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# Rectifiable sets and the Traveling Salesman Problem*

Peter W. Jones
Department of Mathematics, Yale University, New Haven, CT 06520, USA

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## §1. Introduction

Let $K \subset \mathbb{C}$ be a bounded set. In this paper we shall give a simple necessary and sufficient condition for $K$ to lie in a rectifiable curve. We say that a set is a rectifiable curve if it is the image of a finite interval under a Lipschitz mapping. Recall that for a connected set $\Gamma \subset \mathbb{C}$, $\Gamma$ is a rectifiable curve (not necessarily simple) if and only if $l(\Gamma) < \infty$, where $l(\cdot)$ denotes one dimensional Hausdorff measure. This classical result follows from the fact that on any finite graph there is a tour which covers the entire graph and which crosses each edge (but not necessarily each vertex!) at most twice. If $K$ is a finite set then we are essentially reduced to the classical Traveling Salesman Problem (TSP): Compute the length of the shortest Hamiltonian cycle which hits all points of $K$. This is the same, up to a constant multiple, as asking for the infimum of $l(\Gamma)$ where $\Gamma$ is a curve, $K \subset \Gamma$. (Such a $\Gamma$ is called a spanning tree in TSP theory.) For infinite sets $K$, we cannot hope in general to have $K$ be a subset of a Jordan curve. What we should therefore look at is connected sets which contain $K$.

Let $\Gamma_{\min}$ be the shortest (minimal) spanning tree. Then we cannot possibly solve our problem for sets $K$ of infinite cardinality if we cannot find $\Gamma$, $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$, for any finite set $K$. (Here and throughout the paper $C$, $C_0$, $C_1$, $c_0$, etc. denote various universal constants.) While there are several algorithms for computing $l(\Gamma_{\min})$, these algorithms work for finite graphs satisfying the triangle inequality, and do not use the Euclidean properties of $K$. (See [13] for an excellent discussion of some of these algorithms.) Therefore these methods cannot solve our problem for general infinite $K$. We present a method which is a minor modification of a well-known algorithm ("Farthest Insertion" – see [13]) which yields a $\Gamma$ with $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$. The Farthest Insertion algorithm has been extensively studied with large numerical calculations on computers, and is experimentally good in the sense that the $\Gamma$ produced satisfy $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$ for all examples which have

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been calculated. If $K$ contains $N$ points our method could be modified to calculate (up to a constant multiple) $l(\Gamma_{\mathrm{opt}})$ in time $0(N \log N)$, which is what one would expect and which is available for other algorithms [17]. We can thus say that, in a certain sense, our theorem gives a geometric "solution" to the TSP.

A square $Q \subset \mathbb{C}$ is a dyadic square if $Q = [j2^{-n}, (j+1)2^{-n}] \times [k2^{-n}, (k+1)2^{-n}]$, where $j, k, n \in \mathbb{Z}$. We denote by $l(Q) = 2^{-n}$ the sidelength of $Q$. For $\lambda > 0$ we denote by $\lambda Q$ the square with the same center as $Q$, with sidelength $\lambda l(Q)$, and with sides parallel to the axes. For $Q$ a dyadic square let $S_Q$ be an infinite strip (or line in the degenerate case) of smallest possible width which contains $K \cap 3Q$, and let $\omega(Q)$ denote the width of $S_Q$. We then define

$$\beta_K(Q) = \frac{\omega(Q)}{l(Q)},$$

so that $\beta_K(Q)$ measures in a scale invariant way $(0 \leqq \beta_K(Q) \leqq 3)$ the deviation of $K$ from a straight line near $Q$ on scale $l(Q)$. Our main result states that $K$ is contained in a rectifiable curve $\Gamma$ if and only if

$$\beta^2(K) \equiv \sum_Q \beta_K^2(Q)\, l(Q) < \infty \tag{1.1}$$

where the sum is taken over all dyadic squares (or equivalently over all dyadic squares $Q$ with $l(Q) \leqq$ diameter $(K)$). Furthermore, the shortest possible $\Gamma$ has length comparable to diameter $(K)$ plus the sum in (1.1). This is an elementary (if quite lengthy) computation with the Pythagorean theorem if $\beta_K(Q) < \varepsilon_0$ for all $Q$ ($\varepsilon_0$ sufficiently small) or if $K$ already lies in a Lipschitz curve (defined in Section 2). However, if $\beta_K(Q)$ is large for many $Q$ ($K$ is "dispersed") there are technical difficulties to be overcome. Conversely, if $\Gamma$ is a Lipschitz curve it is not hard to prove (and is already in [9]) that $\beta^2(\Gamma)$ is bounded by a multiple of $l(\Gamma)$.

**Theorem 1.** *If $\Gamma \subset \mathbb{C}$ is connected, then*

$$\beta^2(\Gamma) \leqq C_0\, l(\Gamma). \tag{1.2}$$

*Conversely, if $\beta^2(K) < \infty$ there is a connected set $\Gamma$, $K \subset \Gamma$, such that*

$$l(\Gamma) \leqq (1+\delta)\,\mathrm{diameter}\,(K) + C(\delta)\,\beta^2(K). \tag{1.3}$$

**Corollary 1.** *If $K$ is an analytic set and $l(K) < \infty$, then $K$ is totally unrectifiable in the sense of Besicovitch* ([4], [5]) *if and only if*

$$\beta^2(E) = \infty$$

*for every $E \subset K$ with $l(E) > 0$.*

For certain purposes (see e.g. [1]) it is useful to know that the curve constructed for (1.3) can have extra properties. We will show that by taking $\delta$ large enough there is $\Gamma$, $K \subset \Gamma$, such that (1.3) holds, and

(1.4) If $z_1, z_2 \in \Gamma$ there is a connected set $\gamma \subset \Gamma$ such that $z_1, z_2 \in \gamma$ and $l(\gamma) \leqq C_0 |z_1 - z_2|$.

(1.5) If $K \cap Q \neq \phi$ and $l(Q) \leqq$ diameter $(K)$, there is an infinite strip $S_Q$ with width $\beta(Q)\, l(Q)$, with axis $L_Q$, $S_Q \cap K \cap Q \neq \phi$, and such that if $P$ denotes orthogonal projection onto $L_Q$,

$$P(3Q) \subset P(\Gamma \cap S_Q).$$

Condition (1.4) says that $\Gamma$ is uniformly locally connected and (1.5) asserts that $\Gamma$ "crosses" $S_Q \cap 3Q$.

