Recurrence and transience criteria for two cases of stable-like Markov chains

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Abstract

We give recurrence and transience criteria for two cases of time-homogeneous Markov chains on the real line with transition kernel $p(x, dy) = f_x(y - x)dy$, where $f_x(y)$ are probability densities of symmetric distributions and, for large $|y|$, have a power-law decay with exponent $\alpha(x) + 1$, with $\alpha(x) \in (0, 2)$.

If $f_x(y)$ is the density of a symmetric α-stable distribution for negative x and the density of a symmetric β-stable distribution for non-negative x, where $\alpha, \beta \in (0, 2)$, then the chain is recurrent if and only if $\alpha + \beta \geq 2$.

If the function $x \longmapsto f_x$ is periodic and if the set $\{x : \alpha(x) = \alpha_0 := \inf_{x\in\mathbb{R}} \alpha(x)\}$ has positive Lebesgue measure, then, under a uniformity condition on the densities $f_x(y)$ and some mild technical conditions, the chain is recurrent if and only if $\alpha_0 \geq 1$.

Keywords and phrases: characteristics of semimartingale, Feller process, Harris recurrence, Markov chain, Markov process, recurrence, stable distribution, stable-like process, T-model, transience

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}^d$. Let us define $S_n := \sum_{i=1}^n Z_i$ and $S_0 := 0$. The sequence $\{S_n\}_{n\geq 0}$ is called a *random walk* with jumps $\{Z_n\}_{n\in\mathbb{N}}$. The random walk $\{S_n\}_{n\geq 0}$ is said to be *recurrent* if

$$\mathbb{P}\left(\liminf_{n\longrightarrow\infty} |S_n| = 0\right) = 1,$$

and *transient* if

$$\mathbb{P}\left(\lim_{n\longrightarrow\infty} |S_n| = \infty\right) = 1.$$

It is well known that every random walk is either recurrent or transient (see [Dur10, Theorem 4.2.1]). In the case $d = 1$, a symmetric α-stable random walk, i.e., a random walk with jump distribution with characteristic function $\varphi(\xi) = \exp(-\gamma|\xi|^\alpha)$, where $\alpha \in (0, 2]$ and $\gamma \in (0, \infty)$, is recurrent if and only if $\alpha \geq 1$ (see the discussion after [Dur10, Lemma 4.2.12]). For recurrence and transience properties of random walks see [Chu01, Dur10]. In this paper we generalize one-dimensional symmetric α-stable random walks in the way that the index of stability of the jump distribution depends on the current position, and we study the recurrence and transience property of the generalization.

From now on, using the notation from [ST94], we will write SαS for the one-dimensional symmetric α-stable distribution. Let us denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$ and by $\lambda(\cdot)$ the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Furthermore, let us introduce the notation $f(y) \sim g(y)$, when $y \longrightarrow y_0$, for $\lim_{y\longrightarrow y_0} f(y)/g(y) = 1$, where $y_0 \in [-\infty, \infty]$. Recall that if $f(y)$ is the density of SαS distribution with characteristic function $\varphi(\xi) = \exp(-\gamma|\xi|^\alpha)$, where $\alpha \in (0, 2)$ and $\gamma \in (0, \infty)$, then

$$f(y) \sim c_\alpha |y|^{-\alpha-1},$$

when $|y| \longrightarrow \infty$, where $c_1 = \frac{\gamma}{2}$ and $c_\alpha = \frac{\gamma}{\pi}\Gamma(\alpha + 1)\sin\left(\frac{\pi\alpha}{2}\right)$, for $\alpha \neq 1$, (see [ST94, Property 1.2.15]).

Let $\alpha : \mathbb{R} \longrightarrow (0, 2)$ and $c : \mathbb{R} \longrightarrow (0, \infty)$ be arbitrary functions and let $\{f_x\}_{x\in\mathbb{R}}$ be a family of probability densities on $\mathbb{R}$ satisfying:

(i) $x \longmapsto f_x(y)$ is a Borel measurable function for all $y \in \mathbb{R}$ and

(ii) $f_x(y) \sim c(x)|y|^{-\alpha(x)-1}$, for $|y| \longrightarrow \infty$.

We define a Markov chain $\{X_n\}_{n\geq 0}$ on $\mathbb{R}$ by the following transition kernel

$$p(x, dy) := f_x(y - x)dy. \tag{1.1}$$

Transition densities of the chain $\{X_n\}_{n\geq 0}$ are asymptotically equivalent to the densities of symmetric stable distributions. We call the Markov chain $\{X_n\}_{n\geq 0}$ a *stable-like Markov chain.*

For Borel measurable functions $\alpha : \mathbb{R} \longrightarrow (0, 2)$ and $\gamma : \mathbb{R} \longrightarrow (0, \infty)$, let $f_{(\alpha(x),\gamma(x))}(y)$ be the density of a S$\alpha(x)$S distribution given by the following characteristic function $\varphi(x; \xi) = \exp(-\gamma(x)|\xi|^{\alpha(x)})$. A special case of the stable-like chain $\{X_n\}_{n\geq 0}$ is a Markov chain $\{X_n^{\alpha(x)}\}_{n\geq 0}$ given by the following transition kernel

$$p(x, dy) := f_{(\alpha(x),\gamma(x))}(y - x)dy. \tag{1.2}$$

The stable-like chain $\{X_n^{\alpha(x)}\}_{n\geq 0}$ has state dependent stable jumps, i.e., it jumps from the state x by a S$\alpha(x)$S law.

The recurrence and transience problem for the stable-like chain $\{X_n\}_{n\geq 0}$ (the chain given by (1.1)) was already treated in [San12]. Using the Foster-Lyapunov drift criterion for recurrence and transience of Markov chains, under a uniformity condition on the densities $f_x(y)$ and some mild technical conditions (see conditions (C1)-(C5) in [San12]) it is proved that if $\liminf_{|x|\longrightarrow\infty} \alpha(x) > 1$, then the stable-like chain $\{X_n\}_{n\geq 0}$ is recurrent, and if $\limsup_{|x|\longrightarrow\infty} \alpha(x) < 1$, then the stable-like chain $\{X_n\}_{n\geq 0}$ is transient. Results in [San12] give us only sufficient conditions for recurrence and transience. In this paper we treat two special cases of the stable-like chain $\{X_n\}_{n\geq 0}$ not covered in [San12], and give their recurrence and transience criteria. For recurrence and transience properties of Markov chains on general state space see [MT93b].

As already mentioned, we treat only two special cases of stable-like chains:

(i) Let $\alpha, \beta \in (0, 2)$ and $\gamma, \delta \in (0, \infty)$ be arbitrary. Let $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$ be a stable-like chain given by transition densities with following characteristic functions

$$\varphi(x; \xi) = \begin{cases} \exp(-\gamma|\xi|^\alpha), & x < 0 \\ \exp(-\delta|\xi|^\beta), & x \geq 0. \end{cases} \tag{1.3}$$

(ii) Let $\alpha : \mathbb{R} \longrightarrow (0,2)$ and $c : \mathbb{R} \longrightarrow (0,\infty)$ be arbitrary Borel measurable functions and let $\{f_x\}_{x\in\mathbb{R}}$ be an arbitrary family of probability densities on $\mathbb{R}$ with $f_x(-y) = f_x(y)$ for all $x, y \in \mathbb{R}$. Furthermore, let us assume that the function $x \longmapsto f_x$ is a periodic function with period $\tau > 0$ and that the following conditions are satisfied:

(PC1) the function $(x, y) \longmapsto f_x(y)$ is continuous and strictly positive;

(PC2) $f_x(y) \sim c(x)|y|^{-\alpha(x)-1}$, when $|y| \longrightarrow \infty$, for all $x \in \mathbb{R}$;

(PC3) $\lim_{|y|\longrightarrow\infty} \sup_{x\in[0,\tau]} \left| f_x(y)\frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| = 0$;

(PC4) $\inf_{x\in[0,\tau]} c(x) > 0$.

Let $\{X^p_n\}_{n\geq 0}$ be a stable-like chain, called a *periodic stable-like chain*, given by the transition kernel

$$p(x, dy) := f_x(y - x)dy. \tag{1.4}$$

Note that τ-periodicity of the function $x \longrightarrow f_x$ implies τ-periodicity of the functions $\alpha(x)$ and $c(x)$. Indeed, let $x \in \mathbb{R}$ be arbitrary, then, by (PC2), we have:

$$\begin{aligned} 1 &= \lim_{|y|\longrightarrow\infty} f_{x+\tau}(y)\frac{|y|^{\alpha(x+\tau)+1}}{c(x+\tau)} \\ &= \lim_{|y|\longrightarrow\infty} \left(f_x(y)\frac{|y|^{\alpha(x)+1}}{c(x)}\frac{c(x)}{c(x+\tau)}|y|^{\alpha(x+\tau)-\alpha(x)} \right) \\ &= \frac{c(x)}{c(x+\tau)} \lim_{|y|\longrightarrow\infty} |y|^{\alpha(x+\tau)-\alpha(x)}. \end{aligned}$$

Therefore, both stable-like chains $\{X^{(\alpha,\beta)}_n\}_{n\geq 0}$ and $\{X^p_n\}_{n\geq 0}$ satisfy conditions (C1)-(C5) from [San12]. In particular, both stable-like chains $\{X^{(\alpha,\beta)}_n\}_{n\geq 0}$ and $\{X^p_n\}_{n\geq 0}$ are irreducible with respect to the Lebesgue measure (see [San12, Proposition 2.1]). Thus, we have recurrence-transience dichotomy in both cases. Further, together with the τ-periodicity of the function $c(x)$, condition (PC3) implies

$$\sup_{x\in[0,\tau]} c(x) = \sup_{x\in\mathbb{R}} c(x) < \infty \tag{1.5}$$

(see [San12, Remark 1.1]).

From now on, we assume that the stable-like chain $\{X_n\}_{n\geq 0}$ (the chain given by (1.1)) satisfies conditions (C1)-(C5). Note that, in general, this is not the case for the stable-like chain $\{X^{\alpha(x)}_n\}_{n\geq 0}$ given by (1.2) (for sufficient conditions see [San12, Proposition 5.5]). We refer the reader to [San12] for more details about conditions (C1)-(C5).

An example of the periodic stable like-chain $\{X^p_n\}_{n\geq 0}$ satisfying conditions (PC1)-(PC4) is given as follows: Let $\alpha : \mathbb{R} \longrightarrow (0,2)$ be an arbitrary continuous periodic function with period $\tau > 0$ and define the family of density functions $\{f_x\}_{x\in\mathbb{R}}$ on $\mathbb{R}$ by

$$f_x(y) := \begin{cases} \frac{1}{2}\frac{\alpha(x)}{\alpha(x)+1}, & |y| \leq 1 \\ \frac{1}{2}\frac{\alpha(x)}{\alpha(x)+1}|y|^{-\alpha(x)-1}, & |y| \geq 1 \end{cases}$$

for all $x \in \mathbb{R}$. In this case $c(x) = \frac{1}{2}\frac{\alpha(x)}{\alpha(x)+1}$.

Now, let us state the main results of this paper:

Theorem 1.1. *The stable-like chain* $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$ *is recurrent if and only if* $\alpha+\beta\geq 2$.

Theorem 1.2. *If the set* $\{x : \alpha(x) = \alpha_0 := \inf_{x\in\mathbb{R}} \alpha(x)\}$ *has positive Lebesgue measure, then the periodic stable-like chain* $\{X_n^p\}_{n\geq 0}$ *is recurrent if and only if* $\alpha_0 \geq 1$.

As a simple consequence of Theorems 1.1 and 1.2 we get the following well-known recurrence and transience criterion for the random walk case:

Corollary 1.3. *A S*α*S random walk on the real line is recurrent if and only if* $\alpha \geq 1$.

The same problem was already treated, but in continuous-time case, in [Böt11] and [Fra06, Fra07]. In [Böt11] it is proved that the stable-like process $\{X_t^{(\alpha,\beta)}\}_{t\geq 0}$ with the symbol $p(x,\xi) = \gamma(x)|\xi|^{\alpha(x)}$ is recurrent if and only if $\alpha+\beta\geq 2$, where $\alpha : \mathbb{R}\longrightarrow (0,2)$ and $\gamma : \mathbb{R}\longrightarrow (0,\infty)$ are continuously differentiable functions with bounded derivative such that

$$\alpha(x) = \begin{cases} \alpha, & x<-k \\ \beta, & x>k \end{cases} \quad \text{and} \quad \gamma(x) = \begin{cases} \gamma, & x<-k \\ \delta, & x>k \end{cases}$$

for $\alpha,\beta \in (0,2)$, $\gamma,\delta \in (0,\infty)$ and $k>0$. In [Fra06] the author considers the recurrence and transience problem of the stable-like process $\{X_t^p\}_{t\geq 0}$ with the symbol $p(x,\xi) = \gamma(x)|\xi|^{\alpha(x)}$, where $\alpha : \mathbb{R}\longrightarrow (0,2)$ and $\gamma : \mathbb{R}\longrightarrow (0,\infty)$ are continuously differentiable and periodic functions with bounded derivative, and proves that if the set $\{x\in\mathbb{R} : \alpha(x) = \alpha_0 := \inf_{x\in\mathbb{R}}\alpha(x)\}$ has positive Lebesgue measure, then the process is recurrent if and only if $\alpha_0\geq 1$. Both results and technics, in [Böt11] and [Fra06], will be crucial in proving our results.

Now we explain our strategy of proving the main results. In [Böt11] it is proved that the stable-like process $\{X_t^{(\alpha,\beta)}\}_{t\geq 0}$ is recurrent if and only if $\alpha+\beta\geq 2$, and in [BS09] it is proved that $\{X_t^{(\alpha,\beta)}\}_{t\geq 0}$ can be approximated by a sequence of Markov chains $\{X_n^{(m)}\}_{n\geq 0}$, $m\in\mathbb{N}$, such that $\{X_n^{(1)}\}_{n\geq 0} \overset{\mathrm{d}}{=} \{X_n^{(\alpha,\beta)}\}_{n\geq 0}$. In Theorem 1.1 we prove that all chains $\{X_n^{(m)}\}_{n\geq 0}$, $m\in\mathbb{N}$, are either recurrent or transient at the same time and we prove that their recurrence property is equivalent with the recurrence property of the stable-like process $\{X_t^{(\alpha,\beta)}\}_{t\geq 0}$. This accomplishes the proof of Theorem 1.1. In Theorem 1.2 we subordinate the periodic stable-like chain $\{X_n^p\}_{n\geq 0}$ with the Poisson process $\{N_t\}_{t\geq 0}$ with parameter 1 and, following the ideas form [Fra06], prove that the sequence of strong Markov processes $\{n^{-\frac{1}{\alpha_0}} X_{N_{nt}}^p\}_{t\geq 0}$, $n\in\mathbb{N}$, converges in distribution, with respect to the Skorohod topology, to symmetric α_0-stable Lévy process. Furthermore, we prove that all the processes $\{n^{-\frac{1}{\alpha_0}} X_{N_{nt}}^p\}_{t\geq 0}$, $n\in\mathbb{N}$, are either recurrent or transient at the same time, their recurrence property is equivalent with the recurrence property of a symmetric α_0-stable Lévy process and recurrence properties of the process $\{X_{N_t}^p\}_{t\geq 0}$ and the periodic stable-like chain $\{X_n^p\}_{n\geq 0}$ are equivalent. This accomplishes the proof of Theorem 1.2.

Let us remark that the idea of studying recurrence and transience property of a Markov process in terms of the property of the associated Markov chain is studied in [TT79].

The paper is organized as follows. In Section 2 we introduce some preliminary and auxiliary results which will be needed to make the connection with results proved in [Böt11] and [Fra06]. In Sections 3 and 4 we give proofs of Theorems 1.1 and 1.2 and in Section 5 we treat discrete version of the stable-like chains $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$ and $\{X_n^p\}_{n\geq 0}$ and we derive the same recurrence and transience criteria as in Theorems 1.1 and 1.2.

