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RECURRENCE AND TRANSIENCE CRITERIA FOR RANDOM WALK IN A RANDOM ENVIRONMENT

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Oseledec's Multiplicative Ergodic Theorem is used to give recurrence and transience criteria for random walk in a random environment on the integers. These criteria generalize those given by Solomon in the nearest-neighbor case. The methodology for random environments is then applied to Markov chains with periodic transition functions to obtain recurrence and transience criteria for these processes as well.

1. Introduction. Consider the discrete-time stochastic process $\{X(t)\}_{t\geq 0}$ on the integers which is constructed in the following fashion.

Let Z denote the integers, Z^+ the nonnegative integers, and Z^- nonpositive integers. Fix two positive integers, R and L, and let G be the set of all probability measures on $\{-L, \cdots, R\}$. Let $\{e(z; *)\}_{z\in Z}$ be a sequence of iid, G-valued random variables. For each realization of $\{e(z; *)\}_{\in Z}^{z}$ let $X(t)$ be a stationary Markov chain on Z, starting at 0, with transition matrix $P\{z, z+a\}$ given by

$$P\{z, z+a\} = \begin{bmatrix} e(z; a) & -L \leq a \leq R \\ 0 & \text{otherwise.} \end{bmatrix}$$

The vector $e(z, *)$ gives the distribution of the jumps from z for the X-process. When the $\{e(z; *)\}_{z\in Z}$ are fixed, $X(t)$ is simply a Markov chain on the integers. Notice that for each $t \geq 0$,

$$\Pr\{-L \leq X(t+1) - X(t) \leq R\} = 1.$$

The sequence $\{e(z; *)\}_{z\in Z}$ is called a *random environment*, and a realization of $\{e(z; *)\}_{z\in Z}$ is called an *environment*. The process $X(t)$ described above is called a Generalized Random Walk in a Random Environment on the Integers with Bounded Jump Size, which will be abbreviated RWIRE. Generalized Random Walks in a Random Environment on arbitrary abelian groups were studied by Kalikow [Kalikow, 1981].

When $R = L = 1$ and $\Pr\{e(z; 1) + e(z; -1) = 1\}$, $X(t)$ is known as a Random Walk in a Random Environment. This case has been studied extensively and many results are known. [Solomon, 1975] contains this classification theorem.

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THEOREM. *Solomon. For a Random Walk in a Random Environment, let* $d = E\{\log[e(0; 1)/e(0; -1)]\}$.

If $d > 0$ *then* $\lim_{t\to\infty} X(t) = +\infty$ a.s.

If $d = 0$ *then* $-\infty = \liminf_{t\to\infty} X(t) < \limsup_{t\to\infty} X(t) = +\infty$ a.e.

If $d < 0$ *then* $\lim_{t\to\infty} X(t) = -\infty$ a.s.

The object of this paper is to extend this result to RWIRE. The results may be summarized as follows.

Let μ be the measure on G defined by $\mu(A) = \Pr\{e(0; *) \in A\}$. Let $G^L = \{g \in G\colon \operatorname{supp}(g) \subset \{-L, \cdots, 0\}\}$, $G^R = \{g \in G\colon \operatorname{supp}(g) \subset \{0, \cdots, R\}\}$, $G' = \{g \in G\colon \exists\ a < 0 < b \text{ with } g(a)\cdot g(b) > 0\}$. Note that $(G^L \cup G^R) \cap G' = \varnothing$.

THEOREM 5. *Trivial cases.*

If $\mu(G^L) = 0$ *and* $\mu(G^R) > 0$, *then* $\lim_{t\to\infty} X(t) = +\infty$, a.s. $[P]$.

If $\mu(G^R) = 0$ *and* $\mu(G^L) > 0$, *then* $\lim_{t\to\infty} X(t) = -\infty$, a.s. $[P]$.

If $\mu(G^L) > 0$ *and* $\mu(G^R) > 0$, *then* $\exists\ a \le b$ *depending only on e so that for all t,* $a \le X(t) \le b$.

THEOREM 11. *The Zero-One Law. Suppose that* $\mu(G') = 1$. *Then one of the following mutually exclusive possibilities holds.*

1. $\lim_{t\to\infty} X(t) = +\infty$ a.s.
2. $-\infty = \liminf_{t\to\infty} X(t) < \limsup_{t\to\infty} X(t) = +\infty$ a.s.
3. $\lim_{t\to\infty} X(t) = -\infty$ a.s.

The following ergodic theorem is used to generalize Solomon's theorem.

THEOREM 21. *Oseledec's Multiplicative Ergodic Theorem (Oseledec, 1968; Raghunathan, 1979; Walters, 1982). Let* $\{\mathbf{M}_n\}_{n\in Z+,}$*be a stationary ergodic sequence of* $r \times r$ *real matrices on the probability space* $(\Omega, \mathscr{B}, m)$ *and suppose that* $E[\log^+ \|\mathbf{M}_0\|] < \infty$. *Then there are r constants,*

$$-\infty \le d_1 \le d_2 \le \cdots \le d_r < \infty,$$

and a strictly increasing nonrandom sequence of integers,

$$1 = i_1 < i_2 < \cdots < i_p < i_{p+1} = r + 1,$$

satisfying

$$d_{i_{q-1}} < d_{i_q} \quad \text{if} \quad q = 2, 3, \cdots, p; \quad p \le r,$$

$$d_i = d_j \quad \text{if} \quad i_q \le i, \quad j < i_{q+1}; \quad (\text{the } i_q \text{ mark the points of increase of the } d_i);$$

so that for almost every $\omega \in \Omega$:

For every $v \in R^r$, $\lim_{n\to\infty} n^{-1}\log \|\mathbf{M}_{n-1} \cdots \mathbf{M}_1\mathbf{M}_0\mathbf{v}\|$ *exists or is* $-\infty$;

For $q \le p$, $V(q, \omega) \equiv \{\mathbf{v} \in R^r\colon \lim_{n\to\infty} n^{-1}\log \|M_{n-1} \cdots M_1M_0\mathbf{v}\| \le d_{i_q}\}$ *is a random vector subspace of* R^r *with dimension* $i_{q+1} - 1$.

If $V(0, \omega)$ *denotes* $\{\mathbf{0}\}$, *then* $\mathbf{v} \in V(q, \omega)\backslash V(q-1, \omega)$ *implies that*

$$\lim_{n\to\infty} n^{-1}\log \|\mathbf{M}_{n-1} \cdots \mathbf{M}_1\mathbf{M}_0\mathbf{v}\| = \mathbf{d}_{i_q}.$$

The d_j *are called Lyapunov numbers, and they are a generalization of the logarithms of eigenvalues.*

This theorem is applied as follows. If $E\{\log[e(0; -L)]\} > -\infty$ and $E\{\log[e(0; R)]\} > -\infty$, define the iid sequence of real $(R + L) \times (R + L)$ matrices by

$$\mathbf{A}_y(R + L, k) = \frac{\delta(R + 1, k) - e(y; R + 1 - k)}{e(y; -L)}$$

$$\mathbf{A}_y(i, k) = \delta(i + 1, k) \quad \text{if} \quad 1 \le i < R + L.$$

Theorem 22 is stated in terms of the Lyapunov numbers of the sequence $\{\mathbf{A}_{-y}\}_{y \ge 0}$.

THEOREM 22. *Criteria for transience and recurrence. Suppose that* $E[\log(e(0; -L))] > -\infty$ *and* $E[\log(e(0; R))] > -\infty$. *Let* $\{d_i\}_{1 \le i \le R+L}$ *be the* $R + L$ *Lyapunov numbers of the sequence* $\{\mathbf{A}_{-y}\}_{y \in Z^+}$, *in increasing order.*

If $d_R + d_{R+1} > 0$ *then* $P\{\lim_{t\to\infty} X(t) = \infty\} = 1$.

If $d_R + d_{R+1} = 0$ *then* $P\{-\infty = \liminf_{t\to\infty} X(t) < \limsup_{t\to\infty} X(t) = \infty\} = 1$.

If $d_R + d_{R+1} < 0$ *then* $P\{\lim_{t\to\infty} X(t) = -\infty\} = 1$.

Lemma 24 shows that $d_R = 0$ or $d_{R+1} = 0$, so the criterion $d_R + d_{R+1} < 0$ is equivalent to $d_R < 0 = d_{R+1}$, etc. The condition $E[\log(e(0; -L))] > -\infty$ forces $e(0; -L) > 0$ a.s. and the condition $E[\log(e(0; R))] > -\infty$ forces $e(0; R) > 0$ a.s.

In general, $d_R + d_{R+1}$ is not computable. In the case where $R = L = 1$

$$d_R + d_{R+1} = d_1 + d_2 = E\{\log|\det(\mathbf{A}_0)|\} = E\left\{\log\left(\frac{e(0; 1)}{e(0; -1)}\right)\right\},$$

which shows that Theorem 22 is a natural generalization of Solomon's result. In the case where $R + L > 2$, examples are given where $d_R + d_{R+1}$ can be computed, and in the case where $R = 1$, a formula is given for the expected time for the process to enter $[1, \infty)$. Finally, results analogous to Theorems 11 and 22 are obtained for Markov chains on the integers whose transition matrices $p(*, *)$ satisfy

$$p(x, y) = p(x + N, y + N)$$

for some $N > 0$.

2. The definition of Random Walk In a Random Environment. The definition of a Random Environment on an abelian group is given in Kalikow (1981). The following is a summary of that definition in the case where the abelian group is the integers. The following notation is used throughout:

$$Z = \text{the integers}, \quad Z^+ = \{0, 1, 2, \cdots\}, \quad Z^- = \{\cdots, -2, -1, 0\}.$$

Let X be the set of all integer sequences indexed by Z^+ and endowed with the σ-algebra $\mathscr{F}$ generated by the cylinder sets in X. A Random Walk in a Random Environment (RWIRE) on the integers is a discrete-time stochastic process $\{X(t)\}_{t \in Z^+}$ with integer state space. The measure which defines the RWIRE is defined by a two step construction of a measure $P\{*\}$ on $(X, \mathscr{F})$.

The first step is to define the random environment. *For the rest of this paper,* *R and L will be fixed positive integers.* Let G be the set of of all probability measures on $\{-L, \cdots, R\}$ and let μ be a probability measure on G. It is assumed that L and R are minimal in the sense that

$$(1) \qquad \mu\{g \in G\colon g(-L) > 0\} > 0 \quad \text{and} \quad \mu\{g \in G\colon g(R) > 0\} > 0.$$

DEFINITION 1. The *Random Environment e defined by* μ is an integer indexed sequence of iid G-valued random variables, $\{e(z; *)\}_{z\in Z}$, with common distribution μ.

DEFINITION 2. An *environment* is a realization of the random environment e.

The second step is to define $P\{*\}$ so that conditioned on e, $X(t)$ is a Markov chain on Z starting from 0 with respect to $P\{*\}$ conditioned on e. Define the family of random measures $P^{e,j}\{*\}$ on $(X, \mathscr{F})$ satisfying

$$P^{e,j}\{X(0) = j\} = 1,$$

$$P^{e,j}\{X(t+1) = z_t + a \mid X(0) = j, X(1) = z_1, \cdots, X(t) = z_t\}$$
$$= P^{e,j}\{X(t+1) = z_t + a \mid X(t) = z_t\} = e(z_t; a).$$

Conditioned on e, $X(t)$ is a stationary Markov chain on Z starting from j with respect to $P^{e,j}\{*\}$, whose transition matrix is independent of j. Next, define the random measure $P^e\{*\} = P^{e,0}\{*\}$. Then

$$(2) \qquad P^e\{X(0) = 0\} = 1,$$

$$(3) \qquad P^e\{X(t+1) = z_t + a \mid X(0) = 0, X(1) = z_1, \cdots, X(t) = z_t\}$$
$$= P^e\{i; X(t+1) = z_t + a \mid X(t) = z_t\} = e(z_t; a).$$

Finally, define

$$(4) \qquad P\{*\} = E\{P^e\{*\}\}.$$

Then $X(t)$ is a Markov chain with respect to the measure $P\{* \mid e\}$.

(3) and (4) show that the RWIRE is determined by the measure μ.

Since R and L are fixed and $P^e\{X(t+1) = z + a \mid X(t) = z\} = 0$ for all $t \in Z^+$ and $a \notin \{-L, \cdots, R\}$, $X(t)$ can never move more than R steps to the right or L steps to the left in one unit of time.

A RWIRE on Z can be thought of as a measure on the set of all Markov chains on Z. Many of the following theorems are proven by showing that with respect to this measure, almost every Markov chain has a particular given property.