To prove the first theorem we will use a result which is of some independent interest. An $M$ Lipschitz domain of size one centered at the origin is a simply connected domain whose boundary is a Jordan curve described by $r(\theta)e^{i\theta}$, $0 \leqq \theta \leqq 2\pi$, where $\frac{1}{1+M} \leqq r(\theta) \leqq 1$, and where $|r(\theta_1) - r(\theta_2)| \leqq M\,|e^{i\theta_1} - e^{i\theta_2}|$. An $M$ Lipschitz domain is a dilate and translate of one of the above domains. Lipschitz domains play a central rôle in modern one complex variable theory, and their behavior vis-à-vis Cauchy integrals and harmonic measure is well understood.

**Theorem 2.** *If $\Omega$ is a simply connected domain and $l(\partial\Omega) < \infty$, there is a rectifiable curve $\Gamma$ such that $\Omega \backslash \Gamma = \bigcup_j \Omega_j$ is a decomposition of $\Omega$ into disjoint $C_0$ Lipschitz domains, and*

$$\sum_j l(\partial\Omega_j) \leqq C_0\, l(\partial\Omega).$$

We thank Charles Pugh for pointing out that the above theorem fails if we replace $C_0$ by $\varepsilon_0 \ll 1$. Indeed if one tiles (in the above sense) the unit square by $\varepsilon_0$ Lipschitz domains $\Omega_j$, $\sum l(\partial\Omega_j) = \infty$. Now let $\Gamma$ be a bounded connected set and attach to $\Gamma$ a line segment $L$ and circle $S$. Applying Theorem 2 to $\mathbb{C}\backslash(\Gamma \cup L \cup S)$ we obtain

**Corollary 2.** *If $\Gamma$ is connected there is a connected set $\tilde{\Gamma}$ such that $\Gamma \subset \tilde{\Gamma}$, $l(\tilde{\Gamma}) \leqq C_0\, l(\Gamma)$, every bounded component of $\mathbb{C}\backslash\tilde{\Gamma}$ is a $C_0$ Lipschitz domain, and the unbounded component of $\mathbb{C}\backslash\tilde{\Gamma}$ is the complement of a disk. Furthermore, if $x, y \in \tilde{\Gamma}$ there is a subarc $\gamma \subset \tilde{\Gamma}$ such that $x, y \in \gamma$ and $l(\gamma) \leqq C_1|x - y|$. The constant $C_1$ may be taken to be* 3.

The theorem's proof can be modified to yield further structure on $\Gamma$ if (1.1) holds in some uniform sense. A curve $\Gamma$ is said to be Ahlfors-David regular (AD) if

$$\sup_{\substack{z_0 \in \mathbb{C} \\ r > 0}} r^{-1}\, l(\{z \in \Gamma : |z - z_0| \leqq r\}) < \infty .$$

Then one can show that $K$ lies in an AD curve if and only if

$$\sup_Q l(Q)^{-1} \sum_{Q' \subset Q} \beta_K^2(Q')\, l(Q') < \infty , \tag{1.6}$$

and this therefore gives us a characterization of AD curves. If $\Gamma$ is an $\varepsilon$ quasicircle passing through $\infty$, $\beta_\Gamma(Q) \leqq C_0\varepsilon^{1/2}$ for all $Q$. Conversely, our construction can be modified to show that if $\beta_K(Q) \leqq \varepsilon < \varepsilon_0$ for all $Q$, $K$ is a subset of a $\delta(\varepsilon)$ quasicircle passing through $\infty$.

The history behind our proof of Theorem 1 is at first glance a bit surprising, in that it was discovered by a careful study of Cauchy integral operators on Lipschitz curves. If $\Gamma$ is a Lipschitz curve, a famous theorem due to A.P. Calderón [2] (for small Lipschitz constants) and Coifman, McIntosh and Meyer [3] later, (for any Lipschitz curve) asserts that the Cauchy integral operator is bounded on the Lebesgue space $L^2(\Gamma)$. By now there are myriad proofs of this result. The theorem is very old when $\Gamma = \mathbb{R}$, and is equivalent there to the $L^2$ boundedness of the Hilbert transform. This is essentially easy because one can apply Plancherel's theorem. The author had attempted to find a proof of the result for Lipschitz curves by using the intuitive idea that a Lipschitz curve should look like a straight line at most places on most scales. This was accomplished in [9] by noting that (1.6) holds for Lipschitz curves.

Further reflection shows that the above connection between Cauchy integrals and the TSP should not be so surprising. Let $K$ have finite one-dimensional Hausdorff measure, $l(K) < \infty$. An old problem is to decide whether $\gamma(K)$, the analytic capacity of $K$, is zero. In other words, is there a non-constant, bounded holomorphic function on $\mathbb{C}\setminus K$? (See [6] or [15] for more on $\gamma(K)$.) An easy necessary condition is that $l(K) > 0$. An old conjecture is that $\gamma(K) > 0$ if and only if $\operatorname{Fav}(K) > 0$, where $\operatorname{Fav}(\cdot)$ is Favard length,

$$\operatorname{Fav}(K) = \int_0^\pi l(K_\theta)\, d\theta\,,$$

and where $K_\theta$ is the orthogonal projection of $K$ onto the line $\mathbb{R}e^{i\theta}$. This is known to be false for sets where $K$ has non sigma finite length [11, 15], but is still open when $l(K) < \infty$. The idea Murai has used [11, 15] to attack this problem is to use deep estimates on the $L^2$ bounds for Cauchy integrals on Lipschitz curves. Now sets where $0 < l(K) < \infty$ and $\operatorname{Fav}(K) = 0$ play a central rôle in Besicovitch's theory of rectifiable sets (see e.g. [4, 5]). Besicovitch proved this occurs if and only if $l(K \cap \Gamma) = 0$ for every rectifiable curve $\Gamma$. Such sets are called totally irregular. Combining this with Calderón's theorem on the Cauchy integral [2], it follows that $l(K) < \infty$ and $\operatorname{Fav}(K) > 0$ imply $\gamma(K) > 0$. This connection between Cauchy integrals, geometric measure theory, and the TSP seems all the more natural when one considers that the Cauchy integral also plays a vital part in Borsuk's proof of the Jordan curve theorem. In [1] Christopher Bishop and the author use Theorem 1 and the machinery developed for Cauchy integrals to settle some conjectures about harmonic measure.

