

# RECURRENCE, DIMENSION AND ENTROPY

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## ABSTRACT

Let  $(\Sigma_A, T)$  be a topologically mixing subshift of finite type on an alphabet consisting of  $m$  symbols and let  $\Phi: \Sigma_A \rightarrow \mathbf{R}^d$  be a continuous function. Denote by  $\sigma_\Phi(x)$  the ergodic limit  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x)$  when the limit exists. Possible ergodic limits are just mean values  $\int \Phi d\mu$  for all  $T$ -invariant measures. For any possible ergodic limit  $\alpha$ , the following variational formula is proved:

$$h_{\text{top}}(\{x \in \Sigma_A : \sigma_\Phi(x) = \alpha\}) = \sup \left\{ h_\mu : \int \Phi d\mu = \alpha \right\}$$

where  $h_\mu$  denotes the entropy of  $\mu$  and  $h_{\text{top}}$  denotes topological entropy. It is also proved that unless all points have the same ergodic limit, then the set of points whose ergodic limit does not exist has the same topological entropy as the whole space  $\Sigma_A$ .

### 1. Introduction

Let  $T$  be the shift map on  $\Sigma = \{0, 1, \dots, m-1\}^{\mathbf{N}}$  ( $m \geq 2$  an integer). Given an  $m \times m$  matrix  $A$  with entries 0 or 1, we consider the subshift of finite type  $(\Sigma_A, T)$  [4]. We shall always assume that  $A$  is primitive. That means the dynamical system  $(\Sigma_A, T)$  is topologically mixing. Now let  $\Phi$  be a continuous function defined on  $\Sigma_A$  taking values in  $\mathbf{R}^d$ . We consider the ergodic limit, when it exists,

$$\sigma_\Phi(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x).$$

The quantity  $\sigma_\Phi(x)$  is regarded as the *recurrence* of  $x$  relative to  $\Phi$  (the term ‘recurrence’ takes its usual sense when  $\Phi = (1_B, \dots, 1_B)$  where  $1_B$  denotes the characteristic function of a set  $B$ ).

Let  $L_\Phi$  be the set of  $\alpha$  such that  $\alpha = \sigma_\Phi(x)$  for some  $x \in \Sigma_A$ . As a consequence of the Birkhoff ergodic theorem,  $L_\Phi$  is a non-empty compact convex set. In this paper, we investigate the sizes of the sets with given recurrences:

$$E_\Phi(\alpha) = \{x \in \Sigma_A : \sigma_\Phi(x) = \alpha\} \quad (\alpha \in L_\Phi).$$

We also investigate the size of the set of points such that the limit defining  $\sigma_\Phi(x)$  does not exist. The size of the sets in  $\Sigma_A$  will be measured by their topological entropy. Notice that  $h_{\text{top}}$  is well-defined for non-compact invariant sets using Bowen’s definition [3].

Let  $\mathcal{M}_{\text{inv}}$  be the set of all  $T$ -invariant Borel probability measures concentrated on  $\Sigma_A$ . The function  $\Phi: \Sigma_A \rightarrow \mathbf{R}^d$  induces a map  $\Phi_*: \mathcal{M}_{\text{inv}} \rightarrow \mathbf{R}^d$ , called the *projection map*, given by

$$\Phi_*(\mu) = \int_{\Sigma_A} \Phi d\mu \quad (\mu \in \mathcal{M}_{\text{inv}}).$$

We notice that  $L_\Phi = \Phi_*(\mathcal{M}_{\text{inv}})$  (thus  $L_\Phi$  is non-empty, convex and compact). For  $\alpha \in L_\Phi$ , let

$$\mathcal{F}_\Phi(\alpha) = \{\mu \in \mathcal{M}_{\text{inv}} : \Phi_*(\mu) = \alpha\}.$$

We call  $\mathcal{F}_\Phi(\alpha)$  the *fibre of projection*  $\alpha$ .

Received 5 July 1999; revised 25 September 2000.

2000 *Mathematics Subject Classification* 58F03 (primary), 28A78 (secondary).

*J. London Math. Soc.* (2) 64 (2001) 229–244. © London Mathematical Society 2001.

The main results of the present paper are the following two theorems.

**THEOREM A.** *Suppose that  $\Phi: \Sigma_A \rightarrow \mathbf{R}^d$  is a continuous function. For any  $\alpha \in L_\Phi$ , we have the variational formula*

$$h_{\text{top}}(E_\Phi(\alpha)) = \max_{\mu \in \mathcal{F}_\Phi(\alpha)} h_\mu$$

where  $h_\mu$  is the entropy of  $\mu$ . Moreover,  $h_{\text{top}}(E_\Phi(\alpha))$  is an upper semi-continuous function of  $\alpha$ .

**THEOREM B.** *Suppose that  $\Phi: \Sigma_A \rightarrow \mathbf{R}^d$  is a continuous function. Then the set of points  $x$  such that the limit defining  $\sigma_\Phi(x)$  does not exist is of the same topological entropy as that of  $\Sigma_A$  unless all points  $x \in \Sigma_A$  have the same ergodic limit.*

The techniques we use in proving our two theorems are inspired by dimension theory. Recall that  $\Sigma$  is a metric space where a metric is defined by  $d(x, y) = m^{-n}$  for  $x = (x_j)_{j \geq 1}$  and  $y = (y_j)_{j \geq 1}$  where  $n$  is the largest value such that  $x_j = y_j$  ( $1 \leq j \leq n$ ). Different notions of dimensions are then defined on  $\Sigma$ . We shall talk about the Hausdorff dimension  $\dim_{\text{H}}$ , the packing dimension  $\dim_{\text{p}}$  and the upper box dimension  $\dim_{\text{B}}$  (see [8, 15, 17] for a general account of dimensions). We will exploit the fact that, in our purely symbolic setting, topological entropy is related to dimension by

$$\frac{1}{\log m} h_{\text{top}}(E_\Phi(\alpha)) = \dim_{\text{H}}(E_\Phi(\alpha)) = \dim_{\text{p}}(E_\Phi(\alpha)).$$

Theorem A is a variational principle, but it does not follow from the well-known variational principle of Walters [22], since the invariant set  $E_\Phi(\alpha)$  is not compact. We emphasize that Theorem A holds for any continuous function  $\Phi$ . If  $\Phi$  has some regularity like Hölder continuity or summable variation, the result of Theorem A is part of the folklore in multifractal analysis (the thermodynamical formalism is used there but it does not work in our case because there is a lack of differentiability of pressure function and a lack of Gibbs property). Even for these regular functions, discussions of  $E_\Phi(\alpha)$  for boundary points  $\alpha$  of  $L_\Phi$  are scarce, which is actually a subtle problem.

The invariant set studied by Theorem B may be called the divergence set. The Birkhoff ergodic theorem says that the divergence set is of zero measure with respect to any invariant measure. Theorem B states that it is either empty or large in the sense that it has full topological entropy. The result of full topological entropy was obtained for Hölder functions by Barreira and Schmeling [1] (see also [5, 18]). They used the thermodynamical formalism which does not work for merely continuous functions.

The variational formula in Theorem A for those  $\Phi$  depending only on finitely many coordinates was simplified in [10].

The key points in our proof are: the entropy  $h_\mu$  is upper semi-continuous and can be approximated by the conditional entropies of  $\mu$ ; each conditional entropy is the entropy of a Markov measure; the recurrent set  $E_\Phi(\alpha)$  can be approximated by homogeneous Moran sets.

The paper is organized as follows. §2 is devoted to preliminaries. The variational formula is proved in §3 and the divergence set is studied in §4. Finally §5 contains some remarks.

2. Preliminaries

Most of the material in this section is known. We recall it here for convenience, at the same time introducing notation.