DEFINITION 3. *Metadefinition.* Let "*xyz*" be a property that a Markov chain might have. A RWIRE is said to have property "*xyz*" if for almost every outcome of the random environment e, the Markov chain defined by (2) and (3) has property "*xyz*". For example, a RWIRE is transient if the Markov chain defined by (2) and (3) is transient for almost every environment.

Let

$$G^L = \{g \in G: g \text{ is supported on } -L, \cdots, 0\}$$

and

$$G^R = \{g \in G: g \text{ is supported on } 0, \cdots, R\}.$$

DEFINITION 4. Let $Y(t)$ be a Markov chain on Z. A finite subset A of Z is called a *reflecting-to-the-right barrier* for $Y(t)$ if $Y(T) \geq \min(A)$ for some $T \geq 0$ implies $Y(t) \geq \min(A)$ for all $t \geq T$. *Reflecting-to-the-left barriers* are defined similarly. See Solomon (1975, pages 10–11).

THEOREM 5. *Trivial Cases.*

(5) *If* $\mu(G^L) = 0$ *and* $\mu(G^R) > 0$, *then* $\lim_{t\to\infty} X(t) = +\infty$, *a.s.* $[P]$.

(6) *If* $\mu(G^R) = 0$ *and* $\mu(G^L) > 0$, *then* $\lim_{t\to\infty} X(t) = -\infty$, *a.s.* $[P]$.

(7) *If* $\mu(G^L) > 0$ *and* $\mu(G^R) > 0$, *then there exist* $a \leq b$ *depending only on* e *so that for all* t, $a \leq X(t) \leq b$.

REMARK. This disposes of all of the trivial cases of recurrence and transience of RWIRE on Z. The first two statements generalize the one-way mirror example in Solomon (1975). The last statement gives the generalization of the random walk $Z(t)$ which satisfies $Z(t) \equiv 0$ a.s.

PROOF. The main idea of the proof is to show that reflecting barriers occur in almost every environment.

To prove (5) it is sufficient to show that for almost every environment

(8) $X(t)$ can move to the right from every integer,

(9) There are an infinite number of reflecting-to-the-right barriers to the left and the right of 0.

(8) follows from (3) and the hypothesis that $\mu(G^L) = 0$.

To prove (9), let F_j be the event $\{e(z, *) \in G^R: jL \leq z < (j+1)L\}$. If F_j occurs, then the states $\{jL, \cdots, (j+1)L - 1\}$ form a reflecting-to-the-right barrier. Since $\mu(G^R) > 0$ and the F_j are independent, (9), and hence (5), follows from the Borel-Cantelli Lemma. The proof of (6) is the same as that of (5), with the roles of right and left reversed. To prove (7), note that if $\mu(G^L) > 0$ and $\mu(G^R) > 0$, then there is a reflecting-to-the-right barrier to the left of 0 and a reflecting-to-the-left barrier to the right of 0, which implies (7).

3. Irreducibility of RWIRE. In general, a RWIRE will not be irreducible, even if the trivial case of the elements of G being supported only on multiples of a fixed integer is eliminated. For example, if μ is supported on only two elements of G, g_1 and g_2, and g_1 is supported on -2 and 2 and g_2 is supported on -1 and 1, then the event $\{e(-2; *) = e(2; *) = g_2, e(-1; *) = e(1; *) = g_1\}$ has positive probability. When this event occurs, 0 cannot be reached from any other integer.

However, RWIRE has a property that is almost as strong as irreducibility.

DEFINITION 6. Let $\{Y(t)\}_{t\in T}$ be a stochastic process and let $\tau = \inf\{t > 0: Y(t) \in A\}$. $Y(t)$ is said to *enter* A if τ is finite. If $a \in A$, then $Y(t)$ is said to *enter* A *at* a if $Y(\tau) = a$.

DEFINITION 7. A Markov chain $Y(t)$ on a state space S is called *nonreducible* if there is exactly one closed communicating class C, and $s \in S$ implies $\Pr\{Y(t) \text{ enters } C\} = 1$.

DEFINITION 8. Let Q be a transition matrix on a countable set S, and let $C \subset S$. The *restriction of* Q *to* C, written Q_C, is the transition matrix on C defined by

$$Q_C(x, y) = Q(x, y)[\textstyle\sum_{y\in C} Q(x, y)]^{-1} \quad \text{if} \quad Q(x, y) > 0 \quad \text{for some} \quad y \in C,$$

$$Q_C(x, x) = 1 \qquad \text{if} \quad Q(x, y) = 0 \quad \text{for all} \quad y \in C.$$

DEFINITION 9. Let $Q(x, y)$ be a transition matrix on a countable set S and let $B, C \subset S$. Let $a, b \in C$. a is said to *lead to* b *through* C, written $a \rightsquigarrow_C b$, if there exists n such that $Q_C^n(a, b) > 0$. a is said to *lead to* B *through* C, written $a \rightsquigarrow_C B$, if $a \rightsquigarrow_C b$ for some $b \in B$. a and b are said to *communicate through* C if $a \rightsquigarrow_C b$ and $b \rightsquigarrow_C a$. The set C is called *self-communicating* if the matrix Q_C defines an irreducible Markov chain on C. For each $z \in \{-L, \cdots, R\}$, define

$$m(z) = \mu(\{g \in G: g(z) > 0\}).$$

Notice that $m(z) > 0$ iff there is a positive probability that some environment will allow a jump of size z. Therefore, (1) implies that $m(R) > 0$ and $m(-L) > 0$.

THEOREM 10. *Nonreducibility of* RWIRE.

1. RWIRE *is nonreducible iff* $\mu(G^L) = \mu(G^R) = 0$ *and* $gcd(\{z \neq 0: m(z) > 0\}) = 1$.

2. *Suppose that the* RWIRE *is nonreducible. The unique closed communicating class of the* RWIRE *will be random, and will depend only on the environment. It will be denoted by* C_e *and it has the following properties*:

a. *For each* $n \in Z$ $C_e \cap (-\infty, n]$ *contains a self-communicating subset* $C_{e,n}^-$ *satisfying*

$$\text{(10)} \qquad \textit{if } z \in (-\infty, n],\ z \rightsquigarrow_{(-\infty,n]} C_{e,n}^-,$$

$$\text{(11)} \qquad \textit{for any } z, n \in Z^-,\ P^{e,z}(X(t) \textit{ enters } C_{e,n}^- \cup \{1, \cdots, R\}) = 1.$$

b. *For each* $n \in Z$ $C_e \cap [n, \infty)$ *contains a self-communicating subset* $C_{e,n}^+$ *satisfying*

$$\text{(12)} \qquad \textit{if } z \in [n, \infty),\ z \rightsquigarrow_{[n,\infty)} C_{e,n}^+,$$

(13) *for any* $z, n \in Z^+$, $P^{e,z}\{X(t)$ *enters* $C^+_{e,n} \cup \{-L, \cdots, -1\}\} = 1$.

PROOF. *Proof of 1.* If either condition on the RHS (right-hand side; LHS will denote left-hand side) of 1 fails, it is clear that the RWIRE cannot be nonreducible. To prove that the RHS guarantees that the RWIRE is nonreducible, a "block" argument similar to the one used in the proof of Theorem 5 will be used.

Let $d = gcd(R, L)$, and let F_i be the event

(14) $$H_i \equiv \{i(R + L), i(R + L) + d, \cdots, (i + 1)\cdot(R + L) - d\} \text{ is self-communicating,}$$

(15) $$z \in D_i \equiv \{i(R + L), i(R + L) + 1, \cdots, (i + 1)\cdot(R + L) - 1\} \text{ implies } z \rightsquigarrow_{D_i} H_i.$$

Since the D_i are disjoint, the F_i are pairwise independent events with the same probability. A number-theoretic result proved in the appendix, (Corollary 43), shows that $P\{F_0\} > 0$, so that

$$\Pr\{F_j \text{ i.o.: } j \in Z^+\} = 1 \quad \text{and} \quad \Pr\{F_j \text{ i.o.: } j \in Z^-\} = 1.$$

Let H_e denote the union of the H_i for which F_i occurs.

CLAIM 1. Any two elements of H_e communicate. To prove Claim 1, suppose that F_j and F_k occur, with $j < k$. By definition, H_j and H_k are self-communicating. Since $\mu(G^L) = 0$, a step to the right is possible from each state. Therefore, if $z \in H_j$, then $z \rightsquigarrow_{[j(R+L),(k+1)(R+L))} D_k$, so by the definition of F_k, $z \rightsquigarrow_{[j(R+L),(k+1)(R+L))} H_k$. Interchanging the roles of right and left and of j and k shows that if $z \in H_k$, then $z \rightsquigarrow_{[j(R+L),(k+1)(R+L))} H_j$. Therefore, any two elements of H_e communicate. □

CLAIM 2. Let $z \in Z$. Then $z \rightsquigarrow_Z H_e$ w.p.1. To prove Claim 2, note that w.p.1 there exist $j = j(e)$ and $k = k(e)$ such that F_j and F_k occur and $j(R + L) \le z < (k + 1)(R + L)$. Since w.p.1 $X(t)$ has no jumps larger than $\max(R, L)$ and D_j and D_k each consist of $R + L$ consecutive integers, $z \rightsquigarrow_{[j(R+L),(k+1)(R+L))} D_j \cup D_k$ w.p.1. Since for $i = j$ or k there is a positive probability of entering H_i once $X(t)$ enters D_i, $z \rightsquigarrow_Z H_e$ w.p.1. □

For a particular environment, take C_e to be the smallest closed communicating class which contains H_e. Claims 1 and 2 show that C_e is well-defined, and Claim 2 shows that the RWIRE is nonreducible.

PROOF OF 2. Let $H_{e,n}$ denote the union of the H_i for which $i(L + R) \ge n$ and F_i occurs. Notice that if $z \ge n$, then $z \rightsquigarrow_{[n,\infty)} H^+_{e,n}$. Let $C^+_{e,n}$ denote the class of all states in $[n, \infty)$ which may be reached through $[n, \infty)$ from states in $H^+_{e,n}$. More formally, $c \in C^+_{e,n}$ iff there exists $h \in H^+_{e,n}$ such that $h \rightsquigarrow_{[n,\infty)} c$. $C^-_{e,n}$ is defined similarly. $C^-_{e,n}$ and $C^+_{e,n}$ respectively satisfy (10), (11), and (12), (13).

From here on it is assumed that the RWIRE is nonreducible. If

$$gcd(\{z: m(z) > 0\}) \equiv d$$

is greater than 1, but $\mu(G^L) = \mu(G^R) = 0$, Theorems 11 and 22 are still correct, because $X(t)$ will be a nonreducible RWIRE on $\{dz: z \in Z\}$.

4. The zero-one law. Define $G' = \{g \text{ in } G: \exists\, a > 0, b < 0 \text{ such that } g(a) \cdot g(b) > 0\}$. Note that $\mu(G') = 1$ iff $\mu(G^L) = \mu(G^R) = 0$.

THEOREM 11. *The Zero-One Law. Suppose that* $\mu(G') = 1$. *Then one of the following mutually exclusive possibilities holds*:
1. $\lim_{n\to\infty} X(t) = +\infty$ a.s. $[P]$.
2. $-\infty = \liminf_{n\to\infty} X(t) < \limsup_{n\to\infty} X(t) = +\infty$ a.s. $[P]$.
3. $\lim_{n\to\infty} X(t) = -\infty$ a.s. $[P]$.

The proof of this theorem requires several propositions about Markov chains. Let $Y(t)$ be a Markov chain on $Z^- \cup \{1, \cdots, R\}$ with transition matrix $Q(x, y)$. Suppose that Y and Q satisfy

(16) $$Y(t+1) - Y(t) \le R,$$

(17) $$\{1, \cdots, R\} \quad \text{are absorbing states},$$

(18) for each $z \in Z^-$, $P\{Y(t)$ enters $(z, R] \mid Y(0) = z\} > 0$, and for each $n \in Z^-$ there is a set C_n which is a self-communicating subset of $(-\infty, n]$ satisfying

(19) $$\text{if} \quad z \in (-\infty, n], \quad \text{the} \quad z \rightsquigarrow_{(-\infty,n]} C_n,$$

(20) for any $z \in Z^- \cup \{1, \cdots, R\}$,
$$\Pr(Y(t) \text{ enters } C_n \cup \{1, \cdots, R\} \mid Y(0) = z) = 1.$$

Note that if $e = \{e(z; *)\}_{z\in Z}$ defines a nonreducible RWIRE then Theorem 10 shows that conditions (16)–(20) are satisfied a.s. if $Q(x, y) = e(x; y - x)$ for $x \le 0$.

