{2} (\operatorname{tr} H(r)) a_{00} = 0.$$

Since $A_0(t, r) = A(t, r, 0)$, this equation has the solution

$$a_{00}(t,r) = c_0 \tilde{\chi}(t) e^{-\frac{1}{2} \int_{r_0}^r \operatorname{tr} H(\tilde{r}) d\tilde{r}} \cdot U_A w_A$$

where $\tilde{\chi} \in C_0^{\infty}((\tau, T - \tau))$ satisfies $\tilde{\chi} = 1$ on $[2\tau, T - 2\tau]$, $0 \leq \tilde{\chi} \leq 1$ and $\|\tilde{\chi}\|_{W^{k,\infty}(\mathbb{R})} \leq C_k \tau^{-k}$ with C_k independent of τ , and where w is some arbitrary initial vector and c_0 is some initial constant which determines the value of $a_{00}(t, r_0)$. Since it will simplify later calculations, we choose the value

(3.8)
$$c_0 = \frac{\sqrt[4]{\det \operatorname{Im}(H(r_0))}}{\sqrt{\left(\int_{\mathbb{R}^{m-1}} e^{-|y|^2} dy\right)}} = \frac{\sqrt[4]{\det \operatorname{Im}(H(r_0))}}{\pi^{\frac{m-1}{4}}}.$$

We note that the constructed a_{00} is smooth in time, since the connection form A is smooth in time and the transport equation for a_{00} is well-posed. We can then obtain the rest of a_{01}, \ldots, a_{0N} by solving linear first-order ODEs. The sections a_1, \ldots, a_N may then be determined in much the same manner as a_0 , so that the transport equations in (3.6) are satisfied to Nth degree on Γ , and the smooth time-dependence of the amplitude a follows from the smooth time-dependence of the connection form and the well-posedness of the transport equations that determine a.

Thus, we have constructed a function $v = e^{is(\Psi - st)}a$ in U such that:

$$\Psi(r,y) = r + \frac{1}{2}H(r)y \cdot y + \tilde{\Psi}, \text{ where } \tilde{\Psi} = O(|y|^3),$$

$$a(t,r,y) = s^{\frac{m-1}{4}}(a_0 + s^{-1}a_1 + \dots + s^{-N}a_N)\chi(y/\delta'),$$

$$a_0(t,r,0) = c_0\tilde{\chi}(t)e^{-\frac{1}{2}\int_{r_0}^r \operatorname{tr} H(\tilde{r})d\tilde{r}} U_A w.$$

We now turn to establishing the bounds (3.1) for the function v in U. Henceforth, we shall choose N = 5. To begin with, note that

$$\left| e^{is(\Psi - st)} \right| = e^{-s \operatorname{Im} \Psi} = e^{-\frac{1}{2}s \operatorname{Im}(H(r))y \cdot y} e^{-sO(|y|^3)}.$$

Note also that $\text{Im}(H(r))y \cdot y \ge c |y|^2$ for $(r, y) \in U$, where the constant c > 0 depends on H_0 and the value of δ appearing in the definition of U. Thus, it follows for $(r, y) \in U$ that, provided y is sufficiently small, we have

$$\left|e^{is(\Psi(r,y)-st)}\right| \le Ce^{-\frac{1}{4}cs|y|^2}$$

From this fact, together with the definition of $\chi(y/\delta')$ and $\tilde{\chi}(t)$, we can deduce that for r in a compact interval, possibly after decreasing δ' , we have

$$|v(t,r,y)| \le Cs^{\frac{m-1}{4}}e^{-\frac{1}{4}cs|y|^2}\chi(y/\delta'), \quad s \gg 1.$$

As a result, we have that for $s \gg 1$,

(3.9)
$$\|v\|_{L^2((0,T)\times U;E)} \le C \left\|s^{\frac{m-1}{4}}e^{-\frac{1}{4}cs|y|^2}\right\|_{L^2((0,T)\times U;E)} = O(1).$$

Further, since Ψ , a satisfy (3.3) and (3.6) to 5th order on Γ , we deduce from (3.2) that

$$(i\partial_t + \Delta_A + V)v| \le Cs^{\frac{m-1}{4}}e^{-\frac{1}{4}cs|y|^2}\chi(y/\delta')\left(s^2|y|^6 + s^{-5}\right).$$

