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RECTIFYING CURVES AS CENTRODES AND EXTREMAL CURVES

BY

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Abstract. Rectifying curves are introduced in [2] as space curves whose position vector always lies in its rectifying plane. In this article, we establish a surprising simple relationship between rectifying curves and the notion of centrodes in mechanics. Furthermore, we show that rectifying curves are indeed the extremal curves which satisfy the equality case of a general inequality. Further geometric properties of rectifying curves are also presented.

1. Rectifying curves. Let $\mathbb{E}^3$ denote Euclidean three-space, with its inner product $\langle\ ,\ \rangle$ and let S^2 be the unit sphere in $\mathbb{E}^3$ centered at the origin. Consider a unit-speed space curve $\mathbf{x} : I \to \mathbb{E}^3$, where I is a real interval, that has at least four continuous derivatives. Let $\mathbf{t}$ denote $\mathbf{x}'$. It is possible, in general, that $\mathbf{t}'(s) = 0$ for some s; however, we assume that this never happens. Then we can introduce a unique vector field $\mathbf{n}$ and positive function κ so that $\mathbf{t}' = \kappa\mathbf{n}$. We call $\mathbf{t}'$ the *curvature vector field*, $\mathbf{n}$ the *principal normal vector field*, and κ the *curvature* of the given curve. Since $\mathbf{t}$ is a constant length vector field, $\mathbf{n}$ is orthogonal to $\mathbf{t}$. The *binormal vector field* is defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. It is a unit vector field orthogonal to both $\mathbf{t}$

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and $\mathbf{n}$. One defines the *torsion* τ by the equation $\mathbf{b}' = -\tau\mathbf{n}$. The famous Frenet-Serret equations are given by

$$\mathbf{t}' = \kappa\mathbf{n}, \quad \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}, \quad \mathbf{b}' = -\tau\mathbf{n}. \tag{1.1}$$

At each point of the curve, the planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$, and $\{\mathbf{n}, \mathbf{b}\}$ are known as the *osculating plane*, the *rectifying plane*, and the *normal plane*, respectively. A curve in $\mathbb{E}^3$ is called *twisted* if it has nonzero curvature and torsion.

The Frenet-Serret equations can be interpreted kinematically as follows: If a moving point traverses the curve in such a way that s is the time parameter, then the moving frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ moves in accordance with (1.1). This motion contains, apart from an instantaneous translation, an instantaneous rotation with angular velocity vector given by the Darboux vector $\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}$. The direction of the Darboux vector is that of the *instantaneous axis of rotation*, and its length $\sqrt{\kappa^2 + \tau^2}$, denoted by v_{ang}, is called the *angular speed* (see [4, p.12] or [1]). In terms of the Darboux vector, (1.1) can be expressed as

$$\mathbf{t}' = \mathbf{d} \times \mathbf{t}, \quad \mathbf{n}' = \mathbf{d} \times \mathbf{n}, \quad \mathbf{b}' = \mathbf{d} \times \mathbf{b}. \tag{1.2}$$

It is well known that a curve in $\mathbb{E}^3$ lies in a plane through the origin if its position vector lies in its osculating plane at each point, and lies on a sphere centered at the origin if its position vector lies in its normal plane at each point.

Rectifying curves are defined in [2] as space curves whose position vector always lie in its rectifying plane. So, the position vector $\mathbf{x}$ of a rectifying curve satisfies

$$\mathbf{x}(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s) \tag{1.3}$$

for some functions λ and μ. Independent of the choice of Euclidean coordinates on $\mathbb{E}^3$, we may also define a rectifying curve as a curve with $\kappa > 0$ whose rectifying planes always contain a fixed point. If we choose this fixed point to be the origin, we obtain (1.3). Rectifying curves are exactly those space curves whose axis of instantaneous rotation always passes through a fixed point, which we can always choose as the origin (see [2]).