For such a Markov Chain, and any $x \in Z^-$, $y \in Z^-$, and $x < j \le R$ let

(21) $$f_j(x; y) = \begin{cases} \Pr(Y(t) \text{ enters } (x, R] \text{ at } j \mid Y(0) = y) & x \ge y \\ \delta(j, y) & x < y, \end{cases}$$

(22) $$f(x; y) = \begin{cases} \Pr(Y(t) \text{ enters } (x, R] \mid Y(0) = y) & x \ge y \\ 1 & x < y. \end{cases}$$

PROPOSITION 12. *Suppose* $n \in Z^-$. *Then*

(23) $$f(n; y) < 1 \quad \textit{for some} \quad y \le n \quad \textit{iff} \quad f(n; z) < 1 \quad \textit{for all} \quad z \le n,$$

(24) $$f(n; y) < 1 \quad \textit{for some} \quad y \le n \quad \textit{iff} \quad f(x; z) < 1 \quad \textit{for all} \quad z \le x \le 0.$$

PROOF. *Proof of* (23). Suppose that $f(n, y) < 1$. If $y \in C_n$ then for any

$z \in C_n$, $f(n; z) < 1$ since $z \rightsquigarrow_{C_n} y$. Given any other $z \le n$, $f(n; z) < 1$ since $z \rightsquigarrow_{(-\infty,n]} C_n$. If $y \notin C_n$, then since $\Pr(Y(t)$ enters $C_n \cup \{1, \cdots, R\} \mid Y(0) = z) = 1$, there exists $y_0 \in C_n$ such that $f(n; y_0) < 1$. Now apply the result above.

PROOF. *Proof of* (24). Suppose that $f(n; y) < 1$ for some $y \le n \le 0$. By (23), $f(n; z) < 1$ for all $z \le n$. Therefore, $f(n; y) < 1$ for some $y \le n$ implies $f(x; z) < 1$ if $z \le x \le 0$ and $x \ge n$. So suppose that $x < n$, and $f(x; z) = 1$ for some $z \le x$. Then by (23), $f(x; z) = 1$ for all $z \le x$. So starting from y the Markov Chain visits $(x, R]$ i.o. Each time it visits $(x, R]$, it has a probability p of visiting $(n, R]$. Since the cardinality of $\{x + 1, \cdots, n\}$ is finite,

$$p \ge \inf_{a \in \{x+1,n\}} \Pr\{Y(t) \text{ enters } (n, R] \mid Y(0) = a\} > 0.$$

So by the Generalized Borel-Cantelli Lemma (Breiman, 1968, pages 96–98), the Markov Chain visits $(n, R]$ i.o. starting from y, which implies that $f(n; y) = 1$. This is a contradiction.

PROPOSITION 13. *Suppose* $y \le 0$ *and* (16) *holds. Then*

$$f(0; y) \le f(y; y)[\max_{1 \le j \le R}\{f(0; y + j)\}], \tag{25}$$

$$\max_{1 \le j \le R}\{f(0; y + j)\} \le [\max_{1 \le j \le R}\{f(y + j; y + j)\}] \cdot [\max_{1 \le j \le R}\{f(0; y + R + j)\}]. \tag{26}$$

PROOF. *Proof of* (25). Fix $y \le 0$. Then

$$\begin{aligned} f(0; y) &= \textstyle\sum_{j=1}^{R} f_{y+j}(y; y) f(0; y + j) \qquad \text{strong Markov property} \\ &\le \textstyle\sum_{j=1}^{R} f_{y+j}(y; y)[\max_{1 \le j \le R}\{f(0; y + j)\}] \\ &= f(y; y)[\max_{1 \le j \le R}\{f(0; y + j)\}], \end{aligned}$$

which proves (25).

PROOF. *Proof of* (26).

$$\begin{aligned} &\max_{1 \le j \le R}\{f(0; y + j)\} \\ &\qquad \le \max_{1 \le j \le R}\{f(y + j; y + j) \cdot [\max_{1 \le k \le R}\{f(0; y + j + k)\}]\} \quad \text{by (25)} \\ &\qquad \le [\max_{1 \le j \le R}\{f(y + j; y + j)\}] \cdot [\max_{1 \le k \le 2R}\{f(0; y + k)\}] \\ &\qquad = [\max_{1 \le j \le R}\{f(y + j; y + j)\}] \cdot [\max_{1 \le k \le R}\{f(0; y + R + k)\}]. \end{aligned}$$

The last equality follows because (25) implies that

$$f(0; y) \le \max_{1 \le j \le R}\{f(0; y + j)\}.$$

COROLLARY 14.

$$\log[\max_{1 \le j \le R}\{f(0; -nR + j)\}] \le \textstyle\sum_{p=1}^{n} \log[\max_{1 \le j \le R}\{f(-pR + j; -pR + j)\}].$$

Proof. Iterate Proposition 13.

Proposition 15. *Let* $\{Y(t)\}_{n\in Z^+}$ *be a nonreducible Markov chain with state space* S, *and closed communicating class* C. *Suppose* A *is a finite proper subset of* S *and* $A \not\subset C$. *Then for any* $a \in A$, $\Pr\{Y(t) \in S\backslash A \text{ i.o.} \mid Y(0) = a\} = 1$.

Proof. The proof of this proposition is left to the reader.

Proposition 16. *Let* $\{Y(t)\}_{n\in Z^+}$ *be a nonreducible Markov chain on* Z *with* $|Y(t+1) - Y(t)|$ *uniformly bounded.*
1. *If for all* $a \in Z$, $M > 0$, $N > 0$,

$$\Pr\{Y(t) \text{ enters } [a+N, \infty) \mid Y(0) = a\} = 1$$

and

$$\Pr\{Y(t) \text{ enters } (-\infty, a-M] \mid Y(0) = a\} < 1,$$

then $\lim_{t\to\infty} Y(t) = +\infty$ a.s.
2. *If for all* $a \in Z$, $M > 0$, $N > 0$,

$$\Pr\{Y(t) \text{ enters } [a+N, \infty) \mid Y(0) = a\} = 1$$

and

$$\Pr\{Y(t) \text{ enters } (-\infty, a-M] \mid Y(0) = a\} = 1,$$

then $-\infty = \liminf_{t\to\infty} Y(t) < \limsup_{t\to\infty} Y(t) = +\infty$ a.s. $[P]$.
3. *If for all* $a \in Z$, $M > 0$, $N > 0$.

$$\Pr\{Y(t) \text{ enters } [a+N, \infty) \mid Y(0) = a\} < 1$$

and

$$\Pr\{Y(t) \text{ enters } (-\infty, a-M] \mid Y(0) = a\} = 1,$$

then $\lim_{t\to\infty} Y(t) = -\infty$ a.s.

Proof. The proof of this proposition is left to the reader.

Return now to the consideration of the RWIRE $X(t)$.

Theorem 17. *Suppose that* $\mu(G') = 1$. *Then there are strictly positive constants* c_1 *and* c_2 *such that*

$$\text{(27)} \quad \textit{Either} \quad P^{e,b}\{X(t) \text{ enters } (0,\infty)\} = 1 \quad \text{a.s. for all} \quad b \in Z$$

$$\text{(28)} \quad \textit{or} \quad \limsup_{b\to\infty}(\log[P^{e,-b}\{X(t) \text{ enters } (0,\infty)\}])/b \le -c_1 \quad \text{a.s.;}$$

$$\text{(29)} \quad \textit{Either} \quad P^{e,b}\{X(t) \text{ enters } (-\infty,0)\} = 1 \quad \text{a.s. for all} \quad b \in Z,$$

$$\text{(30)} \quad \textit{or} \quad \limsup_{b\to\infty}(\log[P^{e,b}\{X(t) \text{ enters } (-\infty,0)\}])/b \le -c_2 \quad \text{a.s.}$$

Proof. To prove Theorem 17, it is sufficient to prove the first disjunction. To do this, apply Propositions 12 and 13 in the following manner. For each

$x \in Z^-$, set the transition function $Q(x, y)$ of the above propositions equal to $P^e\{X(t+1) = y \mid X(t) = x\}$. Recall that $Q(x, x) = 1$ for $x \geq 1$. Then for $a \leq b$, $f(b; a)$ defined by (22) becomes a random variable with the same distribution as $P^e\{X(t)$ enters $[b - a, \infty)\}$. Note that $\{\max_{1 \leq j \leq R}\{f(-pR + j; -pR + j)\}\}_{p \in Z^+}$ is a stationary ergodic sequence of random variables since the $\{e(z): z \in Z\}$ are iid variables. Combining Corollary 14 and Birkhoff's Ergodic Theorem gives

$$\begin{aligned}\limsup_{n\to\infty} n^{-1}\log[\max_{1\leq j\leq R}\{f(0; -nR + j)\}] &\leq \limsup_{n\to\infty} n^{-1} \textstyle\sum_{p=1}^{n} \log[\max_{1\leq j\leq R}\{f(-pR + j; -pR + j)\}] \\ &= E[\log(\max_{1\leq j\leq R}\{f(-R + j; -R + j)\})] \quad \text{a.s.}\end{aligned}$$

If this last expectation is negative, then (28) holds. If it is zero, then by Proposition 12, $f(0; 0) = 1$ a.s. and (27) holds.

COROLLARY 18. *Suppose* $\mu(G') = 1$.
1. *Either* a. $P\{X(t)$ *enters* $[b, \infty)\} = 1$ *for all* $b \in Z$, *or* b. $\lim_{b\to\infty} P\{X(t)$ *enters* $[b, +\infty)\} = 0$;
2. *Either* a. $P\{X(t)$ *enters* $(-\infty, -b]\} = 1$ *for all* $b \in Z$, *or* b. $\lim_{b\to\infty} P\{X(t)$ *enters* $(-\infty, -b]\} = 0$.

PROOF. Each equation follows from the corresponding equation in Theorem 17. The key observation is that since the $\{e(z)\}_{z\in Z}$ are iid, for any $A \subset Z$ and $b \in Z$,

$$P^e\{X(t) \text{ enters } A\} =_d P^{e,b}\{X(t) \text{ enters } A + b\}. \tag{31}$$

To prove the "a" possibilities, use (31) and the definition of P. To prove the "b" possibilities, use the definition of P, (31), and the Dominated Convergence Theorem.

COROLLARY 19. *Suppose that* $\mu(G') = 1$. (28) *and* (30) *cannot hold simultaneously.*

PROOF. Suppose not. If follows from Proposition 15 that

$$1 \leq P\{X(t) \text{ enters } (-\infty, -b]\} + P\{X(t) \text{ enters } [b, \infty)\}$$

since for almost every environment $X(t)$ is nonreducible. However, if (28) and (30) both hold, the RHS of this inequality goes to 0 as $b \to \infty$, by Corollary 18. This is a contradiction.

LEMMA 20. *Suppose that* $\mu(G') = 1$. *Then for all* $a \in Z$, (28) *implies that* a.s. $P^{e,a-x}\{X(t)$ *enters* $[a, \infty)\} < 1$ *for all* $x > 0$, (30) *implies that* a.s. $P^{e,a+x}\{X(t)$ *enters* $(-\infty, a]\} < 1$ *for all* $x > 0$.

PROOF. This follows from (31) and Proposition 12.

Now the proof of the Zero-One Law is easily completed. Refer to Theorem 17.

By Corollary 19, (28) and (30) cannot occur simultaneously. The proof proceeds by considering the three remaining possible combinations.

If (30) and (27) hold, then by Lemma 20 for almost every environment the conditions of Proposition 16.1 are satisfied. Therefore, $P^e\{\lim_{x\to\infty} X(t) = \infty\} = 1$ a.s. $[P]$, so $P\{\lim_{x\to\infty} X(t) = \infty\} = 1$. This is the first possibility in the Zero-One Law.

If (29) and (28) hold, then interchanging the roles of right and left in the paragraph above shows that the third possibility in the Zero-One Law holds.

Finally, if (27) and (29) hold, then Proposition 16.2 shows that the second possibility in the Zero-One Law holds.

5. The criteria for transience and recurrence. The object of this section is to give criteria for distinguishing possibilities 1., 2., and 3. of the Zero-One Law (Theorem 11). The method will be to give criteria which distinguish the possible combinations in Theorem 17, and the principal result is Theorem 22.

