Recurrence relations and splitting formulas for the domination polynomial

Tomer Kotek*
Department of Computer Science
Technion—Israel Institute of Technology
Haifa, Israel
tkotek@cs.technion.ac.il

James Preen
Mathematics
Cape Breton University
Sydney, NS B1P6L2, Canada
james_preen@capebretonu.ca

Frank Simon† Peter Tittmann Martin Trinks†
Faculty Mathematics / Sciences / Computer Sciences
Hochschule Mittweida - University of Applied Sciences
Mittweida, Germany
{simon,peter,trinks}@hs-mittweida.de

Submitted: Jun 25, 2012; Accepted: Sep 29, 2012; Published: Oct 11, 2012
Mathematics Subject Classifications: 05C69, 05C31

Abstract

The domination polynomial $D(G, x)$ of a graph G is the generating function of its dominating sets. We prove that $D(G, x)$ satisfies a wide range of reduction formulas. We show linear recurrence relations for $D(G, x)$ for arbitrary graphs and for various special cases. We give splitting formulas for $D(G, x)$ based on articulation vertices, and more generally, on splitting sets of vertices.

Keywords: domination polynomial; recurrence relation; splitting formula

1 Introduction

Recurrence relations of graph polynomials have received considerable attention in the literature. Informally, a graph polynomial $f(G, x)$ *satisfies a linear recurrence relation* if

$$f(G, x) = \sum_{i=1}^{k} g_i(x) f(G_i, x),$$

*Supported by the Fein foundation and the graduate school of the Technion.
†Supported by a grant from the European Social Fund.

where G_i are obtained from G using various vertex and edge elimination operations and the $g_i(x)$'s are given rational functions. For example, it is well-known that the independence polynomial satisfies a linear recurrence relation with respect to two vertex elimination operations, the deletion of a vertex and the deletion of vertex's closed neighborhood. Other prominent graph polynomials in the literature satisfy similar recurrence relations with respect to vertex and edge elimination operations, among them the matching polynomial, the chromatic polynomial and the vertex-cover polynomial, see e.g. [11].

In contrast, it is significantly harder to find recurrence relations for the domination polynomial. We show in Theorem 2.4 that $D(G, x)$ does not satisfy any linear recurrence relation which applies only the commonly used vertex operations of deletion, extraction, contraction and neighborhood-contraction. Nor does $D(G, x)$ satisfy any linear recurrence relation using only edge deletion, contraction and extraction.

In spite of this non-existence result, we give in this paper an abundance of recurrence relations and splitting formulas for the domination polynomial.

The domination polynomial was studied recently by several authors, see [1, 2, 3, 5, 6, 7, 8, 9, 10, 12]. The previous research focused mainly on the roots of domination polynomials and on the domination polynomials of various classes of special graphs. In [12] it is shown that computing the domination polynomial $D(G, x)$ of a graph G is NP-hard and some examples for graphs for which $D(G, x)$ can be computed efficiently are given. Some of our results, e.g. Theorem 5.14, lead to efficient schemes to compute the domination polynomial.

An outline of the paper is as follows. In Section 2 we give a recurrence relation for arbitrary graphs. In Section 3 we give simple recurrence steps in special cases, which allow us e.g. to dispose of triangles, induced 5-paths and irrelevant edges. In Section 4 we consider graphs of connectivity 1, and give several splitting formulas for them. In Section 5 we generalize the results of the previous section to arbitrary separating vertex sets. In Section 6 we show a recurrence relation for arbitrary graphs which uses derivatives of domination polynomials.

1.1 Definitions and notations

This paper discusses simple undirected graphs $G = (V, E)$. A vertex subset $W \subseteq V$ of G is a *dominating vertex set* in G, if for each vertex $v \in V$ of G either v itself or an adjacent vertex is in W.

Definition 1.1. Let $G = (V, E)$ be a graph. The *domination polynomial* $D(G, x)$ is given by

$$D(G, x) = \sum_{i=0}^{|V|} d_i(G) x^i \,,$$

where $d_i(G)$ is the number of dominating sets of size i in G.

A graph polynomial $f(G, x)$ is *multiplicative with respect to components*, if

$$f(G^1 \cup G^2, x) = f(G^1, x) \cdot f(G^2, x),$$

where $G^1 \cup G^2$ denotes the disjoint union of the graphs G^1 and G^2. Obviously, the domination polynomial $D(G,x)$ is multiplicative with respect to components.

We use the following notations: The set of vertices adjacent in G to a vertex of a vertex subset $W \subseteq V$ is the *open neighborhood* $N_G(W)$ of W. The *closed neighborhood* $N_G[W]$ additionally includes all vertices of W itself. In case of a singleton set $W = \{v\}$ we write $N_G(v)$ and $N_G[v]$ instead of $N_G(\{v\})$ and $N_G[\{v\}]$, respectively. We omit the subscript when the graph G is clear from the context.

Hence, a vertex subset $W \subseteq V$ of G is a dominating set of G, if all vertices are in the closed neighborhood of W, i.e. $N_G[W] = V$, and the domination polynomial can be stated as a subset expansion:

$$D(G,x) = \sum_{W \subseteq V} [N_G[W] = V] x^{|W|}, \tag{1}$$

where we denote, for any statement C_W on W,

$$[C_W] = \begin{cases} 1 & \text{if } C_W \text{ holds for } W, \\ 0 & \text{if } C_W \text{ does not hold for } W. \end{cases}$$

We generalize this representation of $D(G,x)$ as follows[1]:

Definition 1.2. Let $G = (V,E)$ be a graph. Let $C = C_W$ be a statement, then we denote by $D_C(G,x)$ the generating function for the number of dominating vertex sets of G satisfying C:

$$D_C(G,x) = \sum_{W \subseteq V} [N_G[W] = V][C_W] x^{|W|}.$$

Examples for statements C_W are: (a) $W \neq \varnothing$ (b) $u \in W$, (c) $u \notin W$, and (d) $u \notin N_G[W]$, where u in (b), (c) and (d) is some fixed vertex of G.

In the sequel we use for graphs $G = (V,E)$ the following vertex and edge operations, which are commonly found in the literature. Let $v \in V$ be a vertex and $e = \{u,v\} \in E$ be an edge of G.

— Vertex deletion: $G - v$ denotes the graph obtained from G by removal of v and all edges incident to v.

— Vertex contraction: G/v denotes the graph obtained from G by the removal of v and the addition of edges between any pair of non-adjacent neighbors of v.

— Vertex extraction: $G - N[v]$ denotes the graph $G - N_G[v]$ obtained by deleting all of the vertices in the closed neighborhood of v and the edges incident to them.

— Vertex appending: $G + \{v, \cdot\}$ denotes the graph $(V \cup \{v'\}, E \cup \{v, v'\})$ obtained from G by adding a new vertex v' and an edge $\{v, v'\}$ to G.

[1] This definition can be generalized using the logical framework of [17].

— Edge deletion: $G - e$ denotes the graph obtained from G by simply removing e.

— Edge contraction: G/e denotes the graph obtained from G by removing e and unifying the end-points of e.

— Edge extraction: $G \dagger e$ denotes the graph $G-u-v$. Note that this operation removes all the edges incident to e and e itself.

For the Sections 4 and 5 we introduce the notion of a splitting of a graph.

Definition 1.3. Let $G = (V, E)$ be a graph and $G^1 = (V^1, E^1)$, $G^2 = (V^2, E^2)$ subgraphs of G with $V = V^1 \cup V^2$, $X = V^1 \cap V^2$, $E = E^1 \cup E^2$ and $E^1 \cap E^2 = \varnothing$, then we say that (G^1, G^2, X) is a *splitting of* G at the *separating vertex set* X. In case of $X = \{v\}$ the vertex v is called an *articulation* of G.