In section 2 we recall some known results on rectifying curves from [2]. In section 3, we establish a very simple surprising relationship between rectifying curves and centrodes of space curves. In section 4, we show that rectifying curves are in fact the extremal curves with respect to some special function. In the last section, we provide further geometric properties of rectifying curves.

2. Some known results. We recall some known results on rectifying curves from [2] for later use.

Lemma 1. *Let* $\mathbf{x} = \mathbf{x}(s)$ *be a unit-speed curve in* $\mathbb{E}^3$ *with* $\kappa > 0$*. Then* $\mathbf{x}$ *is a rectifying curve if and only if one the following four statements holds:*

i. *The distance function* $\rho = |\mathbf{x}|$ *satisfies* $\rho^2 = s^2 + c_1 s + c_2$ *for some constants* c_1 *and* c_2*.*

ii. *The tangential component of the position vector of the curve is given by* $\langle \mathbf{x}, \mathbf{t} \rangle = s + b$ *for some constant* b*.*

iii. *The normal component* $\mathbf{x}^N$ *of the position vector of the curve has constant length and the distance function* ρ *is nonconstant.*

iv. *The torsion* τ *is nonzero, and the binormal component of the position vector is constant, i.e.,* $\langle \mathbf{x}, \mathbf{b} \rangle$ *is constant.*

A curve is called a *helix* if its tangent makes a constant angle with a fixed line. A twisted curve in $\mathbb{E}^3$ is a helix if and only if the ratio $\tau : \kappa$ of the curve is a nonzero constant. Similarly, a curve in $\mathbb{E}^3$ with $\kappa > 0$ is congruent to a rectifying curve if and only if the ratio $\tau : \kappa$ of the curve is a nonconstant linear function in arclength function s (see, Theorem 2 of [2]).

For a unit-speed curve on a surface, the length of the surface-tangential component of acceleration is the *geodesic curvature*, denoted by κ_g. Curves with vanishing geodesic curvature are called geodesics.

The following result from [2] plays an important role in this note. Here we provide a simple alternate proof of this useful result.

Theorem A. *Let* $\mathbf{x}$ *be a curve in* $\mathbb{E}^3$ *with* $\kappa > 0$. *Then it is a rectifying curve if and only if it is given by*

$$\mathbf{x}(t) = a\sec(t + t_0)\mathbf{y}(t), \tag{2.1}$$

where a, t_0 *are constants with* $a \neq 0$ *and* $\mathbf{y} = \mathbf{y}(t)$ *is a unit-speed curve in* S^2.

Proof. For a unit-speed curve $\mathbf{y} = \mathbf{y}(t)$ in S^2 and a positive function ρ, we put $\mathbf{x} = \rho\mathbf{y}$. Then we have

$$\mathbf{x} = \rho\mathbf{y}, \;\; \mathbf{x}' = \rho'\mathbf{y} + \rho\mathbf{y}'. \tag{2.2}$$

which implies that the unit tangent vector field $\mathbf{t}$ of $\mathbf{x}$ is given by

$$\mathbf{t} = \frac{\rho'}{v}\mathbf{y} + \frac{\rho}{v}\mathbf{y}', \quad v = \sqrt{\rho^2 + \rho'^2}. \tag{2.3}$$

Hence, the principal normal vector field $\mathbf{n}$ of $\mathbf{x}$ is parallel to

$$\mathbf{t}' = \Big(\frac{\rho'}{v}\Big)'\mathbf{y} + \Big(\frac{2\rho'}{v} - \frac{\rho\rho' + \rho'\rho''}{v^3}\Big)\mathbf{y}' + \frac{\rho}{v}\mathbf{y}''. \tag{2.4}$$