Throughout the paper we use the following notation. We write $\mathbb{Z}_+$ and $\mathbb{R}_+$, respectively, for nonnegative integers and nonnegative real numbers. For $x,y\in\mathbb{R}$ let $x\wedge y := \min\{x,y\}$ and $x\vee y := \max\{x,y\}$. For two functions $f(x)$ and $g(x)$ we write $f(x) = o(g(x))$, when $x\longrightarrow x_0$, if $\lim_{x\longrightarrow x_0} f(x)/g(x) = 0$, where $x_0\in[-\infty,\infty]$. Write $B_b(\mathbb{R})$, $C(\mathbb{R})$, $C_b(\mathbb{R})$, and $C_0(\mathbb{R})$, respectively, for the sets of bounded Borel measurable functions, continuous functions, continuous bounded functions and continuous functions vanishing at infinity. Together with the supnorm $||\cdot||_\infty := \sup_{x\in\mathbb{R}}|\cdot|$, $B_b(\mathbb{R})$, $C_b(\mathbb{R})$ and $C_0(\mathbb{R})$ are a Banach spaces. Furthermore, $(\{X_n\}_{n\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}})$, $(\{X_n^{\alpha(x)}\}_{n\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}})$, $(\{X_n^{(\alpha,\beta)}\}_{n\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}})$ and $(\{X_n^p\}_{n\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}})$ will denote the stable-like chains on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ given by (1.1), (1.2), (1.3) and (1.4), respectively, while $(\{Y_n\}_{n\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}})$ and $(\{Y_t\}_{t\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}})$ will denote an arbitrary Markov chain and an arbitrary càdlàg strong Markov process on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ given by transition kernels $p(x,B)$ and $p^t(x,B)$, for $x\in\mathbb{R}$, $B\in\mathcal{B}(\mathbb{R})$ and $t\in\mathbb{R}_+$, respectively. Using the notation from [Twe94], we use the term *Markov model* and notation $\{Y_t\}_{t\in\mathbb{T}}$, where $\mathbb{T}$ is either $\mathbb{Z}_+$ or $\mathbb{R}_+$, when the result holds regardless of the time set involved. For $x\in\mathbb{R}$, $B\in\mathcal{B}(\mathbb{R})$ and $n\in\mathbb{N}$, let $p^n(x,B) := \mathbb{P}^x(Y_n\in B)$. For $x\in\mathbb{R}$ and $B\in\mathcal{B}(\mathbb{R})$ we put $\eta_B := \sum_{n=0}^{\infty} 1_{\{Y_n\in B\}}$ or $\eta_B := \int_0^\infty 1_{\{Y_t\in B\}}dt$, $\tau_B := \inf\{n\geq 0 : Y_n\in B\}$ or $\tau_B := \inf\{t\geq 0 : Y_t\in B\}$, $Q(x,B) := \mathbb{P}^x(\eta_B=\infty)$, $L(x,B) := \mathbb{P}^x(\tau_B<\infty)$ and $U(x,B) := \mathbb{E}^x(\eta_B)$.

2 Preliminary and auxiliary results

In this section we give some preliminary and auxiliary results needed for proving the main results of this paper.

Definition 2.1. *A Markov model $\{Y_t\}_{t\in\mathbb{T}}$ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is* φ-irreducible *if there exists a probability measure $\varphi(\cdot)$ on $\mathcal{B}(\mathbb{R})$ such that, whenever $\varphi(B)>0$, we have $U(x,B)>0$ for all $x\in\mathbb{R}$.*

Note that the stable-like chains $\{X_n\}_{n\geq 0}$ and $\{X_n^{\alpha(x)}\}_{n\geq 0}$ (the chains given by (1.1) and (1.2)) are $\varphi-$irreducible for any probability measure $\varphi(\cdot)$ on $\mathcal{B}(\mathbb{R})$ which is absolutely continuous with respect to the Lebesgue measure (see [San12, Proposition 2.1]).

In [Twe94, Theorem 2.1] it is shown that the irreducibility measure can always be maximized. If $\{Y_t\}_{t\in\mathbb{T}}$ is a φ-irreducible Markov model on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, then there exists a probability measure $\psi(\cdot)$ on $\mathcal{B}(\mathbb{R})$ such that the model $\{Y_t\}_{t\in\mathbb{T}}$ is ψ-irreducible and $\bar\varphi \ll \psi$, for every irreducibility measure $\bar\varphi(\cdot)$ on $\mathcal{B}(\mathbb{R})$ of the model $\{Y_t\}_{t\in\mathbb{T}}$. The measure $\psi(\cdot)$ is called the *maximal irreducibility measure* and from now on, when we refer to the irreducibility measure we actually refer to the maximal irreducibility measure. For the ψ-irreducible Markov model $\{Y_t\}_{t\in\mathbb{T}}$ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ set $\mathcal{B}^+(\mathbb{R}) = \{B\in\mathcal{B}(\mathbb{R}) : \psi(B)>0\}$. The maximal irreducibility measure for the stable-like chains $\{X_n\}_{n\geq 0}$ and $\{X_n^{\alpha(x)}\}_{n\geq 0}$ is equivalent, in absolutely continuous sense, with the Lebesgue measure (see [San12, Proposition 2.1]).

Recall that a function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is called *lower semicontinuous* if $\liminf_{y\longrightarrow x} f(y)\geq f(x)$ holds for all $x\in\mathbb{R}$.

Definition 2.2. *Let $\{Y_t\}_{t\in\mathbb{T}}$ be a Markov model on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$.*

(i) A set $B\in\mathcal{B}(\mathbb{R})$ is uniformly transient *if there exists a finite constant $M\geq 0$ such that $U(x,B)\leq M$ holds for all $x\in\mathbb{R}$. The model $\{Y_t\}_{t\in\mathbb{T}}$ is* transient *if it is ψ-irreducible and if there exists a countable cover of $\mathbb{R}$ with uniformly transient sets.*

(ii) A set $B\in\mathcal{B}(\mathbb{R})$ is recurrent *if $U(x,B)=\infty$ holds for all $x\in\mathbb{R}$. The model $\{Y_t\}_{t\in\mathbb{T}}$ is* recurrent *if it is ψ-irreducible and if every set $B\in\mathcal{B}^+(\mathbb{R})$ is recurrent.*

(iii) A set $B\in\mathcal{B}(\mathbb{R})$ is Harris recurrent*, or* H-recurrent*, if $Q(x,B)=1$ holds for all $x\in\mathbb{R}$. The model $\{Y_t\}_{t\in\mathbb{T}}$ is* H-recurrent *if it is ψ-irreducible and if every set $B\in\mathcal{B}^+(\mathbb{R})$ is H-recurrent.*

(iv) *The model* $\{Y_t\}_{t\in\mathbb{T}}$ *is called a* T-model *if for some distribution* $a(\cdot)$ *on* $\mathbb{T}$ *there exists a kernel* $T(x, B)$ *with* $T(x,\mathbb{R}) > 0$ *for all* $x \in \mathbb{R}$*, such that the function* $x \longmapsto T(x, B)$ *is lower semicontinuous for all* $B \in \mathcal{B}(\mathbb{R})$*, and*

$$\int_{\mathbb{T}} p^t(x, B)a(dt) \geq T(x, B)$$

holds for all $x \in \mathbb{R}$ *and all* $B \in \mathcal{B}(\mathbb{R})$*.*

Let us remark that the H-recurrence property can be defined in the equivalent way: The model $\{Y_t\}_{t\in\mathbb{T}}$ is H-recurrent if it is ψ-irreducible and if $L(x, B) = 1$ holds for all $x \in \mathbb{R}$ and all $B \in \mathcal{B}^+(\mathbb{R})$ (see [Twe94, Theorem 2.4]). In general, recurrence and H-recurrence properties are not equivalent (see [Twe94, Chapter 9]). Obviously, H-recurrence implies recurrence. In the case of a Markov model which is a λ-irreducible T-model, these two properties are equivalent (see [San12, Proposition 5.3] and [Böt11, Theorem 4.2]).

In the following proposition, by assuming certain continuity properties, we determine "nice" sets for Markov models.

Proposition 2.3. *Let* $\{Y_t\}_{t\in\mathbb{T}}$ *be a* ψ*-irreducible Markov model, then:*

(i) *the model* $\{Y_t\}_{t\in\mathbb{T}}$ *is either recurrent or transient.*

In addition, if we assume that $\{Y_t\}_{t\in\mathbb{T}}$ *is a T-model, then:*

(ii) *the model* $\{Y_t\}_{t\in\mathbb{T}}$ *is H-recurrent if and only if there exists a H-recurrent compact set.*

(iii) *assume the following additional assumption in the continuous-time case: for every compact set* $C \in \mathcal{B}(\mathbb{R})$ *there exists a distribution* $a_C(\cdot)$ *on* $\mathbb{R}_+$*, such that*

$$\inf_{x\in C} \int_0^\infty \mathbb{P}^x(X_t \in B)a_C(dt) > 0 \tag{2.1}$$

holds for all $B \in \mathcal{B}^+(\mathbb{R}^d)$*. Then the model* $\{Y_t\}_{t\in\mathbb{T}}$ *is transient if and only if every compact set is uniformly transient.*

(iv) *under the assumption (2.1) for the continuous-time case, the model* $\{Y_t\}_{t\in\mathbb{T}}$ *is recurrent if and only if there exists a recurrent compact set.*

Proof. (i) The proof is given in [Twe94, Theorem 2.3].

(ii) The proof is given in [MT93b, Proposition 9.1.7] and [MT93a, Theorem 3.3].

(iii) The proof for the discrete-time case is given in [MT93b, Theorems 8.3.5]. If the process $\{Y_t\}_{t\geq 0}$ is transient, then there exists at least one uniformly transient set $B \in \mathcal{B}^+(\mathbb{R})$. By assumption (2.1),

$$\delta_C := \inf_{x\in C} \int_0^\infty \mathbb{P}^x(X_t \in B)a_C(dt) > 0$$

holds for every compact set $C \in \mathcal{B}(\mathbb{R})$. Using the Chapman-Kolmogorov equation we have:

$$\begin{aligned} U(x, B) &= \int_0^\infty U(x, B)a_C(dt) = \int_0^\infty \int_0^\infty p^s(x, B)ds a_C(dt) \\ &\geq \int_0^\infty \int_0^\infty p^{s+t}(x, B)ds a_C(dt) = \int_0^\infty \int_0^\infty \int_{\mathbb{R}} p^s(x, dy)p^t(y, B)ds a_C(dt) \\ &\geq \delta_C \int_0^\infty \int_C p^s(x, dy)ds = \delta_C U(x, C). \end{aligned}$$

(iv) The proof follows directly from (i) and (iii).

□

Now, we derive the recurrence and transience dichotomy by using sample-paths properties of Markov models. Let $B \in \mathcal{B}(\mathbb{R})$ be arbitrary and let $\mathbb{D}(\mathbb{R})$ be the space of real-valued càdlàg functions equipped with the Skorohod topology. In the continuous-time case, define the *set of recurrent paths* by:

$$R(B) := \{\omega \in \mathbb{D}(\mathbb{R}) : \forall n \in \mathbb{N},\ \exists t \geq n \text{ such that } \omega(t) \in B\},$$

and the *set of transient paths* by:

$$T(B) := \{\omega \in \mathbb{D}(\mathbb{R}) : \exists s \geq 0 \text{ such that } \omega(t) \notin B,\ \forall t \geq s\}.$$

It is clear that $T(B) = R(B)^c$, and for any open set $O \subseteq \mathbb{R}$, by the right continuity, $R(O)$ and $T(O)$ are measurable (with respect to the Borel σ-algebra generated by the Skorohod topology). In the discrete-time case, using the same notation, we similarly define the *set of recurrent paths* by:

$$R(B) := \{\omega \in \mathbb{R}^{\mathbb{Z}_+} : \forall n \in \mathbb{N},\ \exists m \geq n \text{ such that } \omega(m) \in B\},$$

and the *set of transient paths* by:

$$T(B) := \{\omega \in \mathbb{R}^{\mathbb{Z}_+} : \exists m \geq 0 \text{ such that } \omega(n) \notin B,\ \forall n \geq m\}.$$

Clearly, $T(B) = R(B)^c$ and for any $B \in \mathcal{B}(\mathbb{R})$, $R(B)$ and $T(B)$ are $\mathcal{B}(\mathbb{R})^{\mathbb{Z}_+}$ measurable.

Proposition 2.4. *Let $\boldsymbol{Y} = \{Y_t\}_{t\in\mathbb{T}}$ be a λ-irreducible T-model, and let us assume (2.1) holds for the continuous-time case. Then the following 0-1 property must be met:*

$$\mathbb{P}^x(\eta_O = \infty) = 0 \quad \textit{for all } x \in \mathbb{R} \textit{ and all open bounded sets } O \subseteq \mathbb{R}$$

or

$$\mathbb{P}^x(\eta_O = \infty) = 1 \quad \textit{for all } x \in \mathbb{R} \textit{ and all open bounded sets } \quad O \subseteq \mathbb{R}.$$

In particular, the model $\boldsymbol{Y}$ is recurrent if and only if $\mathbb{P}^x_{\boldsymbol{Y}}(R(O)) = 1$ for all $x \in \mathbb{R}$ and all open bounded sets $O \subseteq \mathbb{R}$, and it is transient if and only if $\mathbb{P}^x_{\boldsymbol{Y}}(T(O)) = 1$ for all $x \in \mathbb{R}$ and all open bounded sets $O \subseteq \mathbb{R}$.

Proof. The 0-1 property in the discrete-time case follows from [San12, Proposition 5.3] and [MT93b, Theorems 6.2.5 and 8.3.5]. The claim in the continuous-time case follows from Proposition 2.3, [Twe94, Theorem 5.1] and [Böt11, Theorem 4.2]. Now, the characterization by sample paths easily follows from the 0-1 property and [MT93a, Theorem 3.3]. □

As already mentioned, the stable-like chains $\{X_n\}_{n\geq 0}$ and $\{X_n^{\alpha(x)}\}_{n\geq 0}$ (the chains given by (1.1) and (1.2)) are λ-irreducible and, by [San12, Proposition 5.2], the stable-like chain $\{X_n\}_{n\geq 0}$ is a T-model. In the following proposition we give sufficient conditions for the stable-like chain $\{X_n^{\alpha(x)}\}_{n\geq 0}$ to be a T-model.

Proposition 2.5. *Let $\alpha : \mathbb{R} \longrightarrow (0,2)$ and $\gamma : \mathbb{R} \longrightarrow (0,\infty)$ be continuous functions. Then the stable-like chain $\{X_n^{\alpha(x)}\}_{n\geq 0}$ is a T-model. In particular, $\{X_n^{\alpha(x)}\}_{n\geq 0}$ is H-recurrent if and only if it is recurrent.*

Proof. Let us define $a(\cdot) := \delta_1(\cdot)$ and $T(x,B) := p(x,B)$ for $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$. We prove that the function $x \longmapsto T(x,B)$ is lower semicontinuous for every $B \in \mathcal{B}(\mathbb{R})$. Let $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$ be arbitrary and such that $\lambda(B) < \infty$. By the dominated convergence theorem and continuity of the functions $\alpha(x)$ and $\gamma(x)$ we have

$$\begin{aligned}\lim_{y\longrightarrow x} p(y,B) &= \lim_{y\longrightarrow x}\int_B f_{(\alpha(y),\gamma(y))}(z-y)dz\\ &= (2\pi)^{-1}\lim_{y\longrightarrow x}\int_B\int_{\mathbb{R}}\cos(\xi(z-y))e^{-\gamma(y)|\xi|^{\alpha(y)}}d\xi dz\\ &= \int_B\int_{\mathbb{R}}\cos(\xi(z-x))e^{-\gamma(x)|\xi|^{\alpha(x)}}d\xi dz\\ &= p(x,B).\end{aligned}$$

Let $B \in \mathcal{B}(\mathbb{R})$ be arbitrary, then, by Fatou's lemma, we have

$$\liminf_{y\longrightarrow x} p(y,B) = \liminf_{y\longrightarrow x}\sum_{n\in\mathbb{Z}} p(y, B\cap(n,n+1]) \geq \sum_{n\in\mathbb{Z}} p(x, B\cap(n,n+1]) = p(x,B). \tag{2.2}$$

□

Recall that a Markov model $\{Y_t\}_{t\in\mathbb{T}}$ is said to satisfy the *C_b-Feller property* if for all $f \in C_b(\mathbb{R})$ and all $t \in \mathbb{T}$ the function $x \longmapsto \int_{\mathbb{R}} p^t(x,dy)f(y)$ is in the space $C_b(\mathbb{R})$. Furthermore, a Markov model $\{Y_t\}_{t\in\mathbb{T}}$ is said to satisfy the *strong Feller property* if for all $f \in B_b(\mathbb{R})$ and all $t \in \mathbb{T}\setminus\{0\}$ the function $x \longmapsto \int_{\mathbb{R}} p^t(x,dy)f(y)$ is in the space $C_b(\mathbb{R})$. In [MT93b, Proposition 6.1.1] it is shown that the C_b-Feller property (respectively the strong Feller property) of a Markov model is equivalent with the lower semicontinuity of the function $x \longmapsto p^t(x,O)$ (respectively the function $x \longmapsto p^t(x,B)$) for all open sets $O \subseteq \mathbb{R}$ (respectively all Borel sets $B \subseteq \mathbb{R}$) and all $t \in \mathbb{T}\setminus\{0\}$. Note that (2.2) reads that the stable-like chain $\{X_n^{\alpha(x)}\}_{n\geq 0}$ satisfies the C_b-Feller property and the strong Feller property.