Recall the definition of $f_j(0; y)$ from (21). Then according to the Markov property, if $y \in Z^-$, then

$$f_j(0; y) = \sum_{x\in Z} P^{e,y}\{X(t+1) = x\} f_j(0; x) = \sum_{a=-L}^{R} e(y; a) f_j(0; y+a). \tag{32}$$

Now, suppose that $e(y, -L) > 0$. Then (32) can be rewritten as

$$\begin{aligned} f_j(0; y - L) &= \sum_{a=1-L}^{R} \frac{\delta(0, a) - e(y; a)}{e(y; -L)} f_j(0; y+a) \\ &= \sum_{a=1-L}^{R} \mathbf{A}_y(R+L, R+1-a) f_j(0; y+a), \end{aligned} \tag{33}$$

where

$$\mathbf{A}_y(R+L, k) = \frac{\delta(R+1, k) - e(y; R+1-k)}{e(y; -L)}.$$

For $y \in Z$, let $\mathbf{A}_y$ be the $(R+L) \times (R+L)$ real matrix defined by

$$\begin{aligned} \mathbf{A}_y(i, k) &= \delta(i+1, k) && \text{if } 1 \le i < R+L, \\ \mathbf{A}_y(i, k) &= \mathbf{A}_y(R+L, k) && \text{if } i = R+L. \end{aligned} \tag{34}$$

Notice that $\mathbf{A}_y$ only depends on $e(y; *)$.

Let $\mathbf{f}_{y,j}$ be the $(R+L) \times 1$ column vector whose ath coordinate is

$$\mathbf{f}_{y,j}(a) = f_j(0; y+R+1-a). \tag{35}$$

Then for $y \le 0$,

$$\mathbf{f}_{y-1,j} = \mathbf{A}_y \mathbf{f}_{y,j} = \mathbf{A}_y \cdots \mathbf{A}_0 \mathbf{f}_{0,j}. \tag{36}$$

For example, when $L = 2$ and $R = 2$,

$$\mathbf{f}_{0,1} = \begin{bmatrix} 0 \\ 1 \\ \mathbf{f}_1(0; 0) \\ \mathbf{f}_1(0; -1) \end{bmatrix}, \qquad \mathbf{f}_{0,2} = \begin{bmatrix} 1 \\ 0 \\ \mathbf{f}_2(0; 0) \\ \mathbf{f}_2(0; -1) \end{bmatrix},$$

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-e(0; 2)}{e(0; -2)} & \dfrac{-e(0; 1)}{e(0; -2)} & \dfrac{1 - e(0; 0)}{e(0; -2)} & \dfrac{-e(0; -1)}{e(0; -2)} \end{bmatrix}.$$

Let r be a positive integer. Let $\| * \|$ denote the euclidean norm on R^r, and let $\| \mathbf{M} \|$ denote operator norm of an $r \times r$ matrix $\mathbf{M}$, with respect to euclidean norm. If $b \in R$, let $\mathbf{b}$ be the element of R^r whose ath coordinate is b^a. For example, when $r = 4$,

$$\mathbf{b} = \begin{bmatrix} b \\ b^2 \\ b^3 \\ b^4 \end{bmatrix}$$

THEOREM 21. *Oseledec's Multiplicative Ergodic Theorem* (*Oseledec*, 1968; *Raghunathan*, 1979; *Walters*, 1982). *Let* $\{\mathbf{M}_n\}_{n \in Z^+}$ *be a stationary ergodic sequence of* $r \times r$ *real matrices on the probability space* $(\Omega, \mathscr{B}, m)$ *and suppose that* $E[\log^+ \| \mathbf{M}_0 \|] < \infty$. *Then there are r constants,*

$$-\infty \le d_1 \le d_2 \le \cdots \le d_r < \infty,$$

and a strictly increasing nonrandom sequence of integers,

$$1 = i_1 < i_2 < \cdots i_p < i_{p+1} = r + 1,$$

satisfying

$$d_{i_{q-1}} < d_{i_q} \quad \text{if} \quad q = 2, 3, \cdots, p \le r,$$

$$d_i = d_j \quad \text{if} \quad i_q \le i, \ \ j < i_{q+1}; \quad (\text{the } i_q \text{ mark the points of increase of the } d_i);$$

so that for almost every $\omega \in \Omega$:

For every $\mathbf{v} \in R^r$, $\lim_{n\to\infty} n^{-1} \log \| \mathbf{M}_{n-1} \cdots \mathbf{M}_1 \mathbf{M}_0 \mathbf{v} \|$ *exists or is* $-\infty$;

For $q \le p$, $V(q, \omega) \equiv \{\mathbf{v} \in R^r\colon \lim_{n\to\infty} n^{-1} \log \| \mathbf{M}_{n-1} \cdots \mathbf{M}_1 \mathbf{M}_0 \mathbf{v} \| \le d_{i_q}\}$ *is a random vector subspace of* R^r *with dimension* $i_{q+1} - 1$.

If $V(0, \omega)$ *denotes* $\{\mathbf{0}\}$, *then* $\mathbf{v} \in V(q, \omega) \backslash V(q-1, \omega)$ *implies that*

$$\lim_{n\to\infty} n^{-1} \log \| \mathbf{M}_{n-1} \cdots \mathbf{M}_1 \mathbf{M}_0 \mathbf{v} \| = d_{i_q}.$$

The d_j *are called Lyapunov numbers, and they are a generalization of the logarithms of eigenvalues.*

It is now possible to state the criteria for recurrence and transience.

THEOREM 22. *Criteria for transience and recurrence of* RWIRE *on* Z. *Suppose that* $E[\log(e(0; -L))] > -\infty$ *and* $E[\log(e(0; R))] > -\infty$. *Let* $\{d_i\}_{1\le i\le R+L}$ *be the* $R + L$ *Lyapunov numbers of the sequence* $\{\mathbf{A}_{-y}\}_{y\in Z^+}$, *in increasing order.*

If $d_R + d_{R+1} > 0$ *then* $P\{\lim_{t\to\infty} X(t) = \infty\} = 1$;

If $d_R + d_{R+1} = 0$ *then* $P\{-\infty = \liminf_{t\to\infty} X(t) < \limsup_{t\to\infty} X(t) = \infty\} = 1$;

If $d_R + d_{R+1} < 0$ *then* $P\{\lim_{t\to\infty} X(t) = -\infty\} = 1$.

REMARKS. Lemma 24 shows that $d_R = 0$ or $d_{R+1} = 0$, so the criterion $d_R + d_{R+1} < 0$ is equivalent to $d_R < 0 = d_{R+1}$, etc. The condition $E[\log(e(0; -L))] > -\infty$ forces $e(0; -L) > 0$ a.s. and the condition $E[\log(e(0; R))] > -\infty$ forces $e(0; R) > 0$ a.s.

6. The proof of the recurrence and transience criteria.

LEMMA 23. *Suppose that* $E[\log(e(0; -L))] > -\infty$. *Let* $V(0)$ *denote*

$$\{\mathbf{v} \in R^{R+L}: \lim_{n\to\infty} n^{-1}\log \| \mathbf{A}_{1-n} \cdots \mathbf{A}_{-1}\mathbf{A}_0\mathbf{v} \| \le 0\}.$$

1. *If* b *is an eigenvalue of* $\mathbf{A}_y$ *then* $\mathbf{b}$ *is the corresponding eigenvector.*
2. *If* b *is nonrandom and* b *is an eigenvalue of* $\mathbf{A}_0$ *a.s.* $[\mu]$, *then*

$$\lim_{n\to\infty} n^{-1}\log \| \mathbf{A}_{1-n} \cdots \mathbf{A}_{-1}\mathbf{A}_0\mathbf{b} \| = \log |b|;$$

3. $V(0)$ *is a random vector subspace of* R^{R+L} *with nonrandom dimension* $\ge R$.
4. $\dim[V(0)] = R$ *implies that* $P^{e,y}\{X(t)$ *enters* $(0, \infty)\} = 1$ *a.s. for all* $y \in Z$.

PROOF.

1. This follows from the form of $\mathbf{A}_y$. See (34).
2. This follows the definition of eigenvector, Part 1, and the fact that the $\mathbf{A}_y$ are iid.
3. The hypothesis that $E[\log(e(0; -L))] > -\infty$ and the fact that the $\mathbf{A}_y$ are iid make it possible to apply Theorem 21 to the sequence $\{\mathbf{A}_{-y}\}_{y\in Z^+}$. By definition, $V(0)$ is a subspace. To see that the $\dim(V(0)) \ge R$, notice that

$$\{\mathbf{f}_{0,j}\}_1 \le j \le R$$

is a set of R independent vectors contained in $V(0)$. (See (35) and (36)).
4. It follows from Part 2 and the fact that $\mathbf{A}_y\mathbf{1} = \mathbf{1}$ that $\mathbf{1} \in V(0)$. So if the dimension of $V(0)$ is R then $\mathbf{1}$ and $\{\mathbf{f}_{0,j}\}_{1\le j\le R}$ are dependent vectors. (21) and (35) show that the dependence is given by

$$\mathbf{1} = \mathbf{f}_{0,1} + \cdots + \mathbf{f}_{0,R}.$$

The result now follows from the definition of the $\mathbf{f}_{y,j}$ in (35) and Proposition 12. □

It is easy to show that $\mathbf{A}_y^{-1}$ has the form

$$(37)\quad \mathbf{A}_y^{-1}(1, j) = \frac{\delta(R, j) - e(y; R - j)}{e(y; R)}, \quad \mathbf{A}_y^{-1}(i, k) = \delta(i, k + 1) \quad \text{if} \quad i \geq 2.$$

For the remainder of this section it is assumed that μ satisfies $E\{\log(e(0, -L))\} > -\infty$ and $E\{\log(e(0, R))\} > -\infty$, so that Lemma 23 and Theorem 21 apply to $\{\mathbf{A}_{-y}\}_{y\in Z^+}$ and to $\{\mathbf{A}_y^{-1}\}_{y\in Z^+}$.

LEMMA 24. *Let $d_1 \leq \cdots \leq d_R \leq d_{R+1} \leq \cdots \leq d_{R+L}$ be the Lyapunov numbers of the sequence of matrices $\{\mathbf{A}_{-y}\}_{y\in Z^+}$. Then either $d_R = 0$ or $d_{R+1} = 0$.*

PROOF. Define

$$g_j(0; y) = P^{e,y}\{X(t) \text{ enters } [-L, 0) \text{ at } j\} \quad \text{if} \quad 0 \leq y,$$

$$g_j(0; y) = \delta(j, y) \quad \text{if} \quad 0 > y.$$

Then for $y \geq 0$,

$$g_j(0; y) = \textstyle\sum_{x\in Z} P^{e,y}\{X(t + 1) = x\} g_j(0; x) = \sum_{a=-L}^{R} e(y; a) g_j(0; y + a).$$

Therefore

$$g_j(0; y + R) = \sum_{a=-L}^{R-1} \frac{\delta(0, a) - e(y; a)}{e(y; R)} g_j(0; y + a)$$

$$= \textstyle\sum_{a=-L}^{R-1} \mathbf{B}_y(1, R - a) g_j(0; y + a),$$

where

$$\mathbf{B}_y(1, j) = \frac{\delta(R, j) - e(y; R - j)}{e(y; R)}.$$

For $y \in Z$, let $\mathbf{B}_y$ be the $(R + L) \times (R + L)$ real matrix defined by

$$\mathbf{B}_y(i, k) = \delta(i, k + 1) \quad \text{if} \quad i \geq 2, \qquad \mathbf{B}_y(i, k) = \mathbf{B}_y(1, k) \quad \text{if} \quad i = 1,$$

and let $\mathbf{g}_{y,j}$ be the $(R + L) \times 1$ matrix whose αth coordinate is given by

$$\mathbf{g}_{y,j}(a) = g_j(0, y + R - a).$$

(Note that this construction is parallel to that of $\mathbf{f}_{y,j}$ and $\mathbf{A}_y$ in (32)–(36).) Then $\mathbf{B}_y \mathbf{g}_{y,j} = \mathbf{g}_{y+1,j}$, $y \geq 0$ (compare with (36)), and $\mathbf{B}_y = \mathbf{A}_y^{-1}$ (see (37)). Let $W(0)$ denote $\{\mathbf{v} \in R^{R+L}: \lim_{n\to\infty} n^{-1} \log \|\mathbf{B}_{n-1} \cdots \mathbf{B}_1 \mathbf{B}_0 \mathbf{v}\| \leq 0\}$. Substituting $\mathbf{B}_{-y}$ for $\mathbf{A}_y$, $\mathbf{g}_j(0, -y)$ for $\mathbf{f}_j(0, y)$ and L for R in Lemma 23 shows that $\dim[W(0)]$ is constant and at least L.