On the other hand, for the unit-speed curve $\mathbf{y}$, we have

$$\mathbf{y}'' = -\mathbf{y} + \kappa_g\, \mathbf{n}_{\mathbf{y}}, \tag{2.5}$$

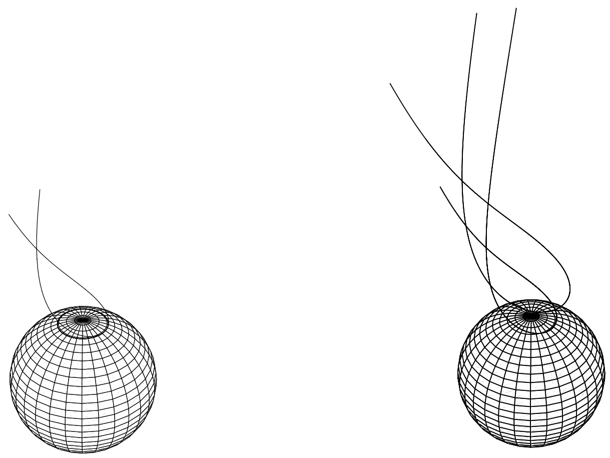
where $\mathbf{n}_{\mathbf{y}} = \mathbf{y} \times \mathbf{y}'$, the *spherical normal vector field*, is a unit vector field tangent to S^2, but normal to $\mathbf{y}$. So, by substituting (2.5) into (2.4), we obtain

$$\kappa v \mathbf{n} = \Big(\Big(\frac{\rho'}{v}\Big)' - \frac{\rho}{v}\Big)\mathbf{y} + \Big(\frac{2\rho'}{v} - \frac{\rho\rho' + \rho'\rho''}{v^3}\Big)\mathbf{y}' + \frac{\kappa_g}{v}\mathbf{x} \times \mathbf{y}'. \tag{2.6}$$

Since $\langle \mathbf{y}, \mathbf{y}\rangle = 1$, we have $\langle \mathbf{x}, \mathbf{y}'\rangle = 0$. So, after taking the scalar product of (2.6) with $\mathbf{x}$, we know that $\mathbf{x}$ is a rectifying curve if and only if the distance function ρ of $\mathbf{x}$ satisfies the differential equation: $(\rho'/v)' = \rho/v$, which is equivalent to

$$\rho\rho'' - 2\rho'^2 - \rho^2 = 0. \tag{2.7}$$

The nontrivial solutions of equation (2.7) are given by $\rho(t) = a \sec(t+t_0)$ with constants $a \neq 0$ and t_0. Hence, $\mathbf{x} = \rho\mathbf{y}$ is a rectifying curve if and only if it is the curve given by (2.1).



a. A rectifying curve over a circle. b. Two rectifying curves over a circle.

3. Rectifying curves as centrodes. For a regular curve $\mathbf{x}$ in $\mathbb{E}^3$ with $\kappa \neq 0$, the curve given by the Darboux vector $\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}$ is called the *centrode* of $\mathbf{x}$. The notion of centrodes plays some important roles in mechanics and joint kinematics (see, for instance, [3, 5, 7, 8]).

The following result establish a very simple relationship between centrodes and rectifying curves.

Theorem 1. *The centrode of a unit speed curve in $\mathbb{E}^3$ with nonzero constant curvature κ and nonconstant torsion τ is a rectifying curve.*

Conversely, every rectifying curve in $\mathbb{E}^3$ is the centrode of some unit speed curve with nonzero constant curvature and nonconstant torsion.

Proof. Assume that $\mathbf{x} = \mathbf{x}(s)$ is a unit-speed curve with nonzero constant curvature and nonconstant torsion. Consider the centrode

$$\mathbf{d}(s) = \tau(s)\mathbf{t}(s) + \kappa(s)\mathbf{b}(s), \tag{3.1}$$

of $\mathbf{x}$. By differentiating the centrode $\mathbf{d}$ with respect to s and applying the Frenet-Serret equations, we have $\mathbf{d}' = \tau'\mathbf{t}$ which implies that the unit tangent vectors of $\mathbf{d}$ and $\mathbf{x}$ at the corresponding points are parallel. So, the first equation in Frenet-Serret equations (1.1) implies that the principal normal vectors of $\mathbf{d}$ and $\mathbf{x}$ at the corresponding points are also parallel. Hence, the binormal vector fields of $\mathbf{d}$ and $\mathbf{x}$ are parallel as well. Therefore, by (3.1), we know that the position vector of the centrode of $\mathbf{x}$ always lies in its rectifying plane. Consequently, by definition, the centrode $\mathbf{d}$ is a rectifying curve.