For  $k \geq 1$ ,  $\Sigma_{A,k}$  denotes the set of finite sequences  $\omega = (x_1, \dots, x_k)$  such that  $a_{x_i, x_{i+1}} = 1$  for all  $1 \leq i < k$ . These sequences  $\omega$  are called (*admissible*) *words of length*  $|\omega| (= k)$ . For  $\omega = (a_1, \dots, a_k) \in \Sigma_{A,k}$ , the *k-cylinder*  $[\omega]$  is defined by  $\{x \in \Sigma_A : x_1 = a_1, \dots, x_k = a_k\}$ . There is a one-to-one correspondence between  $\Sigma_{A,k}$  and the set of *k-cylinders*, so sometimes we shall use  $\Sigma_{A,k}$  to denote the set of all *k-cylinders*. The prefix of length  $n$  of a point  $x \in \Sigma$  will be written as  $x|_n$ .

Let  $\xi_0$  be the partition consisting of all 1-cylinders  $[0], [1], \dots, [m-1]$ . Let  $\xi_n$  be the join of the partitions  $T^{-j}\xi_0$  ( $0 \leq j \leq n$ ). Since  $\xi_0$  is a generator, the entropy  $h_\mu$  of an invariant measure  $\mu \in \mathcal{M}_{\text{inv}}$  can be expressed as [22]

$$h_\mu = \lim_{n \rightarrow \infty} \frac{H_\mu(\xi_n)}{n} \quad \text{where } H_\mu(\xi_n) = - \sum_{A \in \xi_n} \mu(A) \log \mu(A).$$

The *n*th conditional entropy of  $\mu$ , denoted by  $h_\mu^{(n)}$ , is defined by

$$h_\mu^{(0)} = H_\mu(\xi_0), \quad h_\mu^{(n)} = H_\mu(\xi_n) - H_\mu(\xi_{n-1}) \quad (\forall n \geq 1).$$

Using elementary properties of the conditional entropy [22, p. 80], the following proposition may be proved.

PROPOSITION 1. For each  $\mu \in \mathcal{M}_{\text{inv}}$ , we have

$$h_\mu = \lim_{n \rightarrow \infty} h_\mu^{(n)} = \inf_n h_\mu^{(n)}.$$

The entropy  $h_\mu$  is an upper semi-continuous functional defined on  $\mathcal{M}_{\text{inv}}$  with respect to the weak\* topology.

Markov measures on the full shift space were discussed in [7]. We present them here for subshifts of finite type. Markov measures form a special class of invariant measures and they are dense in  $\mathcal{M}_{\text{inv}}$ . A Borel probability measure  $\mu$  on  $\Sigma_A$  is uniquely determined by its values on cylinders. On the other hand, any set function  $\mu$  defined on cylinders satisfying the following conditions: for all  $a \in \Sigma_{A,n}$ , all  $n \geq 1$

$$\sum_{a \in \Sigma_{A,n}} \mu([a]) = 1, \quad \sum_{\epsilon} \mu([a, \epsilon]) = \mu([a])$$

may be uniquely extended to a Borel probability measure on  $\Sigma_A$ . Such a measure  $\mu$  is invariant if and only if for all  $a \in \Sigma_{A,n}$ , all  $n \geq 1$

$$\sum_{\epsilon} \mu([\epsilon, a]) = \mu([a]).$$

These three conditions may be referred to as the normalization condition, the consistency condition and the invariance condition. Let  $\ell \geq 1$  be an integer. By a *Markov measure of order  $\ell$*  or simply  *$\ell$ -Markov measure*, we mean a measure  $\mu \in \mathcal{M}_{\text{inv}}$  having the Markov property

$$\mu([\epsilon_1, \dots, \epsilon_n]) = \mu([\epsilon_1, \dots, \epsilon_{n-1}]) \frac{\mu([\epsilon_{n-\ell}, \dots, \epsilon_n])}{\mu([\epsilon_{n-\ell}, \dots, \epsilon_{n-1}])} \quad (n > \ell).$$

The proof of the following proposition is straightforward.

**PROPOSITION 2.** *Suppose that  $\mu \in \mathcal{M}_{\text{inv}}$  is an  $\ell$ -Markov measure. The entropy of  $\mu$  is*

$$h_\mu = - \sum_{\epsilon_1, \dots, \epsilon_{\ell+1}} \mu([\epsilon_1, \dots, \epsilon_{\ell+1}]) \log \frac{\mu([\epsilon_1, \dots, \epsilon_{\ell+1}])}{\mu([\epsilon_1, \dots, \epsilon_\ell])}.$$

Moreover,  $h_\mu^{(n)} = h_\mu$  for  $n \geq \ell + 1$ .

For  $n \geq 1$ , denote by  $\Delta_n$  the set of probability vectors  $p$  defined on  $\Sigma_{A, n}$  satisfying

$$\sum_\epsilon p(\epsilon, \epsilon_1, \dots, \epsilon_{n-1}) = \sum_\epsilon p(\epsilon_1, \dots, \epsilon_{n-1}, \epsilon).$$

The Markov property is equivalent to

$$\mu([\epsilon_1, \dots, \epsilon_{n+\ell}]) = \mu([\epsilon_1, \dots, \epsilon_{\ell+1}]) \prod_{j=2}^n \frac{\mu([\epsilon_j, \dots, \epsilon_{j+\ell}])}{\mu([\epsilon_j, \dots, \epsilon_{j+\ell-1}])} \quad (n \geq 1).$$

It follows that an  $\ell$ -Markov measure is uniquely determined by the function  $p$  defined on  $\Sigma_{A, \ell+1}$  by

$$p(x_1, \dots, x_{\ell+1}) = \mu([x_1, \dots, x_{\ell+1}])$$

which belongs to  $\Delta_{\ell+1}$ . Conversely, given  $p \in \Delta_k$ , we define for  $n > k$

$$\mu([a_1, \dots, a_n]) = p(a_1, \dots, a_k) \prod_{j=2}^{n-k+1} \frac{p(a_j, \dots, a_{j+k-1})}{\sum_\epsilon p(a_j, \dots, a_{j+k-2}, \epsilon)}.$$

This set function  $\mu$  can be uniquely extended to a  $(k-1)$ -Markov measure. In this way, we get a one-to-one correspondence between  $\Delta_k$  and the set of all  $(k-1)$ -Markov measures.

Let  $p \in \Delta_k$  ( $k \geq 1$ ). If  $p(x) > 0$  for  $x \in \Sigma_{A, k}$ , we write  $p \in \Delta_k^+$ . From the last expression of  $\mu([a_1, \dots, a_n])$  we get the following (see [4] for the definition of Gibbs measures).

**PROPOSITION 3.** *Suppose that  $p \in \Delta_k^+$ . Then the  $(k-1)$ -Markov measure corresponding to  $p$  is the Gibbs measure associated to the potential*

$$\psi(x) = \log p(x_1, \dots, x_k) - \log \sum_\epsilon p(x_1, \dots, x_{k-1}, \epsilon).$$

Let us summarize. Given  $\mu \in \mathcal{M}_{\text{inv}}$ , we have a sequence of Markov measures  $\{\mu_k\}$  which correspond to the functions  $\{p_k \in \Delta_k\}$  induced from  $\mu$  by

$$p_k(a_1, \dots, a_k) = \mu([a_1, \dots, a_k]).$$

These Markov measures approach  $\mu$  in the following sense (Propositions 1 and 2):

$$\mu = w^* - \lim_{k \rightarrow \infty} \mu_k, \quad h_\mu = \lim_{k \rightarrow \infty} h_{\mu_k}.$$

We call  $\mu_k$  the  $k$ th Markov approximation of  $\mu$ . If the support of  $\mu$  is the whole space  $\Sigma_A$ , its Markov approximations are all Gibbs measures then ergodic (Proposition 3).