Since $\mathbf{B}_y = \mathbf{A}_y^{-1}$, Theorem 10.3 of Walters (1982) shows that the Lyapunov numbers for the sequence $\{\mathbf{B}_y\}_{y\in Z^+}$ are $-d_{R+L} \leq \cdots \leq -d_{R+1} \leq -d_R \leq \cdots \leq -d_1$, the negatives of the Lyapunov numbers of the sequence $\{\mathbf{A}_{-y}\}_{y\in Z^+}$. $\text{Dim}[V(0)] \geq R$ implies that $d_R \leq 0$ and $\dim[W(0)] \geq L$ implies that $-d_{R+1} \leq 0$. Therefore, $d_R \leq 0 \leq d_{R+1}$, and the result follows from Lemma 23 since $\mathbf{A}_y \mathbf{1} = \mathbf{1}$ implies that some $d_j = 0$.

The proof of Theorem 11 shows that in order to prove Theorem 22 it suffices to show

$$d_R + d_{R+1} > 0 \quad \text{implies (27) and (30),}$$

$$d_R + d_{R+1} = 0 \quad \text{implies (27) and (29),}$$

$$d_R + d_{R+1} < 0 \quad \text{implies (28) and (29).}$$

The proof of Lemma 24 shows that it then suffices to show

$$d_R + d_{R+1} > 0 \quad \text{implies (27),}$$

$$d_R + d_{R+1} = 0 \quad \text{implies (27) and (29),}$$

$$d_R + d_{R+1} < 0 \quad \text{implies (28),}$$

because an interchange of the roles of right and left shows that the proof that $d_R + d_{R+1} > 0$ implies (27) is the same as the proof that $d_R + d_{R+1} < 0$ implies (29), and that the proof that $d_R + d_{R+1} > 0$ implies (30) is the same as the proof that $d_R + d_{R+1} < 0$ implies (28). Therefore, the proof of Theorem 22 can be completed as follows.

$d_R + d_{R+1} > 0$ implies (27). If $d_R + d_{R+1} > 0$, then $\dim[V(0)] = R$, and so by Lemma 23, (27) holds.

$d_R + d_{R+1} < 0$ implies (28). Let $V'(0)$ denote

$$\{\mathbf{v} \in R^{R+L}: \lim_{n\to\infty} n^{-1} \| \mathbf{A}_{1-n} \cdots \mathbf{A}_{-1}\mathbf{A}_0\mathbf{v} \| < 0\}.$$

The condition $d_R + d_{R+1} < 0$ guarantees that $\dim[V'(0)] = R$. Let $\mathbf{h}_1, \cdots, \mathbf{h}_R$ span $V'(0)$. Each $\mathbf{h}_j$ is an element of R^{R+L}. Let $\mathbf{c}_j$ be the element of R^R formed from the top R entries of $\mathbf{h}_j$. For example, if $R = L = 1$ and $\mathbf{h}_1 = [1, 4]^t$ then $\mathbf{c}_1 = [1]$. (^{t} means matrix transpose.)

CLAIM 1. $\mathbf{c}_1, \cdots \mathbf{c}_R$ are independent. Suppose not. Then there is a vector $\mathbf{h} \in V'(0)$ with the property that $\mathbf{h} \neq \mathbf{0}$ but the first R coordinates of $\mathbf{h}$ are identically 0. Define $h(y)$ by

$$h(y) = (R - y + 1)\text{th coordinate of } \mathbf{h} \quad \text{if} \quad y > -L, \tag{38}$$

$$h(y) = \sum_{a=1-L}^{R} \frac{\delta(0, a) - e(y + L; a)}{e(y + L; -L)} h(y + L + a) \quad \text{if} \quad y \le -L. \tag{39}$$

On the other hand, (38), (39) and a construction parallel to the one found in (33)–(36) show that for $y \le -L$, $h(y)$ is the last coordinate of $\mathbf{A}_{y+L} \cdots \mathbf{A}_{-1}\mathbf{A}_0\mathbf{h}$. $\mathbf{h} \in V'(0)$, so $h(y) \to 0$ exponentially as $y \to -\infty$. On the other hand, (39) may be written as

$$\sum_{a=-L}^{R} e(y + L; a)h(y + L + a) = h(y + L), \quad y \le -L;$$

or by substituting y for $y + L$,

$$\sum_{a=-L}^{R} e(y; a)h(y + a) = h(y), \quad y \le 0.$$

Therefore, $h(y)$ is a bounded function which satisfies

$$(40)\quad \sum_{a=-L}^{R} e(y+L;\,a)h(y+L+a) = h(y+L), \qquad y \le 0;\quad h(1) = \cdots = h(R) = 0,$$

and $\lim_{n\to\infty} h(-n) = 0$.

Now, fix any $z \in Z^-$, and let $Z(t)$ be the Markov chain on $Z^- \cup \{1, \cdots, R\}$ which is absorbed at $\{1, \cdots, R\}$ and which has transition matrix $Q(y, y+a) = e(y, a)$ on Z^-. Suppose that $Z(0) = y \le 0$. $M(t) = h(Z(t))$ is a bounded martingale, so standard martingale convergence theorems (Ash, 1972; Breiman, 1968), show that $h(y) = 0$. Since y was arbitrary, this shows that $\mathbf{h} = \mathbf{0}$, which is a contradiction. □

Since the $\mathbf{c}_j$ are independent, there is a vector $\mathbf{f} \in V'(0)$ with the property that the first R coordinates of $\mathbf{f}$ are 1. A function $f(y)$ which satisfies

$$(41)\quad \sum_{a=-L}^{R} e(y;\,a)f(y+a) = f(y), \quad y \le 0; \quad f(1) = \cdots = f(R) = 1,$$

may be produced from $\mathbf{f}$ in the same way as $h(y)$ was produced from $\mathbf{h}$ in (40), and $\mathbf{f}(y)$ has the property that $\lim_{n\to\infty} \mathbf{f}(-n) = 0$ since $\mathbf{f} \in V'(0)$. Recall the definition of $\mathbf{f}(0; y)$ from (22). $\mathbf{f}(0; y)$ is also a bounded solution to (41).

CLAIM 2. $f(0; y) = f(y)$ for $y \le R$. Since $f(0; y) = f(y) = 1$ for $1 \le y \le R$, it is sufficient to show that $f(0; y) = f(y)$ for $-L < y \le 0$. Suppose that $f(0; a) \ne f(a)$. Let $k(y) = f(0; y) - f(y)$. Let $Z(t)$ be the Markov chain on $\{\cdots, R-1, R\}$ which is absorbed on $\{1, \cdots, R\}$ and governed by the transition function $Q(e; y, y+x) = e(y; x)$ for $y \le 0$, with $Z(0) = a$. Let A be the event that $Z(t)$ is absorbed in $\{1, \cdots, R\}$. Let $M(t) = f(0; Z(t))$. Then $M(t)$ is a bounded martingale and so

$$\begin{aligned} f(0; a) &= E[M(0)] = E[M(\infty)] = E[M(\infty); A] + E[M(\infty); A^c] \\ &= f(0; a) + E[M(\infty); A^c]. \end{aligned}$$

Therefore

$$E[M(\infty); A^c] = 0.$$

Next, define the martingale $M'(t)$ by $M'(t) = k(Z(t))$, $t \in Z^+$. Then

$$(42)\quad f(0; a) - f(a) = E[M'(\infty)] = E[M'(\infty); A^c] = E[M(\infty); A^c] = 0.$$

The third equality in (42) follows from the fact that $f(Z(t))$ converges a.s., and that on A^c, given any $n \in Z^-$, $Z(t) < n$ i.o. □

Claim 2 implies that

$$\limsup_{b\to\infty} b^{-1}\log[P^{e,b}\{X(t) \text{ enters } (0, \infty)\}] \le d_R \quad \text{a.s.,}$$

which implies (28) in Theorem 17.

<u>$d_R + d_{R+1} = 0$ implies (27) and (29).</u> First of all, notice that (28) holds iff

$\dim[V'(0)] = R$. To see this, first suppose that (28) holds. Then $\mathbf{f}_{0,j} \in V'(0)$ for $1 \le j \le R$. Since $\dim(V'(0)) \le R$, it must be that $\dim(V'(0)) = R$. For the converse, see the proof above that $d_R + d_{R+1} < 0$ implies (28). □

Therefore, (28) holds iff $d_R < 0$. Similarly, (30) holds iff $d_{R+1} > 0$. So, the only possibility is that $d_R + d_{R+1} = 0$ implies that neither (28) nor (30) hold.

7. Examples. In general the value $d_R + d_{R+1}$ of Theorem 22 is not calculable. However, there are two facts which make it possible to construct nontrivial examples of Random Walk in a Random Environment on Z for which $d_R + d_{R+1}$ can be calculated. These facts are

$$d_1 + d_2 + \cdots + d_{R+L} = E[\log|\det(\mathbf{A}_0)|] \tag{43}$$

(see Raghunathan, 1979, page 362),

(44) if b is a nonrandom eigenvalue of $\mathbf{A}_0$ then $\log|b| = d_j$ for some j (see Lemma 23).

(43) guarantees that a particular Random Walk in a Random Environment can be classified by Theorem 22 if $R + L - 1$ of the d_j can be calculated, for then one need only subtract their sum from $E[\log|\det(\mathbf{A}_0)|]$ to obtain the remaining d_j and then order $\{d_j\}_{1 \le j \le R+L}$ in order to apply Theorem 22. Since it is already known that either $d_R = 0$ or $d_{R+1} = 0$, it is only necessary to compute $R + L - 2$ additional values of d_j. For example, if $R = L = 1$, there is a complete classification of all RWIRE on Z to which Theorem 22 applies because

$$d_1 + d_2 = E\left\{\log\left[\frac{e(0;1)}{e(0;-1)}\right]\right\}. \tag{45}$$

Compare with Solomon, 1975, equation 1.7.

(44) guarantees that if the support of μ gives rise to an $\mathbf{A}_0$ with $R + L - 1$ nonrandom eigenvalues, then $R + L - 1$ of the d_j are calculable. The following is the main idea in the construction of the examples.

Denote the characteristic polynomial of $\mathbf{A}_0$ by $C(X)$. Notice that $C(X)$ is a polynomial with random coefficients. It is already known that $C(1) = 0$, and there are at most $R + L - 1$ additional roots of $C(X)$. Specifying $R + L - 1$ distinct nonrandom roots of $C(X)$ gives $R + L - 1$ equations for the coefficients of $C(X)$, which then may be solved to give all of the coefficients in terms of just two of the coefficients. Of course, there are restrictions on the roots since the $e(0; j)$ must be nonnegative, and $e(0, L)$ and $e(0; R)$ must be positive. Consider the following special cases.

EXAMPLE 1. $L = 1$ and $R = 2$. Suppose that $e(0; 0) = 0$ a.s. Then

$$C(X) = (X-1)\left\{X^2 - \left[\frac{e(0;1) + e(0;2)}{e(0;-1)}\right]X - \frac{e(0;2)}{e(0;-1)}\right\} = (X-1)M(X).$$

Since $M(0) < 0$ and $M(-1) > 0$, M must have a root in $(-1, 0)$. Call this root

α and let $e'(0; j)$ denote $e(0; j)/e(0; -1)$. Then $e'(0; 1)$ and $e'(0; 2)$ satisfy the equations

$$\begin{bmatrix} \alpha & \alpha + 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e'(0; 1) \\ e'(0; 2) \end{bmatrix} = \begin{bmatrix} \alpha^2 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/e(0; -1) \end{bmatrix}.$$

Solving these equations for $e(0; 1)$ and $e(0; 2)$ gives

$$e(0; 1) = [\alpha + 1] - [\alpha^2 + \alpha + 1]e(0; -1), \tag{46}$$

$$e(0; 2) = [\alpha^2 + \alpha]e(0; -1) - \alpha. \tag{47}$$

Since $e(0; 1) \geq 0$ and $e(0; 2) > 0$, α and $e(0; -1)$ must satisfy

$$0 < e(0; -1) \leq [1 + \alpha]/[\alpha^2 + \alpha + 1] < 1. \tag{48}$$

Having chosen α in $(-1, 0)$, if the measure μ is chosen so that $e(0; *)$ satisfies (46)–(48), then $\{d_1, d_2, d_3\} = \{\log |\alpha|, 0, E[\log(e(0;2)/e(0;-1))] - \log |\alpha|\}$. However,

$$\log\left(\frac{e(0; 2)}{e(0; -1)}\right) - \log |\alpha| = \log\left[-\alpha - 1 + \frac{1}{e(0; -1)}\right] > \log(-\alpha) = \log |\alpha|,$$

so $d_2 + d_3 = E[\log(-\alpha - 1 + 1/e(0; -1))]$. According to Theorem 22, if (46) and (47) hold w.p.1 then

$$E\left[\log\left(-\alpha - 1 + \frac{1}{e(0; -1)}\right)\right] > 0 \quad \text{implies} \quad X(t) \to +\infty \quad \text{a.s.} \quad [P],$$

$$E\left[\log\left(-\alpha - 1 + \frac{1}{e(0; -1)}\right)\right] = 0$$

$$\text{implies} \quad -\infty = \liminf X(t) < \limsup X(t) = +\infty \quad \text{a.s.} \quad [P];$$

$$E\left[\log(-\alpha - 1 + \frac{1}{e(0; 1)})\right] < 0 \quad \text{implies} \quad X(t) \to -\infty \quad \text{a.s.} \quad [P].$$