Conversely, assume that $\alpha = \alpha(s)$ is a unit speed rectifying curve in $\mathbb{E}^3$ defined on an open interval I. Then, from Theorem 2 of [2], we know that the ratio $\tau_\alpha/\kappa_\alpha$ of the torsion and curvature functions of α is a nonconstant linear function in arc length. Thus, without loss of generality, we may assume that

$$\kappa_\alpha = \frac{c\tau_\alpha}{s} \tag{3.2}$$

for some positive number c.

Let $f(s) = \frac{1}{c}\int_{s_0}^{s} \kappa_\alpha(u)du$. Obviously $g = f^{-1}$ exists. By applying the fundamental theorem for space curves, there exists a unit speed curve $\beta = \beta(t)$ whose curvature and torsion functions satisfy $\kappa_\beta(t) = c$ and $\tau_\beta(t) = g(t)$.

Let us consider the centrode of β given by $\mathbf{d}_\beta(t) = g(t)\mathbf{t}_\beta(t) + c\mathbf{b}_\beta(t)$ and its reparametrization $\gamma(s) = \mathbf{d}_\beta(f(s))$. Then

$$\gamma(s) = s\mathbf{t}_\beta(f(s)) + c\mathbf{b}_\beta(f(s)).$$

It follows that $\gamma'(s) = \mathbf{t}_\beta(f(s))$. Thus γ has unit speed and $\mathbf{t}_\gamma(s) = \mathbf{t}_\beta(f(s))$. Differentiating twice gives that the curvature and torsion functions of γ are given by $\kappa_\gamma(s) = cf'(s) = \kappa_\alpha(s)$ and $\tau_\gamma(s) = sf'(s)$. From (3.2) then follows that $\tau_\gamma = \tau_\alpha$.

Therefore the unit speed curves $\alpha(s)$ and γ have the same curvature and torsion functions. Hence, the fundamental theorem for space curves implies that α is congruent to γ. Consequently, the rectifying curve α is the centrode of a space curve with nonzero constant curvature and nonconstant torsion.

The curve $\mathbf{x}$ in Theorem 1 can be replaced by a curve with nonzero constant torsion and nonconstant curvature. In fact, we also have the following.

Theorem 2. *The centrode of a unit speed curve in $\mathbb{E}^3$ with nonconstant curvature and nonzero constant torsion is a rectifying curve.*

Conversely, every rectifying curve in $\mathbb{E}^3$ is the centrode of some unit speed curve with nonconstant curvature and nonzero constant torsion.

We omit the proof since it can be proved in the same way as Theorem 1.

Remark 1. The centrode of a curve with nonzero constant curvature and nonzero constant torsion is a point.

Remark 2. For a regular curve $\mathbf{x}$ with $\kappa \neq 0$, the curves $\mathbf{C}_{\pm} = \mathbf{x} \pm \mathbf{d}$ are called the *co-centrodes* of $\mathbf{x}$. Similar proof as Theorem 1 also yields that, a curve $\mathbf{x}$ with nonzero constant curvature and nonconstant torsion is a rectifying curve if and only if one of its co-centrodes is a rectifying curve.

4. Rectifying curves as extremal curves. If a curve $\mathbf{x}$ is given by $\mathbf{x} = \rho\mathbf{y}$ for some function ρ and curve $\mathbf{y}$ in S^2, then $\mathbf{x}$ is called *a curve over* $\mathbf{y}$ and $\mathbf{y}$ is the *spherical projection* of $\mathbf{x}$.

The next result shows that a rectifying curve is indeed an extremal curve which assumes the minimum value of the function $v^4\kappa^2/\rho^2$ at each point among the curves with the same spherical projection.