It follows, then, that

$$(3.10) \|(i\partial_t + \Delta_A + V)v\|_{L^2((0,T)\times U;E)} \lesssim \left\|s^{\frac{m-1}{4}}e^{-\frac{1}{4}cs|y|^2} \left(s^2|y|^6 + s^{-5}\right)\right\|_{L^2((0,T)\times U;E)} = O(s^{-1}),$$

and by the same method we may further deduce that

(3.11)
$$\|(i\partial_t + \Delta_A + V)v\|_{H^{1,0}((0,T)\times U;E)} \lesssim \|s^{\frac{m-1}{4}}e^{-\frac{1}{4}cs|y|^2} (s^2|y|^6 + s^{-3})\|_{L^2((0,T)\times U;E)} = O(s).$$

Thus, we have established the result of Proposition 3 for v in U. It remains to show that we can construct an approximate Gaussian beam solution v in M by gluing together the approximate solutions which we have constructed in U.

To this end, recall the definition of the sets U_j from Lemma 1. Fix δ' and choose an open cover U_0, \ldots, U_K of $\gamma([a_0, b_0])$, with each U_j corresponding to an interval I_j , as in Lemma 1. We first find a function $v^{(0)} = e^{is(\Psi^{(0)} - st)}a^{(0)}$ in U_0 following the method above, with some fixed initial conditions at r_0 for the ODEs that determine $\Psi^{(0)}$ and $a^{(0)}$. We continue by choosing some $\tilde{r}_1 \in I_0 \cap I_1$ so that $\gamma(\tilde{r}_1) \in U_0 \cap U_1$, and construct $v^{(1)} = e^{is(\Psi^{(1)} - st)}a^{(1)}$ in U_1 again by the above method, choosing our initial conditions for $\Psi^{(1)}$, $a^{(1)}$ at \tilde{r}_1 to be, respectively, the values of $\Psi^{(0)}$ and $a^{(0)}$ at \tilde{r}_1 . In this manner, we can proceed to determine $v^{(K)}$. We choose a partition of unity $\{\rho_j(r)\}$ for $[a_0, b_0]$ corresponding to the family of intervals $\{I_j\}$, and let $\tilde{\rho}_j(r, y) = \rho_j(r)$ in U_j . We can then define

$$v = \sum_{j=0}^{K} \tilde{\rho}_j v^{(j)}.$$

Since the ODEs for the phases and amplitudes have the same initial value in U_j as in U_{j+1} , we can deduce that $v^{(j)} = v^{(j+1)}$ in $U_j \cap U_{j+1}$. Therefore, we conclude that the L^2 -scale bounds (3.9) for v and (3.10)-(3.11) for $(i\partial_t + \Delta_A + V)v$ follow with U = M from the corresponding bounds in U_l for each $v^{(l)}$.

Before we proceed, we note also the following partition, which shall be useful later. Suppose that $p_1, \ldots, p_{K'}$ are the distinct points where the geodesic self-intersects, $0 < r_1 < \cdots < r_K < L$ are the times where the geodesic self-intersects, and $V_1, \ldots, V_{K'}$ are balls centered at $p_1, \ldots, p_{K'}$ respectively. Note that in each V_j , the function v is given by

$$v|_{V_j} = \sum_{\gamma(t_l) = p_j} v^{(l)}.$$

By considering the individual sets $U_l \setminus (\bigcup_{j=1}^{K'} V_j)$, we can then choose a finite cover $\{W_1, \dots, W_{K''}\}$ of the remaining points of $(\bigcup_{l=1}^{K} U_l) \setminus (\bigcup_{j=1}^{K'} V_j)$, where each W_k is a subset of U_{l_k} for some l_k . In particular, we can choose W_k which remain away from $\gamma(r_{l_k})$, so that for small enough δ' it follows that

$$v|_{W_k} = v^{(l_k)}$$

Together, the V_i and W_k cover form a cover

(3.12)
$$\operatorname{supp}(v) \cap M \subset \left(\cup_{j=1}^{K'} V_j \right) \cup \left(\cup_{k=1}^{K''} W_k \right).$$

4. Determination of the Scattering Data

Let us begin by establishing the following useful Lemma, which will aid us in the proof of Theorem 1.

Lemma 2. Given any connection form A in M, there is a gauge-equivalent connection form \tilde{A} with the additional property that $\tilde{A}(\nu)|_{\partial M} = 0$.