In our proof of Theorem A, we need to construct a kind of Cantor set, called a homogeneous Moran set. It is helpful to think of  $\Sigma$  as the interval  $[0, 1]$  and cylinders

as subintervals. Let  $\{n_k\}_{k \geq 1}$  be a sequence of positive integers and  $\{c_k\}_{k \geq 1}$  be a sequence of positive numbers satisfying  $n_k \geq 2$ ,  $0 < c_k < 1$ ,  $n_1 c_1 \leq \delta$  and  $n_k c_k \leq 1$  ( $k \geq 2$ ), where  $\delta$  is some positive number. Let

$$D = \bigcup_{k \geq 0} D_k \text{ with } D_0 = \{\emptyset\}, D_k = \{(i_1, \dots, i_k); 1 \leq i_j \leq n_j, 1 \leq j \leq k\}.$$

If  $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k$ ,  $\tau = (\tau_1, \dots, \tau_m) \in D_m$ , we define  $\sigma * \tau = (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_m)$ . Suppose that  $J$  is an interval of length  $\delta$ . A collection  $\mathcal{F} = \{J_\sigma; \sigma \in D\}$  of subintervals of  $J$  is said to have a *homogeneous Moran structure* if it satisfies:

- (1)  $J_\emptyset = J$ .
- (2) For any  $k \geq 0$  and  $\sigma \in D_k$ ,  $J_{\sigma*1}, J_{\sigma*2}, \dots, J_{\sigma*n_{k+1}}$  are subintervals of  $J_\sigma$  and  $J_{\sigma*i} \cap J_{\sigma*j} = \emptyset$  ( $i \neq j$ ).
- (3) For any  $k \geq 1$  and any  $\sigma \in D_{k-1}$ ,  $1 \leq j \leq n_k$ , we have

$$\frac{|J_{\sigma*j}|}{|J_\sigma|} = c_k$$

where  $|A|$  denotes the length of  $A$ .

If  $\mathcal{F}$  is such a collection,  $E := \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma$  is called a *homogeneous Moran set* determined by  $\mathcal{F}$ .

**PROPOSITION 4 [11, 12].** *For the homogeneous Moran set defined above, we have*

$$\dim_{\text{H}} E \geq \liminf_{n \rightarrow \infty} \frac{\log n_1 n_2 \dots n_k}{-\log c_1 c_2 \dots c_{k+1} n_{k+1}}.$$

To end this section, we point out that the topological entropy of a subshift  $\Sigma_A$  is, up to a multiplicative factor  $\log m$ , equal to the Hausdorff dimension (and its packing) dimension.

### 3. Proof of Theorem A

Although the theorem is stated in terms of entropy, it will be more convenient to use dimension during the proof. Actually, we will prove that

$$\dim_{\text{H}} E_\Phi(\alpha) = \dim_{\text{P}} E_\Phi(\alpha) = \frac{1}{\log m} \max_{\mu \in \mathcal{F}_\Phi(\alpha)} h_\mu.$$

First we prove a formal formula of dimension. The matrix  $A$  being primitive, there is an integer  $M \geq 1$  such that all the entries of  $A^M$  are strictly positive. Then for any  $\omega = (x_j)_{j=1}^n \in \Sigma_{A,n}$  and any  $0 \leq z \leq m-1$ , there are  $0 \leq y_1, \dots, y_{M-1} \leq m-1$  such that

$$(x_1, \dots, x_n, y_1, \dots, y_{M-1}, z) \in \Sigma_{A, n+M}.$$

We call  $\bar{\omega} = (x_1, \dots, x_n, y_1, \dots, y_{M-1})$  an *extension* of  $\omega$  joining  $z$ .

Let  $\alpha \in \mathbf{R}^d$ ,  $n \geq 1$  and  $\epsilon > 0$ . Denote by  $F(\alpha, n, \epsilon)$  the set of all  $n$ -cylinders each of which contains at least a point  $x$  such that

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x) - \alpha \right| < \epsilon.$$

Let  $f(\alpha, n, \epsilon)$  be the cardinality of  $F(\alpha, n, \epsilon)$ .

We shall use  $S_n(\Phi, x)$  to denote the partial sum  $\sum_{j=0}^{n-1} \Phi(T^j x)$  and use  $A_n(\Phi, x)$  to denote the average  $n^{-1} S_n(\Phi, x)$ . We shall write  $V_n(\Phi) = \sum_{j=1}^n \text{var}_j(\Phi)$  where  $\text{var}_j(\Phi) = \sup_{x|_n=y|_n} |\Phi(x) - \Phi(y)|$  ( $|\cdot|$  denoting the Euclidean norm).

PROPOSITION 5. For  $\alpha \in L_\Phi$ , we have

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{\log f(\alpha, n, \epsilon)}{\log m^n} = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log f(\alpha, n, \epsilon)}{\log m^n} (= : \Lambda_\Phi(\alpha)).$$

The function  $\Lambda_\Phi : L_\Phi \rightarrow [0, 1]$  is concave and upper semi-continuous.

*Proof.* Without loss of generality, assume that  $|\Phi(x)| \leq 1$  (for all  $x$ ) where  $|\cdot|$  denotes the Euclidean norm. We want to show that  $\log f(\alpha, n, \epsilon)$ , as a sequence of  $n$ , has a kind of subadditivity. More precisely, for any  $\epsilon > 0$ , there is an  $N$  such that

$$[f(\alpha, n, \epsilon)]^p \leq f(\alpha, (n+M)p, 2\epsilon) \quad (\forall n \geq N, \forall p \geq 1).$$

In fact, suppose that  $\{\omega_1, \dots, \omega_p\} \subset F(\alpha, n, \epsilon)$ . Let  $\omega = \bar{\omega}_1 \dots \bar{\omega}_p$  where  $\bar{\omega}_k$  is an extension of  $\omega_k$  joining the leading alphabet of  $\omega_{k+1}$  (with convention  $\omega_{p+1} = \omega_1$ ). Let  $x_k \in [\omega_k]$  ( $1 \leq k \leq p$ ) be a point such that

$$\left| \sum_{j=0}^{n-1} \Phi(T^j x_k) - n\alpha \right| < n\epsilon.$$

Let  $x$  be a point in  $[\omega]$ . We have

$$S_{(n+M)p}(\Phi, x) - (n+M)p\alpha = \sum_{k=0}^{p-1} \sum_{j=0}^{n+M-1} (\Phi(T^j z_k) - \alpha)$$

where  $z_k$  is a point in  $\Sigma_A$  with  $z_k|_{n+M} = \bar{\omega}_k$  so that  $z_k \in [\bar{\omega}_k] \subset [\omega_k]$ . The above inner sum over  $j$  is bounded by

$$\left| \sum_{j=0}^{n+M-1} (\Phi(T^j x_k) - \alpha) \right| + \sum_{j=0}^{n+M-1} |\Phi(T^j z_k) - \Phi(T^j x_k)| \leq n\epsilon + 2M + V_n(\Phi) + 2M.$$

It follows that

$$|A_{n+M}(\Phi, x) - \alpha| \leq \frac{n}{n+M}\epsilon + \frac{4M}{n+M} + \frac{V_n(\Phi)}{n+M}.$$

Since  $\Phi$  is continuous,  $n^{-1}V_n(\Phi)$  tends to zero as  $n \rightarrow \infty$ . Therefore  $|A_{n+M}(\Phi, x) - \alpha| \leq 2\epsilon$  for sufficiently large  $n \geq N$  and for all  $p \geq 1$ . Then  $[\omega]$ , which contains  $x$ , is in  $F(\alpha, (n+M)p, 2\epsilon)$ . Notice that different choices  $\{\omega_1, \dots, \omega_p\}$  give rise to different  $\omega$ . Thus we get the desired subadditivity. By using this subadditivity, it is easy to get

$$\limsup_{n \rightarrow \infty} \frac{\log f(\alpha, n, \epsilon)}{\log m^n} \leq \liminf_{n \rightarrow \infty} \frac{\log f(\alpha, n, 2\epsilon)}{\log m^n}$$

from which the equality of the two limits follows.