EXAMPLE 2. $L = 2$ and $R = 2$.
Again, suppose that $e(0; 0) = 0$ a.s. Then

$$C(X) = (X - 1)\left(X^3 + \frac{e(0; -2) + e(0; -1)}{e(0; 2)} X^2 - \frac{e(0; 1) + e(0; 2)}{e(0; -2)} X - \frac{e(0; 2)}{e(0; -2)}\right)$$

$$= (X - 1)M(X).$$

$M(-1) = (e(0; -2) + e(0; -1))/e(0; -2)$ and so $M(-1) = 0$ iff $e(0; -1) = e(0; 1) = 0$, which is really just the nearest neighbor case, i.e. $L = R = 1$. To avoid this well-understood example, suppose that $M(-1) > 0$. Furthermore, $M(0) < 0$, so M must have one root in $(-1, 0)$, call it α, and another root in $(-\infty, -1)$, call it β. Let $e'(0; j)$ denote $e(0; j)/e(0; -2)$. Then $e'(0; -1)$, $e'(0; 1)$ and $e'(0; 2)$

satisfy the set of equations

$$\begin{bmatrix} -\beta^2 & \beta & \beta+1 \\ -\alpha^2 & \alpha & \alpha+1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e'(0; -1) \\ e'(0; 1) \\ e'(0; 2) \end{bmatrix} = \begin{bmatrix} \beta^3+\beta^2 \\ \alpha^3+\alpha^2 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{e(0; -2)} \end{bmatrix}.$$

Solving these equations for $e(0; -1)$, $e(0; 1)$ and $e(0; 2)$ gives

$$e(0; -1) = \frac{1}{\alpha+\beta+1} - e(0; -2)\left[1 + \frac{\alpha^2+\alpha\beta+\beta^2}{\alpha+\beta+1}\right],$$

$$e(0; 1) = e(0; -2)\left[\frac{\alpha^2+(\alpha\beta)^2+\beta^2}{\alpha+\beta+1} + \alpha\beta\right] + \frac{\alpha+\alpha\beta+\beta}{\alpha+\beta+1},$$

$$e(0; 2) = -\frac{\alpha\beta}{\alpha+\beta+1}\,[1 + e(0; -2)(\alpha+\beta+\alpha\beta)].$$

These three formulae become less complicated with the substitutions $a = \alpha + \beta$ and $b = \alpha\beta$:

$$e(0; -1) = \frac{1}{a+1} - e(0; -2)\left[a + \frac{1-b}{a+1}\right]; \tag{49}$$

$$e(0; 1) = e(0; -2)\left[a + b - 1 + \frac{(b-1)^2}{a+1}\right] + 1 + \frac{b-1}{a+1}; \tag{50}$$

$$e(0; 2) = -\frac{b}{a+1}\,[1 + e(0; -2)(a+b)]. \tag{51}$$

Since $e(0; 2) > 0$, $e(0; -2) > 0$, and $e(0; 1) + e(0; -1) > 0$, $e(0; 2)$, a and b must satisfy

$$\frac{ab+b}{a+b} - a - b \le \frac{1}{e(0; -2)} < a^2 + a + 1 - b$$

(52) or

$$\frac{ab+b}{a+b} - a - b < \frac{1}{e(0; -2)} \le a^2 + a + 1 - b.$$

The inequality $(ab + b)/(a + b) - a - b < a^2 + a + 1 - b$ reduces to $a + b + 1 < 0$, which is equivalent to $(\alpha + 1)(\beta + 1) < 0$. This shows that every choice of $\beta < -1 < \alpha < 0$ is possible.

The example now continues parallel to Example 1. The aim is to show that

$$d(2) + d(3) = E\left[\log\left[\frac{e(0; 2)}{\alpha\beta e(0; -2}\right]\right].$$

The analysis will be in the same spirit as that of Example, 1, but more complicated. Applying the left hand inequalities in (52), and then (51) gives

$$\frac{e(0;\,2)}{\alpha\beta e(0;\,-2)} = -\left(\frac{1}{e(0;\,-2)} + a + b\right)\Big/ (a+1)$$
$$\geq -\left[a + b + \left(\frac{ab+b}{a+b} - a - b\right)\right]\Big/(a+1)$$
$$= -\frac{b}{a+b} = |\alpha|\,\frac{\beta}{\alpha+\beta+\alpha\beta} > |\alpha|.$$

Similarly,

$$\frac{e(0;\,2)}{\alpha\beta e(0;\,-2)} = -\left[\frac{1}{e(0;\,-2)} + a + b\right]\Big/ (a+1)$$
$$\leq -[a^2 + a + 1 - b + (a+b)]/(a+1)$$
$$= -[a+1] = -\alpha - 1 - \beta < |\beta|.$$

Therefore,

$$\log|\alpha| < \log\left[\frac{e(0;\,2)}{\alpha\beta e(0;\,-2)}\right] < \log|\beta|.$$

Since $d_2 = 0$ or $d_3 = 0$, and $\log|\alpha| < 0 < \log|\beta|$, it must be that

$$d_2 + d_3 = E\left\{\log\left[\frac{e(0;\,2)}{\alpha\beta e(0;\,-2)}\right]\right\}.$$

Therefore, if (49)–(51) hold w.p.1. then

$$E\left[\log\left(\frac{1}{e(0;\,-2)} + \alpha + \beta + \alpha\beta\right)\right] - \log(|\alpha+\beta+1|) > 0$$

implies $X(t) \to +\infty$ a.s. $[P]$,

$$E\left[\log\left(\frac{1}{e(0;\,-2)} + \alpha + \beta + \alpha\beta\right)\right] - \log(|\alpha+\beta+1|) = 0$$

implies $-\infty = \liminf X(t) < \limsup X(t) = +\infty$ a.s. $[P]$,

$$E\left[\log\left(\frac{1}{e(0;\,-2)} + \alpha + \beta + \alpha\beta\right)\right] - \log(|\alpha+\beta+1|) < 0$$

implies $X(t) \to -\infty$ a.s. $[P]$.

8. Hitting times. In the case where $R = 1$, the expected time for the RWIRE $X(t)$ to move from 0 to 1 can be computed explicitly. Let

$$T_n = \inf\{t > 0\colon X(t) = n\}$$

and

$$m(z) = \textstyle\sum_{a=-L}^{1} a e(z;\,a).$$

THEOREM 25. *$E\{T_1\} < \infty$ if and only if $E\{1/e(0; 1)\} < \infty$ and $E\{m(0)/e(0; 1)\} > 0$. Moreover, if $E\{1/e(0; 1)\} < \infty$ and $E\{m(0)/e(0; 1)\} > 0$, then $E\{T_1\} = E\{1/e(0; 1)\}[E\{m(0)/e(0; 1)\}]^{-1}$.*

A proof of Theorem 25 will only be given under the additional hypothesis that $e(0; 1) < 1$ a.s. This extra hypothesis simplifies the proof by eliminating the possibility of transient states which do not communicate with the state 1. The lemmas and propositions used to prove Theorem 25, along with their proofs, can be modified to allow $e(0; 1) = 1$ on a set of positive measure.

To prove Theorem 25, let $X^{\Gamma}(t)$ be the process obtained from $X(t)$ by reflecting $X(t)$ at 1. Given the environment e, $X^{\Gamma}(t)$ has transition function $p^e(x, y)$ satisfying

$$p^e(x, x + a) = e(x; a) \text{ if } x \leq 0 \text{ and } -L \leq a \leq 1, p^e(1, 0) = 1,$$

$$p^e(x, x + a) = 0 \text{ otherwise.}$$

Since the transition mechanism for $X^{\Gamma}(t)$ is the same as that of $X(t)$ on $(-\infty, 0]$, the distribution of the time for $X^{\Gamma}(t)$ to return to 1, starting from 1, is the same as the distribution of $1 + T_1$.

LEMMA 26. *For a fixed environment, the system of equations*

$$v(1) = 1; \tag{53}$$

$$v(y) = \textstyle\sum_{x \leq 1} v(x) p^e(x, y), \quad y \leq 1; \tag{54}$$

has a nonnegative solution $v(*)$, for which

$$\textstyle\sum_{y \leq 1} v(y) < \infty \tag{55}$$

if and only if $E\{T_1 \mid e\} < \infty$. Furthermore, if $E\{T_1 \mid e\} < \infty$ then the solution to (53), (54) *is unique and $E\{T_1 \mid e\} = \sum_{y \leq 0} v(y)$.*

PROOF. See Breiman, 1968, pages 143–145, and recall that given the environment, $X^{\Gamma}(t)$ is a Markov chain. □

Suppose that $e(0; 1) > 0$ a.s. Define the random variables $h(n)$, $n \in Z$ by

$$h(0) = 1; \qquad h(a) = 0,\ a \geq 1; \tag{56}$$

$$h(n - 1) = \textstyle\sum_{r=0}^{L-1} h(n + r) \sum_{p=r+1}^{L} \dfrac{e(n + r; -p)}{e(n + r; 1)}, \quad n \leq 0. \tag{57}$$

For $n \leq 0$, let $\mathscr{F}_n \equiv \sigma\{e(n; *), \cdots, e(0; *)\}$, the sigma field generated by $\{e(n; *), \cdots, e(0; *)\}$.

PROPOSITION 27. *For $n \leq 0$, $h(n - 1) \in \mathscr{F}_n$.*

PROOF. Induction.

PROPOSITION 28. *For* $n \le 0$,

$$E\{h(n-1)\} = \sum_{r=0}^{L-1} E\{h(n+r)\} \sum_{p=r+1}^{L} E\{e(0; -p)/e(0; 1)\}.$$

PROOF. Take expectations of the left and right sides of (57), and use Proposition 27 and the fact that the $e(z; *)$ are iid.

PROPOSITION 29. $E\{m(0)/e(0; 1)\} > 0$ *if and only if* $\sum_{z\le 0} E\{h(z)\} < \infty$.

PROOF. Suppose that $E\{m(0)/e(0; 1)\} > 0$. Then, by the definition of $m(0)$, $E\{e(0; a)/e(0; 1)\} < \infty$ for all $a < 0$. The characteristic polynomial of the equation

$$y(n-1) = \sum_{r=0}^{L-1} y(n+r) \sum_{p=r+1}^{L} E\left\{\frac{e(0; -p)}{e(0; 1)}\right\}$$

is

$$X^L - \sum_{r=1}^{L} X^{L-r} \sum_{p=r}^{L} E\left\{\frac{e(0; -p)}{e(0; 1)}\right\}. \tag{58}$$

It follows from the hypothesis $E\{m(0)/e(0; 1)\} > 0$ that if ρ is a root of (58), then $|\rho| < 1$. Therefore, there exists a constant $K > 0$ and a constant θ, $0 < \theta < 1$, such that $E\{h(n)\} \le K\theta^{-n}$, and so, $\sum_{z\le 0} E\{h(z)\} < \infty$.

On the other hand, suppose that $\sum_{z\le 0} E\{h(z)\} < \infty$. Then, by the definition of $h(-1)$, $E\{e(0; a)/e(0; 1)\} < \infty$ for all $a < 0$, so by Proposition 28,

$$\begin{aligned}\sum_{n\le 0} E\{h(n)\} &= 1 + \sum_{n\le 0} E\{h(n-1)\}\\ &= 1 + \sum_{n\le 0} \sum_{r=0}^{L-1} E\{h(n+r)\} \sum_{p=r+1}^{L} E\left\{\frac{e(0; -p)}{e(0; 1)}\right\}\\ &= 1 + \sum_{n\le 0} E\{h(n)\} \sum_{r=0}^{L-1} \sum_{p=r+1}^{L} E\left\{\frac{e(0; -p)}{e(0; 1)}\right\}.\end{aligned}$$

Therefore,

$$E\left\{\frac{m(0)}{e(0; 1)}\right\} \sum_{z\le 0} E\{h(z)\} = 1, \tag{59}$$

so $E\{m(0)/e(0; 1)\} > 0$.