The paper is organized as follows. In Section 2 we use an argument with conformal mappings to prove Theorem 2. Section 3 uses Theorem 2 to give a proof of (1.2). Section 4 is devoted to a construction of the curve $\Gamma$ which satisfies (1.3)–(1.5). Section 5 is an appendix where we give an elementary proof that (1.2) holds for Lipschitz curves. That result (first proved by Fourier analysis in [9]) is used in Section 2.

The author is grateful to Stephen Semmes for many conversations on condition (1.1). I believed at one time that I had a counter-example to (1.3). When Stephen

Semmes repeatedly asked me to write it down, I found an error, and Theorem 1 soon appeared.

## §2. Proof of Theorem 2

We assume the reader is familiar with corona type constructions as presented e.g. in the book of Garnett [7]. Let $F\colon \mathbb{D} \to \Omega$ be any choice of a Riemann map of the unit disk onto $\Omega$. Since the theorem is trivial if $l(\partial\Omega) = \infty$, we assume $l(\partial\Omega) < \infty$ so that $F'(z)$ is in the Hardy space $H^1$. (See [11].) Let $G(z) = (F'(z))^{1/2}$ so that $G \in H^2$, and apply either Green's formula or Plancherel to obtain

$$\int_{\mathbb{T}} |G(e^{i\theta}) - G(0)|^2\, d\theta = \iint_{\mathbb{D}} |G'(z)|^2 \log\frac{1}{|z|}\, dxdy\,.$$

From this we see that if we set $F' = G^2 = e^{\varphi}$,

$$\iint_{\mathbb{D}} |F'(z)|\,|\varphi'(z)|^2 \log\frac{1}{|z|}\, dxdy \leqq 4\int_{\mathbb{T}} |F'(e^{i\theta})|\, d\theta \leqq 8\, l(\partial\Omega)\,. \tag{2.1}$$

Now it is well known that $\varphi$ is in the Bloch space $B$ and of norm at most 6, i.e.

$$|\varphi'(z)| \leqq 6(1 - |z|^2)^{-1}, \qquad z \in \mathbb{D}\,. \tag{2.2}$$

See e.g. [16]. Let us decompose $T$ into dyadic arcs, i.e. arcs of length $\pi 2^{-n}$. For such a dyadic arc $I$ we associate to it a dyadic "square"

$$Q = Q_I = \left\{ z\colon \frac{z}{|z|} \in I,\ 1 - \pi 2^{-n} \leqq |z| \leqq 1 \right\}.$$

Let $T_Q = \{z \in Q\colon 1 - 2\pi^{-n} \leqq |z| \leqq 1 - \pi 2^{-n-1}\}$ denote the "top half" of $Q$. Let $z_Q$ be the center of $T_Q$.

Fix a dyadic square $Q$ and perform the following stopping time argument. If there is $z \in T_Q$ with $|\varphi(z) - \varphi(z_Q)| \geqq \varepsilon$, stop and let $\mathcal{D}_Q = T_Q$. In this case we say $Q$ is of type 0. Otherwise let $Q_1, Q_2, \ldots$ be those dyadic squares inside $Q$ which satisfy

$$\sup_{z \in T_{Q_j}} |\varphi(z) - \varphi(z_Q)| \geqq \varepsilon\,,$$

and define

$$\mathcal{D}_Q = Q \setminus \bigcup_{j=1}^{\infty} Q_j\,.$$

Then if $\mathcal{D}_Q$ is not of Type 0,

$$|\varphi(z) - \varphi(z_Q)| \leqq \varepsilon, \qquad z \in \mathcal{D}_Q\,. \tag{2.3}$$

By the construction of $\mathcal{D}_Q$ we see that $\partial\mathcal{D}_Q \equiv \gamma_Q$ is a chord arc domain:

If $z_1, z_2 \in \gamma_Q$, there is an arc $\gamma \subset \gamma_Q$ joining $z_1$ to $z_2$ and of length $l(\gamma) \leqq 6\, |z_1 - z_2|$.

Using this procedure we tile $\mathbb{D}$ by regions $\mathscr{D}_Q$ via the usual stopping time arguments. If a region $\mathscr{D}_Q$ so formed is not of Type 0, we say $\mathscr{D}_Q$ is of Type 1 if $l(T \cap \gamma_Q) \geqq \frac{1}{2} l(T \cap \partial Q)$, and we say $\mathscr{D}_Q$ is of Type 2 otherwise. Then if $\mathscr{D}_Q$ is of Type 1 or 2,

$$| F'(z) - F'(z_Q) | \leqq 2\varepsilon | F'(z_Q) |$$

whenever $z \in \mathscr{D}_Q$.

We first bound the Type 0 regions. By normal families and (2.2),

$$\int_{\gamma_Q} |F'(z)| \, ds(z) \leqq C \iint_{T_Q} |F'(z)| \, |\varphi'(z)|^2 \log \frac{1}{|z|} \, dxdy \, .$$

whenever $\mathscr{D}_Q = T_Q$ is of Type 0. Consequently, inequality (2.1) yields

$$\sum_{\text{Type } 0} \int_{\gamma_Q} | F'(z) | ds(z) \leqq C l(\partial \Omega) \, ,$$

because the regions $\mathscr{D}_Q$ are disjoint. Now let $N_0$ be a suitably large integer (in fact, $N_0 = 2$ will do) and divide each $\mathscr{D}_Q = T_Q \in$ Type 0 into $4^{N_0}$ "squares" of essentially equal sidelength. Then if $\tilde{\mathscr{D}}_Q$ is any of these "squares", (2.2) shows that $F(\tilde{\mathscr{D}}_Q)$ is a $C_0$ Lipschitz domain. By our last estimate,

$$\sum_{\mathscr{D}_Q \in \text{Type } 0} \sum_{\tilde{\mathscr{D}}_Q} l(\partial F(\tilde{\mathscr{D}}_Q)) \leqq C_0 \, l(\partial \Omega) \, .$$

We now turn to the regions of Type 1. Since the sets $T \cap \partial Q_j$ and $T \cap \partial Q_k$ can intersect in at most two points if $j \neq k$, and since (2.3) holds,

$$\begin{aligned}
&\sum_{\text{Type } 1} \int_{\gamma_Q} | F'(z) | ds(z) \\
&\leqq \sum_{\text{Type } 1} 12(1 + 2\varepsilon) \int_{T \cap \gamma_Q} | F'(z) | \, ds(z) \\
&\leqq 12(1 + 2\varepsilon) \int_T | F'(z) | \, ds(z) \\
&= 12(1 + 2\varepsilon) \, l(\partial \Omega) \, .
\end{aligned} \tag{2.4}$$