Theorem 3. *Let* $\mathbf{y} = \mathbf{y}(t)$ *be a unit-speed curve in* S^2. *Then, for any positive function* $\rho = \rho(t)$, *the curvature* κ *and the speed* v *of* $\mathbf{x} = \rho\mathbf{y}$, *and the geodesic curvature* κ_g *of* $\mathbf{y}$ *satisfy the inequality:*

$$\kappa_g^2 \leq \frac{v^4\kappa^2}{\rho^2}, \tag{4.1}$$

with the equality sign holding identically if and only if $\mathbf{x}$ *is a rectifying curve.*

Proof. Let $\mathbf{y}$ be a unit-speed curve in S^2 and ρ a positive function. Since the spherical normal vector field $\mathbf{n}_{\mathbf{y}} = \mathbf{y} \times \mathbf{y}'$ of $\mathbf{y}$ is perpendicular to $\mathbf{x}' = \rho'\mathbf{y} + \rho\mathbf{y}'$, we may put

$$\mathbf{y} \times \mathbf{y}' = \cos\gamma\mathbf{n} + \sin\gamma\mathbf{b}, \tag{4.2}$$

for some function γ.

By differentiating (4.2) with respect to the arclength of $\mathbf{y}$ and by applying (2.5) and the Frenet-Serret equations, we discover that

$$\kappa_g\mathbf{y}' = v\kappa\cos\gamma\mathbf{t} + (\gamma' + v\tau)(\sin\gamma\mathbf{n} - \cos\gamma\mathbf{b}). \tag{4.3}$$

Since $\mathbf{y} = \mathbf{y}' \times (\mathbf{y} \times \mathbf{y}')$, we obtain from (4.2) and (4.3) that

$$\kappa_g \mathbf{y} = (\gamma' + v\tau)\mathbf{t} - v\kappa \cos\gamma(\sin\gamma \mathbf{n} - \cos\gamma \mathbf{b}). \tag{4.4}$$

From $\langle \mathbf{x}, \mathbf{x} \rangle = \rho^2$, we have $\langle \mathbf{y}, \mathbf{t} \rangle = \rho'/v$. Hence, after taking the scalar product of (4.4) with $\mathbf{t}$, we get $\kappa_g \rho' = v(\gamma' + v\tau)$. Substituting this into (4.3) and (4.4) ensures that

$$v\kappa_g \mathbf{y} = \rho' \kappa_g \mathbf{t} - v^2 \kappa \cos\gamma(\sin\gamma \mathbf{n} - \cos\gamma \mathbf{b}), \tag{4.5}$$

$$v\kappa_g \mathbf{y}' = v^2 \kappa \cos\gamma \mathbf{t} + \rho' \kappa_g (\sin\gamma \mathbf{n} - \cos\gamma \mathbf{b}). \tag{4.6}$$

Both equations yield

$$\rho^2 \kappa_g^2 = v^4 \kappa^2 \cos^2\gamma, \tag{4.7}$$

which implies inequality (4.1). Clearly, the equality sign of (4.1) holds if and only if we have $\sin\gamma = 0$. It follows from (4.2) that condition $\sin\gamma = 0$ is equivalent to the condition: $\mathbf{y} \times \mathbf{y}' = \pm \mathbf{n}$.

On the other hand, from $\mathbf{x} = \rho \mathbf{y}$, we find

$$\mathbf{x}' = \rho' \mathbf{y} + \rho \mathbf{y}', \quad \mathbf{x}'' = \rho'' \mathbf{y} + 2\rho' \mathbf{y}' + \rho \mathbf{y}'', \tag{4.8}$$

which ensures that

$$\begin{aligned} (\mathbf{x}' \times \mathbf{x}'') \times \mathbf{x}' &= \langle \mathbf{x}', \mathbf{x}' \rangle \mathbf{x}'' - \langle \mathbf{x}', \mathbf{x}'' \rangle \mathbf{x}' \\ &= \rho(\rho\rho'' - \rho'^2)\mathbf{y} - \rho'(\rho\rho'' - 2\rho'^2 - \rho^2)\mathbf{y}' + \rho(\rho^2 + \rho'^2)\mathbf{y}''. \end{aligned} \tag{4.9}$$