Unfortunately, the stable-like chain $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$ (the chain given by (1.3)) does not satisfy the C_b-Feller property and the strong Feller property ($\liminf_{y\longrightarrow 0} p(y,O) \geq p(0,O)$ does not hold for some open sets $O \subseteq \mathbb{R}$). We introduce its "continuous" and in the recurrence and transience sense equivalent version: Let $k > 0$ be arbitrary and let $\{\bar{X}_n^{(\alpha,\beta)}\}_{n\geq 0}$ be the stable-like Markov chain defined by transition densities with following characteristic functions

$$\bar{\varphi}(x;\xi) = \exp(-\bar{\gamma}(x)|\xi|^{\bar{\alpha}(x)}),$$

where functions $\bar{\alpha} : \mathbb{R} \longrightarrow (0,2)$ and $\bar{\gamma} : \mathbb{R} \longrightarrow (0,\infty)$ are continuous functions such that

$$\bar{\alpha}(x) = \begin{cases} \alpha, & x < -k \\ \beta, & x > k \end{cases} \quad \text{and} \quad \bar{\gamma}(x) = \begin{cases} \gamma, & x < -k \\ \delta, & x > k. \end{cases}$$

Proposition 2.6. *The stable-like chain $\{\bar{X}_n^{(\alpha,\beta)}\}_{n\geq 0}$ is recurrent if and only if the stable-like chain $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$ is recurrent.*

Proof. By [San12, Propostion 5.4], it suffices to prove that condition (C5) holds, i.e., that there exists $l > 0$ such that for all compact sets $C \subseteq [-l,l]^c$ with $\lambda(C) > 0$, we have

$$\inf_{x\in[-k,k]}\int_{C-x} f_{(\bar{\alpha}(x),\bar{\gamma}(x))}(dy) > 0.$$

Let us take $l = k$. Without loss of generality, let $C \subseteq (k, \infty)$ be a compact set with $\lambda(C) > 0$. Then by symmetry and bell-shaped property of densities $f_{(\bar{\alpha}(x),\bar{\gamma}(x))}(y)$ (see [Gaw84, Theorem 1]), we have

$$\inf_{x\in[-k,k]} \int_{C-x} f_{(\bar{\alpha}(x),\bar{\gamma}(x))}(y)dy \geq \inf_{x\in[-k,k]} \int_{C+k} f_{(\bar{\alpha}(x),\bar{\gamma}(x))}(y)dy.$$

Let us assume that $\inf_{x\in[-k,k]} \int_{C-x} f_{(\bar{\alpha}(x),\bar{\gamma}(x))}(y)dy = 0$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq [-k, k]$, such that $\lim_{n\longrightarrow\infty} x_n = x_0 \in [-k, k]$ and

$$\lim_{n\longrightarrow\infty} \int_{C+k} f_{(\bar{\alpha}(x_n),\bar{\gamma}(x_n))}(y)dy = (2\pi)^{-1} \lim_{n\longrightarrow\infty} \int_{C+k} \int_{\mathbb{R}} \cos(\xi y)e^{-\bar{\gamma}(x_n)|\xi|^{\bar{\alpha}(x_n)}} d\xi dy = 0.$$

Now, by the dominated convergence theorem and continuity of the functions $\bar{\alpha}(x)$ and $\bar{\gamma}(x)$ we have

$$\begin{aligned} 0 &= (2\pi)^{-1} \lim_{n\longrightarrow\infty} \int_{C+k} \int_{\mathbb{R}} \cos(\xi y)e^{-\bar{\gamma}(x_n)|\xi|^{\bar{\alpha}(x_n)}} d\xi dy \\ &= (2\pi)^{-1} \int_{C+k} \int_{\mathbb{R}} \cos(\xi y)e^{-\bar{\gamma}(x_0)|t|^{\bar{\alpha}(x_0)}} d\xi dy = \int_{C+k} f_{(\bar{\alpha}(x_0),\bar{\gamma}(x_0))}(y)dy. \end{aligned}$$

This is impossible since $\lambda(C) > 0$. □

For a Markov process $\{Y_t\}_{t\geq 0}$ we define a family of operators $\{P_t\}_{t\geq 0}$ on $B_b(\mathbb{R})$ by $P_t f(x) := \mathbb{E}^x(f(Y_t))$. Since $\{Y_t\}_{t\geq 0}$ is a Markov process, the family $\{P_t\}_{t\geq 0}$ forms a *semigroup* of linear operators on $(B_b(\mathbb{R}), ||\cdot||_\infty)$, i.e., $P_t \circ P_s = P_{t+s}$ and $P_0 = I$. Furthermore, the semigroup $\{P_t\}_{t\geq 0}$ is *contractive* ($||P_t f||_\infty \leq ||f||_\infty$ for all $f \in B_b(\mathbb{R})$) and *positivity preserving* ($P_t f \geq 0$ whenever $f \geq 0$, $f \in B_b(\mathbb{R})$). The process $\{Y_t\}_{t\geq 0}$ is said to be a C_0-*Feller process* if the semigroup $\{P_t\}_{t\geq 0}$ forms a *Feller semigroup*. This means that:

(i) the family $\{P_t\}_{t\geq 0}$ is a semigroup of linear operators on the space $C_0(\mathbb{R})$;

(ii) the family $\{P_t\}_{t\geq 0}$ is *strongly continuous*, i.e., $\lim_{t\longrightarrow 0} ||P_t f - f||_\infty = 0$.

The *infinitesimal generator* $\mathcal{A}$ of the semigroup $\{P_t\}_{t\geq 0}$ is defined by

$$\mathcal{A}f := \lim_{t\longrightarrow 0} \frac{P_t f - f}{t}$$

on $\mathcal{D}_\mathcal{A} := \{f \in B_b(\mathbb{R}) : \lim_{t\longrightarrow 0} \frac{P_t f - f}{t}$ exists in supnorm$\}$. If the set of smooth functions with compact support $C_c^\infty(\mathbb{R})$ is contained in $\mathcal{D}_\mathcal{A}$ and $\mathcal{A}(C_c^\infty(\mathbb{R})) \subseteq C(\mathbb{R})$, then $\mathcal{A}|_{C_c^\infty(\mathbb{R})}$ is a *pseudo-differential operator*, i.e., it can be written in the form

$$\mathcal{A}|_{C_c^\infty(\mathbb{R})} f(x) = -\int_{\mathbb{R}} p(x,\xi)e^{ix\xi}\widehat{f}(\xi)d\xi, \tag{2.3}$$

where $\widehat{f}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-ix\xi}f(x)dx$ is the Fourier transform of $f(x)$ (see [Cou66, Theorem 3.4]). The function $p : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ is called the *symbol* of the pseudo-differential operator. It is measurable and locally bounded in (x, ξ) and continuous and negative definite as a function of ξ. Hence, by [Jac01, Theorem 3.7.7], $\xi \longmapsto p(x, \xi)$ has for each x the Lévy-Khinchine representation, i.e.,

$$p(x,\xi) = a(x) - ib(x)\xi + \frac{1}{2}c(x)\xi^2 - \int_{\mathbb{R}} \left(e^{iy\xi} - 1 - iy\xi 1_{\{z:|z|\leq 1\}}(y)\right) \nu(x, dy), \tag{2.4}$$

where $a(x) \geq 0$, $b(x) \in \mathbb{R}$ and $c(x) \geq 0$ are Borel measurable functions and $\nu(x, \cdot)$ is a Borel kernel on $\mathbb{R} \times \mathcal{B}(\mathbb{R})$, such that $\nu(x, \{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge y^2)\nu(x, dy) < \infty$ holds for all $x \in \mathbb{R}$. The quadruple $(a(x), b(x), c(x), \nu(x, \cdot))$ is called the *Lévy-quadruple* of the pseudo-differential operator $\mathcal{A}|_{C_c^\infty(\mathbb{R})}$. In the following we assume, without loss of generality, that every Feller process has càdlàg paths (see [RY99, Theorem III.2.7]).

Proposition 2.7. *Let $a \neq 0$ be arbitrary and let $\{N_t^\kappa\}_{t\geq 0}$ be the Poisson process with parameter $\kappa > 0$ independent of a Markov chain $\{Y_n\}_{n\geq 0}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then the process $\{Y_t^{(a,\kappa)}\}_{t\geq 0}$, defined by $Y_t^{(a,\kappa)} := aY_{N_t^\kappa}$, is*

(i) a strong Markov process with the strongly continuous semigroup $\{P_t^{(a,\kappa)}\}_{t\geq 0}$ and the infinitesimal generator

$$\mathcal{A}^{(a,\kappa)} f(x) = \kappa \int_{\mathbb{R}} (f(y) - f(x)) p\left(a^{-1}x, a^{-1}dy\right)$$

with the domain $\mathcal{D}_{\mathcal{A}^{(a,\kappa)}} = B_b(\mathbb{R}^d)$;

(ii) λ-irreducible and recurrent (respectively H-recurrent) if and only if the chain $\{Y_n\}_{n\geq 0}$ is λ-irreducible and recurrent (respectively H-recurrent).

Proof. (i) First, note that if $\{Y_n\}_{n\geq 0}$ is a Markov chain with respect to the family of probability measures $\{\mathbb{P}^x\}_{x\in\mathbb{R}^d}$, then $\{aY_n\}_{n\geq 0}$ is a Markov chain with respect to the family of probability measures $\{\mathbb{Q}^x := \mathbb{P}^{a^{-1}x}\}_{x\in\mathbb{R}}$. Hence, the process $\{Y_t^{(a,\kappa)}\}_{t\geq 0}$ is a strong Markov process. Clearly, its transition kernel is given by

$$p^t(x, dy) = e^{-\kappa t} \sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} p^n \left(a^{-1}x, a^{-1}dy\right).$$

Now, the claim easily follows.

(ii) The equivalence of λ-irreducibility and recurrence between the process $\{Y_t^{(a,\kappa)}\}_{t\geq 0}$ and the chain $\{Y_n\}_{n\geq 0}$ easily follows from the definition and the fact that the exponential distribution has finite all moments. In the case of H-recurrence we have

$$L^{(a,\kappa)}(x, B) = \mathbb{Q}^x(\tau_B^{(a,\kappa)} < \infty) = \mathbb{P}^{a^{-1}x}(\tau_{a^{-1}B} < \infty) = L(a^{-1}x, a^{-1}B).$$

Hence, the process $\{Y_t^{(a,\kappa)}\}_{t\geq 0}$ is H-recurrent if and only if the chain $\{Y_n\}_{n\geq 0}$ is H-recurrent. □

It is natural to expect that if the functions $\alpha : \mathbb{R} \longrightarrow (0, 2)$ and $\gamma : \mathbb{R} \longrightarrow (0, \infty)$ are continuous, the process $\{Y_t^{\alpha(x)}\}_{t\geq 0} := \{aX_{N_t^\kappa}^{\alpha(x)}\}_{t\geq 0}$ is a C_0-Feller process. We need the following lemma.

Lemma 2.8. *Let $0 < \varepsilon < 2$ and $C > 0$ be arbitrary, and let $\alpha : \mathbb{R} \longrightarrow (\varepsilon, 2)$ and $\gamma : \mathbb{R} \longrightarrow (0, C)$ be arbitrary functions. Furthermore, let $\{f_{(\alpha(x),\gamma(x))}\}_{x\in\mathbb{R}}$ be a family of $S\alpha(x)S$ densities given by the following characteristic functions $\varphi(x; \xi) = \exp(-\gamma(x)|\xi|^{\alpha(x)})$. Then the following uniformity condition holds*

$$\lim_{b\longrightarrow\infty} \sup_{x\in\mathbb{R}} \int_b^\infty f_{(\alpha(x),\gamma(x))}(y)dy = 0. \tag{2.5}$$

Moreover,

$$\lim_{|y|\longrightarrow\infty}\sup_{\{x\in\mathbb{R}:\,\alpha(x)<1\}}\left|f_{(\alpha(x),\gamma(x))}(y)\frac{|y|^{\alpha(x)+1}}{c(x)}-1\right|=0,$$

where

$$c(x)=\frac{\gamma(x)}{\pi}\Gamma(\alpha(x)+1)\sin\left(\frac{\pi\alpha(x)}{2}\right).$$

Proof. Let $0<\rho<\varepsilon$ be arbitrary and let $\{Z_x\}_{x\in\mathbb{R}}$ be a family of random variables with S$\alpha(x)$S distributions with densities $\{f_{(\alpha(x),\gamma(x))}\}_{x\in\mathbb{R}}$. Then we have

$$\sup_{x\in\mathbb{R}}\int_b^\infty f_{(\alpha(x),\gamma(x))}(y)dy=\sup_{x\in\mathbb{R}}\mathbb{P}(Z_x\geq b)\leq\sup_{x\in\mathbb{R}}\mathbb{P}(|Z_x|\geq b)\leq\frac{1}{b^\rho}\sup_{x\in\mathbb{R}}\mathbb{E}|Z_x|^\rho.$$

Since $\sup_{x\in\mathbb{R}}\mathbb{E}|Z_x|^\rho$ is finite (see [Sat99, page 163]), the first claim easily follows.

To prove the second part of lemma we use [Zol86, Theorem 2.4.2]. Let us fix $x\in\mathbb{R}$ with $\alpha(x)<1$. Then, for all $|y|\geq1$, we have

$$\begin{aligned}&\left|f_{(\alpha(x),\gamma(x))}(y)\frac{|y|^{\alpha(x)+1}}{c(x)}-1\right|\\&=\frac{1}{\Gamma(\alpha(x)+1)\sin\left(\frac{\pi\alpha(x)}{2}\right)}\left|\sum_{n=2}^\infty(-1)^{n+1}\frac{\Gamma(n\alpha(x)+1)}{n!}\sin\left(\frac{n\pi\alpha(x)}{2}\right)\left(\frac{\gamma(x)}{|y|^{\alpha(x)}}\right)^{n-1}\right|\\&\leq\frac{1}{\Gamma(\varepsilon+1)\sin\left(\frac{\pi\varepsilon}{2}\right)}\sum_{n=1}^\infty\left(\frac{C}{|y|^\varepsilon}\right)^n.\end{aligned}$$

Now, by taking $\sup_{\{x\in\mathbb{R}:\,\alpha(x)<1\}}$ and letting $|y|\longrightarrow\infty$, we get the desired result. □

Proposition 2.9. *Let $0<\varepsilon<2$ and $C>0$ be arbitrary, let $\alpha:\mathbb{R}\longrightarrow(\varepsilon,2)$ and $\gamma:\mathbb{R}\longrightarrow(0,C)$ be continuous functions. Furthermore, let $a\neq0$ be arbitrary and let $\{N_t^\kappa\}_{t\geq0}$ be the Poisson process with parameter $\kappa>0$ independent of the stable-like chain $\{X_n^{\alpha(x)}\}_{n\geq0}$ (the chain given by (1.2)). Then the process $\{Y_t^{\alpha(x)}\}_{t\geq0}:=\{aX_{N_t^\kappa}^{\alpha(x)}\}_{t\geq0}$ is*

(i) a C_0-Feller process with the symbol

$$p(x,\xi)=a^{-1}\kappa\left(1-\int_{\mathbb{R}}e^{i\xi y}f_{(\alpha(a^{-1}x),\gamma(a^{-1}x))}(a^{-1}y)dy\right)$$

and the Lévy quadruple $(0,0,0,a^{-1}\kappa f_{(\alpha(a^{-1}x),\gamma(a^{-1}x))}(a^{-1}y)dy)$, and it satisfies the C_b-Feller property and the strong Feller property;

(ii) a T-model.

Proof. By Proposition 2.7, the semigroup of the process $\{Y_t^{\alpha(x)}\}_{t\geq0}$ is given by

$$P_t^{\alpha(x)}f(x)=e^{-\kappa t}\sum_{n=0}^\infty\frac{(\kappa t)^n}{n!}\int_{\mathbb{R}}p^n\left(a^{-1}x,a^{-1}dy\right)f(y),$$

for $f \in B_b(\mathbb{R})$, and the generator

$$\mathcal{A}^{\alpha(x)} f(x) = a^{-1}\kappa \int_{\mathbb{R}} (f(y+x) - f(x)) f_{(\alpha(a^{-1}x),\gamma(a^{-1}x))}(a^{-1}y)dy \tag{2.6}$$

with the domain $\mathcal{D}_{\mathcal{A}^{\alpha(x)}} = B_b(\mathbb{R})$. Furthermore, it is shown that the semigroup is strongly continuous.

(i) The C_b-Feller property easily follows from (2.2) and Fatou's lemma. Now, let us show that $P_t^{\alpha(x)}(C_0(\mathbb{R})) \subseteq C_0(\mathbb{R})$ for all $t \in \mathbb{R}_+$. For $f \in C_0(\mathbb{R})$, by the C_b-Feller property, $P_t^{\alpha(x)} f \in C_b(\mathbb{R})$ for all $t \in \mathbb{R}_+$. Next we show that $P_t^{\alpha(x)} f(x)$ vanishes at infinity for all $f \in C_0(\mathbb{R})$ and all $t \in \mathbb{R}_+$. Let $f \in C_0(\mathbb{R})$ and $\epsilon > 0$ be arbitrary such that $||f||_\infty \leq M$, for some $M \geq 0$. Since $C_c(\mathbb{R})$ is dense in $(C_0(\mathbb{R}), ||\cdot||_\infty)$, there exists $f_\epsilon \in C_c(\mathbb{R})$ such that $||f - f_\varepsilon||_\infty < \epsilon$. We have

$$\begin{aligned}\left|\int_{\mathbb{R}} p(a^{-1}x, a^{-1}dy) f(y)\right| &\leq \int_{\mathbb{R}} p(a^{-1}x, a^{-1}dy)|f(y)| < \int_{\mathbb{R}} p(a^{-1}x, a^{-1}dy)|f_\epsilon(y)| + \epsilon \\ &= a^{-1} \int_{\operatorname{supp} f_\epsilon - x} f_{\alpha(a^{-1}x),\gamma(a^{-1}x))}(a^{-1}y)|f_\epsilon(y+x)|dy + \epsilon \\ &\leq a^{-1}(M+\epsilon) \int_{\operatorname{supp} f_\epsilon - x} f_{\alpha(a^{-1}x),\gamma(a^{-1}x))}(a^{-1}y)dy + \epsilon.\end{aligned}$$

Since $\operatorname{supp} f_\epsilon$ is a compact set, by applying Lemma 2.8, the function $x \longmapsto \int_{\mathbb{R}} p(a^{-1}x, a^{-1}dy)f(y)$ is a $C_0(\mathbb{R})$ function. Thus, by the dominated convergence theorem we have the claim, i.e., the process $\{Y_t^{\alpha(x)}\}_{t\geq 0}$ is a C_0-Feller process.

