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Rectilinear Glass-Cut Dissections of Rectangles to Squares

Jurek Czyzowicz

Departement d'Informatique
Universite du Québec à Hull
Hull, Québec J8X 3X7, Canada
czyzowic@uqah.uquebec.ca

Evangelos Kranakis

Carleton University
School of Computer Science
Ottawa, ON, K1S 5B6, Canada
kranakis@scs.carleton.ca

Jorge Urrutia

Cubiculo 222, Instituto de Matematicas
Universidad Nacional Autonoma de Mexico
D.F., Mexico City, Mexico
urrutia@math.unam.mx

Abstract

We study a problem of dissecting a rectangle into a minimum number of pieces which may be reassembled into a square. The dissection is made using only rectilinear glass-cuts, i.e., vertical or horizontal straight-line cuts separating pieces into two.

1 Introduction

A *glass-cut* of a rectangle is a cut by a straight-line segment that separates the rectangle into two pieces. A *rectilinear* glass-cut is a glass-cut that is either vertical or horizontal. A rectilinear glass-cut *dissection* of a rectangle R to a rectangle R' is a sequence of rectilinear glass-cuts on R such that the resulting pieces can be reassembled to form the rectangle R'. Clearly, a sequence of n rectlinear glass-cuts produces $n+1$ pieces (see Figure 1).

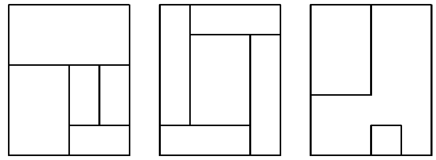

Figure 1: Three dissections of a rectangle: the leftmost is a rectilinear glass-cut while the other two are not.

We address the problem of finding rectilinear glass-cut dissections with minimum number of pieces of a rectangle R into a square S of the same area. Because of scaling, without loss of generality we will suppose that S is a unit square and R is a rectangle of width r and height $\frac{1}{r}$ (i.e., a $r \times \frac{1}{r}$ rectangle). It is known that the problem has no solution for an irrational value of r (see Stillwell [5]). Therefore we suppose that $r = \frac{m}{n}$, where m and n are relatively prime positive integers with $m > n$. For the purpose of ease of analysis we will scale the problem to the equivalent version of dissection of an $m^2 \times n^2$ rectangle into an $mn \times mn$ square, and the glass-cuts are at integer positions. Our dissection algorithm in Section 2 cuts such a rectangle into a number of pieces never exceeding $m + 1$ which may be reassembled to form the square.

Dissections of unit area rectangles into a unit square are always possible if the cuts are not necessarily rectilinear. Namely, a unit area rectangle of dimensions $a \times b$ can always be dissected to a unit square using at most $\lceil a/b \rceil + 2$ pieces. This beautiful result, due to Montucla, is described in [2]. We illustrate with an example.

Example 1.1 *The rectangle of dimensions* 25×9 *can be dissected to a square of dimensions* 15×15 *using four pieces as depicted in Figure 2.*

A simple rectilinear glass-cut dissection of an $\frac{m}{n} \times \frac{n}{m}$ rectangle to a unit square can be obtained as follows. Dissect the rectangle into n rectangles of dimension $\frac{m}{n} \times \frac{1}{m}$. Lay these n rectangles into a single rectangle of width $1/m$ and height m. Dissect this new rectangle into m rectangles of width $1/m$ and height 1. These $m + n - 1$ rectangular pieces can now be assembled to form a unit square. We illustrate with an example.

Example 1.2 *There is a rectilinear glass-cut dissection of a* 25×9 *rectangle into a* 15×15 *square with seven rectangular pieces. The pieces are illustrated in Figure 3.*

2 A New Dissection Algorithm

Definition 2.1 *Let* $p(m, n)$ *be the minimum number of pieces in dissecting the* $m^2 \times n^2$ *rectangle into the* $mn \times mn$ *square. For convenience we also define*

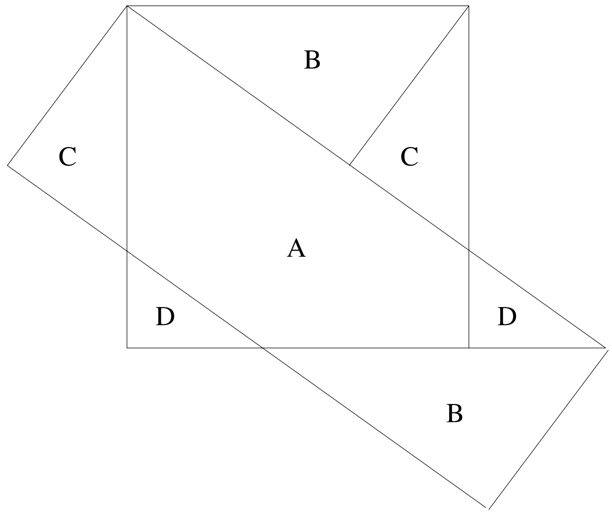

Figure 2: Montucla's dissection of a 25×9 rectangle into a 15×15 square using the four pieces A, B, C, D.