Observe that our definition of a separating vertex set and an articulation is more general than the usual one, as the number of components of the graph G does not necessarily increase, when the vertex set X is removed from G.

Other notation will be introduced as needed.

2 A recurrence relation for arbitrary graphs

For a graph G and a vertex u of G, let $p_u(G, x)$ be the generating function counting those dominating sets for $G - N[u]$ which additionally dominate the vertices of $N(u)$ in G. Equivalently, $p_u(G, x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$. Note $p_u(G, x) = D_{u \notin N_G[W]}(G - u, x)$ from Definition 1.2. Recall that $D_{u \in W}(G, x)$ denotes the generating function of the dominating sets W of G which contain u. $D_{u \notin W}(G, x)$ is defined analogously.

The following theorem will be useful both to prove recurrence relations for arbitrary graphs here and in Section 6, and to give recurrence steps for special cases in Section 3. This result and its two corollaries also appear in [4] but were proved independently.

Theorem 2.1. *Let $G = (V, E)$ be a graph. For any vertex u in G we have*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) - (1 + x)p_u(G, x)\,. \qquad (2)$$

Furthermore, we have

$$D_{u \notin W}(G, x) = D(G - u, x) - p_u(G, x) \qquad (3)$$
$$D_{u \in W}(G, x) = xD(G - N[u], x) + x(D(G/u, x) - p_u(G, x))\,. \qquad (4)$$

Proof. Any dominating set W of $G - u$ is a dominating set of G, unless no neighbor of u in G is in W. The dominating sets of $G - u$ including neither u nor any of its neighbors are enumerated by $p_u(G, x)$. Therefore we have Equation (3).

The dominating sets of G which contain u but not any of its neighbors are counted by $xD(G - N[u], x)$. For every dominating set W counted by $D(G/u, x)$, $W \cup \{u\}$ is a dominating set of G. The polynomial $p_u(G, u)$ counts the dominating sets of G/u which do not contain any neighbor of u in G. The dominating sets of G which contain u and at least one of its neighbors are counted by $x(D(G/u, x) - p_u(G, u))$, and therefore we have Equation (4).

Finally, Equation (2) follows from

$$D(G, x) = D_{u \notin W}(G, x) + D_{u \in W}(G, x) . \qquad \square$$

For every edge $\{u, v\} \in E(G)$, let $p_{u,v}(G, x) = D_{u \notin N_G[W-\{v\}], v \in W}(G - u, x)$ be the generating function counting the dominating sets W in $G - u$ which contain exactly one neighbor of u and that neighbor is v.

Lemma 2.2. *Let $G = (V, E)$ be a graph and let $e = \{u, v\} \in E$. Then*

$$p_u(G - e, x) = p_{u,v}(G, x) + p_u(G, x) .$$

Proof. We can think of $p_u(G - e, x)$ as follows: $p_u(G - e, x)$ counts the dominating sets in $G - u$ which do not contain any member of $N_G(u)$, except possibly v. $p_u(G, x)$ counts the dominating sets of $G - u$ which do not contain any member of $N_G(u)$. Hence, $p_u(G - e, x) - p_u(G, x)$ counts the dominating sets T in $G - w$ which contain v but do not contain any other member $N_G(u)$. $\square$

Using the previous lemma and theorem we can prove that the domination polynomial satisfies a recurrence relation for arbitrary graphs which is based on the edge and vertex elimination operations. The recurrence uses composite operations, e.g. $G - e/u$, which stands for $(G - e)/u$.

Theorem 2.3. *Let $G = (V, E)$ be a graph. For every edge $e = \{u, v\} \in E$,*

$$\begin{aligned} D(G, x) \; &= \; D(G - e, x) + \frac{x}{x-1} \Bigg[D(G - e/u, x) + D(G - e/v, x) \\ &\quad - D(G/u, x) - D(G/v, x) - D(G - N[u], x) - D(G - N[v], x) \\ &\quad + D(G - e - N[u], x) + D(G - e - N[v], x) \Bigg] . \end{aligned}$$

Proof. Let $e = \{v, u\} \in E$. Then $D(G, x) - D(G - e, x)$ counts the dominating sets T of G which are not dominating sets on $G - e$. I.e., T is counted by $D(G, x) - D(G - e, x)$ iff one can choose $a, b \in \{u, v\}$, $a \neq b$, such that $a \in T$, $b \notin T$ and no neighbor of b in G, besides a, belongs to T. That means $D(G, x) - D(G - e, x) = p_{u,v}(G, x) + p_{v,u}(G, x)$.

By Lemma 2.2, $p_{u,v}(G, x) = p_u(G - e, x) - p_u(G, x)$ and $p_{v,u}(G, x) = p_v(G - e, x) - p_v(G, x)$. Thus,

$$D(G, x) = D(G - e, x) + p_u(G - e, x) + p_v(G - e, x) - p_u(G, x) - p_v(G, x) .$$

Using Theorem 2.1,

$$D(G,x) = D(G-e,x) + \frac{1}{1+x} f(G,x,e)$$

where

$$\begin{aligned} f(G,x,e) = {} & xD(G-e/u,x) + D(G-e-u,x) + xD(G-e-N[u],x) - D(G-e,x) \\ & + xD(G-e/v,x) + D(G-e-v,x) + xD(G-e-N[v],x) - D(G-e,x) \\ & - xD(G/u,x) - D(G-u,x) - xD(G-N[u],x) + D(G,x) \\ & - xD(G/v,x) - D(G-v,x) - xD(G-N[v],x) + D(G,x)\,. \end{aligned}$$

The theorem follows using that $D(G-e-u,x) = D(G-u,x)$ and $D(G-e-v,x) = D(G-v,x)$ by rearranging the terms. □

On the other hand, $D(G,x)$ does not satisfy any linear recurrence relation with the four vertex operations of deletion, contraction, extraction and neighborhood-contraction, the latter defined as follows: we denote by $G/N(v)$ the graph obtained from G by removing all of the vertices of $N_G(v)$ and adding edges so that v is made adjacent to every neighbor of $N_G(v)$ in G. Note v is not removed from the graph.

Theorem 2.4. *There do not exist rational functions $a,b,c,d \in \mathbb{R}(x)$ such that for every graph G and vertex v of G it holds that*

$$D(G,x) \;=\; aD(G-v,x) + bD(G/v,x) + cD(G-N[v],x) + dD(G/N(v),x)\,. \quad (5)$$

Proof. Assume there exist a,b,c,d such that Equation (5) holds. We will show that there exist five pairs of graphs and vertices (G_i, v_i) such that $v_i \in V(G_i)$ for which the system of equations with indeterminates a,b,c,d,

$$\begin{aligned} D(G_1,x) &= aD(G_1-v_1,x) + bD(G_1/v_1,x) + cD(G_1-N[v_1],x) + dD(G_1/N(v_1),x) \\ &\;\;\vdots \\ D(G_5,x) &= aD(G_5-v_5,x) + bD(G_5/v_5,x) + cD(G_5-N[v_5],x) + dD(G_5/N(v_5),x) \end{aligned}$$

has no solution. The graphs are: $G_1 = K_1$, $G_2 = K_2$, $G_3 = K_3$, and $G_4 = G_5 = P_3$. For G_1, G_2 and G_3 all of the vertices are symmetric. We choose v_4 as one of the two end-points of P_3, and v_5 as the central vertex in P_3. We have:

G_i (v_i)	$G_i - v_i$	G_i/v_i	$G_i - N[v_i]$	$G_i/N(v_i)$
K_1	$\varnothing$	$\varnothing$	$\varnothing$	K_1
K_2	K_1	K_1	$\varnothing$	K_1
K_3	K_2	K_2	$\varnothing$	K_1
P_3 (v_3 is an end-point)	K_2	K_2	K_1	K_2
P_3 (v_4 is the central vertex)	$K_1 \cup K_1$	K_2	$\varnothing$	K_1

and

$D(G_i, x)$	$D(G_i - v_i, x)$	$D(G_i/v_i, x)$	$D(G_i - N[v_i], x)$	$D(G_i/N(v_i), x)$
x	1	1	1	x
$2x + x^2$	x	x	1	x
$3x + 3x^2 + x^3$	$2x + x^2$	$2x + x^2$	1	x
$x + 3x^2 + x^3$	$2x + x^2$	$2x + x^2$	x	$2x + x^2$
$x + 3x^2 + x^3$	x^2	$2x + x^2$	1	x

but the corresponding system of equations has no solution. □

The proof technique here is general in the sense that it can be extended to other vertex elimination operations. Note, however, that for the specific operations we used here, taking the three graphs G_1, G_2 and G_3 only suffices.