The second part of the proposition easily follows from the relations (2.3), (2.4) and (2.6), and the strong Feller property follows from [SW12, Theorem 1.1]

(ii) The claim follows from [Twe94, Theorem 7.1].

□

Let us recall the notion of characteristics of a semimartingale (see [JS03] or [Sch09]). Let $(\Omega, \mathcal{F}, \{F_t\}_{t\geq 0}, \mathbb{P}, \{S_t\}_{t\geq 0})$, $\{S_t\}_{t\geq 0}$ in the sequel, be a semimatingale and let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a truncation function (i.e., a continuous bounded function such that $h(x) = x$ in a neighborhood of the origin). We define two processes

$$\check{S}(h)_t := \sum_{s\leq t}(\Delta S_s - h(\Delta S_s)) \quad \text{and} \quad S(h)_t := S_t - \check{S}(h)_t,$$

where the process $\{\Delta S_t\}_{t\geq 0}$ is defined by $\Delta S_t := S_t - S_{t-}$ and $\Delta S_0 := S_0$. The process $\{S(h)_t\}_{t\geq 0}$ is a special semimartingale. Hence, it admits the unique decomposition

$$S(h)_t = S_0 + M(h)_t + B(h)_t, \tag{2.7}$$

where $\{M(h)_t\}_{t\geq 0}$ is a local martingale and $\{B(h)_t\}_{t\geq 0}$ is a predictable process of bounded variation.

Definition 2.10. *Let $\{S_t\}_{t\geq 0}$ be a semimartingale and let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the truncation function. Furthermore, let $\{B(h)_t\}_{t\geq 0}$ be the predictable process of bounded variation appearing in (2.7), let $N(\omega, ds, dy)$ be the compensator of the jump measure*

$$\mu(\omega, ds, dy) = \sum_{s:\Delta S_s(\omega)\neq 0} \delta_{(s,\Delta S_s(\omega))}(ds, dy)$$

of the process $\{S_t\}_{t\geq 0}$ and let $\{C_t\}_{t\geq 0}$ be the quadratic co-variation process for $\{S^c_t\}_{t\geq 0}$ (continuous martingale part of $\{S_t\}_{t\geq 0}$), i.e.,

$$C_t = \langle S^c_t, S^c_t\rangle.$$

Then (B, C, N) is called the characteristics *of the semimartingale $\{S_t\}_{t\geq 0}$ (relative to $h(x)$). If we put $\tilde{C}(h)_t := \langle M(h)_t, M(h)_t\rangle$, where $\{M(h)_t\}_{t\geq 0}$ is the local martingale appearing in (2.7), then $(B, \tilde{C}, N)$ is called the* modified characteristics *of the semimartingale $\{S_t\}_{t\geq 0}$ (relative to $h(x)$).*

Proposition 2.11. *Let $a \neq 0$ be arbitrary and let $\{f_x\}_{x\in\mathbb{R}}$ be a family of probability densities on the real line such that $x \longmapsto f_x(y)$ is a Borel measurable function for all $y \in \mathbb{R}$. Let $\{Y_n\}_{n\geq 0}$ be a Markov chain on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$, given by the transition kernel $p(x, dy) := f_x(y - x)dy$. Furthermore, let $\{Y^{(a,\kappa)}_t\}_{t\geq 0}$ be the process defined by $Y^\kappa_t := aY_{N^\kappa_t}$, where $\{N^\kappa_t\}_{t\geq 0}$ is the Poisson process, with respect to the filtration $\{\mathcal{G}_t\}_{t\geq 0}$, with parameter $\kappa > 0$ independent of the chain $\{Y_n\}_{n\geq 0}$. Then the process $\{Y^{(a,\kappa)}_t\}_{t\geq 0}$ is a semimartingale with respect to the filtration $\{\sigma\{\mathcal{F}_\infty \cup \mathcal{G}_t\}\}_{t\geq 0}$, where $\mathcal{F}_\infty = \bigcup_{n=0}^\infty \mathcal{F}_n$, and its characteristics and the modified characteristics, relative to the truncation function $h(x)$, are given by:*

$$\begin{aligned}
B^{(a,\kappa)}_t &= a^{-1}\kappa \int_0^t \int_{\mathbb{R}} h(y) f_{Y_{N^\kappa_s}}(a^{-1}y)dyds,\\
C^{(a,\kappa)}_t &= 0,\\
\tilde{C}^{(a,\kappa)}_t &= a^{-1}\kappa \int_0^t \int_{\mathbb{R}} h^2(y) f_{Y_{N^\kappa_s}}(a^{-1}y)dyds \quad \text{and}\\
N^{(a,\kappa)}(ds, dy) &= a^{-1}\kappa f_{Y_{N^\kappa_s}}(a^{-1}y)dyds.
\end{aligned}$$

Proof. Clearly, the process $\{Y^{(a,\kappa)}_t\}_{t\geq 0}$ is a semimartingale. By Proposition 2.7, the infinitesimal generator of the process $\{Y^{(a,\kappa)}_t\}_{t\geq 0}$ is given by $\mathcal{A}^{(a,\kappa)}f(x) = a^{-1}\kappa \int_{\mathbb{R}}(f(y+x)-f(x))f_{a^{-1}x}(a^{-1}y)dy$, $f \in B_b(\mathbb{R})$. Furthermore, by [EK86, Poposition IV.1.7], for every $f \in B_b(\mathbb{R})$ the process

$$M^f_t := f(Y^{(a,\kappa)}_t) - f(Y^{(a,\kappa)}_0) - \int_0^t \mathcal{A}^{(a,\kappa)} f(Y^{(a,\kappa)}_{s-})ds$$

is a martingale. Let $h(x)$ be the truncation function and let $f \in C^1_b(\mathbb{R})$. Then $\{M^f_t\}_{t\geq 0}$ can be rewritten in the following form

$$\begin{aligned}
M^f_t &= f(Y^{(a,\kappa)}_t) - f(Y^{(a,\kappa)}_0) - a^{-1}\kappa \int_0^t \int_{\mathbb{R}} \left(f(y + Y^{(a,\kappa)}_{s-}) - f(Y^{(a,\kappa)}_{s-})\right) f_{Y_{N^\kappa_{s-}}}(a^{-1}y)dyds\\
&= f(Y^{(a,\kappa)}_t) - f(Y^{(a,\kappa)}_0) - a^{-1}\kappa \int_0^t \int_{\mathbb{R}} f'(Y^{(a,\kappa)}_{s-})h(y) f_{Y_{N^\kappa_{s-}}}(a^{-1}y)dyds\\
&\quad - a^{-1}\kappa \int_0^t \int_{\mathbb{R}} \left(f(y + Y^{(a,\kappa)}_{s-}) - f(Y^{(a,\kappa)}_{s-}) - f'(Y^{(a,\kappa)}_{s-})h(y)\right) f_{Y_{N^\kappa_{s-}}}(a^{-1}y)dyds.
\end{aligned}$$

Now, from [JS03, Proposition II.2.17 and Theorem II.2.42], the claim follows. □

We refer the reader to [JS03, Sch98, Sch09] for more details about characteristics of a semimartingale and connection with Feller processes.

As we know, the recurrence property of SαS random walk, given by the characteristic function $\varphi(\xi) = \exp(-\gamma|\xi|^\alpha)$, depends only on the index of stability $\alpha \in (0, 2]$ and it does not depend on the scaling constant $\gamma \in (0, \infty)$. In the following proposition we show that this is also the case with the stable-like chain $\{X^{\alpha(x)}_n\}_{n\geq 0}$ (the chain given by (1.2)).

Proposition 2.12. *Let $\{X_n^{\alpha(x)}\}_{n\geq 0}$ be the stable-like chain defined in Proposition 2.9. Furthermore, let $c > 0$ be arbitrary and let $\{X_n^{(\alpha(x),c)}\}_{n\geq 0}$ be the stable-like chain which we get by replacing the scaling function $\gamma(x)$ by the scaling function $c\gamma(x)$. Then the stable-like chain $\{X_n^{(\alpha(x),c)}\}_{n\geq 0}$ is recurrent if and only if the stable-like chain $\{X_n^{\alpha(x)}\}_{n\geq 0}$ is recurrent.*

Proof. Let $\{N_t^1\}_{t\geq 0}$ be the Poisson process with parameter 1 independent of the stable-like chain $\{X_n^{(\alpha(x),c)}\}_{n\geq 0}$. Let us define the process $\mathbf{X}^c = \{X_t^c\}_{t\geq 0}$ by $X_t^c := X_{N_t}^{(\alpha(x),c)}$. By Proposition 2.11, the process $\mathbf{X}^c$ has the modified characteristics (relative to the truncation function $h(x)$) given by:

$$\begin{aligned}
B_t^c &= \int_0^t \int_{\mathbb{R}\setminus\{0\}} h(y) f_{(\alpha(X_s^c),c\gamma(X_s^c))}(y)dyds,\\
C_t^c &= 0,\\
\tilde{C}_t^c &= \int_0^t \int_{\mathbb{R}\setminus\{0\}} h^2(y) f_{(\alpha(X_s^c),c\gamma(X_s^c))}(y)dyds \quad \text{and}\\
N^c(ds,dy) &= f_{(\alpha(X_s^c),c\gamma(X_s^c))}(y)dyds.
\end{aligned}$$

Let $c_0 > 0$ be arbitrary and fixed and let us show that

$$\mathbf{X}^c \xrightarrow{\mathrm{d}} \mathbf{X}^{c_0}, \quad \text{when} \quad c \longrightarrow c_0,$$

where $\xrightarrow{\mathrm{d}}$ denotes the convergence in the space of càdlàg functions equipped with the Skorohod topology. We only have to check assumptions 4.3, 4.7, 4.9, 4.10, 4.11 and 4.12 from [JS03, Theorem IX.4.8]. Assumptions 4.3, 4.7, 4.10 and 4.12 can be easily verified by use of [Dur10, Theorem 3.3.5], continuity assumption of the functions $\alpha(x)$ and $\gamma(x)$, the dominated convergence theorem and Propositions 2.9 and 2.11, while assumption 4.9 follows from Lemma 2.8. To verify assumption 4.11 we have to show that

$$\lim_{c\longrightarrow c_0} \sup_{x\in[a,b]} \left| \int_{\mathbb{R}} g(y) \left(f_{(\alpha(x),c_0\gamma(x))}(y) - f_{(\alpha(x),c\gamma(x))}(y) \right) dy \right| = 0$$

holds for all $g \in C_b(\mathbb{R})$ and all $[a,b] \subseteq \mathbb{R}$. If that would not be the case, then there would exist $g \in C_b(\mathbb{R})$, $[a,b] \subseteq \mathbb{R}$, $\delta > 0$ and sequences $\{c_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq [a,b]$ with limits c_0 and $x_0 \in [a,b]$, respectively, such that

$$\left| \int_{\mathbb{R}} g(y) \left(f_{(\alpha(x_n),c_0\gamma(x_n))}(y) - f_{(\alpha(x_n),c_n\gamma(x_n))}(y) \right) dy \right| > \delta \tag{2.8}$$

holds for all $n \in \mathbb{N}$. Let $M \geq 0$ be such that $||g(x)||_\infty \leq M$ and let $R > 0$ be arbitrary. We have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} g(y) \left(f_{(\alpha(x_n),c_0\gamma(x_n))}(y) - f_{(\alpha(x_n),c_n\gamma(x_n))}(y) \right) dy \right|\\
&\leq \left| \int_{-R}^{R} g(y) \left(f_{(\alpha(x_n),c_0\gamma(x_n))}(y) - f_{(\alpha(x_n),c_n\gamma(x_n))}(y) \right) dy \right|\\
&\quad + \left| \int_{|y|\geq R} g(y) \left(f_{(\alpha(x_n),c_0\gamma(x_n))}(y) - f_{(\alpha(x_n),c_n\gamma(x_n))}(y) \right) dy \right|.
\end{aligned}$$

From continuity of the functions $\alpha(x)$ and $\gamma(x)$ and from [Ush99, Corollary 1.2.4], we have

$$\lim_{n\longrightarrow\infty} \left| \int_{-R}^{R} g(y) \left(f_{(\alpha(x_n),c_0\gamma(x_n))}(y) - f_{(\alpha(x_n),c_n\gamma(x_n))}(y) \right) dy \right| = 0.$$

Furthermore, by Lemma 2.8 we have

$$\lim_{R\longrightarrow\infty}\sup_{n\in\mathbb{N}}\left|\int_{|y|\geq R} g(y)\left(f_{(\alpha(x_n),c_0\gamma(x_n))}(y)-f_{(\alpha(x_n),c_n\gamma(x_n))}(y)\right)dy\right|$$
$$\leq M\lim_{R\longrightarrow\infty}\sup_{n\in\mathbb{N}}\int_{|y|\geq R} f_{(\alpha(x_n),c_0\gamma(x_n))}(y)+M\lim_{R\longrightarrow\infty}\sup_{n\in\mathbb{N}}\int_{|y|\geq R} f_{(\alpha(x_n),c_n\gamma(x_n))}(y)=0.$$

Hence,

$$\lim_{n\longrightarrow\infty}\left|\int_{\mathbb{R}} g(y)\left(f_{(\alpha(x_n),c_0\gamma(x_n))}(y)-f_{(\alpha(x_n),c_n\gamma(x_n))}(y)\right)dy\right|=0,$$

what is in contradiction with (2.8). The locally uniform convergence of the other two characteristics easily follows from [Dur10, Theorem 3.3.5], the continuity assumption of the functions $\alpha(x)$ and $\gamma(x)$ and the dominated convergence theorem.

Let $x\in\mathbb{R}$ be arbitrary and let $O\subseteq\mathbb{R}$ be an arbitrary open bounded set. Since the process $\mathbf{X}^{c_0}$ satisfies the C_b-Feller property, by [Fra06, Lemmas 2 and 3], we have $\mathbb{P}^x_{\mathbf{X}^{c_0}}(\partial R(O))=0$. Here ∂A denotes the boundary of the set A. Therefore, by [Bil99, Theorem 2.1], we have

$$\lim_{c\longrightarrow c_0}\mathbb{P}^x_{\mathbf{X}^c}(R(O))=\mathbb{P}^x_{\mathbf{X}^{c_0}}(R(O))$$

for all $x\in\mathbb{R}$ and for all open bounded sets $O\subseteq\mathbb{R}$. Hence, for all $x\in\mathbb{R}$ and all open bounded sets $O\subseteq\mathbb{R}$, the function

$$c\longmapsto\mathbb{P}^x_{\mathbf{X}^c}(R(O))$$

is a continuous function on $(0,\infty)$. Note that (2.1) is satisfied if for the distribution $a_C(\cdot)$ we take $a_C(\cdot):=\delta_{t_0}(\cdot)$, where $t_0>0$ is arbitrary. Since the processes $\mathbf{X}^c$ are λ-irreducible T-models, by Proposition 2.4, $\mathbb{P}^x_{\mathbf{X}^c}(R(O))=1$ for all $c\in(0,\infty)$, all $x\in\mathbb{R}$ and all open bounded sets $O\subseteq\mathbb{R}$, or $\mathbb{P}^x_{\mathbf{X}^c}(R(O))=0$ for all $c\in(0,\infty)$, all $x\in\mathbb{R}$ and all open bounded sets $O\subseteq\mathbb{R}$. This means, again by Proposition 2.4, that all processes $\mathbf{Y}^c$, $c\in(0,\infty)$, are either recurrent or transient at the same time. Now, by Proposition 2.7, the desired result follows. □

3 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Let the function $p:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{C}$ be given by $p(x,\xi)=\gamma(x)|\xi|^{\alpha(x)}$, for some functions $\alpha:\mathbb{R}\longrightarrow(0,2)$ and $\gamma:\mathbb{R}\longrightarrow(0,\infty)$. In [Bas88] it is shown that if the functions $\alpha(x)$ and $\gamma(x)$ satisfy:

(i) $0<\inf_{x\in\mathbb{R}}\alpha(x)\leq\sup_{x\in\mathbb{R}}\alpha(x)<2$ and $0<\inf_{x\in\mathbb{R}}\gamma(x)\leq\sup_{x\in\mathbb{R}}\gamma(x)<\infty$,

(ii) $\beta(z)=o(1/|\ln(z)|)$, when $z\longrightarrow 0$, where $\beta(z):=\sup_{|x-y|\leq z}|\alpha(x)-\alpha(y)|$,

(iii) $\int_0^1\frac{\beta(z)}{z}dz<\infty$, i.e., the function $\alpha(x)$ is Dini continuous and

(iv) $\gamma\in C(\mathbb{R})$,

then the function (symbol) $p(x,\xi)=\gamma(x)|\xi|^{\alpha(x)}$ defines C_b-Feller process on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ called a *stable-like process*. Note that if the function $\alpha(x)$ is Lipschitz continuous, i.e., if there exists $L>0$, such that $|\alpha(x)-\alpha(y)|\leq L|x-y|$ holds for all $x,y\in\mathbb{R}$, then it is also Dini continuous and condition (ii) is satisfied. Write $C_b^1(\mathbb{R})$ for the set of bounded continuously differentiable functions with bounded derivative. Clearly, $\alpha\in C_b^1(\mathbb{R})$ implies Lipschitz continuity of $\alpha(x)$. Furthermore, by [SW12, Theorems 1.1 and 3.3], $\alpha,\gamma\in C_b^1(\mathbb{R})$ imply that the corresponding stable-like process is a C_0-Feller process and it satisfies the C_b-Feller property and the strong Feller property.