Thus, by substituting (2.5) into (4.9), we discover that

$$(\mathbf{x}' \times \mathbf{x}'') \times \mathbf{x}' = (\rho\rho'' - 2\rho'^2 - \rho^2)(\rho \mathbf{y} - \rho' \mathbf{y}') + \rho(\rho^2 + \rho'^2)\kappa_g \, \mathbf{y} \times \mathbf{y}'. \tag{4.10}$$

Because the binormal vector field of $\mathbf{x}$ is in the direction of $\mathbf{x}' \times \mathbf{x}''$, the vector field $(\mathbf{x}' \times \mathbf{x}'') \times \mathbf{x}'$ is in the direction of the principal normal vector field $\mathbf{n}$ of $\mathbf{x}$. Hence, by applying (4.10) we know that the principal normal vector field $\mathbf{n}$ of $\mathbf{x}$ is parallel to the spherical normal vector field $\mathbf{n_y}$ of $\mathbf{y}$ if and only if ρ satisfies (2.7). Since the nontrivial solutions of (2.7) are given by $\rho(t) = a\sec(t+b)$ for constants $a \neq 0$ and b, Theorem A implies that $\mathbf{x} = \rho\mathbf{y}$ is a rectifying curve if and only if $\mathbf{n_y}$ is parallel to $\mathbf{n}$. Consequently, we know that the equality sign of (4.1) holds identically if and only if $\mathbf{x}$ is a rectifying curve.

A space curve is a circular helix if and only if it has nonzero constant curvature and nonzero constant torsion. On the other hand, Theorem 3 enables us to obtain the following classification of curves with nonzero constant curvature and linear torsion in terms of spiral type rectifying curves.

Corollary 1. *A curve* $\mathbf{x}$ *in* $\mathbb{E}^3$ *has nonzero constant curvature and nonconstant linear torsion in arclength* s *if and only if it is congruent to a rectifying curve over a unit speed spiral type curve* $\mathbf{y}$ *in* S^2 *with geodesic curvature* $\kappa_g = c\sec^3 t$ *for some nonzero constant* c.

Proof. If a curve $\mathbf{x}$ in $\mathbb{E}^3$ has nonzero constant curvature and nonconstant linear torsion in arclength s, then Theorem 2 of [2] implies that $\mathbf{x}$ is congruent to a rectifying curve. Thus, after applying a suitable translation in t and a suitable rigid motion, we have $\mathbf{x} = a(\sec t)\mathbf{y}(t)$ for some unit-speed curve $\mathbf{y} = \mathbf{y}(t)$ in S^2 according to Theorem A. From these we have $v^4/\rho^2 = a^2\sec^6 t$. Hence, by applying Theorem 3, we have $\kappa_g = c\sec^3 t$, where c is either the constant given by $a\kappa$ or by $-a\kappa$.

Conversely, if $\mathbf{x}$ is a rectifying curve over a unit speed curve $\mathbf{y} = \mathbf{y}(t)$ in S^2 with geodesic curvature $\kappa_g = c\sec^3 t$, $c \neq 0$, then $\mathbf{x} = a\sec(t+t_0)\,\mathbf{y}(t)$ for some nonzero constant a. From this we find $v^4/\rho^2 = a^2\sec^6(t+t_0)$.

Since $\mathbf{x}$ is a rectifying curve, Theorem 3 implies that $\kappa_g^2 = v^4\kappa^2/\rho^2$. Hence, we get $\kappa^2 = c^2/a^2$ which is a nonzero constant. Therefore, Theorem

2 of [2] implies that the torsion of $\mathbf{x}$ is a nonconstant linear function in arclength.