Proof. Let us use boundary normal coordinates in a neighbourhood of ∂M (see e.g. [14, Section 2.1]) given by $(y,r) \in \partial M \times [0,\varepsilon]$, with $\varepsilon > 0$. Then for $u \in C^{\infty}(M)$ it holds that $\partial_r u|_{\partial M} = -\partial_{\nu} u|_{\partial M}$. Further, in these coordinates, we can decompose the connection form as $A = A(\nu)dr + A(\partial_{u^{\alpha}})dy^{\alpha}$. We fix some $\mu \in C^{\infty}([0,\varepsilon];\mathbb{R})$ such that $\mu(r) = 1$ for r near 0 and $\mu(r) = 0$ for r near ε . We may then define a local gauge $G \in C^{\infty}((0,T) \times \partial M \times [0,\varepsilon];\mathbb{C}^{n \times n})$ via

$$G(r, \cdot) = e^{-r\mu(r)A(\nu)(r, \cdot)}$$

where the dependence on t and y is left implicit. Note that, since A is skew-Hermitian, it follows that G is unitary, and since $G(r) = \text{Id for } r \text{ near } \varepsilon$, we can extend G to an element of $C^{\infty}((0,T) \times M; U(n))$. Moreover, observe that

$$G|_{\partial M} = \mathrm{Id}, \quad \partial_r G|_{\partial M} = -A(\nu)|_{\partial M}$$

Let \tilde{A} denote the gauge transform of A by G, that is

$$\tilde{A} = G^{-1}AG + G^{-1}dG$$

and observe that

$$\tilde{A}(\nu)|_{\partial M} = A(\nu)|_{\partial M} - A(\nu)|_{\partial M} = 0.$$

Since the data Λ_{A_j,V_j} and C_{A_j} are gauge invariant, Lemma 2 implies that we may assume without any loss of generality that $A_1(\nu)|_{\partial M} = A_2(\nu)|_{\partial M} = 0$ in the proofs of Theorems 1 and 2. With this in mind, we proceed to the proof of Theorem 1.

For j = 1, 2, we construct Gaussian beam solutions u_j of the Schrödinger equations

(4.1)
$$(i\partial_t + \Delta_{A_j} + V_j)u_j(t, x) = 0 \text{ in } (0, T) \times M, u_1(0, \cdot) = u_2(T, \cdot) = 0 \text{ in } M.$$

To this end, we fix some geodesic $\gamma_{x,\theta}$ in \widehat{M} for $x \in \partial M$, and choose a system of Fermi coordinates along $\gamma_{x,\theta}$ as in Lemma 1. Using the work of the previous section, we construct approximate solutions v_j in M having the form $e^{is(\Psi-st)}a^{(A_j)}$ away from the self-intersections of $\gamma_{x,\theta}$. We can turn these v_j into exact solutions $u_j = v_j + R_j$ by solving

$$(i\partial_t + \Delta_{A_j} + V_j)R_j = -(i\partial t + \Delta_{A_j} + V_j)v_j \text{ in } (0,T) \times M,$$

$$R_j = 0 \text{ on } (0,T) \times \partial M,$$

$$R_1(0,\cdot) = R_2(T,\cdot) = 0 \text{ in } M.$$

Note that for $s \gg 1$, (3.10) and the energy estimate (2.4) yield

(4.2)
$$\|R_j\|_{L^2((0,T)\times M;E)} \le C \|(i\partial_t + \Delta_{A_j} + V_j)v_j\|_{L^2((0,T)\times M;E)} = O(s^{-1}),$$

whereas (3.11) and the energy estimate (2.14) yield

$$||R_j||_{H^{0,2}((0,T)\times M;E)} \le C ||(i\partial_t + \Delta_{A_j} + V_j)v_j||_{H^{1,0}((0,T)\times M;E)} = O(s).$$

Interpolating between (4.2) and the above, we conclude that for $s \gg 1$ we have

(4.3)
$$\|R_j\|_{H^{0,1}((0,T)\times M;E)} + s \|R_j\|_{L^2((0,T)\times M;E)} = O(1).$$

We then set $\phi_j = u_j$ on $(0,T) \times \partial M$ and consider $\omega \in H^{1,2}((0,T) \times M; E)$ the solution of

(4.4)
$$(i\partial_t + \Delta_{A_2(t)} + V_2(t, x))\omega(t, x) = 0 \text{ in } (0, T) \times M,$$
$$\omega(t, x) = \phi_1 \text{ on } (0, T) \times \partial M,$$
$$\omega(0, \cdot) = 0 \text{ in } M.$$

We observe that the difference $\omega - u_1$ solves the following Schrödinger equation:

(4.5)
$$(i\partial_t + \Delta_{A_2} + V_2)(\omega - u_1) = 2(A_1 - A_2, du_1)_g + Qu_1 \text{ in } (0, T) \times M \omega - u_1 = 0 \text{ on } (0, T) \times \partial M, \omega(0, x) - u_1(0, x) = 0 \text{ in } M,$$

where $Qu_1 = (V_1 - V_2)u_1 + (A_1, A_1u_1)_g - (A_2, A_2u_1)_g - (d^*A_1)u_1 + (d^*A_2)u_1.$

 \Box

Taking the Hermitian inner product of the above equation with u_2 , we deduce that

(4.6)
$$\int_{0}^{T} \int_{M} \langle 2(A_{1} - A_{2}, du_{1})_{g} + Qu_{1}, u_{2} \rangle_{E} \, dV_{g} dt = \int_{0}^{T} \int_{\partial M} \langle \partial_{\nu}(\omega - u_{1}), u_{2} \rangle_{E} \, d\sigma_{g} dt.$$

As a result of Lemma 2, we may assume without loss of generality that $A_1(\nu)|_{\partial M} = A_2(\nu)|_{\partial M} = 0$. Therefore, the right-hand side of the above is bounded by

$$(4.7) \left| \int_{0}^{T} \int_{\partial M} \left\langle \partial_{\nu}(\omega - u_{1}), u_{2} \right\rangle_{E} d\sigma_{g} dt \right| \leq C \left\| \left(\Lambda_{A_{1},V_{1}} - \Lambda_{A_{2},V_{2}} \right) \phi_{1} \right\|_{L^{2}((0,T) \times \partial M;E)} \left\| \phi_{2} \right\|_{L^{2}((0,T) \times \partial M;E)} \\ \leq C \left\| \Lambda_{A_{1},V_{1}} - \Lambda_{A_{2},V_{2}} \right\| \left\| \phi_{1} \right\|_{H^{\frac{9}{4},\frac{3}{2}}((0,T) \times \partial M;E)} \left\| \phi_{2} \right\|_{L^{2}((0,T) \times \partial M;E)},$$

which vanishes when $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_2}$. On the other hand, since $du_1 = is(d\Psi)u_1 + e^{is(\Psi - st)}da^{(A_1)} + dR_1$, the left-hand side of (4.6) can be written as

$$\begin{split} \int_{0}^{T} \int_{M} \left\langle 2(A_{1} - A_{2}, du_{1})_{g} + Qu_{1}, u_{2} \right\rangle_{E} dV_{g} dt &= is \int_{(0,T) \times M} \left\langle 2\left((A_{1} - A_{2}), (d\Psi)u_{1}\right)_{g}, u_{2} \right\rangle_{E} dV_{g} dt \\ &+ \int_{(0,T) \times M} \left\langle 2\left((A_{1} - A_{2}), e^{is(\Psi - st)}da^{(A_{1})} + dR_{1}\right)_{g} + Qu_{1}, u_{2} \right\rangle_{E} dV_{g} dt. \end{split}$$

We can divide the above by s and use the bounds (4.3) and (3.9) to deduce that

(4.8)
$$\left| \int_{(0,T)\times M} \left\langle \left((A_1 - A_2), (d\Psi)u_1 \right)_g, u_2 \right\rangle_E dV_g dt \right| \\ \leq s^{-1} \left| \int_{(0,T)\times M} \left\langle \left((A_1 - A_2), du_1 \right)_g + Qu_1, u_2 \right\rangle_E dV_g dt \right| + O(s^{-1}).$$

By combining (4.6), (4.7), and (4.8), we conclude that when $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_2}$ it follows that

$$\left| \int_{(0,T)\times M} \left\langle \left((A_1 - A_2), (d\Psi)u_1 \right)_g, u_2 \right\rangle_E dV_g dt \right| \le O(s^{-1}).$$

Thus, we can let $s \to \infty$ in the right-hand side of the above to conclude that

(4.9)
$$\lim_{s \to \infty} \int_{(0,T) \times M} \left\langle \left((A_1 - A_2), (d\Psi) u_1 \right)_g, u_2 \right\rangle_E dV_g dt = 0.$$

We now make use of the following Lemma:

Lemma 3.