It is evident that  $0 \leq \Lambda_\Phi(\alpha) \leq 1$ . Let  $\alpha, \beta \in L_\Phi$ . Let  $p, q$  be two positive integers. By subadditivity, for large  $n$  we have

$$[f(\alpha, n, \epsilon)]^p [f(\beta, n, \epsilon)]^q \leq f(\alpha, (n+M)p, 2\epsilon) f(\beta, (n+M)q, 2\epsilon).$$

Let  $u \in F(\alpha, (n+M)p, 2\epsilon)$  and  $v \in F(\beta, (n+M)q, 2\epsilon)$ . Take a point  $x \in [uvw]$  where  $w \in \Sigma_{A, M-1}$  such that  $uw$  is an extension of  $u$  joining the leading alphabet of  $v$ . As above, we can get

$$\begin{aligned} & |S_{(p+q)(n+M)+M}(\Phi, x) - (n+M)p\alpha - (n+M)q\beta| \\ & \leq 2\epsilon(n+M)(p+q) + V_{(n+M)p}(\Phi) + V_{(n+M)q}(\Phi) + M. \end{aligned}$$

It follows that if  $n$  is sufficiently large,  $uvw \in F((p\alpha + q\beta)/(p + q), (n + M)(p + q) + M, 3\epsilon)$ . Consequently, for large  $n$  we have

$$f(\alpha, np, 2\epsilon)f(\beta, nq, 2\epsilon) \leq f\left(\frac{p\alpha + q\beta}{p + q}, n(p + q), 3\epsilon\right).$$

By the equality of the two limits that we have already proved, we can get

$$\frac{p}{p + q}\Lambda_\Phi(\alpha) + \frac{q}{p + q}\Lambda_\Phi(\beta) \leq \Lambda_\Phi\left(\frac{p}{p + q}\alpha + \frac{q}{p + q}\beta\right).$$

This gives the rational concavity of the (bounded) function  $\Lambda_\Phi$ . However, the concavity of  $\Lambda_\Phi$  is a consequence of its rational concavity and its upper semi-continuity that we prove below.

Given  $\alpha \in L_\Phi$ . For any  $\eta > 0$ , there is  $\epsilon > 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{\log f(\alpha, n, \epsilon)}{\log m^n} < \Lambda_\Phi(\alpha) + \eta.$$

As above, it can be proved that for  $\beta \in L_\Phi$  with  $|\beta - \alpha| < \epsilon/3$  we have

$$F(\beta, n, \epsilon/3) \subset F(\alpha, n, \epsilon)$$

when  $n$  is sufficiently large. It follows that  $f(\beta, n, \epsilon/3) \leq f(\alpha, n, \epsilon)$ . Therefore

$$\Lambda_\Phi(\beta) \leq \liminf_{n \rightarrow \infty} \frac{\log f(\beta, n, \epsilon/3)}{\log m^n} \leq \liminf_{n \rightarrow \infty} \frac{\log f(\alpha, n, \epsilon)}{\log m^n} \leq \Lambda_\Phi(\alpha) + \eta.$$

This establishes the upper semi-continuity of  $\Lambda_\Phi$  at  $\alpha$ . □

**PROPOSITION 6.** For  $\alpha \in L_\Phi$ , we have

$$\dim_{\mathbb{H}} E_\Phi(\alpha) = \dim_{\mathbb{P}} E_\Phi(\alpha) = \Lambda_\Phi(\alpha).$$

*Proof.* Step 1: For  $\alpha \in L_\Phi$ , we have  $\dim_{\mathbb{P}} E_\Phi(\alpha) \leq \Lambda_\Phi(\alpha)$ .

Let

$$G(\alpha, k, \epsilon) = \bigcap_{n=k}^{\infty} \{x \in \Sigma : |A_n(\Phi, x) - \alpha| < \epsilon\}.$$

It is clear that for any  $\epsilon > 0$ ,

$$E_\Phi(\alpha) \subset \bigcup_{k=1}^{\infty} G(\alpha, k, \epsilon).$$

Notice that if  $n \geq k$ ,  $G(\alpha, k, \epsilon)$  is covered by the union of all cylinders  $[\omega]$  with  $\omega \in F(\alpha, n, \epsilon)$  whose total number is  $f(\alpha, n, \epsilon)$ . Therefore we have the following estimate

$$\overline{\dim}_{\mathbb{B}} G(\alpha, k, \epsilon) \leq \limsup_{n \rightarrow \infty} \frac{\log f(\alpha, n, \epsilon)}{\log m^n} \quad (\forall \epsilon > 0, \forall k \geq 1).$$

On the other hand, by using the  $\sigma$ -stability of the packing dimension, we have

$$\dim_{\mathbb{P}} E_\Phi(\alpha) \leq \dim_{\mathbb{P}} \left( \bigcup_{k=1}^{\infty} G(\alpha, k, \epsilon) \right) \leq \sup_k \dim_{\mathbb{P}} G(\alpha, k, \epsilon) \leq \sup_k \overline{\dim}_{\mathbb{B}} G(\alpha, k, \epsilon).$$

This, together with Proposition 5, leads to the desired result.

Step 2: For  $\alpha \in L_\Phi$ , we have  $\dim_{\mathbb{H}} E_\Phi(\alpha) \geq \Lambda_\Phi(\alpha)$ .

Given  $\delta > 0$ . By Proposition 5, there are  $\ell_j \uparrow \infty$  and  $\epsilon_j \downarrow 0$  such that

$$f(\alpha, \ell_j, \epsilon_j) > m^{\ell_j(\Lambda_\Phi(\alpha) - \delta/2)}.$$

Write simply  $F_{\ell_j} = F(\alpha, \ell_j, \epsilon_j)$  and  $f_{\ell_j} = f(\alpha, \ell_j, \epsilon_j)$ . Define a new sequence  $\{\ell_j^*\}$  in the following manner

$$\underbrace{\ell_1, \dots, \ell_1}_{N_1}; \underbrace{\ell_2, \dots, \ell_2; \dots}_{N_2}; \underbrace{\ell_j, \dots, \ell_j; \dots}_{N_j}; \dots$$

where  $N_j$  is defined recursively by

$$N_j = 2^{\ell_{j+1} + N_{j-1}} \quad (j \geq 2); \quad N_1 = 1.$$

Denote  $n_j = f_{\ell_j^*}$  and  $c_j = m^{-\ell_j^*}$ . Define

$$\Theta^* = \prod_{j=1}^{\infty} F_{\ell_j^*}.$$

Observe that  $\Theta^*$  is a homogeneous Moran set in  $\Sigma$ . More precisely  $\Theta^*$  is constructed as follows. At level 0, we have only the initial cylinder  $\Sigma$ . In step  $j$ , cut a cylinder of level  $j-1$  into  $m^{\ell_j^*}$  cylinders and pick up  $n_j$  ones. By Proposition 4, we have

$$\begin{aligned} \dim_{\mathbb{H}} \Theta^* &\geq \liminf_{k \rightarrow \infty} \frac{\log(n_1 \dots n_k)}{-\log(c_1 \dots c_k c_{k+1} n_{k+1})} \\ &\geq \liminf_{k \rightarrow \infty} \frac{\log(f_{\ell_1^*} \dots f_{\ell_k^*})}{\log(2^{\ell_1^* + \dots + \ell_k^* + \ell_{k+1}^*})} \\ &= \liminf_{k \rightarrow \infty} \frac{\log(f_{\ell_1^*} \dots f_{\ell_k^*})}{\log(2^{\ell_1^* + \dots + \ell_k^*})} \\ &\geq \Lambda_\Phi(\alpha) - \delta. \end{aligned}$$

However  $\Theta^*$  is a set in  $\Sigma$ , not necessarily in  $\Sigma_A$ . Based on  $\Theta^*$ , we are going to construct a set  $\Theta^{**}$  which is not only in  $\Sigma_A$  but also in  $E_\Phi(\alpha)$ , and which is of the same Hausdorff dimension as  $\Theta^*$ .