The Type 2 regions are a little trickier to bound. The following is the main idea of this section: Use the $L^2$ bounds of (2.1) to bound the Type 2 regions. Let $I_1, I_2, \ldots$ be the horizontal line segments in $\gamma_Q \backslash \partial Q$. Then by hypothesis,

$$\sum_{j=1}^{\infty} l(I_j) \geqq \frac{1}{12} l(\gamma_Q) \, . \tag{2.5}$$

By normal families and estimate (2.2) there is $\delta > 0$ such that

$$l(\{ z \in I_j \colon |\varphi(z) - \varphi(z_Q)| \geqq \delta \}) \geqq \delta l(I_j) \, . \tag{2.6}$$

(Because $\sup_{z \in T(Q_j)} |\varphi(z) - \varphi(z_Q)| \geqq \varepsilon$.) Now let $\omega$ denote harmonic measure for the region $\mathscr{D}_Q$ with respect to the base point $z_Q$, and let $g(z)$ denote Green's function for $\mathscr{D}_Q$ with pole at $z_Q$ so that

$$d\omega = \frac{1}{2\pi}\frac{\partial g}{\partial n}\,ds\,.$$

Then since $\mathscr{D}_Q$ is a chord-arc domain, the results of Jerison and Kenig [8] show that $\omega$ is an $A_\infty$ weight, i.e. there are constants $A, \eta, \eta' > 0$ so that

$$A^{-1}\left(\frac{l(E)}{l(\gamma_Q)}\right)^{\eta} \leqq \omega(E) \leqq A\left(\frac{l(E)}{l(\gamma_Q)}\right)^{\eta'} \tag{2.7}$$

for any Borel set $E \subset \gamma_Q$. (The usual definition of $A_\infty$ uses only the second inequality, but it is well known that this is equivalent to the first [7].) Combining (2.3) with (2.5)–(2.7) we see that

$$\int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega \geqq c\,|F'(z_Q)|\,,$$

where $c = c(\delta, A, \eta) > 0$. Green's formula is valid for chord arc domains, so that

$$2\pi\int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega = \iint_{\mathscr{D}_Q} |G'(z)|^2\,g(z)\,dxdy\,.$$

An elementary argument with the maximum principle shows that if $I = T \cap \partial Q$,

$$g(z) \leqq C\,l(I)^{-1}\log\frac{1}{|z|}$$

whenever $z \in \mathscr{D}_Q$ and $|z - z_Q| \geqq \frac{1}{16}\,l(I)$. Applying the maximum principle again we obtain

$$\begin{aligned}\int_{\gamma_Q} |F'(z)|\,ds(z) &\leqq C\,l(\gamma_Q)\int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega\\ &\leqq C'\,l(\gamma_Q)\iint_{\mathscr{D}_Q} |G'(z)|^2\,l(\gamma_Q)^{-1}\log\frac{1}{|z|}\,dx\,dy\\ &= C''\iint_{\mathscr{D}_Q} |F'(z)|\,|\varphi'(z)|^2\log\frac{1}{|z|}\,dx\,dy\,.\end{aligned}$$

Since the regions $\mathscr{D}_Q$ are disjoint, it follows from (2.1) that

$$\begin{aligned}\sum_{\text{Type 2}}\int_{\gamma_Q} |F'(z)|\,ds(z) &\leqq C\iint_{\mathscr{D}} |F'(z)|\,|\varphi'(z)|^2\log\frac{1}{|z|}\,dx\,dy \qquad (2.8)\\ &\leqq C'\,l(\partial\Omega)\,.\end{aligned}$$

This almost finishes the proof of the theorem, because if $\varepsilon$ is small enough each domain $F(\mathscr{D}_Q)$ will have boundary a (7) chord-arc curve. (This follows immediately from (2.3).) Furthermore by estimates (2.4) and (2.8) on the Type 1 and Type 2 domains,

$$\sum l(\partial F(\mathscr{D}_Q)) \leqq C\ell(\partial\Omega)\,.$$

We thus need only show that each Type 1 or 2 region $\mathscr{D}_Q$ can be decomposed into disjoint $C_0$ Lipschitz domains $\Omega_j$ with $\sum_j l(\partial\Omega_j) \leqq C_0 l(\gamma_Q)$. For then $F(\Omega_j)$ will be a $C_1$ Lipschitz domain if $\varepsilon$ is small enough, and

$$\sum_j l(\partial F(\Omega_j)) \leqq C\, l(\partial F(\mathscr{D}_Q)\,.$$

The construction is a little easier to describe on the upper half plane $\mathbb{R}^2_+ = \{z = x + iy\colon y > 0\}$. We are given a dyadic square $Q \subset \mathbb{R}^2_+$ with side $I \subset \mathbb{R}$. The region $\mathscr{D}_Q$ is constructed by removing from $Q$ disjoint squares $Q_1, Q_2, \ldots$ with (disjoint) sides $I_j \subset \mathbb{R}$. Let $\tilde{I}_j$ denote the top side of $Q_j$. We now build a Cantor type tree $T_j$. Put $\tilde{I}_j$ in $T_j$ and add on line segments on the left and right of $I_j$ down to $\mathbb{R}$ and of length $\sqrt{2}\, l(I_j)$. We obtain a nice "tree" $T^1_j$ with two "roots" of length $\sqrt{2}\, l(I_j)$ touching $\mathbb{R}$ at angle $\pi/4$. Let $z_1$ and $z_2$ be the two points in $T^1_j$ of height $1/4\, l(I_j)$ so that $\operatorname{Re} z_j = x_j = k_j(1/4\, l(I_j))$, $k_j \in \mathbb{Z}, j = 1, 2$. (In other words, $x_1$ and $x_2$ are appropriate dyadic rationals.) Attach to $z_1$ and $z_2$ the line segments to $\mathbb{R}$ (of angle $\pi/4$ with $\mathbb{R}$) which are not already in $T^1_j$ and call this $T^2_j$. $T^2_j$ is obtained by sprouting the roots of $T^1_j$. Continue sprouting, by dropping by a factor of $1/4$ each time and obtain $T^1_j, T^2_j, \ldots, T^n_j, \ldots$ Then $T^n_j$ has $2^n$ "roots" of length $\sqrt{2}\, 4^{-n}\, l(I_j)$ and each terminates at a point $k\, 4^{-n}\, l(I_j)$ where $k \in \mathbb{Z}$. Setting $T_j = \bigcup_{k=1}^{\infty} T^k_j$ we have

$$l(T_j) = (1 + 3\sqrt{2})\, l(I_j)\,.$$

It is an exercise to see that every component of $\mathscr{D}_Q \backslash \bigcup_j T_j$ is a $C_0$ Lipschitz domain. □

We remark that the idea of using Littlewood–Paley estimates to impose bi-Lipschitz structures on sets has been used before in [10] to give a certain quantitative version of Sard's theorem.