Similarly, $D(G, x)$ does not satisfy any linear recurrence relation with the three edge operations of deletion, contraction and extraction.

Theorem 2.5. *There do not exist rational functions $a, b, c \in \mathbb{R}(x)$ such that for every graph G and edge e of G it holds that*

$$D(G, x) \quad = \quad aD(G - e, x) + bD(G/e, x) + cD(G \dagger e, x) \,.$$

The proof of Theorem 2.5 is similar to that of Theorem 2.4 with $G_1 = K_2$, $G_2 = P_3$, $G_3 = K_3$ and $G_4 = P_4$, and where the edge associated with G_4 is the middle edge.

3 Simple recurrence steps for special cases

The domination polynomial of a graph G can be computed entirely using the recurrence relation in Theorem 2.3 and the multiplicativity of $D(G, x)$ with respect to disjoint union. However, in many cases, a simpler recurrence step will suffice.

Proposition 3.1. *Let $G = (V, E)$ be a graph. If $u, v \in V$ and $N[v] \subseteq N[u]$ then*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) \tag{6}$$

and

$$D(G, x) = (1 + x)D(G - u, x) + D_{u \in W}(G - (N[v] \backslash \{u\}), x) \,. \tag{7}$$

Proof. For Equation (6) we use Theorem 2.1. In this case $p_u(G, x) = 0$ since if neither u nor a vertex adjacent to it is in the dominating set, then v (and its neighbors except u) can not be dominated.

For Equation (7) we partition the dominating sets of G into two types, counted by the two terms of Equation (7). $(1 + x)D(G - u, x)$ counts the dominating sets of G which contain at least one vertex from $N_G[v] \backslash \{u\}$. $D_{u \in W}(G - (N[v] \backslash \{u\}), x)$ counts the dominating sets of G which contain no element of $N_G[v]$ except for u. □

Proposition 3.2. *Let $G = (V, E)$ be a graph. If $u, w \in V$ and $N(w) = N(u)$ then*

$$D(G, x) = xD(G/u, x) + D(G - u, x) - xD(G - N[u] - w, x)\,.$$

Proof. In this case w is of degree 0 in $G - N[u]$ and hence w must exist in any dominating set for $G - N[u]$. Since $N(w) = N(u)$ we know that $N(u)$ is dominated by every dominating set for $G - N[u]$. Thus we have $p_u(G, x) = D(G - N[u], x)$. It remains to notice that $D(G - N[u], x) = xD(G - N[u] - w, x)$ since w is an isolated vertex in $G - N[u]$. The result follows from Theorem 2.1. □

Here we generalize Proposition 3.2:

Proposition 3.3. *Let $G = (V, E)$ be a graph. If $u, w \in V$ and $N(w) \subseteq N(u)$ then,*

$$\begin{aligned} D(G, x) &= xD(G/u, x) + D(G - u, x) + xD(G - N[w], x) \\ &\quad -x^2 D(G - N[w]/u, x) - xD(G - N[w] - u, x)\,. \end{aligned} \tag{8}$$

Proof. Applying Theorem 2.1 on $G - N[w]$ and G, both with the vertex u, and rearranging the terms we get

$$\begin{gathered} D(G - N[w], x) - xD(G - N[w]/u, x) - D(G - N[w] - u, x) = \\ xD(G - N[w] - N[u], x) - (1 + x)p_u(G - N[w], x) \end{gathered}$$

and

$$D(G, x) - xD(G/u, x) - D(G - u, x) = xD(G - N[u], x) - (1 + x)p_u(G, x)\,.$$

Using that $G - N[u] = \{w\} \cup (G - N[w] - N[u])$ we have $D(G - N[u], x) = xD(G - N[w] - N[u], x)$ and $p_u(G, x) = xp_u(G - N[w], x)$, and the proposition follows. □

Example 3.4 (Vertices of degree 1)**.** Let $G = (V, E)$ be a graph. Let v be a vertex of degree 1 in a graph G and let u be its neighbor. Then $N[v] = \{u, v\} \subseteq N[u]$ and so, by Proposition 3.1:

$$\begin{aligned} D(G, x) &= xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) \\ &= xD(G/u, x) + xD(G - u - v, x) + xD(G - N[u], x) \\ &= x\left(D(G/u, x) + D(G - u - v, x) + D(G - N[u], x)\right)\,. \end{aligned}$$

Example 3.5 (Vertices of degree $|V(G)| - 2$)**.** Let $G = (V, E)$ be a graph. Let u be such a vertex adjacent to all vertices in G except for w. If there exists a vertex $v \in N(u)$ such that $\{v, w\} \notin E$ then $N[v] \subseteq N[u]$ and so Proposition 3.1 will apply:

$$\begin{aligned} D(G, x) &= xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) \\ &= xD(G/u, x) + D(G - u, x) + x^2\,. \end{aligned}$$

Otherwise, $N(w) = N(u)$ and hence by Proposition 3.2

$$\begin{aligned} D(G, x) &= xD(G/u, x) + D(G - u, x) - D(G - N[u], x) \\ &= xD(G/u, x) + D(G - u, x) - x \\ &= x((1 + x)^{|V|-1} - 1) + D(G - u, x) - x \\ &= x(1 + x)((1 + x)^{|V|-2} - 1) + D(G - u, x)\,. \end{aligned}$$

3.1 Removing triangles

Theorem 2.1 can be used to give a recurrence relation which removes triangles. We define a new operation on edges incident to a vertex u: we denote by $G \odot u$ the graph obtained from G by the removal of all edges between any pair of neighbors of u. Note u is not removed from the graph. This recurrence relation is useful on graphs which have many triangles.

Proposition 3.6. *Let $G = (V, E)$ be a graph and $u \in V$. Then*

$$D(G, x) = D(G - u, x) + D(G \odot u, x) - D(G \odot u - u, x)\,.$$

Proof. First note that, since the operation $\odot u$ only removes the edges between vertices in $N(u)$, these relations will hold:

$$(G \odot u)/u \cong G/u\ , \quad p_u(G, x) = p_u(G \odot u, x)\ , \quad (G \odot u) - N[u] \cong (G - N[u])\,.$$

Using these relations with Theorem 2.1, we have

$$\begin{aligned} D(G, x) - D(G - u, x) &= xD(G/u, x) + xD(G - N[u], x) - (1 + x)p_u(G, x) \\ &= xD((G \odot u)/u, x) + xD((G \odot u) - N[u], x) \\ &\quad -(1 + x)p_u(G \odot u, x)\,. \end{aligned}$$

Now we apply Theorem 2.1 to $G \odot u$ and get that the right-hand side of the latter equation equals

$$D(G \odot u, x) - D((G \odot u) - u, x)\,,$$

and the proposition follows. □

3.2 Induced 5-paths

In [9] the following recurrence formula was proved in terms of edge contractions:

Theorem 3.7. *Suppose G is a graph which contains five vertices u, v, w, y, z that form a path in that order such that the degrees of v, w and y are 2. Then*

$$D(G, x) = x\left(D(G/w, x) + D(G/v/w, x) + D(G/v/w/y, x)\right)\,. \tag{9}$$

Note that the order of contracting v, w and y in $D(G/v/w/y, x)$ does not matter. The same is true for deleting v, w and y in $D(G - v - w - y, x)$.