LEMMA 30. *Suppose that* $e(0; 1) > 0$ a.s. *Then* $h(n)$ *is a nonnegative solution to the system of linear equations*

$$f(a) = 0, \quad a \ge 1; \qquad f(-1) = \sum_{a=1}^{L} \frac{e(0; -a)}{e(0; 1)}; \qquad f(0) = 1;$$

$$f(n-1) - f(n) = f(n) \sum_{a=1}^{L} \frac{e(n; -a)}{e(n; 1)} - \sum_{a=1}^{L} f(n+a) \frac{e(n+a; -a)}{e(n+a; 1)}, \quad n \le -1.$$

PROOF. The proof is left to the reader.

COROLLARY 31. *For* $n \le 0$ *let* $v(n) = h(n)/e(n; 1)$, *and let* $v(1) = 1$. *Then* $v(*)$ *is nonnegative and satisfies* (53) *and* (54). *Furthermore, if* $E\{m(0)/e(0; 1)\} > 0$ *and* $E\{1/e(0; 1)\} < \infty$ *then*

$$\sum_{n \le 0} E\{v(n)\} = E\left\{\frac{1}{e(0; 1)}\right\}\left[E\left\{\frac{m(0)}{e(0; 1)}\right\}\right]^{-1}, \tag{60}$$

and $v(*)$ *satisfies* (55).

PROOF. By definition, $v(*)$ satisfies (53), and direct computation shows that for $y = 0$ and $y = 1$, $v(*)$ satisfies (54). So suppose that $y \le -1$. Define $u(y) \equiv h(y)/e(y; 1)$. Note that for $y \le 0$, $u(y) = v(y)$. Lemma 30 shows that for $y \le -1$,

$$\begin{aligned} v(y) = u(y) &= \textstyle\sum_{a=-1}^{L} u(y+a)e(y+a; -a) = \sum_x u(x)e(x; y-x) \\ &= \textstyle\sum_{x \le 0} u(x)e(x; y-x) = \sum_{x \le 0} v(x)e(x; y-x) \\ &= \textstyle\sum_{x \le 0} v(x)p^e(x, y) = \sum_{x \le 1} v(x)p^e(x, y), \end{aligned}$$

where the last equality follows because $p^e(1, y) = 0$ for $y \le -1$. Therefore, $v(*)$ satisfies (54).

Proposition 27 shows that $E\{v(n)\} = E\{h(n)\}E\{1/e(0; 1)\}$. Therefore, Proposition 29 and (59) show that if $E\{m(0)/e(0; 1)\} > 0$ and $E\{1/e(0; 1)\} < \infty$ then $v(*)$ satisfies (60) and therefore, (55).

PROOF OF THEOREM 25. If $e(0; 1) = 0$ with positive probability then Theorem 25 is clearly true. So suppose that $e(0; 1) > 0$ a.s. $[\mu]$.

It follows from Corollary 31 and Lemma 26 that if $E\{1/e(0; 1)\} < \infty$ and $E\{m(0)/e(0; 1)\} > 0$ then $E\{T_1\} = E\{1/e(0; 1)\}[E\{m(0)/e(0; 1)\}]^{-1} < \infty$.

On the other hand, if $E\{T_1\} < \infty$, then $\sum_{n \le 0} E\{v(n)\} < \infty$ by Lemma 26 and Corollary 31. The definition of $v(*)$ shows that this implies that $E\{1/e(0; 1)\} < \infty$ and $\sum_{z \le 0} E\{h(z)\} < \infty$. Proposition 29 then shows that $E\{m(0)/e(0; 1)\} > 0$. □

In the case when $R = 1$ and $E\{T_1\} < \infty$ the random variables $T_{n+1} - T_n$, $n \ge 0$, form a stationary ergodic sequence. This observation gives the following corollary to Theorem 25.

COROLLARY 32. *If* $E\{1/e(0; 1)\} < \infty$ *and* $E\{m(0)/e(0; 1)\} > 0$, *then*

$$\lim_{n \to \infty} \frac{T_n}{n} = E\left\{\frac{1}{e(0; 1)}\right\}\left[\mathrm{E}\left\{\frac{m(0)}{e(0; 1)}\right\}\right]^{-1} \quad \text{a.s.}$$

PROOF. Apply Birkhoff's Ergodic Theorem.

9. Periodic environments. The methods of Sections 3, 4, 5, and 6 apply equally well to the following type of Markov chain.

DEFINITION 33. A Markov chain on the integers with transition matrix $P(*, *)$ is called a Random Walk in a Periodic Environment (RWIPE) if there exists an integer $N > 0$ such that for all $x, y \in Z$.

(61) $$P(x, y) = P(x + N, y + N).$$

The least positive integer N satisfying (61) is called the period.

A Random Walk in a Periodic Environment with period 1 is an ordinary random walk.

Let $P(*, *)$ be the transition matrix of a Random Walk in a Periodic Environment with period N, and let $e(x; a) = P(x, x + a)$. Then $e(x; *)$ is a probability measure on Z for each $x \in Z$, and

(62) $$e(x; a) = e(x + N; a)$$

for all $x, a \in Z$. Fix $L > 0$ and $R > 0$. Suppose that for all x,

(63) $$e(x; a) = 0 \quad \text{if} \quad a > R \quad \text{or} \quad a < -L,$$

and suppose there are two relatively prime positive integers, a and b, such that for all x,

(64) $$e(x; a) > 0 \quad \text{and} \quad e(x; -b) > 0.$$

THEOREM 34. *The zero-one law for periodic environments. Let $X(t)$ be a Random Walk in a Periodic Environment with period N, and suppose that* (63) *and* (64) *are satisfied. Then one of the following mutually exclusive possibilities holds*:

1. $\lim_{n\to\infty} X(t) = +\infty$ a.s.
2. $-\infty = \liminf_{n\to\infty} X(t) < \limsup_{n\to\infty} X(t) = +\infty$ a.s.
3. $\lim_{n\to\infty} X(t) = -\infty$ a.s.

PROOF. Adapt the line of argument in Sections 3 and 4 for random environments to periodic environments. There are two important changes to be made. To adapt Theorem 17 and its proof, note that the sequence

$$\{\max_{1\le j\le R}\{f(-pR + j; -pR + j)\}\}_{p\in Z^+}$$

is periodic instead of stationary and ergodic, so Birkhoff's Ergodic Theorem is no longer needed to prove the theorem. Regarding nonreducibility, (64) insures that the RWIPE is irreducible and that (16)–(20) are satisfied.

In order to give transience and recurrence criteria, proceed as in Section 5. Suppose that in addition to (63) and (64),

(65) $$e(x; R) > 0 \quad \text{and} \quad e(x; -L) > 0$$

for all $x \in Z$. Let

$$\mathbf{A}_x(R + L, k) = \frac{\delta(R + 1, k) - e(x; R + 1 - k)}{e(x; -L)},$$

and for each $x \in Z$, let $\mathbf{A}_x$ be the $(R + L) \times (R + L)$ real matrix defined by

$$\mathbf{A}_x(i, k) = \delta(i + 1, k) \quad \text{if} \quad 1 \le i < R + L,$$

$$\mathbf{A}_x(i, k) = \mathbf{A}_x(R + L, k) \quad \text{if} \quad i = R + L.$$

Let

$$\mathbf{M} = \mathbf{A}_{Ny+1}\mathbf{A}_{Ny+2} \cdots \mathbf{A}_{N(y+1)} \tag{66}$$

for any $y \in Z$. (62) insures that the sequence of matrices $\{\mathbf{A}_x\}_{x\in Z}$ is periodic with period N, so $\mathbf{M}$ is well-defined. Let

$$\{d_i\}_{1\le i\le R+L} \tag{67}$$

be the logarithms of the absolute values of the $R + L$ (not necessarily distinct) eigenvalues of $\mathbf{M}$ in nondecreasing order.

THEOREM 35. *Criteria for transience and recurrence of* RWIPE *on* Z. *Let* $X(t)$ *be a Random Walk in a Periodic Environment with period* N, *and suppose that* (63), (64), *and* (65) *are satisfied. Let* $\{d_i\}_{1\le i\le R+L}$ *be as in* (67).

If $d_R + d_{R+1} > 0$ *then* $P\{\lim_{t\to\infty} X(t) = \infty\} = 1$;

If $d_R + d_{R+1} = 0$ *then* $P\{-\infty = \liminf_{t\to\infty} X(t) < \limsup_{t\to\infty} X(t) = \infty\} = 1$;

If $d_R + d_{R+1} < 0$ *then* $P\{\lim_{t\to\infty} X(t) = -\infty\} = 1$.

PROOF. Adapt the line of argument in Sections 5 and 6 for random environments to periodic environments. There are two important changes to be made. Instead of using Oseledec's Multiplicative Ergodic Theorem one uses the fact that the matrices $\mathbf{A}_x$ are periodic. To prove that $d_R = 0$ or $d_{R+1} = 0$ (Lemma 24) one must show that if λ is an eigenvalue of $\mathbf{M}$ with multiplicity $n(\lambda)$ then λ^{-1} is an eigenvalue of $\mathbf{A}_{N-1}^{-1} \cdots \mathbf{A}_1^{-1}\mathbf{A}_0^{-1}$ with multiplicity $n(\lambda)$. This follows from the observations that since the $\mathbf{A}_x$ have period N,

$$\mathbf{M}\mathbf{A}_0^{-1}\mathbf{A}_{N-1}^{-1} \cdots \mathbf{A}_1^{-1} = \mathbf{I},$$

and that $\mathbf{A}_0^{-1}\mathbf{A}_{N-1}^{-1} \cdots \mathbf{A}_1^{-1}$ and $\mathbf{A}_{N-1}^{-1} \cdots \mathbf{A}_1^{-1}\mathbf{A}_0^{-1}$ have the same characteristic polynomial.

REMARKS. It is clear that (64) is not a necessary condition. For example, it could be replaced by the condition that there be some x such that $e(x; a) > 0$ for $-L \le a \le R$ and that the period N satisfies $N < R + L$. The problem of finding necessary and sufficient conditions for irreducibility and nonreducibility seems to be very complicated here.

In the case of ordinary random walk, $N = 1$ and $\mathbf{M} = \mathbf{A}_0$. If $C(X) \equiv \det(\mathbf{M} - X\mathbf{I})$ denotes the characteristic polynomial of the matrix $\mathbf{M}$, then

$$e(0; -L)C(X) = (-1)^{R+L}X^R[-1 + \textstyle\sum_{j=-R}^{L} e(0; -j)X^j] \tag{68}$$

and

(69) $$(-1)^{R+L+1}e(0; -L)C'(1) = \sum_{j=-L}^{R} je(0; j).$$

Suppose that (63)–(65) hold. Then (68), (69), Theorem 35 and the usual classification of a random walk as transient or recurrent in terms of the RHS of (69) combine to show that:

If $\sum_{j=-L}^{R} je(0; j) > 0$ then $C(X)$ has a single root at $X = 1$, $R - 1$ roots of modulus less than 1 and L roots of modulus greater than 1;

If $\sum_{j=-L}^{R} je(0; j) = 0$ then $C(X)$ has a double root at $X = 1$, $R - 1$ roots of modulus less than 1 and $L - 1$ roots of modulus greater than 1;

If $\sum_{j=-L}^{R} je(0; j) < 0$ then $C(X)$ has a single root at $X = 1$, R roots of modulus less than 1 and $L - 1$ roots of modulus greater than 1.

APPENDIX

Elementary number theory. Two number-theoretic propositions will be proved which permit the construction of special nonreducible Markov Chains.

If $f: D \to D$ is a function, $f^{(n)}: D \to D$ will denote the n-fold composition of f with itself. For example, when $n = 2$, $f^{(2)}(x) = f(f(x))$.

PROPOSITION 36. *Let a, b, and c be positive integers satisfying*

(70) $$gcd(a, b, c) = 1;$$

(71) $$a > c.$$

Let

$$d_1 = gcd(a, b),$$

let

$$D = \{0, 1, \cdots, a + b - 1\}$$

and let

$$C = \{0, d_1, 2d_1, \cdots, a + b - d_1\}.$$

Then there exists a function $f: D \to D$ satisfying

(72) $$f(x) = y \quad \text{implies that} \quad y - x \in \{a, -b, c\},$$

(73) $$\text{for each} \quad x \in D \quad \text{there exists an } n \text{ such that} \quad f^{(n)}(x) \in C,$$

(74) $$f(C) = C,$$

(75) $$\text{given} \quad c_1, c_2 \in C, \quad \text{there exists an } m \text{ such that} \quad f^{(m)}(c_1) = c_2.$$

REMARK. (72) implies that f has no fixed points and (75) implies that for any $c \in C$, $\{f^{(j)}(c), j = 1, \cdots, (a + b)/d_1\} = C$.