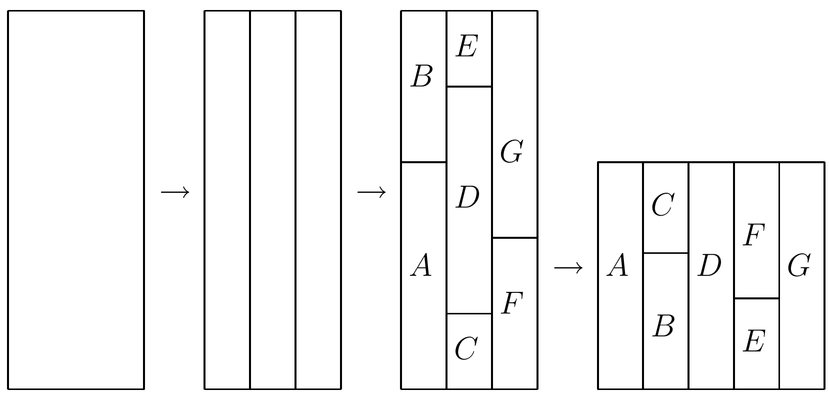

Figure 3: Dissection of a 25×9 rectangle into a 15×15 square. The dissection uses seven rectangular pieces $A, B, C, D, E.F.G$ that can be assembled to form the unit square.

$p(m, 0) = 0.$

It is easy to prove that $p(m, 1) = m$. In the sequel we assume that $n > 1$. First we prove the following lemma.

Lemma 2.1 *If $m > n$ then*

$$p(m, n) \leq 2 \cdot \left\lfloor \frac{m}{n} \right\rfloor + p(n, m \bmod n).$$

PROOF. We start with a rectangle R of dimensions $m^2 \times n^2$. The dissection is in two steps.

Step 1: In the first step we dissect the original rectangle R with vertical glass-cuts (see Figure 4). Each piece is a rectangle with dimensions $(mn) \times n^2$,

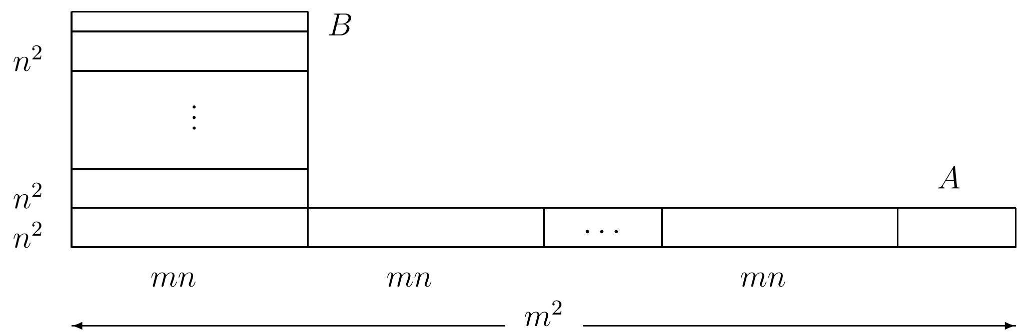

Figure 4: Step 1 in the dissection of an $m^2 \times n^2$ rectangle into an $mn \times mn$ square.

which gives rise to $\lfloor m^2/(mn) \rfloor = \lfloor m/n \rfloor$ such rectangles. It also leaves two "surplus" rectangles to be dissected: one, denoted by A, with dimensions $(mn) \times (mn - \lfloor m/n \rfloor n^2)$ (this is part of the $m^2 \times n^2$ rectangle) and one, denoted by B, with dimensions $(m^2 - \lfloor m/n \rfloor mn) \times n^2$ (this is part of the $mn \times mn$ square).