We can verify this result using Theorem 2.1 as follows:

Proof. Suppose the five vertices in the induced path are u, v, w, y and z in order along the path. We apply Theorem 2.1 to the central vertex, w:

$$D(G, x) = xD(G/w, x) + D(G - w, x) + xD(G - v - w - y, x) - (1 + x)p_w(G, x)\,.$$

Note that the first term is the same as in Equation (9), so it remains to prove that

$$xD(G-v-w-y,x)-(1+x)p_w(G,x)+D(G-w,x) = xD(G/v/w,x)+xD(G/v/w/y,x)\,. \tag{10}$$

Let $H := G - N[w]$, and let S be a dominating set for H. We wish to extend S to a dominating set for each of the graphs in Equation (10) by considering whether or not v and/or y must or may be added to S. For S to dominate $N(w)$ it must include both u and z. We break the proof here into 3 cases, dependent on how many of u and z are in S.

When u, say, is in S it will dominate v and so v can either be in S or out of it, giving a factor of $(1+x)$ to multiply the domination polynomial of H by. If u is not in S then v must be in S in order for S to be a dominating set, giving a factor of x for H. We tabulate the respective contributions for vertices v and y in the different graphs, substituting $q(x) := xD(G-v-w-y,x) - (1+x)p_w(G,x)$:

$\lvert S \cap \{w,y\}\rvert$	$q(x)$	$D(G-w,x)$	$xD(G/v/w,x)$	$xD(G/v/w/y,x)$
2	-1	$(1+x)^2$	$x(1+x)$	x
1	x	$x(1+x)$	$x(1+x)$	x
0	x	x^2	x^2	x

Table 1: Table of contributions from vertices v and y

For each of these rows we can see that Equation (10) is satisfied by adding both pairs of columns. Since all of the possibilities for S fall into one of these three cases, the proof is complete. □

3.3 Irrelevant edges

The easiest recurrence relation one might think of is to remove an edge and to compute the domination polynomial of the graph arising instead of the one for the original graph. Indeed, for the domination polynomial of a graph there might be such irrelevant edges, that can be deleted without changing the value of the domination polynomial at all. We characterize edges possessing this property.

Definition 3.8. Let $G = (V,E)$ be a graph. An *irrelevant edge* is an edge $e \in E$ of G, such that

$$D(G,x) = D(G-e,x). \tag{11}$$

Definition 3.9. Let $G = (V,E)$ be a graph. A vertex $v \in V$ of G is *domination-covered*, if every dominating set of $G - v$ includes at least one vertex adjacent to v in G.

In other words, each dominating set of $G - v$ is a dominating set of G and the dominating sets of $G - v$ are exactly those dominating sets of G not including the vertex v.

Theorem 3.10. *Let $G = (V, E)$ be a graph. A vertex $v \in V$ of G is domination-covered if and only if there is a vertex $u \in N_G[v]$ such that $N_G[u] \subseteq N_G[v]$.*

In case the condition of the theorem above is satisfied, we say that the vertex v is domination-covered by the vertex u.

Proof. First we prove that, if there is such a vertex u, then v is domination-covered. To dominate u, either u or a vertex adjacent to u must be in each dominating set of $G - v$. Because every vertex adjacent to u is also adjacent to v in G, the statement follows.

Secondly we prove that, if v is domination-covered, there is such a vertex u. Assume that v is domination-covered, but there is no such vertex, that means each vertex adjacent to v in G has at least one neighbor not adjacent to v in G. Then the set $V \setminus N[v]$ is a dominating set of $G - v$, but not of G, which contradicts the assumption. □

Theorem 3.11. *Let $G = (V, E)$ be a graph. An edge $e = \{u, v\} \in E$ is an irrelevant edge in G if and only if u and v are domination-covered in $G - e$.*

Proof. We start with the graph $G - e$ and show that no additional dominating set arises by inserting the edge e if and only if u and v are domination-covered in $G - e$.

First we prove, that if both vertices are domination-covered, then e is an irrelevant edge. Assume e is not irrelevant, that means there is a vertex subset W which is a dominating set in G but not in $G - e$. Consequently the edge e must be "used", meaning that w.l.o.g. $u \in W$ and $N_{G-e}[v] \cap W = \varnothing$. But then v is not domination-covered in $G - e$, which contradicts the assumption.

Second we prove, that if e is an irrelevant edge, then both vertices are domination-covered. Assume at least one vertex is not domination-covered in $G - e$, say u. Then $V \setminus N_{G-e}[v]$ is a dominating set in G but not in $G - e$, which contradicts the assumption. □

The theorem above can be applied to the calculation of the domination polynomial of corona graphs $G \circ H$.

For two graphs $G = (V, E)$ and H, the *corona* $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H, named $H_1, \ldots, H_{|V|}$, by adding edges between the ith vertex of G and all vertices of H_i for $1 \leqslant i \leqslant |V|$.

Theorem 3.12. *Let $G = (V, E)$ and $H = (W, F)$ be non-empty graphs. Then*

$$D(G \circ H, x) = [D(K_1 \circ H, x)]^{|V|} = \left[x(1 + x)^{|W|} + D(H, x)\right]^{|V|}. \tag{12}$$

Proof. In the corona of two non-empty graphs, each vertex $u \in V$ of G is adjacent to all vertices of the corresponding copy of H, therefore the vertex u is domination-covered (by each vertex of the according copy of H). Thus, we can delete all edges in E (from the original graph G) in the corona and the arising graph is the disjoint union of $|V|$ copies of the corona $K_1 \circ H$, which proves the first identity. In each of these coronas a dominating vertex set either includes u (the vertex originally in G) and an arbitrary subset of the $|W|$ vertices from the copy of H. Or, it does not include u and equals exactly a dominating vertex set of (the copy of) H, which proves the second identity. □

This result generalizes the statement for a special corona graph $H = K_1$ as given in [1, Theorem 1].

4 Articulations

For many graph polynomials, it is possible to calculate the polynomial of a graph with an articulation by calculating the polynomial of some graphs related to those graphs arising from separating the graph at the articulation. Here we will prove that this idea is also workable for the domination polynomial.

Throughout this section we will be considering three polynomials from Definition 1.2: $D_{v\in W}(G,x)$, $D_{v\notin W}(G,x)$ and $D_{v\notin N_G[W]}(G-v,x) = p_v(G,x)$. The first and the second polynomials count exactly those dominating vertex sets of G including v and not including v, respectively. The third polynomial counts the vertex subsets of $G - N[v]$ which dominate the vertices of $V \setminus \{v\}$ in G. Alternatively, this polynomial is also $p_v(G,x)$ (from Section 2) and counts all dominating vertex sets of $G - v$ that do not include any neighbor of v (in G).