EXAMPLE. $a = 15$, $b = 10$, $c = 6$, $C = \{0, 5, 10, 15, 20\}$

$$\begin{array}{ccccccccccccc}
21 & & & & 17 & & 23 & & & & 19 & & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
11 & \rightarrow & 1 & \rightarrow & 7 & \rightarrow & 13 & \rightarrow & 3 & \rightarrow & 9 & \rightarrow & 15 \\
 & & & & & & & & & & & & \downarrow \\
 & & & & & & & & & & & & 5 \\
 & & & & & & & & & & & & \downarrow \\
16 & & 22 & & & & 18 & & 24 & & & & 20 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
6 & \rightarrow & 12 & \rightarrow & 2 & \rightarrow & 8 & \rightarrow & 14 & \rightarrow & 4 & \rightarrow & 10 \\
 & & & & & & & & & & & & \downarrow \\
 & & & & & & & & & & & & 0
\end{array}$$

PROOF. Proposition 36 will follow from the following two lemmas.

For x a nonnegative integer and y a positive integer, $(x)\bmod(y)$ will denote the least nonnegative integer r satisfying $x = ny + r$.

LEMMA 37. *Let a, b, and c be positive integers which satisfy* (70) *and* (71). *Let*

$$d_2 = gcd(b, c), \tag{76}$$

and define

$$y(j, n) = (nc + j)\mathrm{mod}(b + c)$$

for

$$n \in B \equiv \left\{0, 1, \cdots, \left(\frac{b + c}{d_2}\right) - 1\right\}. \tag{77}$$

For fixed j, define

$$A_j = \{(j)\mathrm{mod}(d_2) + kd_2 \colon k \in B\}. \tag{78}$$

Then

$$y(j, *) \quad \textit{is a one-to-one map from } B \textit{ onto } A_j. \tag{79}$$

PROOF. To show that this function is into A_j, note that for any integer n,

$$d_2 \mid [(nc + j)\mathrm{mod}(b + c) - (j)\mathrm{mod}(d_2)].$$

Therefore, for each n there exists an integer k such that

$$y(j, n) = (j)\mathrm{mod}(d_2) + kd_2. \tag{80}$$

Since

$$0 \le y(j, n) \le b + c - 1,$$

(80) shows that $y(j, n) \in A_j$.

To show that the function is one-to-one, and consequently onto, suppose that $y(j, n) = y(j, m)$. Then

(81) $$(b + c) \mid (nc - mc).$$

(81) implies that

(82) $$\left(\frac{nc - mc)}{d_2}\right)\text{mod}\left(\frac{b + c}{d_2}\right) = 0.$$

(76) implies that c/d_2 and $(b + c)/d_2$ are relatively prime. Therefore,

(83) $$\left(\frac{b + c}{d_2}\right) \Bigg| \; |n - m|.$$

(77) implies that

(84) $$|n - m| < \frac{b + c}{d_2}.$$

Combining (83) and (84) shows that $n = m$, so $y(j, *)$ is one-to-one. □

LEMMA 38. *If* $0 \le n < d_2$, *then* $y(nd_1, 0) = nd_1$.

PROOF. (70) implies that d_1 and d_2 are relatively prime. Therefore, $d_1 d_2 \mid b$. If $0 \le n < d_2$, then $nd_1 < b + c$, and so $(nd_1) \bmod(b + c) = nd_1$.

LEMMA 39. *If* $0 \le m, n < d_2$ *and* $m \ne n$, *then* $A_{nd_1} \cap A_{md_1} = \varnothing$.

PROOF. (70) implies that d_1 and d_2 are relatively prime. Lemma 39 now follows from the fact that in the ordering defined by (78), the consecutive elements of any fixed A_{nd_1} are d_2 units apart and $(nd_1)\text{mod}(d_2) = (md_1)\text{mod}(d_2)$iff $m = n$. (Compare this with (81)–(84).)

LEMMA 40. $\bigcup_{k=0}^{d_2-1} A_{kd_1} = \{0, 1, \cdots, b + c - 1\}$.

PROOF. From Corollary 39 and the fact that $\#(A_j) = (b + c)/d_2$ it follows that

(85) $$\#(\bigcup_{k=0}^{d_2-1} A_{kd_1}) = d_2\left(\frac{b + c}{d_2}\right) = b + c.$$

It follows from (78) that

(86) $$\bigcup_{k=0}^{d_2-1} A_{kd_1} \subset \{0, 1, \cdots, b + c - 1\}.$$

Combining (85) and (86) proves Lemma 40.

PROOF OF PROPOSITION 36. Lemmas 37 and 40 will be used to define a function f which satisfies (72)–(75).

Step 1. By Lemma 40, $\{A_{kd_1}\}_{0\le k<d_2}$ is a partition of the set $\{0, 1, \cdots, b + c - 1\}$. If $x \in A_{kd_1}$ for some k, then there exists an integer n such that $x = y(kd_1, n)$. Define $g(x) = y(kd_1, n + 1)$ for any $n \in Z$ such that $y(kd_1, n) = x$. The definition of $y(j, n)$ and (79) show that for fixed j, $y(j, n + 1)$ is uniquely determined by $y(j, n)$, so this definition of $g(x)$ is unambiguous. For *any* integers n and j,

$$[(n + 1)c + j]\mathrm{mod}(b + c) - [nc + j]\mathrm{mod}(b + c) \in \{c, -b\},$$

so $g(x) = y$ implies that $y - x = c$ or $-b$.

Finally, note that the restriction of g to A_{kd_1} is a one-to-one function onto A_{kd_1}. Lemmas 37 and 38 show that $kd_1 \in A_{kd_1}$, and consequently that for each $x \in A_{kd_1}$ there exists an m such that $g^{(m)}(x) = kd_1 \in C$. Therefore, for any $x \in \{0, 1, \cdots, b + c - 1\}$, there is an m such that $g^{(m)}(x) \in C$.

Step 2. $z(n) = (na)\mathrm{mod}(a + b)$ defines a one-to-one map from $\{0, 1, \cdots, ((a + b)/d_1) - 1\}$ onto C. (Change c to a and set $j = 0$ in Lemma 37.) If $x \in C$, define $f(x) = y$ if there exists an $n \in Z$ such that $z(n) = x$ and $z(n + 1) = y$. If $x \in \cup_{k=0}^{d_2-1} A_{kd_1}\backslash C$, define $f(x) = g(x)$.

Step 3. Finally, for $x \in \{b + c, \cdots, b + a - 1\}\backslash C$, define $f(x) = x - b$. This finishes the construction since $b < b + c$ guarantees that for any $x \in \{b + c, \cdots, b + a - 1\}\backslash C$ there exists an n such that $f^{(n)}(x) \in \{0, 1, \cdots b + c - 1\} \cup C$. It is straightforward to check that f so defined has the desired properties. □

PROPOSITION 41. *In Proposition* 36 (72) *may be replaced by*

$$f(x) = y \quad \textit{implies that} \quad y - x \in \{-a, b, -c\}.$$

PROOF. To prove that this modified version of Proposition 36 holds as well, it is necessary to make three modifications to the proof of the original proposition. Note that the function g defined in Step 2 is invertible, and that f is invertible when restricted to C. So to prove the modified version, replace g by its inverse in Step 2, replace f with its inverse on C in Step 2, and in Step 3, for $x \in \{b + c, \cdots, b + a - 1\}\backslash C$, define $f(x) = x - a$.

PROPOSITION 42. *Let* $\{a_j\}_{j=1}^{r}$ *and* $\{b_j\}_{j=1}^{m}$ *be sets of distinct positive integers satisfying*

$$a_r \ge a_j \quad \text{and} \quad b_m \ge b_j$$

$$gcd\{a_1, \cdots, a_r, b_1, \cdots, b_m\} = 1. \tag{87}$$

Let

$$d = gcd\{a_r, b_m\},$$

let

$$D = \{0, 1, \cdots, a_r + b_m - 1\}$$

and let

$$C' = \{0, d, 2d, \cdots, a_r + b_m - d\}.$$

Then there exists a function $f: D \to D$ *satisfying*

(88) $$f(x) = y \quad \text{implies that} \quad y - x \in \{a_j\}_{j=1}^{r} \cup \{-b_j\}_{j=1}^{m},$$

(89) $$\text{for each} \quad x \quad \text{there exists an } n \text{ such that} \quad f^{(n)}(x) \in C',$$

(90) $$f(C') = C',$$

(91) $$\text{given} \quad c_1, c_2 \in C', \quad \text{there exists an } m \text{ such that} \quad f^{(m)}(c_1) = c_2.$$

REMARK. Propositions 36 and 41 are special cases of Proposition 42.

PROOF. By induction on $N \equiv r + m$. If $N = 2$, Proposition 42 follows from the construction in Step 2 of the proof of Proposition 36. If $N = 3$, Proposition 42 follows from Proposition 36 or 41. Without loss of generality (if $r = 1 < m$, Proposition 41 and its proof may be substituted for Proposition 36 and its proof in the following argument), suppose that $r > 1$ and suppose that the result is true for $N - 1$. If $gcd\{a_2, \cdots, a_r, b_1, \cdots, b_m\} = 1$ the proof is finished. So suppose that

$$d_3 \equiv gcd\{a_2, \cdots, a_r, b_1, \cdots, b_m\} > 1.$$

Let

$$d_4 = gcd\{a_1, b_m\}.$$

Then (87) implies that

$$gcd\{d_3, d_4\} = 1.$$

Substitute d_3 for d_1, d_4 for d_2, a_1 for c and b_m for b in Step 1 of the proof of Proposition 36. This gives a function g from $\{0, 1, \cdots a_1 + b_m - 1\}$ to $\{0, 1, \cdots a_1 + b_m - 1\}$ with the properties that $g(x) = y$ implies that $y - x = a_1$ or $-b_m$, and that for each $x \in \{0, 1, \cdots a_1 + b_m - 1\}$ there exists an n such that $g^{(n)}(x) \in \{0, d_3, \cdots, a_1 + b_m - d_3\}$.

The induction hypothesis applied to $\{a_i/d_3\}_{i=2}^{r}$ and $\{b_i/d_3\}_{i=1}^{m}$ gives a function f from $\{0, d_3, 2d_3, \cdots, a_r + b_m - d_3\}$ to $\{0, d_3, 2d_3, \cdots, a_r + b_m - d_3\}$ which satisfies (88)–(91). So by analogy with Step 2 of the proof of Proposition 36, set

$$f(x) = g(x) \quad \text{for} \quad x \in \{0, 1, \cdots, a_1 + b_m - 1\} \backslash \{0, d_3, 2d_3, \cdots, a_r + b_m - d_3\}.$$

Finally, repeat Step 3 of the proof of Proposition 36 for

$$x \in \{a_1 + b_m, \cdots, a_r + b_m - 1\} \backslash \{0, d_3, 2d_3, \cdots, a_r + b_m - d_3\}. \quad \square$$

COROLLARY 43. *Suppose that* $\mu(G^L) = \mu(G^R) = 0$ *and* $gcd(\{z \neq 0: m(z) > 0\}) = 1$. *Let* F_i *be the event defined by* (14) *and* (15). *Then* $\Pr\{F_0\} > 0$.

PROOF. Let $\{a_j\}_{j=1}^{r} = \{z > 0: m(z) > 0\}$ and let $\{b_j\}_{j=1}^{m} = \{z > 0: m(-z) > 0\}$. Note that $a_r = R$ and $b_m = L$, and the a_j and b_j satisfy the hypotheses of Proposition 42. Let $f: \{0, 1, \cdots, R + L - 1\} \to \{0, 1, \cdots, R + L - 1\}$ satisfy (88)–(91) for this choice of the a_j and b_j. Let G_0 be the event that $e(x; a) > 0$ if $f(x) = x + a$ for $x \in \{0, 1, \cdots, R + L - 1\}$. By the choice of the a_j and b_j, (88), and the fact that for each x, G_0 only requires $e(x; a) > 0$ for *one* a, G_0 occurs with positive probability.

CLAIM. $G_0 \subset F_0$. The key idea here is that if G_0 occurs and $x \in \{0, 1, \cdots, R + L - 1\}$, then $f(x) = y$ implies that $y \in \{0, 1, \cdots, R + L - 1\}$ and $x \rightsquigarrow_{[0,R+L-1]} y$. Then, since $a_r = R$ and $b_m = L$, if G_0 occurs (90) and (91) show that (14) occurs, and (89) shows that (15) occurs. Therefore, $G_0 \subset F_0$.

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