We extend words in  $F_{\ell_j^*}$  in the following manner. Let

$$A = \{a : x = x|_{\ell_j^* - 1} a, x \in F_{\ell_j^*}\}, \quad B = \{b = x|_1, x \in F_{\ell_{j+1}^*}\}.$$

That is,  $A$  is the set of the last alphabets of the words in  $F_{\ell_j^*}$  and  $B$  is the set of the first alphabets of the words in  $F_{\ell_{j+1}^*}$ . For each pair  $(a, b) \in A \times B$ , take one (and only one) word  $w \in \Sigma_{A, M}$  such that  $awb$  is admissible. We call  $w$  a bridge word. For  $y = u_1 u_2 \dots \in \Theta^*$ , define

$$y^* = u_1 w_1 u_2 w_2 \dots u_j w_j u_{j+1} w_{j+1} \dots$$

where  $w_j$  is the bridge word as defined above. We define  $\Theta^{**}$  as the set of sequences  $y^*$ . By considering the map sending  $y$  to  $y^*$  which is nearly bi-Lipschitz, we have  $\dim_{\mathbb{H}} \Theta^{**} = \dim_{\mathbb{H}} \Theta^*$ . On the other hand,  $\Theta^{**} \subset E_\Phi(\alpha)$ . In fact, let  $\ell_j^{**} = \ell_j^* + M$ . For large  $n$  ( $> \ell_1$ ), there is a unique integer  $J(n)$  such that

$$\sum_{i=1}^{j(n)} \ell_i^{**} \leq n < \sum_{i=1}^{J(n)+1} \ell_i^{**}.$$



The choice of  $N_j$  implies that  $\ell_{k+1} = o(N_k)$ . It follows that

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{J(n)} \ell_i^{**}}{\sum_{i=1}^{J(n)+1} \ell_i^{**}} = 1.$$

Let  $x = (x_n) \in \Theta^{**}$ . Let  $\Lambda_j$  be the set of integers between  $\ell_1^{**} + \dots + \ell_{j-1}^{**} + 1$  and  $\ell_1^{**} + \dots + \ell_j^{**}$ . We have

$$\begin{aligned} S_n(\Phi, x) &= \sum_{j=1}^{J(n)} \sum_{k \in \Lambda_j} \Phi(T^k x) + O(\ell_{J(n)+1}^{**}) \\ &= \sum_{j=1}^{J(n)} \ell_j^{**}(\alpha + o(1)) + o(n) \\ &= \alpha n + o(n). \end{aligned}$$

It follows that  $x \in E_\Phi(\alpha)$ . □

We now prove the lower estimate

$$\dim_{\text{H}} E_\Phi(\alpha) \geq \sup_{\mu \in \mathcal{F}_\Phi(\alpha)} \frac{h_\mu}{\log m}.$$

Since the entropy  $h_\mu$  is upper semi-continuous as a function of  $\mu$  (see Proposition 1) and  $\mathcal{F}_\Phi(\alpha)$  is compact, the above supremum is attained. Let  $\mu_0$  be a measure in  $\mathcal{F}_\Phi(\alpha)$  which attains the supremum. Take an invariant measure  $\nu$  with support  $\Sigma_A$ . For any  $\epsilon > 0$ , consider  $\mu_\epsilon = (1 - \epsilon)\mu_0 + \epsilon h_\nu$  as an approximation of  $\mu_0$ . Let  $\alpha_\epsilon = \int \Phi d\mu_\epsilon$ . We have

$$|\alpha_\epsilon - \alpha| \leq 2\epsilon \|\Phi\|$$

where  $\|\Phi\| = \sup_x |\Phi(x)|$ . Since the support of the invariant measure  $\mu_\epsilon$  is the whole space  $\Sigma_A$ , we can find a sequence of ergodic Markov measures  $\mu_\epsilon^{(k)}$  such that  $\mu_\epsilon^{(k)}$  tends to  $\mu_\epsilon$  in the weak\* topology and  $h_{\mu_\epsilon^{(k)}}$  tends to  $h_{\mu_\epsilon}$  as  $k \rightarrow \infty$  (see Propositions 2 and 3). Let  $\alpha_\epsilon^{(k)} = \int \Phi d\mu_\epsilon^{(k)}$ . We have

$$|\alpha_\epsilon^{(k)} - \alpha_\epsilon| \leq \epsilon \quad (k \geq k(\epsilon)).$$

Since  $\Lambda_\Phi(\cdot)$  is upper semi-continuous (Proposition 5), for any  $\eta$ , when  $\epsilon$  is sufficiently small and  $k$  is sufficiently large, we have

$$\Lambda_\Phi(\alpha) \geq \Lambda_\Phi(\alpha_\epsilon^{(k)}) - \eta \geq \frac{h_{\mu_\epsilon^{(k)}}}{\log m} - \eta$$

where for the second inequality we used the fact that  $\Lambda_\Phi(\alpha) \geq h_\mu / \log m$  for any ergodic measure  $\mu \in \mathcal{F}_\Phi(\alpha)$ . In fact, the ergodicity implies that  $\mu(E_\Phi(\alpha)) = 1$ . It follows that  $\dim_{\text{H}} \mu \leq \dim_{\text{H}} E_\Phi(\alpha)$ . However  $\dim \mu = h_\mu / \log m$  by the Shannon–McMillan–Breiman theorem [19, p. 261] (see [9] for  $\dim_{\text{H}} \mu$ ). Letting  $k \rightarrow \infty$  gives

$$\Lambda_\Phi(\alpha) \geq \frac{h_{\mu_\epsilon}}{\log m} - \eta = \frac{(1 - \epsilon)h_{\mu_0} + \epsilon h_\nu}{\log m} - \eta.$$

Now let  $\epsilon \rightarrow 0$  and then  $\eta \rightarrow 0$ . We get  $\Lambda_\Phi(\alpha) \geq h_{\mu_0} / \log m$ .

We now prove the upper estimate

$$\dim_{\text{P}} E_\Phi(\alpha) \leq \sup_{\mu \in \mathcal{F}_\Phi(\alpha)} \frac{h_\mu}{\log m}.$$

Given any  $\epsilon > 0$ . If  $k \geq 1$  is sufficient large, we can find a function  $\Phi_k$  depending only on the first  $k$  coordinates of  $x \in \Sigma_A$  such that

$$|\Phi_k(x) - \Phi(x)| < \frac{\epsilon}{2} \quad (\forall x).$$

Thus, for  $x \in E_\Phi(\alpha)$  we have

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{n} S_n(\Phi_k, x) - \alpha \right| < \epsilon.$$

It follows that

$$E_\Phi(\alpha) \subset \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} B(n, k, \epsilon) \quad (\forall k = k(\epsilon) \text{ large and } \epsilon > 0)$$

where

$$B(n, k, \epsilon) = \{x \in \Sigma : |n^{-1} S_n(\Phi_k, x) - \alpha| < \epsilon\}.$$

In the following, we will show that for each  $j \in \mathbb{N}$

$$\overline{\dim}_B \bigcap_{n=j}^{\infty} B(n, k, \epsilon) \leq \sup_{|\Phi_*(\mu) - \alpha| < 3\epsilon} \frac{h_\mu}{\log m}.$$

Take all cylinders of the form  $[x_1, \dots, x_{n+k-1}]$  which intersect  $B(n, k, \epsilon)$ . They constitute a cover of  $(n+k-1)$ -cylinders of  $B(n, k, \epsilon)$ . Denote by  $T(n, k, \epsilon)$  the number of cylinders contained in this cover. We are going to estimate this number.