$$\lim_{s \to \infty} \int_{(0,T) \times M} \left\langle \left((A_1 - A_2), (d\Psi) u_1 \right)_g, u_2 \right\rangle_E dV_g dt$$
$$= \int_0^T \tilde{\chi}^2 \int_0^{\rho_+(x,\theta)} \left\langle \left(A_1 \left(\gamma'_{x,\theta}(r) \right) - A_2 \left(\gamma'_{x,\theta}(r) \right) \right) U_{A_1} w_1, U_{A_2} w_2 \right\rangle_E dr dt.$$

Before proving the above result, we first conclude the proof of Theorem 1. Since $\tau \in (0, T/4)$ in the definition of $\tilde{\chi}$ is arbitrary, we deduce that

(4.10)
$$\int_{0}^{\rho_{+}(y,\theta)} \left\langle \left(A_{1}(\gamma_{x,\theta}'(r)) - A_{2}(\gamma_{x,\theta}'(r)) \right) U_{A_{1}}w_{1}, U_{A_{2}}w_{2} \right\rangle_{E} dr = 0.$$

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Note that the scattering data C_{A_j} takes values in U(n), so that we can define $C_{A_j}^{-1}$ as the matrix inverse of C_{A_j} . Making use of (1.1), (1.5), and (1.6), it can be shown that

$$(4.11) \left\langle \left(C_{A_2}^{-1} C_{A_1} - \mathrm{Id} \right) w_1, w_2 \right\rangle_E = \int_0^{\rho_+(y,\theta)} \partial_r \left\langle U_{A_1} w_1, U_{A_2} w_2 \right\rangle_E dr = \int_0^{\rho_+(y,\theta)} \left\langle \nabla_{\gamma'_{x,\theta}}^{A_2} U_{A_1} w_1, U_{A_2} w_2 \right\rangle_E dr + \int_0^{\rho_+(y,\theta)} \left\langle U_{A_1} w_1, \nabla_{\gamma'_{x,\theta}}^{A_2} U_{A_2} w_2 \right\rangle_E dr = \int_0^{\rho_+(y,\theta)} \left\langle \left(A_2 (\gamma'_{x,\theta}(r)) - A_1 (\gamma'_{x,\theta}(r)) \right) U_{A_1} w_1, U_{A_2} w_2 \right\rangle_E dr.$$

However, (4.10) and (4.11) imply that $\langle (C_{A_2}^{-1}C_{A_1} - \mathrm{Id})w_1, w_2 \rangle_E = 0$. Since w_1, w_2 were arbitrary, we conclude that $C_{A_2}^{-1}C_{A_1} = \mathrm{Id}$, whence $C_{A_1} = C_{A_2}$ and the proof of Theorem 1 is complete. It remains for us to provide a proof of Lemma 3.

Proof of Lemma 3. By considering a partition of unity subordinate to the open cover (3.12), it is sufficient to consider $A_1 - A_2$ compactly supported in $(V_j \cap M)$ or $(W_k \cap M)$. We shall first prove the latter case. Recall that $a^{(A_j)} = s^{\frac{m-1}{4}}(a_0 + O(s^{-1}))\chi(y/\delta')$. We can observe that

$$\begin{split} \lim_{s \to \infty} & \int_{(0,T) \times M} \left\langle \left((A_1 - A_2), (d\Psi) u_1 \right)_g, u_2 \right\rangle_E dV_g dt \\ &= \lim_{s \to \infty} \int_0^T \int_0^{\rho_+(x,\theta)} \int_{\mathbb{R}^{m-1}} e^{-s \operatorname{Im}(H(r))y \cdot y} e^{sO(|y|^3)} s^{\frac{m-1}{2}} \chi^2(y/\delta') \times \\ & \left[\left\langle \left(A_1(t,r,y) - A_2(t,r,y), d\Psi(t,r,y) a_0^{(A_1)}(t,r,y) \right)_g, a_0^{(A_2)}(t,r,y) \right\rangle_E + O(s^{-1}) \right] |g(r,y)|^{\frac{1}{2}} dy dr dt. \end{split}$$

We make the substitution $y \mapsto s^{-\frac{1}{2}}y$ above, and recall that $d\Psi|_{y=0} = dr$ is dual via the Riemannian metric to $\gamma'_{x,\theta}(r)$. Then, since $\operatorname{Im}(H)$ is positive definite and δ' is small, we note that the exponential term involving $\operatorname{Im}(H)$ dominates the others. Thus, we can conclude that the right-hand side of the above is given by

$$\int_{0}^{T} \int_{0}^{\rho_{+}(x,\theta)} \left(\int_{\mathbb{R}^{m-1}} e^{-\operatorname{Im}(H(r))y \cdot y} dy \right) \left\langle \left(A_{1}(\gamma'_{x,\theta}) - A_{2}(\gamma'_{x,\theta}) \right) a_{0}^{(A_{1})}(t,r,0), a_{0}^{(A_{2})}(t,r,0) \right\rangle_{E} |g(r,0)|^{\frac{1}{2}} dr dt.$$