Remark 3. Applying Frenet-Serret's equations, we may show that either of the co-centrodes $C_{\pm}$ of a curve $\mathbf{x}$ is a fixed point if and only if $\mathbf{x}$ is a rectifying curve with $\kappa = c_1 \neq 0$ and $\tau = c_2 \mp s,\ c_1, c_2 \in \mathbf{R}$, where s is an arclength function of $\mathbf{x}$.

5. Further geometric properties of rectifying curves. Given a curve $\mathbf{y}$ in S^2 and a positive function ρ, the curve $\hat{\mathbf{x}} = \rho^{-1}\mathbf{y}$ is called the *reciprocal* of $\mathbf{x} = \rho\mathbf{y}$. For a given regular curve γ in $\mathbb{E}^3$, $\Gamma_\gamma := \gamma'/||\gamma'||$ is called the *Gauss map* and $\gamma^* = \gamma \times \mathbf{t}_\gamma$ is called the *dual curve* of γ.

Proposition 1. *Let $\mathbf{y} = \mathbf{y}(t)$ be a unit-speed curve in S^2 and $\rho = \rho(t)$ a nonconstant positive function. Then the following eight statements are equivalent:*

a. $\mathbf{x} = \rho\mathbf{y}$ *is a rectifying curve.*
b. *The reciprocal $\hat{\mathbf{x}}$ of $\mathbf{x}$ is of constant speed.*
c. *The dual curve $\mathbf{x}^*$ of $\mathbf{x}$ is spherical, i,e., $|\mathbf{x}^*|$ is a nonzero constant.*
d. *The principal normal vector field $\mathbf{n}$ of $\mathbf{x}$ is parallel to the spherical normal vector field $\mathbf{n}_\mathbf{y}$ of the spherical projection $\mathbf{y}$ of $\mathbf{x}$.*
e. *The speed v of $\mathbf{x}$ is proportional to the squared distance function ρ^2 of $\mathbf{x}$.*
f. *The spherical normal vector field $\mathbf{n}_\mathbf{y}$ is tangent to the Gauss map of $\mathbf{x}$.*
g. *The speed v of $\mathbf{x}$ and the speed v_G of the Gauss map G satisfy $vv_G = \rho\kappa_g$.*
h. *The Gauss map of $\mathbf{x}$ is given by $G(t) = \sin(t + t_0)\mathbf{y}(t) + \cos(t + t_0)\mathbf{y}'(t)$ for some constant t_0.*

Proof. Let $\mathbf{y}$ be a unit-speed curve in S^2 and ρ a positive function. Since the reciprocal of $\mathbf{x} = \rho\mathbf{y}$ is $\hat{\mathbf{x}} = \mathbf{y}/\rho$, the velocity vector of $\hat{\mathbf{x}}$ satisfies

$$\hat{\mathbf{x}}' = -\frac{\rho'}{\rho^2}\mathbf{y} + \frac{1}{\rho}\mathbf{y}'. \tag{5.1}$$

Hence, the reciprocal $\hat{\mathbf{x}}$ has constant speed if and only if ρ satisfies the equation:

$$\left(\frac{\rho'}{\rho^2}\right)^2 + \frac{1}{\rho^2} = \frac{1}{a^2} \tag{5.2}$$

for a constant $a \neq 0$. Since the solutions of (5.2) are given by $\rho = \pm a$ or $\rho = \pm a \sec(t + b)$ with constant b, Theorem A implies that statement (a) and statement (b) are equivalent.

The equivalence of statement (a) and statement (d) are already done in the proof of Theorem 3.

A direct computation shows that the distance function ρ^* of the dual curve $\mathbf{x}^*$ is ρ^2/v. Hence, statements (c) and (e) are equivalent,

It follows from $v = \sqrt{\rho^2 + \rho'^2}$ that the speed v is proportional to ρ^2 if and only if ρ satisfies the equation $\rho^2 = a\sqrt{\rho^2 + \rho'^2}$ for some positive number a. Solving this differential equation for nonconstant solutions gives $\rho(t) = a \sec(t + b)$, $b \in \mathbf{R}$. Thus, statement (a) and statement (e) are also equivalent.