## §3. Proof of (1.2)

We may assume that $l(\Gamma) < \infty$ and the conclusions of Corollary 2 hold for $\Gamma$, because if $\Gamma \subset \tilde{\Gamma}$, $\beta^2(\Gamma) \leqq \beta^2(\tilde{\Gamma})$. We denote by $\Omega_j$ the components of $\mathbb{C} \backslash \Gamma$ and set $\Gamma_j = \partial\Omega_j$, $d_j =$ diameter $(\Gamma_j)$. For a dyadic square $Q$ let $\mathscr{F}(Q) = \{\Gamma_j\colon \Gamma_j \cap 4Q \neq \phi, d_j \geqq l(Q)\}$ and let $G(Q) = \{\Gamma_j\colon \Gamma_j \cap 5Q \neq \phi, d_j < l(Q)\}$. Also let $Q^*$ be the dyadic double of $Q$.

**Lemma 3.1.**

$$\beta^2_\Gamma(Q) \leqq C_1 \sum_{\mathscr{F}(Q)} \beta^2_{\Gamma_j}(Q^*) + C_1\, l(Q)^{-2} \sum_{G(Q)} \text{Area}\,(\Omega_k)\,.$$

*Proof.* The lemma is immediate if $\mathscr{F}(Q) = \phi$, for then either $\beta(Q) = 0$, or $\sum_{G(Q)} \text{Area}\,(\Omega_k) \geqq 9 l(Q)^2$. So suppose $\Gamma_0 \in \mathscr{F}(Q)$. By scaling we may assume $l(Q) = 1$.

Let $L$ be a line such that

$$d \equiv \sup_{z \in \Gamma_0 \cap 5Q} \text{distance}(z, L) \leqq \beta_{\Gamma_0}(Q^*)\, l(Q^*)\,.$$

Let $z_0 \in \Gamma \cap 3Q$ maximize distance $(z, \Gamma_0)$ and let distance $(z_0, \Gamma_0) = d_0$. Denote by $z_1$ the closest point in $\Gamma_0$ to $z_0$, and let $I = [z_0, z_1]$. We define $z_2 = 1/2(z_0 + z_1)$ to be the midpoint of $I$.

*Case 1.* $\mathscr{F}(Q)$ contains three or more curves. Then there is $\Gamma_j \in \mathscr{F}(Q)$ such that $\beta_\Gamma^2(Q^*) \geqq C_1$, so

$$\sum_{\mathscr{F}(Q)} \beta_{\Gamma_j}^2(Q^*) \geqq C_1\,.$$

*Case 2.* $\mathscr{F}(Q)$ contains only $\Gamma_0$. Then $B\left(z_2, \frac{d_0}{2}\right) \subset \bigcup_{G(Q)} \bar{\Omega}_k$. Consequently,

$$\begin{aligned}
\beta_\Gamma^2(Q) &\leqq (d + d_0)^2 \\
&\leqq 2\beta_{\Gamma_0}^2(Q^*) + 2d_0^2 \\
&\leqq 2\beta_{\Gamma_0}^2(Q^*) + \frac{8}{\pi} \sum_{G(Q)} \text{Area}(\Omega_k)\,.
\end{aligned}$$

*Case 3.* $\mathscr{F}(Q) = \{\Gamma_0 \Lambda\, \Gamma_j\}$.

We may as well assume $\beta_{\Gamma_j}(Q^*) < \varepsilon_0, j = 0, 1$, for otherwise there is nothing to prove. Let $d_1 = \sup_{z \in \Gamma_1 \cap 4Q} \text{distance}(z, \Gamma_0)$. Then if $\varepsilon_0$ is small enough $\Gamma \cap 3Q$ is trapped between $\Gamma_0$ and $\Gamma_1$, and

$$\beta_\Gamma(Q) \leqq \beta_{\Gamma_0}(Q^*) + \beta_{\Gamma_1}(Q^*)) + d_1\,.$$

Since $\Gamma_1$ is a $C_0$ Lipschitz curve there is $z_3 \in 9/2Q \backslash (\Omega_0 \cup \Omega_1)$ such that distance $(z_3, \Gamma_j) \geqq c_0 d_1$, $j = 0, 1$. But then

$$\begin{aligned}
\beta_\Gamma^2(Q) &\leqq C_0(\beta_{\Gamma_0}^2(Q^*) + \beta_{\Gamma_1}^2(Q^*)) + C_0 d_1^2 \\
&\leqq C_0(\beta_{\Gamma_0}^2(Q^*) + \beta_{\Gamma_0}^2(Q^*)) + C_1 \sum_{G(Q)} \text{Area}(\Omega_k)\,.
\end{aligned}$$

It is now an easy matter to finish the proof of (1.2). By the results of [9] (or our section 5),

$$\sum_Q \beta_{\Gamma_j}^2(Q)\, l(Q) \leqq C_1\, l(\Gamma_j)$$

for any $C_0$ Lipschitz curve. By Lemma 3.1 it is enough to estimate the sum

$$\sum_Q l(Q)^{-1} \sum_{G(Q)} \text{Area}(\Omega_k)\,.$$

Now for each $n \in \mathbb{Z}$ such that $d_j < 2^{-n}$ there are at most $C_1$ squares $Q$ such that $l(Q) = 2^{-n}$ and $\Gamma_j \in G(Q)$. Consequently, we may estimate the above sum by reversing the order of summation to obtain

$$\sum_{\Gamma_k} \text{Area}\,(\Omega_k) \sum_{\substack{Q \\ \Omega_k \in G(Q)}} l(Q)^{-1}$$

$$\leqq C \sum_{\Gamma_k} \text{Area}\,(\Omega_k) \sum_{m=0}^{\infty} (2^m d_k)^{-1}$$