Proof of Theorem 1.1. Let $k > 0$ be arbitrary and let $\mathbf{X}^{(\alpha,\beta)} = \{X_t^{(\alpha,\beta)}\}_{t\geq 0}$ be the stable-like process on $\mathbb{R}$ which corresponds to the symbol $p(x,\xi) = \gamma(x)|\xi|^{\alpha(x)}$, where the functions $\alpha, \gamma \in C_b^1(\mathbb{R})$ are such that

$$\alpha(x) = \begin{cases} \alpha, & x < -k \\ \beta, & x > k \end{cases} \quad \text{and} \quad \gamma(x) = \begin{cases} \gamma, & x < -k \\ \delta, & x > k. \end{cases}$$

By [Kol00, Theorem 5.1], transition kernel $\mathbb{P}^x(X_t^{(\alpha,\beta)} \in dy)$ is absolutely continuous with respect to the Lebesgue measure, and by [SW12, Theorem 3.3], $\mathbb{P}^x(X_t^{(\alpha,\beta)} \in B) > 0$ holds for all $x \in \mathbb{R}$, all $t \in \mathbb{R}_+$ and all $B \in \mathcal{B}(\mathbb{R})$ with $\lambda(B) > 0$. Therefore, the stable-like process $\mathbf{X}^{(\alpha,\beta)}$ is $\lambda-$irreducible and, by [Twe94, Theorem 7.1], it is a T-model. Hence, from [Böt11, Theorem 4.2], H-recurrence and recurrence properties of the stable-like process $\mathbf{X}^{(\alpha,\beta)}$ are equivalent. Furthermore, from [Böt11, Corollary 5.5], the stable-like process $\mathbf{X}^{(\alpha,\beta)}$ is recurrent if and only if $\alpha + \beta \geq 2$.

By [BS09], the stable-like process $\mathbf{X}^{(\alpha,\beta)}$ can be approximated by a sequence of Markov chains, i.e., for a sequence of Markov chains $\{X_n^{(m)}\}_{n\geq 0}$, $m \in \mathbb{N}$, on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by a sequence of transition kernels $p_m(x,dy)$, $m \in \mathbb{N}$, such that

$$\int_{\mathbb{R}} e^{i\xi y} p_m(x,dy) = e^{i\xi x - \frac{1}{m}\eta(x,\xi)} = e^{i\xi x - \frac{\gamma(x)}{m}|\xi|^{\alpha(x)}},$$

we have that

$$\mathbf{X}^{(m)} \xrightarrow{\text{d}} \mathbf{X}^{(\alpha,\beta)}, \quad \text{as} \quad m \longrightarrow \infty,$$

where $\mathbf{X}^{(m)} = \{X_{\lfloor mt \rfloor}^{(m)}\}_{t\geq 0}$. Again, $\xrightarrow{\text{d}}$ denotes convergence in distribution in the space of càdlàg functions equipped with the Skorohod topology. By Proposition 2.12, the chains $\{X_n^{(m)}\}_{n\geq 0}$, $m \in \mathbb{N}$, are either recurrent or transient at the same time. Hence, the rest of proof is devoted to prove that this dichotomy is equivalent with the recurrence-transience dichotomy of the stable like process $\mathbf{X}^{(\alpha,\beta)}$.

Since the stable-like process $\mathbf{X}^{(\alpha,\beta)}$ is a C_b-Feller process, by [Fra06, Lemmas 2 and 3] we have $\mathbb{P}^x_{\mathbf{X}^{(\alpha,\beta)}}(\partial R(O)) = 0$ for all $x \in \mathbb{R}$ and all open bounded sets $O \subseteq \mathbb{R}$. Therefore, by [Bil99, Theorem 2.1], we have

$$\lim_{m\longrightarrow\infty} \mathbb{P}^x_{\mathbf{X}^{(m)}}(R(O)) = \mathbb{P}^x_{\mathbf{X}^{(\alpha,\beta)}}(R(O)) \tag{3.1}$$

for all $x \in \mathbb{R}$ and for all open bounded sets $O \subseteq \mathbb{R}$.

Let us assume that $\alpha + \beta \geq 2$. Hence, the stable-like process $\mathbf{X}^{(\alpha,\beta)}$ is recurrent. Note that assumption (2.1) follows if for the distribution $a_C(\cdot)$ we take $a_C(\cdot) := \delta_{t_0}(\cdot)$, where $t_0 > 0$ is arbitrary, and apply the strong Feller property. Hence, by the 0-1 property (Proposition 2.4) $\mathbb{P}^x_{\mathbf{X}^{(\alpha,\beta)}}(R(O)) = 1$ holds for all $x \in \mathbb{R}$ and all open bounded sets $O \subseteq \mathbb{R}$. From (3.1), for any starting point $x \in \mathbb{R}$ and any open bounded set $O \subseteq \mathbb{R}$ there exists $m_0 \geq 1$ such that $\mathbb{P}^x_{\mathbf{X}^{(m_0)}}(R(O)) > 0$, i.e., $\mathbb{P}^x\left(\sum_{n=0}^{\infty} 1_{\{X_n^{(m_0)} \in O\}} = \infty\right) > 0$. But, since the stable-like chain $\{X_n^{(m_0)}\}_{n\geq 0}$ is $\lambda-$irreducible T-model, by 0-1 property,

$$\mathbb{P}^x\left(\sum_{n=0}^{\infty} 1_{\{X_n^{(m_0)} \in O\}} = \infty\right) = 1$$

holds for all $x \in \mathbb{R}$ and all open bounded sets $O \subseteq \mathbb{R}$, i.e., the stable like chain $\{X_n^{(m_0)}\}_{n\geq 0}$ is recurrent. Now, by applying Proposition 2.12, all stable-like chains $\{X_n^{(m)}\}_{n\geq 0}$, $m \in \mathbb{N}$, are recurrent. Therefore, since $\{\bar{X}_n^{(\alpha,\beta)}\}_{n\geq 0} \overset{\mathrm{d}}{=} \{X_n^{(1)}\}_{n\geq 0}$ (recall that the stable-like chain $\{\bar{X}_n^{(\alpha,\beta)}\}_{n\geq 0}$ is defined in Proposition 2.6), by Propositions 2.6 the stable-like chain $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$ is recurrent.

Let us now show that the recurrence property of the stable-like chain $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$ implies $\alpha+\beta \geq 2$. Let us assume that this is not the case, i.e., let us assume that $\alpha+\beta<2$. Hence, the stable-like process $\mathbf{X}^{(\alpha,\beta)}$ is transient, i.e., $\mathbb{P}^x_{\mathbf{X}^{(\alpha,\beta)}}(T(O))=1$ holds for all $x\in\mathbb{R}$ and all open bounded sets $O\subseteq\mathbb{R}$. Now, by (3.1), we have

$$\lim_{m\longrightarrow\infty} \mathbb{P}^x_{\mathbf{X}^{(m)}}(T(O)) = \mathbb{P}^x_{\mathbf{X}^{(\alpha,\beta)}}(T(O)) = 1.$$

Hence, for any starting point $x\in\mathbb{R}$ and any open bounded set $O\subseteq\mathbb{R}$, there exists $m_0\geq 1$ such that $\mathbb{P}^x_{\mathbf{X}^{(m_0)}}(T(O))>0$. Therefore, $\mathbb{P}^x\left(\sum_{n=0}^{\infty} 1_{\{X_n^{(m_0)}\in O\}}=\infty\right)<1$. Again, by the 0-1 property, we have

$$\mathbb{P}^x\left(\sum_{n=0}^{\infty} 1_{\{X_n^{(m_0)}\in O\}}=\infty\right)=0$$

for all $x\in\mathbb{R}$ and all open bounded sets $O\subseteq\mathbb{R}$. Hence, the stable like chain $\{X^{(m_0)}\}_{n\geq 0}$ is transient. Therefore, by Proposition 2.12, all the stable-like chains $\{X_n^{(m)}\}_{n\geq 0}$, $m\in\mathbb{N}$, are transient. Since $\{\bar{X}_n^{(\alpha,\beta)}\}_{n\geq 0} \overset{\mathrm{d}}{=} \{X_n^{(1)}\}_{n\geq 0}$, by Proposition 2.6, the stable-like chain $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$ is also transient. But this is in contradiction with recurrence assumption of the stable-like chain $\{X_n^{(\alpha,\beta)}\}_{n\geq 0}$. Hence, we have proved the desired result. □

4 Proof of Theorem 1.2

In this section we give a proof of Theorem 1.2. Recall that the functions $x\longmapsto f_x$, $\alpha(x)$ and $c(x)$ are τ-periodic, the function $(x,y)\longmapsto f_x(y)$ is continuous and strictly positive and $\alpha(x)$ and $c(x)$ are Borel measurable. Let us put $\Lambda:=\tau\mathbb{Z}$ and let $\Pi_\Lambda:\mathbb{R}\longrightarrow\mathbb{R}/\Lambda$ be the covering map. We denote by $\{X_n^{\Lambda p}\}_{n\geq 0}$ the process on $\mathbb{R}/\Lambda$ obtained by projection of the stable-like chain $\{X_n^p\}_{n\geq 0}$ (the chain given by (1.4)) with respect to $\Pi_\Lambda(x)$. By [Kol11, Proposition 3.8.8], the process $\{X_n^{\Lambda p}\}_{n\geq 0}$ is a Markov chain on $\mathbb{R}/\Lambda$ with transition density function

$$p^\Lambda(x,y)=\sum_{k\in\Lambda} p(z_x,z_y+k)=\sum_{k\in\Lambda} f_{z_x}(z_y-z_x+k)$$

for all $x,y\in\mathbb{R}/\Lambda$, where z_x and z_y are arbitrary points in $\Pi_\Lambda^{-1}(\{x\})$ and $\Pi_\Lambda^{-1}(\{y\})$, respectively. Furthermore, by [BLP78, Theorem III.3.1], the chain $\{X_n^{\Lambda p}\}_{n\geq 0}$ possesses an invariant measure $\pi(\cdot)$, with $\pi(\mathbb{R}/\Lambda)<\infty$, and there exist constants $C>0$ and $c>0$, such that for all τ-periodic functions $f\in B_b(\mathbb{R})$ we have

$$\int_{\mathbb{R}/\Lambda} f(z_x)\pi(dx)=0 \quad\Longrightarrow\quad \left|\left|\int_{\mathbb{R}} p_n(\cdot,dy)f(y)\right|\right|_\infty \leq C||f||_\infty e^{-cn} \quad \text{for all} \quad n\in\mathbb{N}.$$

Since $\pi(\mathbb{R}/\Lambda)<\infty$, without loss of generality, we assume that $\pi(\mathbb{R}/\Lambda)=1$. Following the ideas from the proof of [Fra06, Theorem 1], we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\{X_n^{\Lambda p}\}_{n\geq 0}$ be as above. Let us suppose that the set $\{x\in\mathbb{R}:\alpha(x)=\alpha_0:=\inf_{x\in\mathbb{R}}\alpha(x)\}$ has positive Lebesgue measure. By λ-irreducibility of the stable-like chain

$\{X^p_n\}_{n\geq 0}$, this is equivalent with $\pi(\Pi_\Lambda(\{x\in\mathbb{R} : \alpha(x)=\alpha_0 := \inf_{x\in\mathbb{R}}\alpha(x)\}))>0$. Indeed, since $\pi(\cdot)$ is the invariant measure of the chain $\{X^{\Lambda p}_n\}_{n\geq 0}$,

$$\int_{\mathbb{R}/\Lambda} p^\Lambda(x,B)\pi(dx)=\pi(B)$$

holds for all $B\in\mathcal{B}(\mathbb{R}/\Lambda)$, where $\mathcal{B}(\mathbb{R}/\Lambda)$ denotes the Borel σ-algebra with respect to the quotient topology. Let us put $A:=\{x\in\mathbb{R} : \alpha(x)=\alpha_0 := \inf_{x\in\mathbb{R}}\alpha(x)\}$ and $B:=\Pi_\Lambda(A)$. We have

$$\pi(B)=\int_{\mathbb{R}/\Lambda} p^\Lambda(x,B)\pi(dx)=\int_{\mathbb{R}/\Lambda} p(z_x,\Pi^{-1}_\Lambda(B))\pi(dx)=\int_{\mathbb{R}/\Lambda} p(z_x,A)\pi(dx).$$

Now, if $\lambda(A)>0$, then $p(z_x,A)>0$ for all $z_x\in\mathbb{R}$. Therefore, $\pi(B)>0$ as well. On the other hand, if $\lambda(A)=0$, then $p(z_x,A)=0$ for all $z_x\in\mathbb{R}$. Hence, $\pi(B)=0$.

In the sequel (because of τ-periodicity) we use the abbreviation $\alpha(x)$ and $c(x)$, for $\alpha(z_x)$ and $c(z_x)$, where $x\in\mathbb{R}/\Lambda$ and $z_x\in\Pi^{-1}_\Lambda(\{x\})$ are arbitrary.

Let $\{N^1_t\}_{t\geq 0}$ be the Poisson process with parameter 1 independent of the periodic stable-like chain $\{X^p_n\}_{n\geq 0}$ and let us define a λ-irreducible Markov process $\mathbf{Y}^p := \{X^p_{N_t}\}_{t\geq 0}$. By Proposition 2.7, the semigroup of the process $\mathbf{Y}^p$ is given by

$$P_tf(x)=e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{\mathbb{R}} p^n(x,dy)f(y)$$

for $f\in B_b(\mathbb{R})$ and $t\in\mathbb{R}_+$. Hence, for every τ-periodic function $f\in B_b(\mathbb{R})$ we have

$$||P_tf||_\infty \leq C||f||_\infty e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}e^{-cn}=C||f||_\infty e^{-t(1-e^{-c})}. \tag{4.1}$$

Let us define the sequence of semimartingales $\mathbf{Y}^p_n := \{n^{-\frac{1}{\alpha_0}}X^p_{N_{nt}}\}$, $n\in\mathbb{N}$. Now, we prove that the sequence of processes $\mathbf{Y}^p_n$, $n\in\mathbb{N}$, converges in distribution to a symmetric α_0-stable Lévy process $\mathbf{L}=\{L_t\}_{t\geq 0}$ with the modified characteristics (relative to the truncation function $h(x)$)

$$\begin{aligned}B^0_t &= \Theta t\int_{\mathbb{R}}(h(y)-y1_{|y|\leq 1})\frac{dy}{|y|^{\alpha_0+1}},\\ \tilde{C}^0_t &= \Theta t\int_{\mathbb{R}} h^2(y)\frac{dy}{|y|^{\alpha_0+1}} \quad \text{and}\\ N^0(ds,dy) &= \Theta\frac{dyds}{|y|^{\alpha_0+1}},\end{aligned}$$

where $\Theta := \int_{\mathbb{R}/\Lambda} 1_{\{\alpha(x)=\alpha_0\}}c(x)\pi(dx)$ (see [Sch98, Theorem 3.5]). Without loss of generality, we take all the processes $\mathbf{Y}^p_n$, $n\in\mathbb{N}$, and $\mathbf{L}$ to be defined on the same probability spaces $(\Omega,\mathcal{F},\{\mathbb{P}^x\}_{x\in\mathbb{R}})$. In order to prove this convergence, by [JS03, Theorem VIII.2.17] it suffices to show that initial distributions of $\mathbf{Y}^p_n$ converge to initial distribution of $\mathbf{L}$ (what is trivially satisfied) and the modified characteristics $(B^n,\tilde{C}^n,N^n)$ of the processes $\mathbf{Y}^p_n$, $n\in\mathbb{N}$, converge in probability to the modified characteristics $(B^0,\tilde{C}^0,N^0)$, when $n\longrightarrow\infty$. By Proposition 2.11, the modified characteristics $(B^n,\tilde{C}^n,N^n)$ of the process $\mathbf{Y}^p_n$ are given by

$$\begin{aligned}B^n_t &= n^{1+\frac{1}{\alpha_0}}\int_0^t\int_{\mathbb{R}} h(y)f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha_0}}y\right)dyds,\\ \tilde{C}^n_t &= n^{1+\frac{1}{\alpha_0}}\int_0^t\int_{\mathbb{R}} h^2(y)f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha_0}}y\right)dyds \quad \textit{and}\\ N^n(ds,dy) &= n^{1+\frac{1}{\alpha_0}}f_{X^p_{ns}}\left(n^{\frac{1}{\alpha_0}}y\right)dyds.\end{aligned}$$

Note that (PC4), (1.5) and $\lambda(\{x \in \mathbb{R} : \alpha(x) = \alpha_0 := \inf_{x\in\mathbb{R}} \alpha(x)\}) > 0$, i.e., $\pi(\Pi_\Lambda(\{x \in \mathbb{R} : \alpha(x) = \alpha_0 := \inf_{x\in\mathbb{R}} \alpha(x)\})) > 0$, imply $0 < \Theta < \infty$, therefore the above α_0-stable Lévy process characteristics are well defined.