Evaluating the integral in y and using that |g(r,0)| = 1, we rewrite the above integral in the form

$$\int_{0}^{T} \tilde{\chi}^{2} \Big(\int_{\mathbb{R}^{m-1}} e^{-|y|^{2}} dy \Big) \int_{0}^{\rho_{+}(x,\theta)} \frac{|c_{0}|^{2} e^{-\int_{r_{0}}^{r} \operatorname{tr} \operatorname{Re}(H(s)) ds}}{\sqrt{\det \operatorname{Im}(H(r))}} \left\langle \Big(A_{1}(\gamma_{x,\theta}') - A_{2}(\gamma_{x,\theta}') \Big) U_{A_{1}} w_{1}, U_{A_{2}} w_{2} \right\rangle_{E} dr dt.$$

We now use the fact that, according to [14, Lemma 2.58], matrices which solve the Riccati equation (3.5) have the property

$$\det \operatorname{Im}(H(r)) = \det \operatorname{Im}(H(r_0))e^{-2\int_{r_0}^r \operatorname{tr}\operatorname{Re}(H(s))ds}.$$

This fact, together with our choice of constant c_0 in (3.8) is sufficient to conclude that

$$\lim_{s \to \infty} \int_{(0,T) \times M} \left\langle \left((A_1 - A_2), (d\Psi) u_1 \right)_g, u_2 \right\rangle_E dV_g dt$$
$$= \int_0^T \tilde{\chi}^2 \int_0^{\rho_+(y,\theta)} \left\langle \left(A_1(\gamma'_{x,\theta}(r)) - A_2(\gamma'_{x,\theta}(r)) \right) U_{A_1} w_1, U_{A_2} w_2 \right\rangle_E dr dt$$

when $A_1 - A_2$ is compactly supported in $W_k \cap M$. For $A_1 - A_2$ compactly supported in $V_j \cap M$, we can write $v = \sum_{\gamma(t_l)=p_j} v^{(l)}$. Thus, the limit has the form

$$\begin{split} \lim_{s \to \infty} \int_{(0,T) \times M} \left\langle \left((A_1 - A_2), (d\Psi) u_1 \right)_g, u_2 \right\rangle_E dV_g dt \\ &= \sum_{\gamma(t_l) = p_j} \lim_{s \to \infty} \int_{(0,T) \times M} \left\langle \left((A_1 - A_2), (d\Psi) v_1^{(l)} \right)_g, v_2^{(l)} \right\rangle_E dV_g dt \\ &+ \sum_{\substack{l \neq l' \\ \gamma(t_l) = \gamma(t_{l'}) = p_j}} \lim_{s \to \infty} \int_{(0,T) \times M} \left\langle \left((A_1 - A_2), (d\Psi) v_1^{(l)} \right)_g, v_2^{(l')} \right\rangle_E dV_g dt. \end{split}$$

We observe that the first sum converges to the required limit by the same computions used to prove the limit in $W_k \cap M$, and the second sum vanishes via stationary phase arguments as in the proof of [11, Proposition 3.1], thus completing the proof of Lemma 3.

5. Proof of Gauge Equivalence

In what follows, we denote by $\varphi_r(x,v)$ the geodesic flow given by $\varphi_r(x,v) = (\gamma_{x,v}(r), \gamma'_{x,v}(r)) \in TM$. Additionally, we denote by X the geodesic vector field on M, which satisfies $\partial_r(F \circ \varphi_r) = X(F) \circ \varphi_r$, for any function $F: SM \mapsto \mathbb{C}^n$.

For a function $F: SM \to \mathbb{C}^n$ and a connection 1-form $B: TM \mapsto \mathbb{C}^{n \times n}$, we can consider the transport equation:

(5.1)
$$\begin{aligned} Xw + Bw &= -F \text{ in } SM \\ w|_{\partial_{-}SM} = 0, \end{aligned}$$

where B acts on $w: SM \to \mathbb{C}^n$ by multiplication at each point. In the present work, we shall consider only the cases $F(x,\theta) = f(x)$ or $F(x,\theta) = \alpha_j(x)\theta^j$, where $f, \alpha_j: M \to \mathbb{C}^n$. Using the transport equation (5.1), we can define the attenuated ray transform of F with attenuation due to B (see e.g. [21, Section 1]), via:

(5.2)
$$I_B F = w|_{\partial_+ SM}.$$

We can now proceed to the proof of Theorem 2 below.