$$\leqq 2C \sum_{\Gamma_k} \text{Area}\,(\Omega_k)\, d_k^{-1}$$

$$\leqq C' \sum l(\Gamma_k) \leqq 2C'\, l(\Gamma)\,.$$

## §4. Construction of $\Gamma$. (Proof of (1.3)–(1.5).)

We build sets $\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n \subset K$ with the following properties:

$$|z_j - z_k| \geqq 2^{-n}, \qquad z_j, z_k \in \mathscr{L}_n, \qquad j \neq k\,.$$

and

$$\inf_{z_j \in \mathscr{L}_n} |z - z_j| \leqq 2^{-n}, \qquad z \in K\,.$$

These sets may need to be slightly perturbed at various stages of the construction, but the two properties listed above will still hold. We may assume $K \subset [0,1]^2$, and by scaling we may also assume $\mathscr{L}_0 = \{z_0, z_1\}$ where $|z_0 - z_1| = \sup_{z, w \in K} |z - w|$. We define $\Gamma_0 = [2z_0 - z_1, 2z_1 - z_0]$ to be a line segment containing $z_0, z_1$, and extending beyond those points. We let $A \geqq 1$ be a constant to be fixed later, $A = 2^{k_0}$, and then let $\varepsilon_0 > 0$ be small. The value of $\varepsilon_0$ is determined later. Suppose by induction that $\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}$ have been formed and let $x_0 \in \mathscr{L}_n \backslash \mathscr{L}_{n-1}$. If $x_0 \in \Gamma_{n-1}$ we do no construction about $x_0$. Let $Q$ be that dyadic cube containing $x_0$ with $l(Q) = A2^{-n}$. We call the collection of all such cubes $\mathscr{D}_n$. By a translation and rotation we may assume $x_0 \geqq 0$ and its nearest neighbor in $\mathscr{L}_{n-1}$ is the origin. Let $W = \{z\colon 0 < |z| \leqq A2^{-n+1}, |\arg z| \leqq 2\pi/3\}$, and let $W^* = \{z\colon 0 < |z| \leqq A2^{-n+1}, |\pi\text{-}\arg z| \leqq 2\pi/3\}$, so that $W \cup W^* = \mathscr{D} = \{z\colon 0 < |z| \leqq A2^{-n+1}\}$. We assume by induction that the following properties hold:

(P1) Let $\{y_1, \ldots, y_N\} = \mathscr{L}_{n-1} \cap \{\mathscr{D} \cup \{0\}\}$ and arrange the points so that $\operatorname{Re} y_1 \leqq \operatorname{Re} y_2 \leqq \ldots \leqq \operatorname{Re} y_N$. Then $\Gamma_{n-1}$ contains the segments $[y_j, y_{j+1}]$, $1 \leqq N - 1$.

(P2) If $x_0 \notin \Gamma_{n-1}$ and $\mathscr{L}_{n-1} \cap W = \phi$ there is $\theta$, $|\theta| \leqq 2\pi/3$, such that $[0, A2^{-n+1} e^{i\theta}] \subset \Gamma_{n-1}$. If $x_0 \notin \Gamma_{n-1}$ and $\mathscr{L}_{n-1} \cap W^* = \phi$, there is $\psi$, $|\psi - \pi| \leqq 2\pi/3$, such that $[0, A2^{-n+1} e^{i\psi}] \subset \Gamma_{n-1}$.

*Case 1.* $\beta(Q) \geqq \varepsilon_0$. Connect $x_0$ to all points in $\mathscr{L}_n \cap \{|x - x_0| \leqq C2^{-n}\}$ by (straight) line segments. Also add on the line segment $[0, 4Ax_0]$. Then the amount of length added to $\Gamma_{n-1}$ is bounded by

$$C \sum_{Q \in \mathscr{S}_n} \ell(Q) \leqq C' \sum_{Q \in \mathscr{S}_n} \beta^2(Q)\, l(Q)$$

because each Case 1 $Q$ has $\beta(Q)^2 \geqq \varepsilon_0^2$.

For the rest of the cases we assume $\beta(Q) < \varepsilon_0$.

*Case 2.* $\mathscr{L}_{n-1} \cap W \neq \phi$, $\mathscr{L}_{n-1} \cap W^* \neq \phi$. Let $y_1 \in W$ minimize $|z|$, $z \in \mathscr{L}_{n-1} \cap W$, and let $y_{-1} \in W^*$ minimize $|z|$, $z \in \mathscr{L}_{n-1} \cap W^*$. Let $\{x_1, \ldots, x_M\}$ be those points in $\mathscr{L}_n \cap W$ such that $\mathrm{Re}\, y_{-1} \leqq \mathrm{Re}\, x_j \leqq \mathrm{Re}\, y_1$, and label the points so that $\mathrm{Re}\, x_1 \leqq \mathrm{Re}\, x_2 \leqq \ldots \leqq \mathrm{Re}\, x_M$. Replace the segments $[y_{-1}, 0]$, $[0, y_1] \subset \Gamma_{n-1}$ with $[x_1, x_2], \ldots, [x_{M-1}, x_M]$. Then by the Pythagorean theorem, the amount of length added to $\Gamma_n$ is bounded by

$$C\beta^2(Q)\, l(Q)\,.$$

Since $\beta(Q) < \varepsilon_0$, properties (P1) and (P2) are maintained.

*Case 3.* $\mathscr{L}_{n-1} \cap W \neq \phi \cong \mathscr{L}_{n-1} \cap W^* = \phi$, $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} = \phi$. Let $y_1 \in \mathscr{L}_{n-1} \cap W$ minimize $|z|$, $z \in \mathscr{L}_{n-1} \cap W$. Then $[0, y_1] \subset \Gamma_{n-1}$. Let $\{0, x_1, x_2, \ldots, x_M, y_1\}$ be the points in $\mathscr{L}_n \cap W$ between 0 and $y_1$, arranged by increasing real parts. Replace $[0, y_1]$ by $[0, x_1]$, $[x_1, x_2], \ldots, [x_M, y_1]$. Then by Pythagoras the amount of length added is bounded by