Definition 4.1. Let $G = (V,E)$ be a graph and $v \in V$. We define the following two vectors

$$\mathbf{u}_v(G) = \begin{pmatrix} D_{v\notin W}(G,x) \\ D_{v\notin N_G[W]}(G-v,x) \\ \frac{1}{x} D_{v\in W}(G,x) \end{pmatrix}, \qquad \mathbf{d}_v(G) = \begin{pmatrix} D(G-v,x) \\ D(G,x) \\ D(G+\{v,\cdot\},x) \end{pmatrix}$$

and the matrices $\mathbf{P}$ and $\mathbf{Q}$ by

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ x & 0 & 1+x \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}.$$

The domination polynomial of a graph G with an articulation v can be computed from the vectors defined above for the graphs arising from the splitting at v as shown below:

Theorem 4.2. *Let $G = (V,E)$ be a graph with splitting $(G^1, G^2, \{v\})$ and articulation $v \in V$. Then*

$$D(G,x) = \mathbf{u}_v(G^1)^T \mathbf{Q} \mathbf{u}_v(G^2). \tag{13}$$

Proof. Expanding equation (13) we see that matrix $\mathbf{Q}$ combines all of the different ways that dominating sets can arise in G. For instance, the first two rows of $\mathbf{Q}$ give the expression

$$D_{v\notin W}(G^1,x)\left(D_{v\notin W}(G^2,x) + D_{v\notin N_{G^2}[W]}(G^2-v,x)\right) + D_{v\notin W}(G^1,x) D_{v\notin N_{G^2}[W]}(G^2-v,x)$$

which corresponds to saying that if v is not in a dominating set for G then it must either be dominated by a vertex in both G^1 and G^2 or only one of them. If v is in W in G^1 it must also be in W in G^2 so only the $(3,3)$ entry of $\mathbf{Q}$ will be involved. However v will be counted twice and so we will have to multiply by x to get the expression

$$\frac{1}{x} D_{v\in W}(G^1,x) D_{v\in W}(G^2,x)$$

as required. □

Theorem 4.3. *Let $G = (V, E)$ be a graph and $v \in V$. Then*

$$\mathbf{d}_v(G) = \mathbf{P}\mathbf{Q}\mathbf{u}_v(G)\,.$$

Proof. The first row of the stated matrix equation gives

$$D(G - v, x) = D_{v \notin W}(G, x) + D_{v \notin N_G[W]}(G, x)\,,$$

which is a valid equation, as we are counting all dominating vertex sets of the graph $G - v$. For the remaining rows we will utilise Theorem 4.2. First, let us consider the splitting $(G^1, G^2, \{v\})$ of G with $G^1 = (\{v\}, \varnothing)$ and $G^2 = G = (V, E)$ and observe that

$$\mathbf{u}_v(G^1)^T = \begin{pmatrix} 0, & 1, & 1 \end{pmatrix}\,,$$

which equals the second row of the matrix $\mathbf{P}$ as required.

Now choose $G^1 = (\{v, v'\}, \{\{v, v'\}\})$ with $v' \notin V$ and $G^2 = G = (V, E)$ and, similarly,

$$\mathbf{u}_v(G^1)^T = \begin{pmatrix} x, & 0, & 1 + x \end{pmatrix}\,,$$

Theorem 4.2 again proves that the final row of the stated matrix equation is valid. □

Lemma 4.4. *Under the requirements of Definition 4.1 we obtain the matrix*

$$\mathbf{M} = \mathbf{P}\mathbf{Q}\mathbf{P}^T = \begin{pmatrix} 1 & 1 & x \\ 1 & x & 2x + x^2 \\ x & 2x + x^2 & x + 3x^2 + x^3 \end{pmatrix}\,,$$

which is expressible as domination polynomials of paths

$$\mathbf{M} = \begin{pmatrix} D(P_0, x) & D(P_0, x) & D(P_1, x) \\ D(P_0, x) & D(P_1, x) & D(P_2, x) \\ D(P_1, x) & D(P_2, x) & D(P_3, x) \end{pmatrix}\,.$$

Theorem 4.5. *Let $G = (V, E)$ be a graph with splitting $(G^1, G^2, \{v\})$ and articulation $v \in V$. Then the domination polynomial of G can be expressed as follows:*

$$D(G, x) = \mathbf{d}_v(G^1)^T \mathbf{M}^{-1} \mathbf{d}_v(G^2)\,.$$

Proof. By Theorem 4.2 we have

$$D(G, x) = \mathbf{u}_v(G^1)^T \mathbf{Q}\mathbf{u}_v(G^2) \tag{14}$$

and Theorem 4.3 gives

$$\mathbf{u}_v(G^i) = \mathbf{Q}^{-1}\mathbf{P}^{-1}\mathbf{d}_v(G^i) \tag{15}$$

for $i = 1, 2$. Substitution of Equation (15) into Equation (14) gives the result. □

Another recurrence relation for graphs with an articulation can be obtained using Theorem 2.1. This recurrence relation uses only the graphs G^1 and G^2 and their subgraphs.

Theorem 4.6. *Let $G = (V, E)$ be a graph with splitting $(G^1, G^2, \{v\})$ and articulation $v \in V$. Then*

$$\begin{aligned} D(G,x) &= xD(G/v,x) + D(G^1 - v,x) \cdot D(G^2 - v,x) \\ &\quad + xD(G^1 - N[v],x) \cdot D(G^2 - N[v],x) \\ &\quad - \frac{1}{1+x}\left[\left(xD(G^1/v,x) + D(G^1 - v,x) + xD(G^1 - N[v],x) - D(G^1,x)\right)\right. \\ &\quad \left.\cdot \left(xD(G^2/v,x) + D(G^1 - v,x) + xD(G^2 - N[v],x) - D(G^2,x)\right)\right] . \end{aligned}$$

Proof. By Theorem 2.1 we have, for the graphs G, G^1 and G^2 and vertex v:

$$\begin{aligned} (1+x)p_v(G,x) &= xD(G/v,x) + D(G - v,x) + xD(G - N[v],x) - D(G,x) \qquad (16) \\ (1+x)p_v(G^1,x) &= xD(G^1/v,x) + D(G^1 - v,x) + xD(G^1 - N[v],x) - D(G^1,x) \\ (1+x)p_v(G^2,x) &= xD(G^2/v,x) + D(G^2 - v,x) + xD(G^2 - N[v],x) - D(G^2,x) . \end{aligned}$$

$p_v(G, x)$ counts exactly those dominating sets of $G - N[v]$ which dominate $N(v)$. Since v is an articulation, each such set must be a disjoint union of dominating sets for G^i which each dominates $N(v) \cap G^i = N_{G^i}(v)$ for $i = 1, 2$. Hence

$$p_v(G,x) = p_v(G^1,x) \cdot p_v(G^2,x) . \qquad (17)$$

The graphs $G - v$ (respectively $G - N[v]$) are the disjoint union of $G^1 - v$ and $G^2 - v$ (respectively $G^1 - N[v]$ and $G^2 - N[v]$). Hence,

$$\begin{aligned} D(G - v,x) &= D(G^1 - v,x) \cdot D(G^2 - v,x) \qquad (18) \\ D(G - N[v],x) &= D(G^1 - N[v],x) \cdot D(G^2 - N[v],x) . \end{aligned}$$

From Equations (16), (17) and (18) we get the desired result by rearranging the terms. □

5 Arbitrary separating vertex sets

We will now extend the results in Section 4 from articulations to arbitrary separating vertex sets. Theorem 5.14, the Splitting Formula Theorem, will then be applied to a simple separating set of size 2 in Subsection 5.1.

Definition 5.1. Let S be a finite set. We denote by $\mathrm{Mat}(S, S)$ the set of all square matrices with $|S|$ rows and columns, where the rows and columns are indexed by the elements of S. Similarly, we denote by $\mathrm{Mat}(S)$ the set of all vectors with $|S|$ rows indexed by the elements in S.