Proof of Theorem 2. In order to prove Theorem 2, we further assume that M is either i) 2-dimensional and simple, or ii) of dimension $m \ge 3$ with strictly convex boundary and admits the existence of a smooth strictly convex function $\phi : M \to \mathbb{R}$. We note that this second condition is conjectured to be true for all simple manifolds (amongst others, see e.g. [21, Section 2] for discussion), although the question is still open at present.

Consider the candidate gauge $G(t) = U_{A_1(t)}(U_{A_2(t)})^{-1}$. Note that G(t) is unitary, $G(t) : SM \to U(n)$, and depends smoothly on time. Further, since $C_{A_1} = C_{A_2}$, it holds that $G(t)|_{\partial_+SM} = \text{Id}$, and therefore that $G(t)|_{\partial SM} = \text{Id}$. We can observe that

(5.3)
$$\begin{aligned} XG + A_1G - GA_2 &= 0\\ G|_{\partial SM} &= \mathrm{Id} \,. \end{aligned}$$

It remains to show that G(t) depends only on the base-point $x \in M$, and further that G(t) is smooth in M. Note that (5.3) is equivalent to

(5.4)
$$X(G - \mathrm{Id}) + A_1(G - \mathrm{Id}) - (G - \mathrm{Id})A_2 = A_2 - A_1 (G - \mathrm{Id})|_{\partial SM} = 0$$

We can henceforth fix some $t \in (0,T)$ and define a new connection form B via $B(W) = A_1W - WA_2$ for $W: SM \to \mathbb{C}^{n \times n}$. Then (5.4) becomes

(5.5)
$$X(G - \mathrm{Id}) + B(G - \mathrm{Id}) = A_2 - A_1$$
$$(G - \mathrm{Id})|_{\partial SM} = 0.$$

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We can interpret the above equation as a ray transform (see e.g. [21, Section 1]), as $I_B(A_1 - A_2) = 0$, where I_B denotes the ray transform with attenuation due to the connection form B. We emphasize here that A_1 , A_2 and B are smooth, whereas we have not yet shown that G(t) in (5.5) is smooth.

We now apply either [20, Theorem 1.3] if M is 2-dimensional and simple, or [21, Theorem 1.6] if M is of dimension $m \ge 3$ with strictly convex boundary and admits a smooth strictly convex function. Thus, we conclude that $A_2(t) - A_1(t) = \nabla^{B(t)}p(t)$ for some smooth function $p(t) : M \to \mathbb{C}^{n \times n}$, and (5.5) then implies that $G(t) = \mathrm{Id} + p(t)$. Hence, we have shown that G(t) depends only on the base-point $x \in M$, and the smoothness of G(t) in M follows from the smoothness of p(t). Therefore, G satisfies all the necessary conditions to be a gauge, and it follows from (5.3) that A_1 is gauge-equivalent to A_2 via the gauge-transform $A_2 = G^{-1}dG + G^{-1}A_1G$, as required.

We now turn to showing unique determination of the potential. Note that since $A_2 = G^{-1}dG + G^{-1}A_1G$, it remains only to show that $V_2 = G^{-1}V_1G + iG^{-1}\partial_tG$. We define $V_3 = G^{-1}V_1G + iG^{-1}\partial_tG$. By gauge invariance, it holds that $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_3}$. Thus, by the assumption $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_2}$ it follows that

(5.6)
$$\Lambda_{A_2,V_3} = \Lambda_{A_2,V_2}.$$

It remains to show that condition (5.6) implies that $V_2 = V_3$. Note that we can take $A_1 = A_2 = A$ in (4.6), and use the fact that $\Lambda_{A,V_3} = \Lambda_{A,V_2}$ to deduce that

$$\int_{(0,T)\times M} \left\langle (V_3 - V_2)u_1, u_2 \right\rangle_E \circ \varphi_r(y,\theta) dV_g dt = 0.$$

We can let $s \to \infty$ in the left-hand side above, and applying the argument of Lemma 3 we deduce that

$$\int_0^T \tilde{\chi}^2(t) \int_0^{\rho_+(y,\theta)} \langle (V_3 - V_2) U_A w_1, U_A w_2 \rangle_E \circ \varphi_r(y,\theta) dr dt = 0$$