For  $\omega \in \Sigma_{A, n+k-1}$ , denote by  $N_\omega(\epsilon_1, \dots, \epsilon_k)$  the number of words of the form  $\epsilon_1 \dots \epsilon_k$  appearing in  $\omega$ . The family  $\{N_\omega(\epsilon_1, \dots, \epsilon_k)\}$  with varying  $\epsilon_1, \dots, \epsilon_k$  will be called the  $k$ -distribution of  $\omega$ . Let  $D = \{n(\epsilon_1, \dots, \epsilon_k)\}$  be the  $k$ -distribution of some  $\omega$  such that  $[\omega]$  intersects  $B(n, k, \epsilon)$ . This means that

$$\begin{aligned} n(\epsilon_1, \dots, \epsilon_k) &= N_\omega(\epsilon_1, \dots, \epsilon_k) \\ \left| \frac{1}{n} \sum_{\epsilon_1, \dots, \epsilon_k} n(\epsilon_1, \dots, \epsilon_k) \Phi_k(\epsilon_1, \dots, \epsilon_k) - \alpha \right| &< \epsilon. \end{aligned}$$

We denote by  $\mathcal{T}(D)$  the collection of all such cylinders  $[\omega]$  and by  $\Gamma(D)$  its cardinality. Let  $\mathcal{D}_k$  be the set of all possible such distributions  $D$ . The cover is then decomposed into disjoint families  $\mathcal{T}(D)$  with  $D \in \mathcal{D}_k$ . It is clear that the cardinality of  $\mathcal{D}_k$ , the number of families, is at most  $n^{m^k}$ . Then we have

$$T(n, k, \epsilon) = \sum_{D \in \mathcal{D}_k} \Gamma(D) \leq n^{m^k} \max_{D \in \mathcal{D}_k} \Gamma(D).$$

Thus

$$\frac{\log T(n, k, \epsilon)}{\log m^n} \leq \max_{D \in \mathcal{D}_k} \frac{\log \Gamma(D)}{\log m^n} + O\left(\frac{\log n}{n}\right).$$

(The constant in ‘ $O$ ’ depends upon  $k$ .)

Since  $n(\epsilon_1, \dots, \epsilon_k) = N_\omega(\epsilon_1, \dots, \epsilon_k)$ ,  $\{n(\epsilon_1, \dots, \epsilon_k)/n\}$  defines a probability vector. For any  $\eta > 0$ , there is a number  $N = N(\eta)$  such that when  $n > N$  we can find  $p \in \Delta_k^+$  such that (see [10, Lemma 3])

$$\left| \frac{n(\epsilon_1, \dots, \epsilon_k)}{n} - p(\epsilon_1, \dots, \epsilon_k) \right| < \eta, \quad p(\epsilon_1, \dots, \epsilon_k) > \frac{\eta}{m^{k+1}}.$$

Consider the Markov measure  $\nu_p$  corresponding to  $p$ . For any cylinder  $[\omega] \in \mathcal{T}(D)$  with  $\omega = (x_i)_{i=1}^{n+k-1}$ , we have  $N_\omega(\epsilon_1, \dots, \epsilon_k) = n(\epsilon_1, \dots, \epsilon_k)$ , so we have

$$\begin{aligned} \nu_p([\omega]) &= \frac{p(x_1, \dots, x_k)}{t(x_1, \dots, x_k)} \prod_{\epsilon_1, \dots, \epsilon_k} t(\epsilon_1, \dots, \epsilon_k)^{n(\epsilon_1, \dots, \epsilon_k)} \\ &\geq \frac{\eta}{m^{k+1}} \prod_{\epsilon_1, \dots, \epsilon_k} t(\epsilon_1, \dots, \epsilon_k)^{n(\epsilon_1, \dots, \epsilon_k)} \end{aligned}$$

where

$$t(a_1, \dots, a_k) = \frac{p(a_1, \dots, a_k)}{\sum_\epsilon p(a_1, \dots, a_{k-1}, \epsilon)}.$$

Let  $a = a(D)$  denote the right-hand side of the above inequality. Then

$$a\Gamma(D) \leq \nu_p\left(\bigcup_{[\omega] \in \mathcal{T}(D)} [\omega]\right) \leq 1.$$

Combining the last two expressions gives

$$\Gamma(D) \leq \frac{1}{a} \leq \frac{m^{k+1}}{\eta} \prod_{\epsilon_1, \dots, \epsilon_k} t(\epsilon_1, \dots, \epsilon_k)^{-n(\epsilon_1, \dots, \epsilon_k)}.$$

Then

$$\begin{aligned} \frac{\log \Gamma(D)}{\log m^n} &\leq O\left(\frac{|\log \eta|}{n}\right) - \sum_{\epsilon_1, \dots, \epsilon_k} \frac{n(\epsilon_1, \dots, \epsilon_k)}{n} \log_m t(\epsilon_1, \dots, \epsilon_k) \\ &\leq O\left(\frac{|\log \eta|}{n}\right) - \sum_{\epsilon_1, \dots, \epsilon_k} p(\epsilon_1, \dots, \epsilon_k) \log_m t(\epsilon_1, \dots, \epsilon_k) + O(\eta |\log \eta|). \end{aligned}$$

Note that  $\Phi_*(\nu_p)$  is near  $\alpha$  in the sense that

$$\begin{aligned} |\Phi_*(\nu_p) - \alpha| &\leq \left| \sum_{\epsilon_1, \dots, \epsilon_k} p(\epsilon_1, \dots, \epsilon_k) \Phi_k(\epsilon_1, \dots, \epsilon_k) - \alpha \right| + \epsilon \\ &\leq \left| \sum_{\epsilon_1, \dots, \epsilon_k} \frac{n(\epsilon_1, \dots, \epsilon_k)}{n} \Phi_k(\epsilon_1, \dots, \epsilon_k) - \alpha \right| + m^k \eta \|\Phi_k\| + \epsilon \\ &\leq m^k \eta \|\Phi_k\| + 2\epsilon. \end{aligned}$$

Now we can conclude that

$$\frac{\log \Gamma(D)}{\log m^n} \leq O\left(\frac{|\log \eta|}{n}\right) + O(\eta |\log \eta|) + \sup_{\mu \in \Delta_k, |\Phi_*(\mu) - \alpha| < m^k \eta \|\Phi_k\| + 2\epsilon} \frac{h_\mu}{\log m}.$$

Since the right-hand side is independent of  $D$ , we get

$$\frac{\log T(n, k, \epsilon)}{\log m^n} \leq O\left(\frac{|\log \eta| + \log n}{n}\right) + O(\eta |\log \eta|) + \sup_{|\Phi_*(\mu) - \alpha| < m^k \eta \|\Phi_k\| + 2\epsilon} \frac{h_\mu}{\log m}.$$