Definition 5.2. Let S^1, S^2 be finite sets. Given two matrices $\mathbf{U} \in \mathrm{Mat}(S^1, S^1)$ and $\mathbf{V} \in \mathrm{Mat}(S^2, S^2)$ with

$$\mathbf{U} = (u(x, y))_{x,y \in S^1}$$
$$\mathbf{V} = (v(x, y))_{x,y \in S^2}$$

we define the Kronecker product

$$\mathbf{U} \otimes \mathbf{V} = (u(x, y)v(r, s))_{(x,r) \in S^1 \times S^2, (y,s) \in S^1 \times S^2}$$

of $\mathbf{U}$ and $\mathbf{V}$ being an element in $\mathrm{Mat}(S^1 \times S^2, S^1 \times S^2)$.

Proposition 5.3 (see [16]). *Let U, V be finite sets and $\mathbf{U} \in \mathrm{Mat}(U, U)$, $\mathbf{V} \in \mathrm{Mat}(V, V)$ be invertible matrices. We have that*

$$(\mathbf{U} \otimes \mathbf{V})^{-1} = \mathbf{U}^{-1} \otimes \mathbf{V}^{-1}. \tag{19}$$

We will denote by 2^X the power set of a given set X.

Definition 5.4. Let G be a graph with splitting (G^1, G^2, X). We define the set $R(X)$ as follows:

$$R(X) = \{(A_X, B_X) \in 2^X \times 2^X : A_X \subseteq B_X\},$$

Note that $R(X)$ has $3^{|X|}$ elements and for all $v \in X$ the three-element set

$$\begin{aligned} R(\{v\}) &= \{(A_v, B_v) \in 2^{\{v\}} \times 2^{\{v\}} : A_v \subseteq B_v\} \\ &= \{(\varnothing, \varnothing), (\varnothing, \{v\}), (\{v\}, \{v\})\}. \end{aligned}$$

In the following section we will use the bijection f which has these two properties when dealing with Kronecker products of matrices in $\mathrm{Mat}(R(\{v\}), R(\{v\}))$ for $v \in X$.

$$\prod_{v \in X} R(\{v\}) \mapsto R(X) \quad , \qquad \prod_{v \in X} (A_v, B_v) \mapsto \left(\bigcup_{v \in X} A_v, \bigcup_{v \in X} B_v \right),$$

Assume now that U and V are finite and disjoint sets. Under this restriction, if $\mathbf{A}$ and $\mathbf{B}$ have rows and columns indexed by $R(U)$ and $R(V)$, then $\mathbf{A} \otimes \mathbf{B}$ has rows and columns indexed by $R(U \cup V)$, following the above bijection.

Finally, for every vertex v, the rows and columns of the matrices $\mathbf{P}_v, \mathbf{Q}_v, \mathbf{M}_v \in \mathrm{Mat}(R(\{v\}), R(\{v\}))$ defined in an analogous way to the matrices in Section 4 will always be given with this specific ordering of the elements in $R(\{v\})$:

$$(\varnothing, \{v\}), (\varnothing, \varnothing), (\{v\}, \{v\}).$$

For example we have in the third row and column of matrix $\mathbf{P}_v$, from Section 4, the entry $1 + x$, that is corresponding to the row and the column indexed by $(\{v\}, \{v\})$.

Definition 5.5. Let (G^1, G^2, X) be a splitting of G with $G^i = (V^i, E^i)$. We define the Kronecker products

$$\mathbf{P}_X = \bigotimes_{v\in X} \mathbf{P}_v \ , \quad \mathbf{Q}_X = \bigotimes_{v\in X} \mathbf{Q}_v \ , \quad \mathbf{M}_X = \bigotimes_{v\in X} \mathbf{M}_v \, .$$

Proposition 5.6. *The matrices* $\mathbf{P}_X$ *and* $\mathbf{Q}_X$ *are invertible.*

Proof. For every $v \in X$ the matrices $\mathbf{P}_v$ and $\mathbf{Q}_v$ are invertible, as $\det(\mathbf{P}_v) \neq 0$ and $\det(\mathbf{Q}_v) \neq 0$. Furthermore we conclude by Proposition 5.3

$$\mathbf{P}_X^{-1} = \bigotimes_{v\in X} \mathbf{P}_v^{-1} \ , \quad \mathbf{Q}_X^{-1} = \bigotimes_{v\in X} \mathbf{Q}_v^{-1} \ ,$$

which proves the claim. □

Proposition 5.7. *We have*

$$\mathbf{M}_X = \mathbf{P}_X \mathbf{Q}_X \mathbf{P}_X^T. \tag{20}$$

Proof. Observe that we have the following chain of equalities

$$\begin{aligned}
\mathbf{P}_X \mathbf{Q}_X \mathbf{P}_X^T &= \bigotimes_{v\in X} \mathbf{P}_v \cdot \bigotimes_{v\in X} \mathbf{Q}_v \cdot \bigotimes_{v\in X} \mathbf{P}_v^T && \text{(Definition)} \\
&= \bigotimes_{v\in X} (\mathbf{P}_v \mathbf{Q}_v \mathbf{P}_v^T) && \text{(Property of the Kronecker product)} \\
&= \bigotimes_{v\in X} \mathbf{M}_v && \text{(Lemma 4.4)} \\
&= \mathbf{M}_X \, .
\end{aligned}$$
□

Definition 5.8. Let $G = (V, E)$ be a graph with splitting (G^1, G^2, X) with $G^i = (V^i, E^i)$. We define

$$D(G^i, (A, B), x) = \sum_{W^i \subseteq V^i} [W^i \cap X = A, N^i \cap X = B][N^i \setminus X = V^i \setminus X] x^{|W^i \setminus X|}$$

for all $(A, B) \in R(X)$ with $N^i := N_{G^i}[W^i]$ and $i = 1, 2$. Furthermore we define the vectors $\mathbf{u}_X(G^i) \in \mathrm{Mat}(R(X))$ by

$$\mathbf{u}_X(G^i) = (D(G^i, (A, B), x))_{(A,B)\in R(X)} \tag{21}$$

for $i = 1, 2$.

Remark 5.9. The reader should observe that the vector $\mathbf{u}_X(G^i)$ is the direct generalization of the vector $\mathbf{u}_v(G^i)$ in Section 4. Indeed, for an articulation $\{v\}$ as a singleton vertex separating set $X = \{v\}$, we obtain the following three identities

$$\begin{aligned}
D(G, (\varnothing, \varnothing), x) &= D_{v \notin N_G[W]}(G - v, x) \, , \\
D(G, (\varnothing, \{v\}), x) &= D_{v \notin W}(G, x) \, , \\
D(G, (\{v\}, \{v\}), x) &= D_{v \in W}(G, x)/x \, .
\end{aligned}$$

In other words we have $\mathbf{u}_{\{v\}}(G^i) = \mathbf{u}_v(G^i)$.

Lemma 5.10. *Let $G = (V, E)$ be a graph with splitting (G^1, G^2, X), with $G^i = (V^i, E^i)$ for $i = 1, 2$. We have, for all $W \subseteq V$,*

$$|W| = |W^1 \setminus X| + |(W^1 \cap X) \cup (W^2 \cap X)| + |W^2 \setminus X|$$

and

$$[N_G(W) = V] = [N^1 \setminus X = V^1 \setminus X][(N^1 \cup N^2) \cap X = X][N^2 \setminus X = V^2 \setminus X]$$

where $W^i := W \cap V^i$ and $N^i := N_{G^i}[W^i]$ for $i = 1, 2$.

Proof. The first claim follows directly by partitioning of W into three disjoint sets

$$W = (W^1 \setminus X) \cup ((W^1 \cap X) \cup (W^2 \cap X)) \cup (W^2 \setminus X)\,.$$

Similarly we obtain

$$N_G[W] = (N^1 \setminus X) \cup ((N^1 \cap X) \cup (N^2 \cap X)) \cup (N^2 \setminus X)\,,$$

which obviously implies the second claim. □

We will now give the generalization of Theorem 4.2 to arbitrary separating vertex sets.