Since the choice of $\tilde{\chi}$ is arbitrary, we conclude that for $V = V_3 - V_2$ and for each $t \in (0, T)$ and $(y, \theta) \in \partial_+ SM$ we have

$$\int_{0}^{\rho_{+}(y,\theta)} \langle VU_{A}w_{1}, U_{A}w_{2} \rangle_{E} \circ \varphi_{r}(y,\theta) dr = 0,$$

where $\varphi_r(y,\theta)$ is the geodesic flow defined by $\varphi_r(y,\theta) = (\gamma_{y,\theta}(r), \gamma'_{y,\theta}(r))$. Then, by linearity, we conclude that

(5.7)
$$\int_0^{\rho_+(y,\theta)} U_A^{-1} V U_A \circ \varphi_r(y,\theta) dr = 0.$$

In order to finish the proof, we wish to interpret (5.7) as an attenuated ray transform.

For $W \in C^{\infty}(SM; \mathbb{C}^{n \times n})$, we can define the map BW = AW - WA = [A, W]. Since $A(x, v) = A_j(x)v^j$, we can observe that $BW(x, v) = [A_j(x), W]v^j$.

Recall that the inner product $\langle \cdot, \cdot \rangle_E$ on the trivial bundle $E = M \times \mathbb{C}^n$ induces an inner product on the endomorphism bundle $\operatorname{End}(E) = M \times \mathbb{C}^{n \times n}$ via $\langle X, Y \rangle_{\operatorname{End}(E)} = \operatorname{tr}(X^*Y)$. Note that we can regard the 1-form B as a connection form on $\operatorname{End}(E)$. Further, by the cyclic property of trace, we observe that

$$\langle BX,Y\rangle_{\operatorname{End}(E)} = \operatorname{tr}(-X^*AY + AX^*Y) = \operatorname{tr}(-X^*AY + X^*YA) = \langle X, -BY\rangle_{\operatorname{End}(E)},$$

and that B is, therefore, a unitary connection on the endomorphism bundle.

Letting X once again denote the geodesic vector field, we consider ω the solution of the transport equation

(5.8)
$$X\omega + B\omega = -V \text{ in } SM, \quad \omega = 0 \text{ on } \partial_{-}SM.$$

Using (5.8), we now observe that

$$X(U_{A}^{-1}\omega U_{A}) = U_{A}^{-1}A\omega U_{A} + U_{A}^{-1}(-V - B\omega)U_{A} - U_{A}^{-1}\omega AU_{A} = -U_{A}^{-1}VU_{A}$$

The above then implies that $\partial_r[(U_A^{-1}\omega U_A)\circ\varphi_r(y,\theta)] = -U_A^{-1}VU_A\circ\varphi_r(y,\theta)$, using the definitions of the geodesic vector field and geodesic flow. We can integrate this last expression to obtain

(5.9)
$$-\int_0^{\rho_+(y,\theta)} U_A^{-1} V U_A \circ \varphi_r(y,\theta) dr = (U_A^{-1} \omega U_A) \circ \varphi_{\rho_+(y,\theta)}(y,\theta) - (U_A^{-1} \omega U_A) \circ \varphi_0(y,\theta).$$

Note that $\varphi_{\rho_+(y,\theta)}(y,\theta) \in \partial_- SM$ and $\varphi_0(y,\theta) = (y,\theta) \in \partial_+ SM$. By recalling that $U_A = Id$ on $\partial_+ SM$ and $\omega = 0$ on $\partial_- SM$, we observe that the right-hand side of (5.9) is just $-\omega|_{\partial_+ SM}$.

Therefore, (5.7) and (5.8) tell us that

$$I_B V = \omega|_{\partial_+ SM} = \int_0^{\rho_+(y,\theta)} U_A^{-1} V U_A \circ \varphi_r(y,\theta) dr = 0.$$

Hence, if M is 2-dimensional and simple, we can apply [20, Theorem 1.3] to conclude that the above implies V = 0. On the other hand, if M is of dimension $m \ge 3$ with strictly convex boundary and admits a smooth strictly convex function, we instead apply the result of [21, Theorem 1.6] to conclude that V = 0. Thus, it holds that $V_2 = G^{-1}V_1G + iG^{-1}\partial_t G$, and hence (A_1, V_1) is gauge-equivalent to (A_2, V_2) , as required. \Box

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