Step 2: In the second step we rotate the rectangle B 90 degrees counterclockwise, The resulting rectangles have dimensions $mn \times rn$ and $n^2 \times rm$, where $r = m - \lfloor m/n \rfloor n$, We now perform the following dissection (see Figure 5). We dissect A into $\lfloor mn/n^2 \rfloor = \lfloor m/n \rfloor$ rectangles each of dimension $n^2 \times rn$. The remaining rectangle in A is in fact an $rn \times rn$ square. These pieces are placed in B one on top of the other. It is easy to see that the remaining rectangle has dimensions $n^2 \times r^2$. If R' is the rectangle with dimensions $n^2 \times r^2$ we see that the original dissection problem of converting the rectangle R into a square has been transformed into the problem of converting the rectangle R' into a square at an extra cost of $2\lfloor m/n \rfloor$ rectangles. This completes the proof of Lemma 2.1. ■

Lemma 2.1 gives an algorithm for computing a dissection of the $m^2 \times n^2$ rectangl into an $mn \times mn$ square. Consider the sequence of integers generated

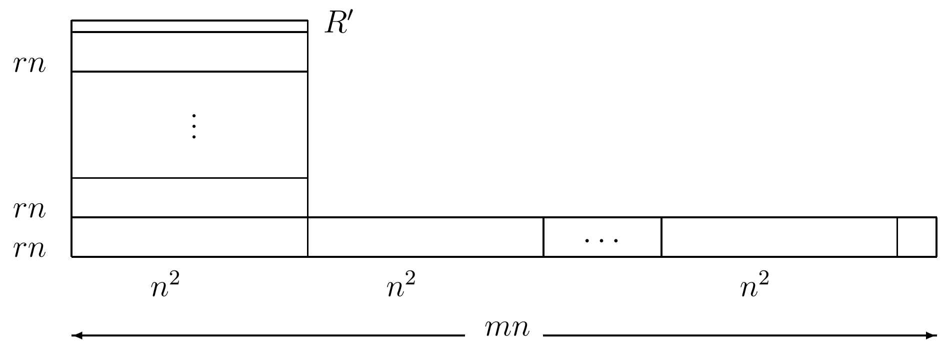

Figure 5: Step 2 of the dissection. We rotate the rectangle B and dissect. The remaining rectangle R' has dimensions $n^2 \times r^2$.

by the Euclidean algorithm: $r_0 = m, r_1 = n$ and

$$\begin{array}{lll} r_0 & = q_0 r_1 + r_2 & 0 \le r_2 < r_1 \\ r_1 & = q_1 r_2 + r_3 & 0 \le r_3 < r_2 \\ \vdots & & \vdots \\ r_i & = q_i r_{i+1} + r_{i+2} & 0 \le r_{i+2} < r_{i+1} \\ \vdots & \vdots & \\ r_k & = q_k r_{k+1} & r_{k+2} = 0, \end{array}$$

where $r_{k+1} = \gcd(m, n) = 1$ and $k \in O(\log n)$. If we iterate Lemma 2.1 k times then we obtain a dissection consisting of

$$p(m, n) \le 2 \sum_{i=0}^{k-1} \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor + p(r_k, r_{k+1})$$

rectangular pieces. In computing the last term $p(r_k, r_{k+1})$ note that by the Euclidean algorithm $r_k = q_k r_{k+1}$ and hence we have to dissect a rectangle of dimensions $(q_k r_{k+1})^2 \times r_{k+1}^2$ into a square of dimensions $q_k r_{k+1} \times q_k r_{k+1}$. It is now easy to see that this last dissection can be accomplished in exactly $q_k = r_k / r_{k+1} = r_k$ rectangular pieces each of dimensions $q_k r_{k+1} \times r_{k+1}$. To sum up we have proved the following theorem.

Theorem 2.1 *An $\frac{m}{n} \times \frac{n}{m}$ rectangle can be dissected into a unit square using only rectilinear glass-cuts, and the number of pieces does not exceed*

$$2 \sum_{i=0}^{k-1} \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor + r_k, \tag{1}$$

where $r_0 = m > r_1 = n > \cdots > r_{k+1} = \gcd(m,n) = 1$ *is the sequence of remainders produced by the computation of* $\gcd(m,n)$ *using the Euclidean algorithm.* ■

We illustrate the previous method with an example.