Recall that for a Borel measurable function $g : \mathbb{R} \longrightarrow \mathbb{R}$ and a random measure $\mu(\omega, ds, dx)$ on $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R})$, the $*$-product is defined by

$$g * \mu_t(\omega) := \begin{cases} \int_{[0,t]\times\mathbb{R}} g(x)\mu(\omega, ds, dx), & \int_{[0,t]\times\mathbb{R}} |g(x)|\mu(\omega, ds, dx) < \infty \\ \infty, & \text{otherwise,} \end{cases}$$

(see [JS03, Definition II.1.3] for details). Let $g \in C_b(\mathbb{R})$ vanish in a neighborhood of the origin. We have

$$\begin{aligned} g * N^n_t &= \int_0^t \int_{\mathbb{R}} g(y) N^n(ds, dy) \\ &= \int_0^t \int_{\mathbb{R}} g(y) n^{1+\frac{1}{\alpha_0}} f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha_0}} y\right) dyds \\ &= \int_0^t \int_{\mathbb{R}} g\left(n^{\frac{1}{\alpha(Y^p_{ns})}-\frac{1}{\alpha_0}} y\right) n^{1+\frac{1}{\alpha(Y^p_{ns})}} f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha(Y^p_{ns})}} y\right) dyds \\ &= \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(X^p_{ns})=\alpha_0\}} g\left(n^{\frac{1}{\alpha(Y^p_{ns})}-\frac{1}{\alpha_0}} y\right) n^{1+\frac{1}{\alpha(Y^p_{ns})}} f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha(Y^p_{ns})}} y\right) dyds \\ &\quad + \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y^p_{ns})>\alpha_0\}} g\left(n^{\frac{1}{\alpha(Y^p_{ns})}-\frac{1}{\alpha_0}} y\right) n^{1+\frac{1}{\alpha(Y^p_{ns})}} f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha(Y^p_{ns})}} y\right) dyds \\ &= \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y^p_{ns})=\alpha_0\}} g(y) \frac{c(Y^p_{ns})}{|y|^{\alpha_0+1}} dyds \qquad (4.2) \\ &\quad + \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y^p_{ns})=\alpha_0\}} g(y) \left(n^{1+\frac{1}{\alpha_0}} f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha_0}} y\right) - \frac{c(Y^p_{ns})}{|y|^{\alpha_0+1}}\right) dyds \qquad (4.3) \\ &\quad + \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y^p_{ns})>\alpha_0\}} g(y)\, n^{1-\frac{\alpha(Y^p_{ns})}{\alpha_0}} \frac{c(Y^p_{ns})}{|y|^{\alpha(Y^p_{ns})+1}} dyds \qquad (4.4) \\ &\quad + \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y^p_{ns})>\alpha_0\}} g(y) \left(n^{1+\frac{1}{\alpha_0}} f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha_0}} y\right) - n^{1-\frac{\alpha(Y^p_{ns})}{\alpha_0}} \frac{c(Y^p_{ns})}{|y|^{\alpha(Y^p_{ns})+1}}\right) dyds. \qquad (4.5) \end{aligned}$$

Let $0 < \varepsilon < 1$ be arbitrary. Then, by (PC3), there exists $y_\varepsilon \geq 1$, such that

$$(1-\varepsilon)\frac{c(x)}{|y|^{\alpha(x)+1}} < f_x(y) < (1+\varepsilon)\frac{c(x)}{|y|^{\alpha(x)+1}} \qquad (4.6)$$

holds for all $|y| \geq y_\varepsilon$ and all $x \in \mathbb{R}$. Since the function $g(x)$ vanishes in a neighborhood of the origin, by (4.6) and the dominated convergence theorem, (4.3) and (4.5) converge to 0, $\mathbb{P}^x$-a.s., when $n \longrightarrow \infty$. Let us prove that (4.4) converges in $L^2(\Omega, \mathbb{P}^x)$ to 0, when $n \longrightarrow \infty$. We define

$$U_n(z) := \int_{\mathbb{R}} g(y) \left(1_{\{\alpha(z)>\alpha_0\}} n^{1-\frac{\alpha(z)}{\alpha_0}} \frac{c(z)}{|y|^{\alpha(z)+1}} - \int_{\mathbb{R}/\Lambda} 1_{\{\alpha(x)>\alpha_0\}} n^{1-\frac{\alpha(x)}{\alpha_0}} \frac{c(x)}{|y|^{\alpha(x)+1}} \pi(dx)\right) dy.$$

By τ-periodicity of functions $\alpha(x)$ and $c(x)$, the function $U_n(z)$ is τ-periodic and

$$\int_{\mathbb{R}/\Lambda} U_n(z)\pi(dz) = 0.$$

Using integration by parts formula, Markov property and (4.1), we have

$$\mathbb{E}^x\left[\left(\int_0^t U_n(Y^p_{ns})\right)^2\right] = 2\int_0^t\int_0^s \mathbb{E}^x[U_n(Y^p_{ns})U_n(Y^p_{nr})]drds$$
$$= 2\int_0^t\int_0^s \mathbb{E}^x[\mathbb{E}^x[U_n(Y^p_{ns})|\mathcal{F}_{nr}]U_n(Y^p_{nr})]drds$$
$$= 2\int_0^t\int_0^s \mathbb{E}^x[P_{n(s-r)}U_n(Y^p_{nr})U_n(Y^p_{nr})]drds$$

$$= 2\int_0^t\int_0^s Ce^{-n(1-e^{-c})(s-r)}||U_n||^2_\infty = \frac{2C||U_n||^2_\infty}{n(1-e^{-c})}\int_0^t(1-e^{-n(1-e^{-c})s})ds \le \frac{C||U_n||^2_\infty}{n(1-e^{-c})}. \tag{4.7}$$

Note that, by (1.5),

$$|U_n(z)| \le \sup_{z\in\mathbb{R}}\left(\int_\mathbb{R}|g(y)|\frac{c(z)}{|y|^{\alpha(z)+1}}dy + \int_\mathbb{R}|g(y)|\int_{\mathbb{R}/\Lambda}\frac{c(x)}{|y|^{\alpha(x)+1}}\pi(dx)dy\right) < \infty,$$

i.e., $||U_n||_\infty$ remains bounded as n grows. Hence

$$\lim_{n\longrightarrow\infty}\mathbb{E}^x\left[\left(\int_0^t U_n(Y^p_{ns})\right)^2\right] = 0.$$

Furthermore,

$$\left(\mathbb{E}^x\left[\left(\int_0^t\int_\mathbb{R} 1_{\{\alpha(Y^p_{ns})>\alpha_0\}}g\,(y)\,n^{1-\frac{\alpha(Y^p_{ns})}{\alpha_0}}\frac{c(Y^p_{ns})}{|y|^{\alpha(Y^p_{ns})+1}}dyds\right)^2\right]\right)^{\frac{1}{2}}$$
$$\le \left(\mathbb{E}^x\left[\left(\int_0^t U_n(Y^p_{ns})\right)^2\right]\right)^{\frac{1}{2}}$$
$$+\left(\mathbb{E}^x\left[\left(\int_0^t\int_\mathbb{R}\int_{\mathbb{R}/\Lambda}1_{\{\alpha(x)>\alpha_0\}}n^{1-\frac{\alpha(x)}{\alpha_0}}g(y)\frac{c(x)}{|y|^{\alpha(x)+1}}\pi(dx)dy\right)^2\right]\right)^{\frac{1}{2}}. \tag{4.8}$$

By the dominated convergence theorem, (4.8) converges to zero, when $n\longrightarrow\infty$, i.e., (4.4) converges in $L^2(\Omega,\mathbb{P}^x)$ to 0, when $n\longrightarrow\infty$. Now, let us prove that (4.2) converges in $L^2(\Omega,\mathbb{P}^x)$ to

$$g*N^0_t = t\int_\mathbb{R}\int_{\mathbb{R}/\Lambda}1_{\{\alpha(x)=\alpha_0\}}g(y)\frac{c(x)}{|y|^{\alpha_0+1}}\pi(dx)dy,$$

when $n\longrightarrow\infty$. We define

$$U(z) := \int_\mathbb{R} g(y)\left(1_{\{\alpha(z)=\alpha_0\}}\frac{c(z)}{|y|^{\alpha_0+1}} - \int_{\mathbb{R}/\Lambda}1_{\{\alpha(x)=\alpha_0\}}\frac{c(x)}{|y|^{\alpha_0+1}}\pi(dx)\right)dy.$$

By τ-periodicity of functions $\alpha(x)$ and $c(x)$, the function $U(z)$ is τ-periodic and

$$\int_{\mathbb{R}/\Lambda}U(z)\pi(dz) = 0.$$

Hence, in the same way as for (4.4), it can be shown that $g * N_t^n$ converges in probability to $g * N_t^0$. In the same way one can prove that B_t^n converges in probability to B_t^0, when $n \longrightarrow \infty$.

At the end, let us show that $\widetilde{C}_t^n$ converges in probability to $\widetilde{C}_t^0$, when $n \longrightarrow \infty$. Recall that the truncation function $h(x)$ is a bounded Borel measurable function satisfying $h(x) = x$ in a neighborhood of the origin. Let $\delta > 0$ be small enough and such that $h(x) = x$ for all $x \in (-\delta, \delta)$. We have

$$
\begin{aligned}
\widetilde{C}_t^n &= \int_0^t \int_{\mathbb{R}} h^2(y) n^{1+\frac{1}{\alpha_0}} f_{Y_{ns}^p}\left(n^{\frac{1}{\alpha_0}} y\right) dyds \\
&= \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} h^2(y)\, n^{1+\frac{1}{\alpha_0}} f_{Y_{ns}^p}\left(n^{\frac{1}{\alpha_0}} y\right) dyds \\
&\quad + \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y_{ns}^p)>\alpha_0\}} h^2(y)\, n^{1+\frac{1}{\alpha_0}} f_{Y_{ns}^p}\left(n^{\frac{1}{\alpha_0}} y\right) dyds \\
&= \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} h^2(y) \frac{c(Y_{ns}^p)}{|y|^{\alpha_0+1}} dyds \qquad (4.9) \\
&\quad + \int_0^t \int_{(-\delta,\delta)^c} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} h^2(y) \left(n^{1+\frac{1}{\alpha_0}} f_{Y_{ns}^p}\left(n^{\frac{1}{\alpha_0}} y\right) - \frac{c(Y_{ns}^p)}{|y|^{\alpha_0+1}}\right) dyds \qquad (4.10) \\
&\quad + \int_0^t \int_{(-\delta,\delta)} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} y^2 \left(n^{1+\frac{1}{\alpha_0}} f_{Y_{ns}^p}\left(n^{\frac{1}{\alpha_0}} y\right) - \frac{c(Y_{ns}^p)}{|y|^{\alpha_0+1}}\right) dyds \qquad (4.11) \\
&\quad + \int_0^t \int_{\mathbb{R}} 1_{\{\alpha(Y_{ns}^p)>\alpha_0\}} h^2(y)\, n^{1-\frac{\alpha(Y_{ns}^p)}{\alpha_0}} \frac{c(Y_{ns}^p)}{|y|^{\alpha(Y_{ns}^p)+1}} dyds \qquad (4.12) \\
&\quad + \int_0^t \int_{(-\delta,\delta)^c} 1_{\{\alpha(Y_{ns}^p)>\alpha_0\}} h^2(y) \left(n^{1+\frac{1}{\alpha_0}} f_{Y_{ns}^p}\left(n^{\frac{1}{\alpha_0}} y\right) - n^{1-\frac{\alpha(Y_{ns}^p)}{\alpha_0}} \frac{c(Y_{ns}^p)}{|y|^{\alpha(Y_{ns}^p)+1}}\right) dyds \qquad (4.13) \\
&\quad + \int_0^t \int_{(-\delta,\delta)} 1_{\{\alpha(Y_{ns}^p)>\alpha_0\}} y^2 \left(n^{1+\frac{1}{\alpha_0}} f_{Y_{ns}^p}\left(n^{\frac{1}{\alpha_0}} y\right) - n^{1-\frac{\alpha(Y_{ns}^p)}{\alpha_0}} \frac{c(Y_{ns}^p)}{|y|^{\alpha(Y_{ns}^p)+1}}\right) dyds. \qquad (4.14)
\end{aligned}
$$

By (4.6) and the dominated convergence theorem, (4.10) and (4.13) converge to 0 $\mathbb{P}^x$-a.s., when $n \longrightarrow \infty$. Let us prove that (4.11) converges to 0 $\mathbb{P}^x$-a.s., when $n \longrightarrow \infty$ and $\delta \longrightarrow 0$, respectively. By using (4.6), we have

$$
\begin{aligned}
&\int_0^t \int_{(-\delta,\delta)} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} y^2 \left(n^{1+\frac{1}{\alpha_0}} f_{Y_{ns}^p}\left(n^{\frac{1}{\alpha_0}} y\right) - \frac{c(Y_{ns}^p)}{|y|^{\alpha_0+1}}\right) dyds \\
&= \int_0^t \int_{(-n^{\frac{1}{\alpha_0}}\delta,\, n^{\frac{1}{\alpha_0}}\delta)} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} y^2 n^{1-\frac{2}{\alpha_0}} f_{Y_{ns}^p}(y)\, dyds \\
&\quad - \int_0^t \int_{(-\delta,\delta)} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} |y|^{1-\alpha_0} c(Y_{ns}^p) dyds \\
&= \int_0^t \int_{(-y_\varepsilon, y_\varepsilon)} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} y^2 n^{1-\frac{2}{\alpha_0}} f_{Y_{ns}^p}(y)\, dyds \\
&\quad + \int_0^t \int_{(-n^{\frac{1}{\alpha_0}}\delta,\, -y_\varepsilon)\cup(y_\varepsilon,\, n^{\frac{1}{\alpha_0}}\delta)} 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} y^2 n^{1-\frac{2}{\alpha_0}} f_{Y_{ns}^p}(y)\, dyds \\
&\quad + \frac{2}{2-\alpha_0} \delta^{2-\alpha_0} \int_0^t 1_{\{\alpha(Y_{ns}^p)=\alpha_0\}} c(Y_{ns}^p) ds
\end{aligned}
$$

$$
\begin{aligned}
\leq & n^{1-\frac{2}{\alpha_0}}\int_0^t\int_{(-y_\varepsilon,y_\varepsilon)}1_{\{\alpha(Y^p_{ns})=\alpha_0\}}y^2f_{Y^p_{ns}}(y)\,dyds\\
&+(1+\varepsilon)n^{1-\frac{2}{\alpha_0}}\int_0^t\int_{(-n^{\frac{1}{\alpha_0}}\delta,-y_\varepsilon)\cup(y_\varepsilon,n^{\frac{1}{\alpha_0}}\delta)}1_{\{\alpha(Y^{(n)}_s)=\alpha_0\}}y^2\frac{c(Y^p_{ns})}{|y|^{\alpha_0+1}}dyds\\
&+\frac{2}{2-\alpha_0}\delta^{2-\alpha_0}\int_0^t 1_{\{\alpha(Y^p_{ns})=\alpha_0\}}c(Y^p_{ns})ds\\
=& n^{1-\frac{2}{\alpha_0}}\int_0^t\int_{(-y_\varepsilon,y_\varepsilon)}1_{\{\alpha(Y^p_{ns})=\alpha_0\}}y^2f_{Y^p_{ns}}(y)\,dyds\\
&+(1+\varepsilon)\frac{2}{2-\alpha_0}\delta^{2-\alpha_0}\int_0^t 1_{\{\alpha(Y^p_{ns})=\alpha_0\}}c(Y^p_{ns})ds\\
&-(1+\varepsilon)n^{1-\frac{2}{\alpha_0}}\frac{2}{2-\alpha_0}y_\varepsilon^{2-\alpha_0}\int_0^t 1_{\{\alpha(Y^p_{ns})=\alpha_0\}}c(Y^p_{ns})ds\\
&+\frac{2}{2-\alpha_0}\delta^{2-\alpha_0}\int_0^t 1_{\{\alpha(Y^p_{ns})=\alpha_0\}}c(Y^p_{ns})ds.
\end{aligned}
$$

Now, by (1.5) and the dominated convergence theorem, we have

$$
\lim_{\delta\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_0^t\int_{(-\delta,\delta)}1_{\{\alpha(Y^p_{ns})=\alpha_0\}}y^2\left(n^{1+\frac{1}{\alpha_0}}f_{Y^p_{ns}}\left(n^{\frac{1}{\alpha_0}}y\right)-\frac{c(Y^p_{ns})}{|y|^{\alpha_0+1}}\right)dyds=0\quad\mathbb{P}^x\text{-a.s.}
$$

In completely the same way one can prove that (4.14) converges to 0 $\mathbb{P}^x$-a.s., when $n\longrightarrow\infty$ and $\delta\longrightarrow 0$, respectively. In order to prove that (4.12) converges in $L^2(\Omega,\mathbb{P}^x)$ to 0, when $n\longrightarrow\infty$, we define

$$
V_n(z):=\int_{\mathbb{R}}h^2(y)\left(1_{\{\alpha(z)>\alpha_0\}}n^{1-\frac{\alpha(z)}{\alpha_0}}\frac{c(z)}{|y|^{\alpha(z)+1}}-\int_{\mathbb{R}/\Lambda}1_{\{\alpha(x)>\alpha_0\}}(x)n^{1-\frac{\alpha(x)}{\alpha_0}}\frac{c(x)}{|y|^{\alpha(x)+1}}\pi(dx)\right)dy
$$

and proceed as for (4.4). It remains to prove that (4.9) converges in $L^2(\Omega,\mathbb{P}^x)$ to $\tilde{C}^0_t$, when $n\longrightarrow\infty$. Let us define

$$
V(z):=\int_{\mathbb{R}}h^2(y)\left(1_{\{\alpha(z)=\alpha_0\}}\frac{c(z)}{|y|^{\alpha_0+1}}-\int_{\mathbb{R}/\Lambda}1_{\{\alpha(x)=\alpha_0\}}\frac{c(x)}{|y|^{\alpha_0+1}}\pi(dx)\right)dy.
$$

By τ-periodicity of the functions $\alpha(x)$ and $c(x)$, the function $V(z)$ is τ-periodic and

$$
\int_{\mathbb{R}/\Lambda}V(z)\pi(dz)=0.
$$

Hence, by repeating the same calculation as for (4.4), we have the claim. Therefore, by [JS03, Theorem VIII.2.17], we have proved that the sequence of processes $\mathbf{Y}^p_n$ converges in distribution to symmetric α_0-stable Lévy process $\mathbf{L}$ with the compensator (Lévy measure) $N^0(ds,dy)$.