$$C\beta^2(Q)\, l(Q)\,,$$

and properties (P1) and (P2) are maintained at $x_1, \ldots, x_M$ because $\beta(Q) < \varepsilon_0$. We now add on some line segments near 0 and $y_1$ to assure properties (P1) and (P2) hold there. First suppose $x_{-1} \in K \cap W^*$ maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq A2^{-n}\}$. If $|x_{-1}| < 8A^{-1}2^{-n}$ add to $\Gamma_n$ the line segment $[-2^{-n+1}, 0]$. If $|x_{-1}| \geqq 8A^{-1}2^{-n}$, add to $\Gamma_n$ the line segment $\left[2^{-n+1}\dfrac{x_{-1}}{|x_{-1}|}, 0\right]$. In either case the amount of length added is bounded by $2^{-n+1}$. Let $z_0, z_1, \ldots, z_N = x_1$ be the points in $\mathscr{L}_{n+k_0} \cap \{|z| \leqq 2^{-n}\}$, arranged by increasing real parts, where $2^{k_0} = 2A^2$. We add on to $\Gamma_n$ the line segments $[z_0, z_1], \ldots, [z_{N-1}, z_N]$, so that at stages $n+1$, $n+2, \ldots, n+k_0-1$ no constructions need be performed in $\{|z| \leqq 2^{-n}\}$. Since $\beta(Q) < \varepsilon_0$, properties (P1), (P2) are preserved for $z_1, \ldots, z_N$ at stage $n+k_0$ by the choice of $[-2^{-n+1}, 0]$ $\left[\text{respectively } 2^{-n+1}\dfrac{x_{-1}}{|x_{-1}|}, 0\right]$. A similar construction is performed at $y_1$.

Let $E = [0, A2^{-n+1}e^{i\theta}] \subset \Gamma_{n-1} \cap W^*$ be the line segment assured by hypothesis (P2) and let

$$I_Q = [A2^{-n-1}e^{i\theta}, A2^{-n}e^{i\theta}]\,.$$

Then the construction will show that $I_Q$ is not altered at any future stage, $I_Q \subset \bigcap_{k=0}^{\infty} \Gamma_{n+k}$. If $A$ is large enough,

(4.1) The amount of length added is bounded by $1/2\, l(I_Q)$.

Furthermore, if $I_Q$ and $I_{Q'}$ are any intervals so formed at any stage of the construction,

$$I_Q \cap I_{Q'} = \phi . \tag{4.2}$$

*Case 4.* $\mathscr{L}_{n-1} \cap W \neq \phi$, $\mathscr{L}_{n-1} \cap W^* = \phi$, $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} \neq \phi$. First suppose $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} = \{x_{-1}\}$. We may assume, by changing $\mathscr{L}_n$ if necessary, that $x_{-1}$ maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$. The construction is as in Case 3, but we add the segment $[2x_{-1}, 0]$. Then estimates (4.1) and (4.2) hold and as in the argument of Case 3 properties (P1) and (P2) will hold at future stages. If $\mathscr{L}_n \cap W^* \cap \{|x| \leqq 2^{-n+1}\}$ contains two points $x_{-1}, x_{-2}$, we may assume $x_{-1}$ maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$. Then we add $[2x_{-1}, x_{-1}]$, and $[x_{-1}, x_{-2}]$, $[x_{-2}, 0]$. Estimates (4.1) and (4.2) hold as do properties (P1) and (P2) at future stages. The case where $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$ contains two points is treated similarly.

*Case 5.* $\mathscr{L}_{n-1} \cap W = \phi \cap \mathscr{L}_{n-1} \cap W^* \neq \phi$. We assume $x_0$ maximizes $|z|$, $z \in K \cap W \cap \{|z| \leqq 2^{-n+1}\}$. Let $\{y_1, y_2, \ldots, y_N = x_0\}$ be the points in $(\mathscr{L}_n \cap W^*) \cup \{|z| \leqq 2^{-n+1}\}$ arranged by increasing real parts, and as in Case 3 replace arcs of $\Gamma_{n-1}$ in that region by $[y_1, y_2], \ldots, [y_{N-1}, y_N]$. As in Case 3 we add on $[x_0, 2x_0]$ and line segments in $W \cap \{|z| \leqq 2^{-n+1}\}$ so that $\mathscr{L}_{n+k_0} \cap W \cap \{|z| \leqq 2^{-n+1}\} \subset \Gamma_n$. We also choose $I_Q$ as in Case 3. Then (4.1) and (4.2) hold and as in Case 3, (P1) and (P2) are preserved.

*Case 6.* $\mathscr{L}_{n-1} \cap W = \phi$, $\mathscr{L}_{n-1} \cap W^* = \phi$. Let $\{y_1, \ldots, y_N\}$ be the points in $\mathscr{L}_{n+k_0} \cap \{|z| \leqq 2^{-n+1}\}$ arranged by increasing real part. We may assume either $y_1 = 0$ or $y_1$ maximizes $|z|$, $z \in K \cap W^* \cap \{2^{-(n+k_0)} \leqq |z| \leqq 2^{-n+1}\}$, and we may assume $y_N$ maximizes $|z|$, $z \in K \cap W \cap \{|z| \leqq 2^{-n+1}\}$. Add on the line segments $[2y_1, y_1]$, $[2y_N, y_N]$ and $[y_j, y_{j+1}]$, $1 \leqq j \leqq N-1$. Let $I_Q$ be as in Case 3. Then (4.1) and (4.2) hold and (P1) and (P2) are preserved.

*Remark.* By the choice of $\Gamma_0$ only Case 1 or Case 2 constructions can happen at $z_0$ and $z_1$ until stage $k_0$, $2^{k_0} = A$. Therefore (P1) and (P2) will always hold at $z_0, z_1$.

To conclude the proof we note that by Cases 1–6 and estimate (4.1) the quantity $l(\Gamma_n) - l(\Gamma_{n-1})$ is bounded by

$$C \sum_{Q \in \mathscr{D}_n} \beta^2(Q)\, l(Q) + \frac{1}{2} \sum_{Q \in \mathscr{D}_n} l(I_Q) .$$

Summing from $n = 1$ to $N$ we obtain

$$\begin{aligned} l(\Gamma_N) - l(\Gamma_0) &\leqq C \sum_Q \beta^2(Q)\, l(Q) + \frac{1}{2} \sum_{l(Q) \leqq 2^{-N}} l(I_Q) \\ &\leqq C \sum_Q \beta^2(Q)\, l(Q) + \frac{1}{2} l(\Gamma_N) , \end{aligned}$$

the final inequality being a consequence of (4.2). Therefore

$$l(\Gamma_N) \leqq 2 l(\Gamma_0) + C \sum_Q \beta^2(Q)\, l(Q) ,$$

and taking limits we obtain the first part of the theorem. We note that the segments $[2z_0 - z_1, z_0]$, $[z_1, 2z_1 - z_0] \subset \Gamma_0$ are never altered at any stage of the construction. By throwing them away and taking $A$ large enough we could build $\Gamma$ so that

$$l(\Gamma) \leqq (1 + \delta)\,\text{diameter}(K) + C_\delta \sum_Q \beta_K^2(Q)\, l(Q)\,,$$

but then (1.5) would not hold for $Q = [0, 1]^2$.