Let  $n \rightarrow \infty$  then let  $\eta \rightarrow 0$ , we get

$$\overline{\dim}_B \bigcap_{n=j}^{\infty} B(n, k, \epsilon) \leq \limsup_{n \rightarrow \infty} \frac{\log T(n, k, \epsilon)}{\log m^n} \leq \sup_{|\Phi_*(\mu) - \alpha| \leq 3\epsilon} \frac{h_\mu}{\log m}.$$

By the  $\sigma$ -stability of the packing dimension and the inequality  $\dim_P \leq \overline{\dim}_B$  (see [17]), we have

$$\begin{aligned} \dim_P(E_\Phi(\alpha)) &\leq \dim_P\left(\bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} B(n, k, \epsilon)\right) \\ &\leq \sup_j \overline{\dim}_B \bigcap_{n=j}^{\infty} B(n, k, \epsilon) \leq \sup_{|\Phi_*(\mu) - \alpha| \leq 3\epsilon} \frac{h_\mu}{\log m}. \end{aligned}$$

Note that the set of invariant measures  $\mu$  such that  $|\Phi_*(\mu) - \alpha| \leq 3\epsilon$  is compact and that  $h_\mu$  is upper semi-continuous. For any  $\ell \geq 1$ , there is an invariant measure  $\mu_\ell$  such that

$$h_{\mu_\ell} = \sup_{|\Phi_*(\mu) - \alpha| \leq 1/\ell} h_\mu.$$

We can also assume that  $\mu_\ell$  converges to some measure  $\mu_\infty$ . It is clear that  $\mu_\infty \in \overline{\mathcal{F}}_\Phi(\alpha)$ . Thus, by the upper semi-continuity of the entropy,

$$\dim_P E_\Phi(\alpha) \leq \liminf_{\epsilon \rightarrow 0} \sup_{|\Phi_*(\mu) - \alpha| \leq 3\epsilon} \frac{h_\mu}{\log m} = \frac{h_{\mu_\infty}}{\log m}. \quad \square$$

#### 4. Proof of Theorem B

As in the proof of Theorem A, we will use Hausdorff dimension instead of topological entropy.

We first prove the fact that if the limit defining  $\sigma_\Phi(x)$  does not exist for some point  $x$ , it does not exist for points in a set of Hausdorff dimension equal to that of  $\Sigma_A$ . Such points are called *divergent points*.

We start with a method to construct new divergent points by using a given divergent point. Take a subset of words  $W^* \subset \Sigma_{A, M-1}$  such that for any  $t, s \in \{0, 1, \dots, m-1\}$  there is one and only one word  $w \in W^*$  such that  $tw_s \in \Sigma_{A, M+1}$  (maybe there are several choices for a given pair  $(t, s)$ , but we choose only one). Assume that  $\Phi$  takes real values. Without loss of generality, we assume that  $|\Phi(x)| \leq 1$  (for all  $x \in \Sigma_A$ ). Denote by  $\underline{\sigma}_\Phi(x)$  and  $\overline{\sigma}_\Phi(x)$  the lim inf and lim sup of  $n^{-1} \sum_{j=0}^{n-1} \Phi(T^j x)$ .

Suppose there are two points  $u, v \in \Sigma_A$  (maybe  $u = v$ ) such that  $\overline{\sigma}_\Phi(u) > \underline{\sigma}_\Phi(v)$ . Take two numbers  $a, b$  such that

$$\overline{\sigma}_\Phi(u) > a > b > \underline{\sigma}_\Phi(v).$$

For any  $k \geq 1$ , there exists  $\ell_k \geq 1$  such that

$$|\Phi(x) - \Phi(y)| \leq \frac{a-b}{k} \text{ if } x_i = y_i \quad (1 \leq i \leq \ell_k).$$

Take a rapidly increasing sequence  $\{n_i\}_{i \geq 1}$  satisfying

$$\begin{aligned} \frac{1}{n_{2k-1}} \sum_{j=0}^{n_{2k-1}-1} \Phi(T^j(u)) > a, \quad \frac{1}{n_{2k}} \sum_{j=0}^{n_{2k}-1} \Phi(T^j(v)) < b, \\ k = o(n_k), \quad \ell_k = o(n_k), \quad 2(M-1)k \leq \sum_{j=1}^k n_j, \quad \sum_{j=1}^{k-1} n_j = o(n_k). \end{aligned}$$

Fix an integer  $q \geq 1$ . Let  $x = x_1 x_2 \dots \in \Sigma_A$ . We cut  $x$  into words of lengths  $\{qn_k\}$  where  $\{n_k\}$  is as above. This means that

$$x = \bar{x}_1 \bar{x}_2 \dots \bar{x}_k \dots \text{ with } |\bar{x}_k| = qn_k.$$

Define  $\bar{u}_k$  to be the prefix of  $u$  with  $|\bar{u}_k| = n_{2k-1}$  and  $\bar{v}_k$  to be the prefix of  $v$  with  $|\bar{v}_k| = n_{2k}$ . Denote

$$B_{2k-1} = \bar{u}_k, \quad B_{2k} = \bar{v}_k.$$

Now construct a new point  $x^* \in \Sigma_A$  as follows

$$x^* = \bar{x}_1 W'_1 B_1 W''_1 \bar{x}_2 W'_2 B_2 W''_2 \dots \bar{x}_k W'_k B_k W''_k \dots$$

where  $W'_k$  and  $W''_k$  are admissible words in  $W^*$ .

**PROPOSITION 7.** *Suppose that  $\bar{\sigma}_\Phi(u) > \underline{\sigma}_\Phi(v)$  for some  $u, v \in \Sigma_A$  (maybe  $u = v$ ). If  $\underline{\sigma}_\Phi(x) = \bar{\sigma}_\Phi(x)$ , then  $\bar{\sigma}_\Phi(x^*) > \underline{\sigma}_\Phi(x^*)$  where  $x^*$  is constructed as above from  $x$ .*

*Proof.* Notice that the word  $\bar{x}_k W'_k B_k W''_k$  is of length

$$L_k = qn_k + (M-1) + n_k + (M-1) = (q+1)n_k + 2(M-1).$$

Let  $N_k = \sum_{j=1}^k L_j = (q+1) \sum_{j=1}^k n_j + 2k(M-1)$  and let  $M_k = q \sum_{j=1}^k n_j$ . Consider the partial sum

$$\sum_{j=0}^{N_{2k}-1} \Phi(T^j x^*) = \sum_{j=0}^{N_{2k-1}-1} \Phi(T^j x^*) + \sum_{j=N_{2k-1}}^{N_{2k}-1} \Phi(T^j x^*).$$

The first sum on the right is bounded by  $N_{2k-1}$  (since 1 is an upper bound of  $\Phi$ ). For the second sum, we have

$$\begin{aligned} \sum_{j=N_{2k-1}}^{N_{2k}-1} \Phi(T^j x^*) &= \sum_{j=0}^{L_{2k}-1} \Phi(T^j(\bar{x}_{2k} W'_{2k} B_{2k} W''_{2k} \dots)) \\ &\leq \sum_{j=M_{2k-1}}^{M_{2k}-\ell_k-1} \Phi(T^j x) + (qn_{2k} - \ell_k) \cdot \frac{a-b}{k} + (\ell_k + M-1) \\ &\quad + \sum_{j=0}^{n_{2k}-\ell_k-1} \Phi(T^j v) + (n_{2k} - \ell_k) \cdot \frac{a-b}{k} + (\ell_k + M-1) \\ &= \sum_{j=M_{2k-1}}^{M_{2k}-\ell_k-1} \Phi(T^j x) + \sum_{j=0}^{n_{2k}-\ell_k-1} \Phi(T^j v) \\ &\quad + \frac{(a-b)((q+1)n_{2k} - 2\ell_k)}{k} + 2(\ell_k + M-1) \\ &=: A_k + B_k + C_k + D_k \end{aligned}$$