Now, let us prove that the periodic stable-like chain $\{X^p_n\}_{n\geq 0}$ is recurrent if and only if $\alpha_0\geq 1$. By [Fra06, Lemmas 2 and 3], the set of recurrent paths $R(O)$ is a continuity set for the probability measure $\mathbb{P}^x_{\mathbf{L}}(\cdot)$ for all $x\in\mathbb{R}$ and all open bounded sets $O\subseteq\mathbb{R}$. Furthermore, since $\mathbf{L}$ is a λ-irreducible T-model (note that (2.1) is trivially satisfied), by Proposition 2.4, $\mathbf{L}$ is recurrent if and only if $\mathbb{P}^x_{\mathbf{L}}(R(O))=1$ for all $x\in\mathbb{R}$ and all open bounded sets $O\subseteq\mathbb{R}$, and it is transient if and only if $\mathbb{P}^x_{\mathbf{L}}(T(O))=1$ for all $x\in\mathbb{R}$ and all open bounded sets $O\subseteq\mathbb{R}$.

Let $x \in \mathbb{R}$ be an arbitrary starting point and let $O \subseteq \mathbb{R}$ be an arbitrary open bounded set. By [Bil99, Theorem 2.1], we have

$$\lim_{n\longrightarrow\infty} \mathbb{P}^x_{\mathbf{Y}^p_n}(R(O)) = \mathbb{P}^x_{\mathbf{L}}(R(O)). \tag{4.15}$$

If the stable-like chain $\{X^p_n\}_{n\geq 0}$ is recurrent, since it is λ-irreducible T-model, it is H-recurrent as well. Hence, by Proposition 2.7, all the processes $\mathbf{Y}^p_n$, $n \in \mathbb{N}$, are H-recurrent. This implies

$$\mathbb{P}^x_{\mathbf{Y}^n_n}(R(O)) = 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

Therefore, by (4.15), $\mathbb{P}_{\mathbf{L}}(R(O)_x) = 1$, i.e., $\mathbf{L}$ is recurrent.

Let us assume that the periodic stable-like chain $\{X^p_n\}_{n\geq 0}$ is transient. Then, by Proposition 2.4, $\mathbb{P}^x(\tau_O < \infty) = 0$ for all $x \in \mathbb{R}$ and all open bounded sets $O \subseteq \mathbb{R}$. Hence, by Proposition 2.7, $\mathbb{P}^x(\tau^n_O < \infty) = 0$, i.e.,

$$\mathbb{P}^x_{\mathbf{Y}^p_n}(R(O)) = 0$$

for all $n \in \mathbb{N}$, all $x \in \mathbb{R}$ and all open bounded sets $O \subseteq \mathbb{R}$. Therefore, by (4.15), $\mathbb{P}^x_{\mathbf{L}}(R(O)) = 0$, i.e., $\mathbf{L}$ is transient. Finally, by [Sat99, Corollary 37.17], $\mathbf{L}$ is recurrent if and only if $\alpha_0 \geq 1$. This accomplishes the proof. □

Remark 4.1. (i) In Theorem 1.2 we assume that the densities $\{f_x\}_{x\in\mathbb{R}}$ satisfy $f_x(-y) = f_x(y)$ for all $x, y \in \mathbb{R}$ and $f_x(y) \sim c(x)|y|^{-\alpha(x)-1}$, when $|y| \longrightarrow \infty$. This assumptions can be relaxed. Let $\alpha : \mathbb{R} \longrightarrow (0, 2)$ and $c_+, c_- : \mathbb{R} \longrightarrow (0, \infty)$ be Borel measurable functions and let $\{f_x\}_{x\in\mathbb{R}}$ be an arbitrary family of probability densities on $\mathbb{R}$. Furthermore, let us assume that the function $x \longmapsto f_x$ is a periodic function with period $\tau > 0$ and that the following conditions are satisfied:

(PC1') the function $(x, y) \longmapsto f_x(y)$ is continuous and strictly positive;

(PC2') $f_x(y) \sim c_+(x)y^{-\alpha(x)-1}$, when $y \longrightarrow \infty$, and
$f_x(y) \sim c_-(x)(-y)^{-\alpha(x)-1}$, when $y \longrightarrow -\infty$, for all $x \in \mathbb{R}$;

(PC3') $\displaystyle\lim_{y\longrightarrow\infty} \sup_{x\in[0,\tau]} \left| f_x(y)\frac{y^{\alpha(x)+1}}{c_+(x)} - 1 \right| = 0$ and $\displaystyle\lim_{y\longrightarrow-\infty} \sup_{x\in[0,\tau]} \left| f_x(y)\frac{|y|^{\alpha(x)+1}}{c_-(x)} - 1 \right| = 0;$

(PC4') $\displaystyle\inf_{x\in[0,\tau]} (c_-(x) \wedge c_+(x)) > 0.$

Hence, the densities $\{f_x\}_{x\in\mathbb{R}}$ have two-tail behavior. Let $\{\bar{X}^p_n\}_{n\geq 0}$ be a Markov chain given by the transition kernel $\bar{p}(x, dy) := f_x(y - x)dy$. By completely the same arguments as in the proof of Theorem 1.2, we can deduce recurrence and transience property of the chain $\{\bar{X}^p_n\}_{n\geq 0}$. If the set $\{x \in \mathbb{R} : \alpha(x) = \alpha_0 := \inf_{x\in\mathbb{R}} \alpha(x)\}$ has positive Lebesgue measure, then by subordination of the chain $\{\bar{X}^p_n\}_{n\geq 0}$ with the Poisson process $\{N_t\}_{t\geq 0}$ with parameter 1 (independent of the chain $\{\bar{X}^p_n\}_{n\geq 0}$), one can prove that the process $\{n^{-\frac{1}{\alpha_0}} \bar{X}^p_{N_{nt}}\}_{t\geq 0}$ converges in distribution, with respect to the Skorohod topology, to α_0-stable Lévy process. In general, this α_0-stable Lévy process is not symmetric anymore. Non-symmetry of the densities $\{f_x\}_{x\in\mathbb{R}}$ implies that the α_0-stable Lévy process has a nonzero shift parameter, and two-tail behavior implies that the α_0-stable Lévy process has a nonzero skewness parameter. Hence, by [Sat99, Corollary 37.17], the only recurrent cases are if either $\alpha_0 > 1$ and shift parameter vanishes or $\alpha_0 = 1$ and skewness parameter vanishes.

(ii) As already mentioned, it is shown in [Fra06] that if the functions $\alpha : \mathbb{R} \longrightarrow (0, 2)$ and $\gamma : \mathbb{R} \longrightarrow (0, \infty)$ are continuously differentiable with bounded derivative and periodic and if the set $\{x \in \mathbb{R} : \alpha(x) = \alpha_0 := \inf_{x\in\mathbb{R}} \alpha(x)\}$ has positive Lebesgue measure, then the stable-like process with the symbol $p(x, \xi) = \gamma(x)|\xi|^{\alpha(x)}$ is recurrent if and only if $\alpha_0 \geq 1$. In general, we cannot apply Theorem 1.2 for the discrete-time version of this stable-like process, i.e., for the stable-like chain $\{X_n^{\alpha(x)}\}_{n\geq0}$ (the chain given by (1.2)), since we do not have a proof that its transition densities satisfy condition (PC3). But, by repeating the proof of Theorem 1.1 we deduce: If $\alpha : \mathbb{R} \longrightarrow (0, 2)$ and $\gamma : \mathbb{R} \longrightarrow (0, \infty)$ are continuously differentiable and periodic functions with bounded derivative and if the set $\{x \in \mathbb{R} : \alpha(x) = \alpha_0 := \inf_{x\in\mathbb{R}} \alpha(x)\}$ has positive Lebesgue measure, then the stable-like chain $\{X_n^{\alpha(x)}\}_{n\geq0}$ is recurrent if and only if $\alpha_0 \geq 1$.

Similarly, by repeating the proof of Theorem 1.1, we can prove transience property of the discrete-time version of the stable-like process considered in [SW12], i.e., the process given by the symbol $p(x, \xi) = \gamma(x)|\xi|^{\alpha(x)}$, where $\alpha : \mathbb{R} \longrightarrow (0, 2)$ and $\gamma : \mathbb{R} \longrightarrow (0, \infty)$ are continuously differentiable functions with bounded derivative and such that $\limsup_{|x|\longrightarrow\infty} \alpha(x) < 1$ and $0 < \inf_{x\in\mathbb{R}} \gamma(x) \leq \sup_{x\in\mathbb{R}} \gamma(x) < \infty$.

5 Discrete state case

In this section we derive the same recurrence and transience criteria as in Theorems 1.1 and 1.2 for discrete version of the stable-like chains $\{X_n^{(\alpha,\beta)}\}_{n\geq0}$ and $\{X_n^p\}_{n\geq0}$ (the chains given by (1.3) and (1.4)). Without loss of generality, we treat the case on the state space $\mathbb{Z}$. Let $\alpha : \mathbb{Z} \longrightarrow (0, 2)$ and $c : \mathbb{Z} \longrightarrow (0, \infty)$ be arbitrary functions and let $\{f_i\}_{i\in\mathbb{Z}}$ be a family of probability functions on $\mathbb{Z}$ which satisfies $f_i(j) \sim c(i)|j|^{-\alpha(i)-1}$, when $|j| \longrightarrow \infty$. Let $\{X_n^d\}_{n\geq0}$ be a Markov chain on $\mathbb{Z}$ given by the following transition kernel

$$p(i, j) := f_i(j - i).$$

The chain $\{X_n^d\}_{n\geq0}$ can be understood as a discrete version of the stable-like chain $\{X_n\}_{n\geq0}$, i.e., the probability functions $f_i(j)$ are discrete versions of densities $f_x(y)$. It is clear that if $f_i(j) > 0$ for all $i, j \in \mathbb{Z}$, then the chain $\{X_n^d\}_{n\geq0}$ is irreducible. Therefore, it is either recurrent or transient. If the following conditions are satisfied

(CD1) $f_i(j) \sim c(i)|j|^{-\alpha(i)-1}$, when $|j| \longrightarrow \infty$, for all $i \in \mathbb{Z}$;

(CD2) there exists $k \in \mathbb{N}$ such that

$$\lim_{|j|\longrightarrow\infty} \sup_{i\in\{-k,\dots,k\}^c} \left| f_i(j) \frac{|j|^{\alpha(i)+1}}{c(i)} - 1 \right| = 0,$$

then the chain $\{X_n^d\}_{n\geq0}$ is recurrent if $\liminf_{|i|\longrightarrow\infty} \alpha(i) > 1$, and it is transient if $\limsup_{|i|\longrightarrow\infty} \alpha(i) < 1$ (see [San12]). Note that conditions (CD1) and (CD2) also implies irreducibility of the chain $\{X_n^d\}_{n\geq0}$ in the case when $f_i(j) > 0$ is not satisfied for all $i, j \in \mathbb{Z}$.

5.1 Step case

Let $\{X_n^{d(\alpha,\beta)}\}_{n\geq0}$ be a discrete version of the stable-like chain $\{X_n^{(\alpha,\beta)}\}_{n\geq0}$ given by (1.3), i.e., a special case of the chain $\{X_n^d\}_{n\geq0}$ given by the following step functions

$$\alpha(i) = \begin{cases} \alpha, & i < 0 \\ \beta, & i \geq 0 \end{cases} \quad \text{and} \quad c(i) = \begin{cases} c, & i < 0 \\ d, & i \geq 0, \end{cases}$$

where $\alpha, \beta \in (0,2)$ and $c, d \in (0, \infty)$.

Recall that a random walk $\{S_n\}_{n\geq 0}$ is *attracted* to a random variable X if there exist sequences of real numbers $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$, $B_n > 0$ for all $n \in \mathbb{N}$, such that

$$\frac{S_n}{B_n} - A_n \xrightarrow{\mathrm{d}} X.$$

Here $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution. Furthermore, if $A_n = 0$ for all $n \in \mathbb{N}$, then we say that the random walk $\{S_n\}_{n\geq 0}$ is *strongly attracted* to X. The random variable X can only have a stable distribution (see [IL71, Theorem 2.1.1]). Now, from [GK54, Theorem 35.2] which gives necessary and sufficient conditions in order that a random walk $\{S_n\}_{n\geq 0}$ is attracted to a random variable with stable distribution with the index of stability $\alpha \in (0,2)$, we easily derive:

Proposition 5.1. *Let $\alpha \in (0,2)$ and $c \in (0,\infty)$ be arbitrary and let $f_{(\alpha,c)} : \mathbb{Z} \longrightarrow \mathbb{R}$ be an arbitrary probability function such that $f_{(\alpha,c)}(j) \sim c|j|^{-\alpha-1}$, when $|j| \longrightarrow \infty$. Let us assume that $f_{(\alpha,c)}(-j) = f_{(\alpha,c)}(j)$ holds for all $j \in \mathbb{Z}$ if $\alpha = 1$, and $\sum_{j\in\mathbb{Z}} j f_{(\alpha,c)}(j) = 0$ holds if $\alpha > 1$. Then the random walk $\{S_n\}_{n\geq 0}$ with the jump distribution*

$$\begin{pmatrix} \dots & -1 & 0 & 1 & \dots \\ \dots f_{(\alpha,c)}(-1) & f_{(\alpha,c)}(0) & f_{(\alpha,c)}(1) & \dots \end{pmatrix}$$

is strongly attracted to $S\alpha S$ distribution.

From Proposition 5.1, as a special case of [RF78, Theorem 2], we have:

Theorem 5.2. *If the probability functions $f_{(\alpha,c)}(j) := f_i(j)$, for $i < 0$, and $f_{(\beta,d)}(j) := f_i(j)$, for $i \geq 0$, appearing in the definition of the chain $\{X_n^{d(\alpha,\beta)}\}_{n\geq 0}$, satisfy $f_{(\alpha,c)}(j) = f_{(\alpha,c)}(-j)$ and $f_{(\beta,d)}(j) = f_{(\beta,d)}(-j)$ for all $j \in \mathbb{Z}$, then the chain $\{X_n^{d(\alpha,\beta)}\}_{n\geq 0}$ is recurrent if $\alpha + \beta > 2$, and it is transient if $\alpha + \beta < 2$.*

Note that previous theorem does not say anything about the case when $\alpha + \beta = 2$. This case is not covered by [RF78] and it seems to be much more complicated.

5.2 Periodic case

In this subsection we consider a discrete version of the periodic stable-like chain $\{X_n^p\}_{n\geq 0}$ given by (1.4). Let $\{X_n^{dp}\}_{n\geq 0}$ be a Markov chain on $\mathbb{Z}$ given by

$$\alpha(i) = \begin{cases} \alpha, & i \in 2\mathbb{Z} \\ \beta, & i \in 2\mathbb{Z}+1 \end{cases} \quad \text{and} \quad c(i) = \begin{cases} c, & i \in 2\mathbb{Z} \\ d, & i \in 2\mathbb{Z}+1, \end{cases}$$

where $\alpha, \beta \in (0,2)$ and $c, d \in (0,\infty)$, and let us assume that probability functions $f_{(\alpha,c)}(j) := f_{2i}(j)$ and $f_{(\beta,d)}(j) := f_{2i+1}(j)$, $i \in \mathbb{Z}$, satisfy $f_{(\alpha,c)}(-j) = f_{(\alpha,c)}(j)$ and $f_{(\beta,d)}(-j) = f_{(\beta,d)}(j)$ for all $j \in \mathbb{Z}$. Let us define the following stopping times inductively: $T_0^\alpha := 0$, $T_0^\beta := 0$, $T_n^\alpha := \inf\{k > T_{n-1}^\alpha : X_k^{dp} \in 2\mathbb{Z}\}$ and $T_n^\beta := \inf\{k > T_{n-1}^\beta : X_k^{dp} \in 2\mathbb{Z}+1\}$, for $n \in \mathbb{N}$.