Theorem 5.11. *Let (G^1, G^2, X) be a splitting of G. We have*

$$D(G, x) = \mathbf{u}_X(G^1)^T \mathbf{Q}_X \mathbf{u}_X(G^2). \tag{22}$$

Proof. We have, by Lemma 5.10, that

$$x^{|W|} = x^{|W^1 \setminus X|} x^{|(W^1 \cap X) \cup (W^2 \cap X)|} x^{|W^2 \setminus X|}$$

and

$$[N_G[W] = V] = [N^1 \setminus X = V^1 \setminus X][B^1 \cup B^2 = X][N^2 \setminus X = V^2 \setminus X]$$

for all $W \subseteq V$ with $N^i := N_{G^i}[W^i]$, $B^i := N^i \cap X$ and $i = 1, 2$. A summation over all $W \subseteq V$ and sorting according to Definition 5.8 by $(A, B^i) \in R(X)$ yields

$$D(G, x) = \sum_{\substack{(A,B^1) \in R(X) \\ (A,B^2) \in R(X)}} D(G^1, (A, B^1), x) x^{|A|} [B^1 \cup B^2 = X] D(G^2, (A, B^2), x)\,,$$

which is just the stated matrix equation. □

Definition 5.12. Let $G = (V, E)$ be a graph with splitting (G^1, G^2, X). For every $(A, B) \in R(X)$ we denote by $G^i_{(A,B)}$ the graph

$$G^i_{(A,B)} = G^i + \sum_{v \in A} \{v, \cdot\} - \sum_{v \in B \setminus A} v\,.$$

Furthermore we introduce the vectors

$$\mathbf{d}_X(G^i) = (D(G^i_{(A,B)}, x))_{(A,B) \in R(X)}$$

with $\mathbf{d}_X(G^i) \in \mathrm{Mat}(R(X))$ for $i = 1, 2$. These vectors are again extensions of the vectors $\mathbf{d}_v(G^i) \in \mathrm{Mat}(R(\{v\}))$ already defined in Section 4.

Proposition 5.13. *Let $G=(V,E)$ be a graph and $\varnothing \subset X \subseteq V$. We then have that*

$$\mathbf{d}_X(G) = \mathbf{P}_X \mathbf{Q}_X \mathbf{u}_X(G).$$

Proof. The proof can be accomplished by utilizing Theorem 5.11 and is a straightforward extension of Theorem 4.3 to arbitrary separating vertex sets.

Let $(A,B) \in R(X)$ and consider the splitting $(G^1_{(A,B)}, G^2_{(A,B)}, X)$ of the graph

$$G_{(A,B)} = G + \sum_{v\in A}\{v,\cdot\} - \sum_{v\in B\setminus A} v\,,$$

such that

$$\begin{aligned} G^1_{(A,B)} &= (X \setminus (B\setminus A) \cup \{v' : v \in A\}, \{\{v,v'\} : v \in A\}) \\ G^2_{(A,B)} &= (V \setminus (B\setminus A), \{\{u,v\} \in E : \{u,v\} \subseteq V \setminus (B \setminus A)\})\,. \end{aligned}$$

Now we have

$$D(G_{(A,B)},x) = \mathbf{u}_X(G^1_{(A,B)})^T \mathbf{Q}_X \mathbf{u}_X(G^2_{(A,B)})$$

by Theorem 5.11. Hence we have to show that $\mathbf{u}_X(G^1_{(A,B)})$ equals the row of the matrix $\mathbf{P}_X$ indexed by (A,B). Let $X=\{x_1,\ldots,x_k\}$ and observe that the graph $G^1_{(A,B)}$ can be written as the disjoint union of k graphs, say $H_1,\ldots,H_k$, some of them possibly being null graphs (graphs with no vertices), whenever $x_i \in B \setminus A$. If we pick an element in $\mathbf{u}_X(G^1_{(A,B)})$ being indexed by $(C,D) \in R(X)$ we can therefore conclude

$$D(G^1_{(A,B)},(C,D),x) = \prod_{i=1}^{k} D(H_i,(C\cap\{x_i\}, D\cap\{x_i\}),x)\,, \tag{23}$$

because the domination property is multiplicative in components. The right-hand side of Equation (23) equals the element of the matrix $\mathbf{P}_X$ in the row indexed by (A,B) and the column indexed by (C,D). □

We are now able to state the main theorem of this section which will enables calculation of the domination polynomial of a graph given arbitrary separating vertex sets.

Theorem 5.14 (Splitting Formula Theorem)**.** *Let $G=(V,E)$ be a graph with splitting (G^1,G^2,X). We have*

$$D(G,x) = \mathbf{d}_X(G^1)^T \mathbf{M}_X^{-1} \mathbf{d}_X(G^2)\,.$$

Proof. It is

$$\begin{aligned} D(G,x) &= \mathbf{u}_X(G^1)^T \mathbf{Q}_X \mathbf{u}_X(G^2) && \text{(Theorem 5.11)} \\ \mathbf{d}_X(G^i) &= \mathbf{P}_X \mathbf{Q}_X \mathbf{u}_X(G^i) \qquad i=1,2 && \text{(Proposition 5.13)} \end{aligned}$$

and by Proposition 5.6 we know that the matrices $\mathbf{P}_X$ and $\mathbf{Q}_X$ are invertible, so that

$$\mathbf{u}_X(G^i) = \mathbf{Q}_X^{-1}\mathbf{P}_X^{-1}\mathbf{d}_X(G^i)$$

for $i = 1, 2$. Hence, after substitution, we have

$$\begin{aligned} D(G,x) &= \mathbf{d}_X(G^1)^T_X\mathbf{P}_X^{-1,T}\mathbf{Q}_X^{-1}\mathbf{P}_X^{-1}\mathbf{d}_X(G^2) \\ &= \mathbf{d}_X(G^1)^T(\mathbf{P}_X\mathbf{Q}_X\mathbf{P}_X^T)^{-1}\mathbf{d}_X(G^2) \\ &= \mathbf{d}_X(G^1)^T\mathbf{M}_X^{-1}\mathbf{d}_X(G^2). \qquad \square \end{aligned}$$

5.1 A particular case of the Splitting Formula Theorem

In this subsection we illustrate the use of Theorem 5.14 by applying it to the case of a simple separating set of size 2. Let G be a graph with splitting (G^1, G^2, X) at the separating vertex set $X = \{u, v\}$. In this case we have the two matrices $\mathbf{M}_u \in \mathrm{Mat}(R(\{u\}), R(\{u\}))$ and $\mathbf{M}_v \in \mathrm{Mat}(R(\{v\}), R(\{v\}))$ with

$$\mathbf{M}_u = \begin{array}{l} \\ (\varnothing,\{u\} \\ (\varnothing,\varnothing) \\ (\{u\},\{u\}) \end{array} \begin{array}{c} \begin{array}{ccc} (\varnothing,\{u\}) & (\varnothing,\varnothing) & (\{u\},\{u\}) \end{array} \\ \begin{pmatrix} 1 & 1 & x \\ 1 & x & 2x+x^2 \\ x & 2x+x^2 & x+3x^2+x^3 \end{pmatrix} \end{array}$$

$$\mathbf{M}_v = \begin{array}{l} \\ (\varnothing,\{v\}) \\ (\varnothing,\varnothing) \\ (\{v\},\{v\}) \end{array} \begin{array}{c} \begin{array}{ccc} (\varnothing,\{v\}) & (\varnothing,\varnothing) & (\{v\},\{v\}) \end{array} \\ \begin{pmatrix} 1 & 1 & x \\ 1 & x & 2x+x^2 \\ x & 2x+x^2 & x+3x^2+x^3 \end{pmatrix} \end{array}$$