Since $G = \mathbf{t}$, we have $G' = v\kappa\mathbf{n}$. Therefore, statements (c) and (f) are equivalent. Moreover, we have $v_G = v\kappa$. Hence, $vv_G = \kappa_g \rho$ if and only if $v^2\kappa = \kappa_g \rho$. So, by applying Theorem 3, we see that statements (a) and (g) are equivalent.

From (4.8) we know that the Gauss map of $\mathbf{x}$ is given by

$$G = \frac{\rho'}{v}\mathbf{y} + \frac{\rho}{v}\mathbf{y}'. \tag{5.3}$$

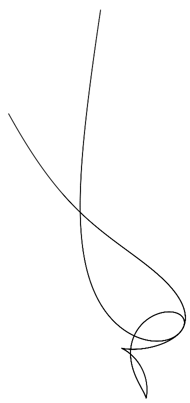
It follows from (5.3) that statement (g) holds if and only if ρ satisfies the following system:

$$\rho = \sqrt{\rho^2 + \rho'^2}\cos(t + t_0), \quad \rho' = \sqrt{\rho^2 + \rho'^2}\sin(t + t_0). \tag{5.4}$$

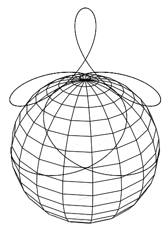
On the other hand, it is easy to verify that the nontrivial solutions of (5.4) are also given by $\rho(t) = a\sec(t + t_0)$ with constants $a \neq 0$ and t_0. Hence, by applying Theorem A again, we conclude that statements (a) and (h) are equivalent as well.

Remark 4. The equivalence of statements (a) and (d) of Proposition 1 implies that $\kappa_g = vv_{ang}$ for a rectifying curve $\mathbf{x} = \rho\mathbf{y}$, where v and v_{ang} are the speed and the angular speed of $\mathbf{x}$ and κ_g the geodesic curvature of $\mathbf{y}$.

Remark 5. The equivalence of statements (a) and (b) of Proposition 1 can be interpreted optically as follows: The curve $\mathbf{x} = \rho\mathbf{y}$ is rectifying if and only if the spherical mirror reflection $\hat{\mathbf{x}} = \rho^{-1}\mathbf{y}$ (with respect to spherical mirror S^2) is of constant speed.



c. A rectifying curve and its reciprocal.



d. A spherical curve and its centrode.

References

1. P. Appell, *Traité de Mécanique Rationnelle*, Vol. 1, 6th ed., Gauthier-Villars, Paris, 1941.

2. B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Monthly, **110** (2003), 147-152.

3. S. Gertzbein, J. Seligman, R. Holtby, K. Chan, N. Ogston, A. Kapasouri, M. Tile and B. Cruickshank, *Centrode patterns and segmental instability in degenerative disk disease*, Spine, **10** (1985), 257-261.

4. D. Laugwitz, *Differential and Riemannian Geometry,* Academic Press, New York, 1965.

5. N. Ogston, G. King, S. Gertzbein, M. Tile, A. Kapasouri, and J. Rubenstein, *Centrode patterns in the lumbar spine-base-line studies in normal subjects*, Spine, **11** (1986), 591-595.

6. P. H. Strubecker, *Differentialgeometrie. I : Kurventheorie der Ebene und des Raumes,* Walter de Gruyter, Berlin, 1964.

7. P. J. Weiler and R. E. Bogoch, *Kinematics of the distal radioulnar joint in rheumatoid-arthritis-an in-vivo study using centrode analysis*, J. Hand Surgery, **20A** (1995), 937-943.

8. H. Yeh and J. I. Abrams, *Principles of Mechanics of Solids and Fluids,* Vol. 1, McGraw-Hall, New York, 1960.

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