where the inequality is obtained by cutting the sum into four sums: the first one is taken over the first  $qn_{2k} - \ell_k$  terms whose sum is bounded by  $\sum_{j=0}^{qn_{2k}-\ell_k-1} \Phi(T^j x) + (qn_{2k} - \ell_k)(a-b)/k$ ; the second one is taken over the next  $\ell_k + M - 1$  terms each of which is controlled by 1; the third sum taken over the next  $n_{2k} - \ell_k$  terms can be written as

$$\sum_{j=0}^{n_{2k}-\ell_k-1} \Phi(T^j(B_{2k} W''_{2k} \dots))$$

which is bounded by  $\sum_{j=0}^{n_{2k}-\ell_k-1} \Phi(T^j v) + (n_{2k} - \ell_k)(a-b)/k$ ; the last sum can be estimated like the second sum. Thus we get

$$\frac{1}{N_{2k}} \sum_{j=0}^{N_{2k}-1} \Phi(T^j x^*) \leq \frac{N_{2k-1}}{N_{2k}} + \frac{A_k}{N_{2k}} + \frac{B_k}{N_{2k}} + \frac{C_k}{N_{2k}} + \frac{D_k}{N_{2k}}.$$

By the properties of  $n_k$ , the quantities  $N_{2k-1}$ ,  $C_k$  and  $D_k$  are all  $o(N_{2k})$  and

$$\lim_{k \rightarrow \infty} \frac{A_k}{N_{2k}} = \frac{q}{q+1} \sigma_\Phi(x), \quad \liminf_{k \rightarrow \infty} \frac{B_k}{N_{2k}} \leq \frac{b}{q+1}.$$

It follows that  $\underline{\sigma}_\Phi(x^*) \leq (q/(q+1))\sigma_\Phi(x) + b/(q+1)$ . In the same way, we can prove that  $\bar{\sigma}_\Phi(x^*) \geq (q/(q+1))\sigma_\Phi(x) + a/(q+1)$ , so that  $\bar{\sigma}_\Phi(x^*) - \underline{\sigma}_\Phi(x^*) > (a-b)/(q+1) > 0$ .  $\square$

Let  $G_\Phi$  be the set of (good) points  $x$  such that  $\underline{\sigma}_\Phi(x) = \bar{\sigma}_\Phi(x)$ . When points  $u$  and  $v$ , the corresponding sequence  $\{n_k\}$  and the number  $q \geq 1$  are fixed as above, to any point  $x \in G_\Phi$  we have associated a (unique) point  $x^*$  in  $\Sigma_A$  (the uniqueness comes from the unique choice in the definition of  $W^*$ ). If we have a subset  $E$  of  $G_\Phi$ , in this way we get a (bad) subset  $E_q$  of  $\Sigma_A$ .

**PROPOSITION 8.** *Keep the same notation as above. For any  $E \subset G_\Phi$ , we have*

$$\dim_{\text{H}} E_q \geq \frac{q}{q+2} \dim_{\text{H}} E.$$

*Proof.* Consider the bijective map  $f: E \rightarrow E_q$  defined by  $f(x) = x^*$ . It suffices to show that  $f^{-1}$  is  $q/(q+2)$ -Hölder [15, p. 139], or equivalently

$$\rho(f(x), f(y)) \geq \rho(x, y)^{1+2/q}.$$

In fact, suppose that  $\rho(x, y) = m^{-r}$ .

- (a) If  $r \leq qn_1$ , we have  $\rho(f(x), f(y)) = \rho(x, y)$ .
- (b) If  $q \sum_{i=1}^{2k-1} n_i < r \leq q \sum_{i=1}^{2k} n_i$  for some  $k \geq 1$ , we have

$$\rho(f(x), f(y)) = m^{-r - \sum_{i=1}^{2k-1} n_i - 2(M-1)(2k-1)} \geq m^{-r - (r/q) - (r/q)} = m^{-r(1+(2/q))}.$$

- (c) If  $q \sum_{i=1}^{2k} n_i < r \leq q \sum_{i=1}^{2k+1} n_i$  for some  $k \geq 1$ , we also have

$$\rho(f(x), f(y)) \geq m^{-r(1+(2/q))}. \quad \square$$

To complete the proof of Theorem B we use the fact that  $\dim_{\text{H}} G_\Phi = \dim_{\text{H}} \Sigma_A$ . To see this, it suffices to notice that the measure of maximal entropy of  $(\Sigma_A, T)$  is the Parry measure and that  $G_\Phi$  has full Parry measure [22, p. 194].  $\square$

### 5. Remarks

(1) Let  $C(\Sigma_A)$  be the set of real-valued continuous functions. Two functions  $f_1, f_2 \in C(\Sigma_A)$  are said to be *cohomologous* if  $f_1 - f_2 = \psi \circ T - \psi + c$  for some  $\psi \in C(\Sigma_A)$  and some constant  $c$ . This is an equivalence relation. Denote by  $\tilde{C}(\Sigma_A)$  the quotient space relative to this relation. Suppose that  $\Phi = (\Phi_1, \dots, \Phi_a)$  is of summable variation. It may be proved that the ergodic limit  $\sigma_\Phi(x)$  is a constant function on  $\Sigma_A$  if and only if  $\Phi_j$  considered as function in the quotient space are of rank zero. More generally, it may also be proved that the dimension of  $L_\Phi$  is the rank of  $\{\Phi_j\}_{1 \leq j \leq a}$  considered as elements in the quotient space.

(2) According to Theorem A, the dimension  $\dim E_\Phi(\alpha)$  (or equivalently the topological entropy  $h_{\text{top}}(E_\Phi(\alpha))$ ) is concave and varies upper semi-continuously on  $L_\Phi$ , then is continuous in the interior of  $L_\Phi$  [20, Section 10]. We wonder if there is always

continuity on the boundary of  $L_\Phi$ . The thermodynamical formalism shows differentiability in the interior for a function  $\Phi$  of summable variation, and analyticity in the interior for a Hölder function  $\Phi$  [21].

(3) When  $\Phi$  is of summable variation, the variational formula (Theorem A), together with the thermodynamical formalism, implies the relation

$$P^*(\alpha) = - \max_{\mu \in \mathcal{F}_\Phi(\alpha)} h_\mu$$

where  $P^*(\alpha)$  is the Legendre transform of the pressure function  $P(\beta)$  (that is, the pressure of the potential  $\langle \beta, \Phi \rangle$ ). It follows that

$$P(\beta) = \inf_{\alpha \in L_\Phi} \left[ \langle \alpha, \beta \rangle + \max_{\mu \in \mathcal{F}_\Phi(\alpha)} h_\mu \right].$$

Then we might get good information on the pressure function whenever the fibres of projection are known. Therefore the fibres of projection are worthy of study. Some related works are by T. Bousch [2], S. Bullett and P. Sentenac [6] and O. Jenkinson [13, 14] (see also [16]).

(4) Generalization to conformal expanding dynamical systems is easy to guess. Generalization to other systems is worthy of study and Liapunov exponents would be involved.

*Acknowledgements.* The paper was finished when the first author visited FIM at ETH-Zurich in Switzerland. He would like to thank Professors O. Lanford III and A.-S. Sznitman for their warm hospitality. The authors thank T. Bousch, F. Gamboa, O. Jenkinson, A. K. Manning and Z. Y. Wen for useful information and enlightening discussions. They also thank an anonymous referee for his (or her) suggestions from which the present version has benefited. D. J. Feng was supported by Tsinghua University Funding Program and J. Wu was supported by NSFC Grant #19971064. The second and third authors were supported by the Special Funds for Major State Basic Research Projects of China.

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