Example 2.1 *There is a five piece rectilinear glass-cut dissection of the* 25×9 *rectangle into a* 15×15 *square. The dissection is depicted in Figures 6 and 7.*

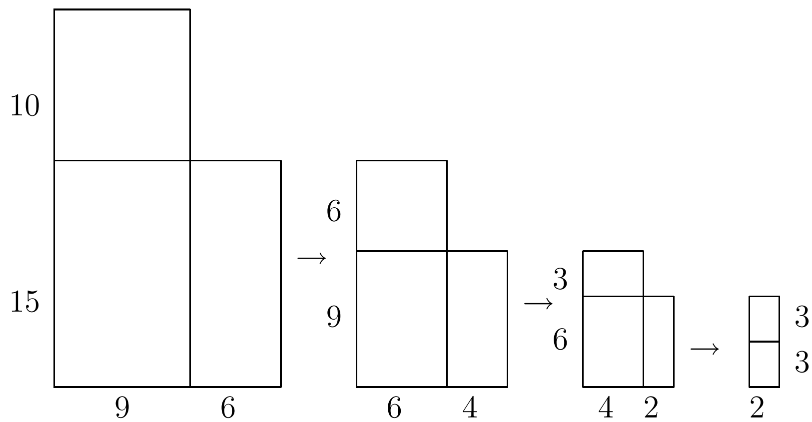

Figure 6: Dissection of a 25×9 rectangle into a 15×15 square using our algorithm. There are five rectangular pieces in the dissection of dimensions $15 \times 9, 9 \times 6, 6 \times 4, 3 \times 2, 3 \times 2$, respectively.

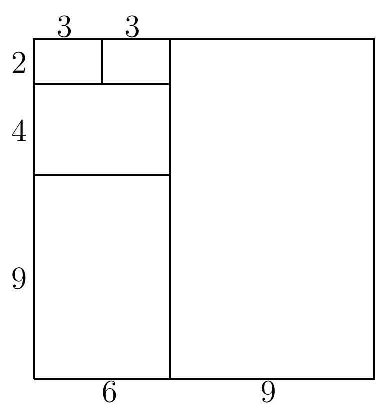

Figure 7: The square of dimensions 15×15 resulting from the dissection by assembling the five rectangles of dimensions $15 \times 9, 9 \times 6, 6 \times 4, 3 \times 2, 3 \times 2$.

Using Formula 1, we can also prove the following upper bound on the number of pieces.

Theorem 2.2 *The number of pieces to dissect an* $\frac{m}{n} \times \frac{n}{m}$ *rectangle to unit square does not exceed* $m + 1$.

PROOF. We show that for $n > 1$ the number of pieces obtained by the previous algorithm never exceeds $m + 1$. If $n = 2$ then it is not hard to see that $p(m, 2) \leq 2\lfloor m/2 \rfloor + p(2, m \bmod 2) \leq m$. Hence without loss of generality we may assume $n \geq 3$.

We now prove by induction on m that $p(m, n) \leq m + 1$. If $n \leq m/3$ then using the induction hypothesis and since $n \geq 3$,

$$\begin{aligned} p(m,n) &\leq 2\lfloor m/n \rfloor + p(n, m \bmod n) \\ &\leq 2(m/n) + n + 1 \\ &\leq 2(m/3) + m/3 + 1 \\ &= m + 1. \end{aligned}$$

Hence without loss of generality we may assume $n > m/3$, which also implies $\lfloor m/n \rfloor = 2$. If also $n \leq m - 4$ then from the induction hypothesis we have

$$\begin{aligned} p(m,n) &\leq 2\lfloor m/n \rfloor + p(n, m \bmod n) \\ &\leq 2 \cdot 2 + n + 1 \\ &\leq 4 + m - 4 + 1 \\ &= m + 1. \end{aligned}$$

Hence, without loss of generality we may assume that $n \geq m - 3$. If $m > 6$ then $m/n < 2$ and $\lfloor m/n \rfloor = 1$. Hence, if also $n \leq m - 2$ then

$$\begin{aligned} p(m,n) &\leq 2\lfloor m/n \rfloor + p(n, m \bmod n) \\ &\leq 2 \cdot 1 + n + 1 \\ &\leq 2 + m - 2 + 1 \\ &= m + 1. \end{aligned}$$

This reduces to the case where $m > 6$ and $n \geq m - 1$. In the case where $m = n + 1$ we can prove directly that $p(m, m - 1) \leq m + 1$. So we only need to consider the cases $6 \geq m > n \geq 3$. Since $\gcd(m, n) = 1$ this leaves only the cases $(6.5), (5, 4), (5, 3), (4, 3)$. In view of Example 2.1 we have that $p(5, 3) \leq 5$. This and the previous observations complete the proof of the theorem. ■

3 Open Problem

We do not know whether or not our algorithm gives the optimal number of pieces. In fact no non-trivial lower bound is known which is valid for all possible rectilinear (and otherwise) dissections. For additional problems see also [1].

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