To show property (1.5) holds, we must add some line segments to $\Gamma$ to form a new curve $\tilde{\Gamma}$. Fix a dyadic cube $Q$ with $\beta(2Q) < \varepsilon_0$ and first suppose that there are points $x_0 \in K \cap Q$ and $x_1, x_2 \in K \cap (5Q \backslash 3Q)$ with the angle between $[x_0, x_1]$ and $[x_0, x_2]$ greater than $\pi/2$. Then the construction yields a subcurve of $\Gamma$ which connects $x_1$ to $x_2$ in $S_Q$. For the other case (where $x_1, x_2 \in K \cap (5Q \backslash 3Q)$ implies the angle between $[x_0, x_1]$ and $[x_0, x_2]$ is less than $\pi/2$), the construction shows there is an arc $I_Q \subset \Gamma \cap 3Q$ such that distance $(I_Q, K) \geqq C\, l(Q)$. Add to $\Gamma$ a line segment $J_Q$ crossing $5Q$ in $S_Q$. Then $l(J_Q) \leqq C l(I_Q)$ and since the $I_Q$ are essentially disjoint from each other,

$$\sum_Q l(J_Q) \leqq C' \sum_Q l(I_Q) \leqq C'\, l(\Gamma)\,.$$

Consequently $l(\tilde{\Gamma}) \leqq C\, l(\Gamma)$.

To show that (1.4) holds, apply Corollary 2 to $\Gamma$ and obtain $\tilde{\tilde{\Gamma}}$ such that $l(\tilde{\tilde{\Gamma}}) \leqq C_0 + C_0 \beta^2(K)$ and such that the conclusions of Corollary 2 hold. Let $\mathbb{C} \backslash \tilde{\tilde{\Gamma}} = \bigcup \Omega_j$, and suppose $x, y \in \tilde{\tilde{\Gamma}}$. Let $I = [x, y]$ and let $I = E \bigcup \bigcup_j I_j$ be a decomposition of $I$ into $E = I \cap \tilde{\tilde{\Gamma}}$ and open intervals $I_j$ which lie in $\Omega_{k(j)}$. Setting $I_j = [x_j, y_j]$, we have $x_j, y_j \in \partial\Omega_{k(j)}$, and consequently there is an arc $\gamma_j \subset \partial\Omega_{k(j)}$ connecting $x_j$ to $y_j$ such that $l(\gamma_j) \leqq C_0 |x_j - y_j|$. Then if $\gamma = E \bigcup \bigcup_j \gamma_j$, $\gamma$ is connected, $\gamma \subset \tilde{\tilde{\Gamma}}$, and $l(\gamma) \leqq l(E) + \sum_j l(\gamma_j) \leqq l(E) + C_0 \sum_j |x_j - y_j| \leqq C_0 |x - y|$.

## §5. Appendix

In this section we show that (1.2) holds when $\Gamma$ is a $C_0$ Lipschitz curve. By using a dilation, we may assume that $\Gamma$ is given by the parametrization $\psi(\theta) = r(\theta) e^{i\theta}$, where $C_0^{-1} \leqq r(\theta) \leqq 1$, and $|r(\theta_1) - r(\theta_2)| \leqq C_0 |\theta_1 - \theta_2|$. Let $\Gamma_n$ be the polygon obtained from the line segments

$$J_j^n = [\psi(j 2^{-n+1}\pi), \psi(j+1) 2^{-n+1}\pi)], \qquad 0 \leqq j \leqq 2^n\,.$$

Then $J_j^n$ splits into two intervals $J_{2j}^{n+1}, J_{2j+1}^{n+1}$ at stage $n+1$. Define

$$\delta_{n,j} = 2^{-n} \sup_{z \in J_{2j}^{n+1} \cup J_{2j+1}^{n+1}} \text{distance}\,(z, J_j^n)\,.$$

Then by elementary geometry,

$$l(J_{2j}^{n+1}) + l(J_{2j+n}^{n+1}) - l(J_j^n) \geqq C (\delta_{n,j})^2\, 2^{-n}$$

Summing from $n = 1$ to $\infty$ we obtain

$$c \sum_{n,j} (\delta_{n,j})^2 2^{-n} \leqq l(\Gamma) . \tag{5.1}$$

$$\sum_{k=n}^{\infty} \sum_{J_m^k \subset \theta_j^n} (\delta_{k,m})^2 2^{-k} \leqq C 2^{-n} .$$

Here we define $\Gamma_j^n = \Gamma \cap \{j 2^{-n+1}\pi \leqq \theta \leqq (j+1) 2^{-n+1}\pi\} \equiv \Gamma \cap \theta_j^n$. Our result will follow if we can show that

$$\tilde{\beta}(\Gamma_j^n) = 2^{-n} \sup_{z \in \Gamma_j^n} \text{distance}\,(z, J_j^n)$$

satisfies

$$\sum_{n,j} \tilde{\beta}(\Gamma_j^n)^2 2^{-n} \leqq C_1 , \tag{5.2}$$

for then we may rotate the dyadic grid through $[0, 2\pi]$ to obtain quantities $\tilde{\beta}_\theta(\Gamma_j^n)$ and note that

$$\sum_{l(Q) = 2^{-n-2}} \beta_\Gamma(Q)^2 \, l(Q) \leqq$$

$$C \int_0^{2\pi} \left\{ \sum_j \tilde{\beta}_\theta(\Gamma_j^n)^2 2^{-n} \right\} d\theta .$$

To prove (5.2) notice that

$$\tilde{\beta}(\Gamma_j^n) \leqq C_1 \sum_{k=1}^{\infty} \sup_{J_m^{n+k} \subset \theta_j^n} 2^{-k} \delta_m^{n+k} .$$

Then (5.2) follows from the above inequality, Minkowski's inequality (or Cauchy–Schwarz), and (5.1).

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