Proposition 5.3. $\mathbb{P}^i(T_n^\alpha < \infty) = \mathbb{P}^i(T_n^\beta < \infty) = 1$ *for all $i \in \mathbb{Z}$ and all $n \in \mathbb{N}$.*

Proof. Let us prove that $\mathbb{P}^i(T_n^\alpha < \infty) = 1$ for all $i \in \mathbb{Z}$ and all $n \in \mathbb{N}$ by induction. Let $i \in \mathbb{Z}$ be arbitrary and let $n = 1$. We have

$$\begin{aligned}\mathbb{P}^i(T_1^\alpha = \infty) =& \mathbb{P}^i(X_k^{pd} \in 2\mathbb{Z}+1,\ \forall k \in \mathbb{N}) = \lim_{k\longrightarrow\infty} \mathbb{P}^i(X_l^{dp} \in 2\mathbb{Z}+1,\ 1 \le l \le k) \\ =& \lim_{k\longrightarrow\infty} \sum_{i_1\in 2\mathbb{Z}+1} p(i,i_1) \sum_{i_2\in 2\mathbb{Z}+1} p(i_1,i_2)\dots \sum_{i_{k-1}\in 2\mathbb{Z}+1} p(i_{k-2},i_{k-1})p(i_{k-1},2\mathbb{Z}+1).\end{aligned}$$

Note that $p(2i+1, 2\mathbb{Z}+1) = \sum_{j\in 2\mathbb{Z}} f_{(\beta,d)}(j) < 1$ for all $i \in \mathbb{Z}$. Therefore, if we put $C := \sum_{j\in 2\mathbb{Z}} f_{(\beta,d)}(j)$ and $C_i := p(i, 2\mathbb{Z}+1)$, we have

$$\mathbb{P}^i(T_1^\alpha = \infty) = \lim_{k\longrightarrow\infty} C_i C^{k-1} = 0,$$

i.e., $\mathbb{P}^i(T_1^\alpha < \infty) = 1$. Let us assume that $\mathbb{P}^i(T_{n-1}^\alpha < \infty) = 1$ and let us prove that $\mathbb{P}^i(T_n^\alpha < \infty) = 1$. By denoting $N := T_{n-1}^\alpha$ and using strong Markov property we have

$$\mathbb{P}^i(T_n^\alpha < \infty) = \mathbb{E}^i[\mathbb{E}^i[1_{\{T_1^\alpha<\infty\}} \circ \theta_N|\mathcal{F}_N]] = \mathbb{E}^i[\mathbb{E}^{X_N}[1_{\{T_1^\alpha<\infty\}}]] = \sum_{j\in 2\mathbb{Z}} \mathbb{E}^i[1_{\{X_N=j\}}] = 1,$$

where θ_n is the shift operator on the canonical state space $\mathbb{Z}^{\{0,1,\dots\}}$. In the completely analogously way we prove that $\mathbb{P}^i(T_n^\beta < \infty) = 1$ for all $i \in \mathbb{Z}$ and all $n \in \mathbb{N}$. □

For $n \ge 0$, let us put $Y_n^\alpha = X_{T_n^\alpha}^{dp}$ and $Y_n^\beta = X_{T_n^\beta}^{dp}$, then, from Proposition 5.3, $\{Y_n^\alpha\}_{n\ge 0}$ and $\{Y_n^\beta\}_{n\ge 0}$ are well defined Markov chains. Let $i \in \mathbb{Z}$ and let us define the following stopping times: $\tau_i := \inf\{n \ge 1 : X_n^{dp} = i\}$, $\tau_i^\alpha := \inf\{n \ge 1 : Y_n^\alpha = i\}$ and $\tau_i^\beta = \inf\{n \ge 1 : Y_n^\beta = i\}$.

Proposition 5.4. *For all $i \in \mathbb{Z}$, $n \in \mathbb{N}$, $j_1,\dots,j_n \in 2\mathbb{Z}$ and all $k_1,\dots,k_n \in 2\mathbb{Z}+1$ we have $\mathbb{P}^i(Y_1^\alpha = j_1,\dots,Y_n^\alpha = j_n) > 0$ and $\mathbb{P}^i(Y_1^\beta = k_1,\dots,Y_n^\beta = k_n) > 0$. In particular, the chains $\{Y_n^\alpha\}_{n\ge 0}$ and $\{Y_n^\beta\}_{n\ge 0}$ are irreducible on their state spaces.*

Proof. The set $2\mathbb{Z}$ is the state space of the chain $\{Y_n^\alpha\}_{n\ge 0}$, and the set $2\mathbb{Z}+1$ is the state space of the chain $\{Y_n^\beta\}_{n\ge 0}$. Let $i \in \mathbb{Z}$ and $j_1 \in 2\mathbb{Z}$ be arbitrary, then we have

$$\begin{aligned}\mathbb{P}^i(Y_1^\alpha = j_1) =& p(i,j_1) + \sum_{i_1\in 2\mathbb{Z}+1} p(i,i_1)p(i_1,j_1) + \sum_{i_1\in 2\mathbb{Z}+1} p(i,i_1) \sum_{1_2\in 2\mathbb{Z}+1} p(i_1,i_2)p(i_2,j_1) + \dots \\ \ge& \sum_{i_1\in 2\mathbb{Z}+1} p(i,i_1)p(i_1,j_1).\end{aligned}$$

If $i \in 2\mathbb{Z}$, then we take $i_1 \in 2\mathbb{Z}+1$ such that $f_{(\alpha,c)}(i_1 - i) > 0$ and $f_{(\beta,d)}(j_1 - i_1) > 0$. Therefore,

$$\mathbb{P}^i(Y_1^\alpha = j_1) \ge f_{(\alpha,c)}(i_1 - i)f_{(\beta,d)}(j_1 - i_1) > 0.$$

If $i \in 2\mathbb{Z}+1$, then we take $i_1 \in 2\mathbb{Z}+1$ such that $f_{(\beta,d)}(i_1 - i) > 0$ and $f_{(\beta,d)}(j_1 - i_1) > 0$. Hence, we have

$$\mathbb{P}^i(Y_1^\alpha = j_1) \ge f_{(\beta,d)}(i_1 - i)f_{(\beta,d)}(j_1 - i_1) > 0.$$

Let $i \in \mathbb{Z}$ and $j_1, j_2 \in 2\mathbb{Z}$ be arbitrary, then we have

$$\mathbb{P}^i(Y_1^\alpha = j_1, Y_2^\alpha = j_2) = \mathbb{P}^i(Y_2^\alpha = j_2|Y_1^\alpha = j_1)\mathbb{P}^i(Y_1^\alpha = j_1) = \mathbb{P}^{j_1}(Y_1^\alpha = j_2)\mathbb{P}^i(Y_1^\alpha = j_1) > 0.$$

Let $n > 2$. Let us suppose that for all $i \in \mathbb{Z}$ and for all $j_1, \ldots j_{n-1} \in 2\mathbb{Z}$ we have

$$\mathbb{P}^i(Y_1^\alpha = j_1, \ldots, Y_{n-1}^\alpha = j_{n-1}) > 0.$$

Let $j_n \in 2\mathbb{Z}$ be arbitrary, then we have

$$\begin{aligned}\mathbb{P}^i(Y_1^\alpha = j_1, \ldots, Y_n^\alpha = j_n) =& \mathbb{P}^i(Y_n^\alpha = j_n | Y_{n-1}^\alpha = j_{n-1}, \ldots, Y_1^\alpha = j_1)\mathbb{P}^i(Y_1^\alpha = j_1, \ldots, Y_{n-1}^\alpha = j_{n-1}) \\ =& \mathbb{P}^{j_{n-1}}(Y_1^\alpha = j_n)\mathbb{P}^i(Y_1^\alpha = j_1, \ldots, Y_{n-1}^\alpha = j_{n-1}) > 0.\end{aligned}$$

Analogously we prove the claim for the chain $\{Y_n^\beta\}_{n\geq 0}$. Let $i, j \in 2\mathbb{Z}$ be arbitrary, then we have

$$\mathbb{P}^i(\tau_j^\alpha < \infty) \geq \mathbb{P}^i(\tau_j^\alpha = 1) = \mathbb{P}^i(Y_1^\alpha = j) > 0.$$

Similarly, for arbitrary $i, j \in 2\mathbb{Z} + 1$ we have

$$\mathbb{P}^i(\tau_j^\beta < \infty) > 0.$$

Hence, the chains $\{Y_n^\alpha\}_{n\geq 0}$ and $\{Y_n^\beta\}_{n\geq 0}$ are irreducible. □

Proposition 5.5. *The Markov chains $\{X_n^{dp}\}_{n\geq 0}$, $\{Y_n^\alpha\}_{n\geq 0}$ and $\{Y_n^\beta\}_{n\geq 0}$ have the same recurrence property.*

Proof. Let $i \in 2\mathbb{Z}$ be arbitrary, then we have

$$\mathbb{P}^i(\tau_i^\alpha = \infty) = \mathbb{P}^i(Y_n^\alpha \in 2\mathbb{Z} \setminus \{i\},\ n \in \mathbb{N}) = \mathbb{P}^i(X_n^{dp} \in \mathbb{Z} \setminus \{i\},\ n \in \mathbb{N}) = \mathbb{P}^i(\tau_i = \infty).$$

Similarly, for arbitrary $i \in 2\mathbb{Z} + 1$ we have $\mathbb{P}^i(\tau_i = \infty) = \mathbb{P}^i(\tau_i^\beta = \infty)$. □

Proposition 5.6. *Chains $\{Y_n^\alpha\}_{n\geq 0}$ and $\{Y_n^\beta\}_{n\geq 0}$ are symmetric random walks with jump distributions $\mathbb{P}^0(Y_1^\alpha \in \cdot)$ and $\mathbb{P}^1(Y_1^\beta - 1 \in \cdot)$.*

Proof. Note first that for arbitrary $i, j \in \mathbb{Z}$ we have

$$\begin{aligned}\mathbb{P}^0(Y_{n+1}^\alpha = 2i - 2j | Y_n^\alpha = 0) =& p(0, 2i - 2j) + \sum_{k_1 \in 2\mathbb{Z}+1} p(0, k_1)p(k_1, 2i - 2j) \\ &+ \sum_{k_1 \in 2\mathbb{Z}+1} p(0, k_1) \sum_{k_2 \in 2\mathbb{Z}+1} p(k_1, k_2)p(k_2, 2i - 2j) + \ldots \\ =& p(2i, 2j) + \sum_{k_1 \in 2\mathbb{Z}+1} p(2j, k_1 + 2j)p(k_1 + 2j, 2i) \\ &+ \sum_{k_1 \in 2\mathbb{Z}+1} p(2j, k_1 + 2j) \sum_{k_2 \in 2\mathbb{Z}+1} p(k_1 + 2j, k_2 + 2j)p(k_2 + 2j, 2i) + \ldots \\ =& \mathbb{P}^0(Y_{n+1}^\alpha = 2i | Y_n^\alpha = 2j).\end{aligned}$$

Let us prove that the random variables $Y_{n+1}^\alpha - Y_n^\alpha$, $n \geq 0$, are symmetric i.i.d. random variables with respect to the probability measure $\mathbb{P}^0(\cdot)$. Let $n \geq 0$. Then we have

$$\begin{aligned}\mathbb{P}^0(Y_{n+1}^\alpha - Y_n^\alpha = 2i) =& \sum_{j\in\mathbb{Z}} \mathbb{P}^0(Y_{n+1}^\alpha = 2i + 2j,\ Y_n^\alpha = 2j) \\ =& \sum_{j\in\mathbb{Z}} \mathbb{P}^0(Y_{n+1}^\alpha = 2i + 2j | Y_n^\alpha = 2j)\mathbb{P}^0(Y_n^\alpha = 2j) = \mathbb{P}^0(Y_1^\alpha = 2i).\end{aligned}$$

Let $n \geq 1$. Then we have

$$
\begin{aligned}
&\mathbb{P}^0(Y^\alpha_{n+1} - Y^\alpha_n = 2i,\ Y^\alpha_n - Y^\alpha_{n-1} = 2j)\\
&= \sum_{k\in\mathbb{Z}} \mathbb{P}^0(Y^\alpha_{n+1} = 2i+2j,\ Y^\alpha_n = 2k,\ Y^\alpha_{n-1} = 2k-2j)\\
&= \sum_{k\in\mathbb{Z}} \mathbb{P}^0(Y^\alpha_{n+1} = 2i+2k | Y^\alpha_n = 2k)\mathbb{P}^0(Y^\alpha_n = 2k | Y^\alpha_{n-1} = 2k-2j)\mathbb{P}^0(Y^\alpha_{n-1} = 2k-2j)\\
&= \mathbb{P}^0(Y^\alpha_1 = 2i)\mathbb{P}^0(Y^\alpha_1 = 2j) = \mathbb{P}^0(Y^\alpha_{n+1} - Y^\alpha_n = 2i)\mathbb{P}^0(Y^\alpha_n - Y^\alpha_{n-1} = 2j).
\end{aligned}
$$

This proves that the random variables $Y^\alpha_{n+1} - Y^\alpha_n$, $n \geq 0$, are i.i.d. random variables. Symmetry is obvious. Completely analogously we prove that the random variables $Y^\beta_{n+1} - Y^\beta_n$, $n \geq 0$, are i.i.d. symmetric random variables with respect to the probability measure $\mathbb{P}^1(\cdot)$. □

Proposition 5.7. *If $\alpha \wedge \beta < 1$, then the chain $\{X^{dp}_n\}_{n\geq 0}$ is transient.*

Proof. Without loss of generality, let us suppose that $\alpha \wedge \beta = \alpha < 1$. By Proposition 5.5, it is enough to prove that the chain $\{Y^\alpha_n\}_{n\geq 0}$ is transient. From Proposition 5.6 we know that the chain $\{Y^\alpha_n\}_{n\geq 0}$ is symmetric random walk on $2\mathbb{Z}$ with respect to the probability measure $\mathbb{P}^0(\cdot)$. For every $i \in \mathbb{Z}$ we have

$$\mathbb{P}^0(Y^\alpha_1 = 2i) = p(0, 2i) + \sum_{j\in 2\mathbb{Z}+1} p(0, j)p(j, 2i) + \ldots \geq f_{(\alpha,c)}(2i).$$

Let $\varphi(\xi)$ be the characteristic function of the distribution $\mathbb{P}^0(Y^\alpha_1 \in \cdot)$. From the symmetry property of the distribution $\mathbb{P}^0(Y^\alpha_1 \in \cdot)$, we have

$$\mathrm{Re}\left(\frac{1}{1-\varphi(\xi)}\right) = \frac{1}{\sum_{j\in\mathbb{Z}}(1-\cos(2j\xi))\mathbb{P}^0(Y^\alpha_1 = 2j)} \leq \frac{1}{\sum_{j\in\mathbb{Z}}(1-\cos(2j\xi))f_{(\alpha,c)}(2j)}.$$

Note that $\sum_{j\in\mathbb{Z}} \cos(2j\xi) f_{(\alpha,c)}(2j)$ is the Fourier transform of the symmetric sub-probability measure on $2\mathbb{Z}$. Using completely the same arguments as in [Spi76, page 88], from [Dur10, Theorem 3.2.9] we get the desired result. □

Let $m \geq 1$, $\alpha_0, \ldots, \alpha_{m-1} \in (0, 2)$ and $c_0, \ldots, c_{m-1} \in (0, \infty)$ be arbitrary. Let $\{X^{dp}_n\}_{n\geq 0}$ be a Markov chain on $\mathbb{Z}$ given by

$$\alpha(i) = \alpha_j \quad \text{and} \quad c(i) = c_j$$

for $i \equiv j \bmod (m)$, i.e., the functions $\alpha : \mathbb{Z} \longrightarrow (0, 2)$ and $c : \mathbb{Z} \longrightarrow (0, \infty)$ are periodic functions with period m. Furthermore, let us suppose that probability functions $f_{(\alpha_i,c_i)}(j)$, $i = 0, \ldots, m-1$, satisfy $f_{(\alpha_i,c_i)}(-j) = f_{(\alpha_i,c_i)}(j)$ for all $j \in \mathbb{Z}$ and $i = 0, \ldots, m-1$. Then, it is not hard to prove that Propositions 5.3, 5.4, 5.5 and 5.6, except perhaps the symmetry property of related chains (random walks) $\{Y^{\alpha_i}_n\}_{n\geq 0}$, $i = 0, \ldots, m$, are also valid in this periodic case. Therefore, analogously as in Proposition 5.7 using

$$\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1-a}{(1-a)^2 + b^2} \leq \frac{1}{1-a}$$

for all $z = a + ib \in \mathbb{C}$ such that $|z| \leq 1$, we have:

Theorem 5.8. *If $\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} < 1$, then the chain $\{X^{dp}_n\}_{n\geq 0}$ is transient.*

Clearly, the above statement should be an if and only if statement, i.e., there is no reason not to believe that $\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} = 1$ implies recurrence of the chain $\{X^{dp}_n\}_{n\geq 0}$. But this case is not covered by [San12] and, again, it seems to be much more complicated.

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