and the inverse matrices

$$\mathbf{M}_u^{-1} = \begin{array}{l} \\ (\varnothing,\{u\}) \\ (\varnothing,\varnothing) \\ (\{u\},\{u\}) \end{array} \begin{array}{c} \begin{array}{ccc} (\varnothing,\{u\}) & (\varnothing,\varnothing) & (\{u\},\{u\}) \end{array} \\ \begin{pmatrix} x^3+3x^2 & x^2+x & -2x \\ x^2+x & -x^3-2x^2-x & x^2+x \\ -2x & x^2+x & 1-x \end{pmatrix} \end{array} \cdot \frac{1}{x(1+x)^2}$$

$$\mathbf{M}_v^{-1} = \begin{array}{l} \\ (\varnothing,\{v\}) \\ (\varnothing,\varnothing) \\ (\{v\},\{v\}) \end{array} \begin{array}{c} \begin{array}{ccc} (\varnothing,\{v\}) & (\varnothing,\varnothing) & (\{v\},\{v\}) \end{array} \\ \begin{pmatrix} x^3+3x^2 & x^2+x & -2x \\ x^2+x & -x^3-2x^2-x & x^2+x \\ -2x & x^2+x & 1-x \end{pmatrix} \end{array} \cdot \frac{1}{x(1+x)^2}.$$

By the definition of the Kronecker product, the matrix $\mathbf{M}_X = \mathbf{M}_u \otimes \mathbf{M}_v$ evaluates to the matrix depicted in appendix A. Observe that the rows and columns of the matrix $\mathbf{M}_X$ are labelled by $R(\{u\}) \times R(\{v\})$ which will be somewhat inconvenient. Hence we assume that for the labelling of the rows and columns the function $f\colon R(\{u\}) \times R(\{v\}) \to R(\{u, v\})$ is applied. This gives the second matrix in Appendix A. The inverse matrix $\mathbf{M}_X^{-1}$ can then be computed in the same vein by $\mathbf{M}_X^{-1} = \mathbf{M}_u^{-1} \otimes \mathbf{M}_v^{-1}$ and is shown in Appendix B.

The vectors $\mathbf{d}_X(G^1)$ and $\mathbf{d}_X(G^2)$ are given by

$$\mathbf{d}_X(G^i) = \begin{array}{l} {\scriptstyle (\varnothing,\{u,v\})} \\ {\scriptstyle (\varnothing,\{v\})} \\ {\scriptstyle (\{u\},\{u,v\})} \\ {\scriptstyle (\varnothing,\{u\})} \\ {\scriptstyle (\varnothing,\varnothing)} \\ {\scriptstyle (\{u\},\{u\})} \\ {\scriptstyle (\{v\},\{u,v\})} \\ {\scriptstyle (\{v\},\{v\})} \\ {\scriptstyle (\{u,v\},\{u,v\})} \end{array} \begin{pmatrix} D(G^i - u - v, x) \\ D(G^i - v, x) \\ D(G^i + \{u,\cdot\} - v, x) \\ D(G^i - u, x) \\ D(G^i, x) \\ D(G^i + \{u,\cdot\}, x) \\ D(G^i + \{v,\cdot\} - u, x) \\ D(G^i + \{v,\cdot\}, x) \\ D(G^i + \{u,\cdot\} + \{v,\cdot\}, x) \end{pmatrix}$$

and the splitting theorem yields

$$D(G,x) = \mathbf{d}_X(G^1)^T \mathbf{M}_X^{-1} \mathbf{d}_X(G^2).$$

Theorem 5.15. *Let $G = (V,E)$ be a graph. For every $e = \{u,v\} \in E$,*

$$\begin{aligned} D(G,x) \;\; = \;\; & \frac{1}{(1+x)^2}\Big[x(1+x)\left(D(G-v,x) + D(G-u,x)\right) \\ & +(1-x)D(G+\{u,\cdot\}-v,x) - (1+x)D(G-e+\{u,\cdot\},x) \\ & +(1-x)D(G-u+\{v,\cdot\},x) - (1+x)D(G-e+\{v,\cdot\},x) \\ & +(1+x)^2 D(G-e,x) + 2D(G-e+\{u,\cdot\}+\{v,\cdot\},x) \\ & -2xD(G-u-v,x)\Big]. \end{aligned}$$

Proof. Assume now that $e = \{u,v\} \in E$ is an edge of $G = (V,E)$ and that we have the splitting $G^1 = G - e$ and $G^2 = (\{u,v\}, \{\{u,v\}\})$ at the separating vertex set $X = \{u,v\}$. The vector $\mathbf{d}_X(G^2)$ can be computed to be

$$\mathbf{d}_X(G^2) = \begin{array}{l} {\scriptstyle (\varnothing,\{u,v\})} \\ {\scriptstyle (\varnothing,\{v\})} \\ {\scriptstyle (\{u\},\{u,v\})} \\ {\scriptstyle (\varnothing,\{u\})} \\ {\scriptstyle (\varnothing,\varnothing)} \\ {\scriptstyle (\{u\},\{u\})} \\ {\scriptstyle (\{v\},\{u,v\})} \\ {\scriptstyle (\{v\},\{v\})} \\ {\scriptstyle (\{u,v\},\{u,v\})} \end{array} \begin{pmatrix} D(P_0,x) \\ D(P_1,x) \\ D(P_2,x) \\ D(P_1,x) \\ D(P_2,x) \\ D(P_3,x) \\ D(P_2,x) \\ D(P_3,x) \\ D(P_4,x) \end{pmatrix} = \begin{array}{l} {\scriptstyle (\varnothing,\{u,v\})} \\ {\scriptstyle (\varnothing,\{v\})} \\ {\scriptstyle (\{u\},\{u,v\})} \\ {\scriptstyle (\varnothing,\{u\})} \\ {\scriptstyle (\varnothing,\varnothing)} \\ {\scriptstyle (\{u\},\{u\})} \\ {\scriptstyle (\{v\},\{u,v\})} \\ {\scriptstyle (\{v\},\{v\})} \\ {\scriptstyle (\{u,v\},\{u,v\})} \end{array} \begin{pmatrix} 1 \\ x \\ x^2+2x \\ x \\ x^2+2x \\ x^3+3x^2+x \\ x^2+2x \\ x^3+3x^2+x \\ x^4+4x^3+4x^2 \end{pmatrix}$$

and a computer algebra system can be used to obtain:

$$\mathbf{M}_X^{-1}\mathbf{d}_X(G^2) = \begin{array}{l} \scriptstyle(\varnothing,\{u,v\}) \\ \scriptstyle(\varnothing,\{v\}) \\ \scriptstyle(\{u\},\{u,v\}) \\ \scriptstyle(\varnothing,\{u\}) \\ \scriptstyle(\varnothing,\varnothing) \\ \scriptstyle(\{u\},\{u\}) \\ \scriptstyle(\{v\},\{u,v\}) \\ \scriptstyle(\{v\},\{v\}) \\ \scriptstyle(\{u,v\},\{u,v\}) \end{array} \begin{pmatrix} (1+x)^2 \\ x(1+x) \\ -(1+x) \\ x(1+x) \\ -2x \\ 1-x \\ -(1+x) \\ 1-x \\ 2 \end{pmatrix} \cdot \frac{1}{(1+x)^2}\,.$$

Hence we compute by the splitting theorem

$$\begin{aligned} D(G,x) \quad = \quad & \frac{1}{(1+x)^2}\Big[x(1+x)\left(D(G^1-v,x)+D(G^1-u,x)\right)+ \\ & (1-x)D(G^1+\{u,\cdot\}-v,x)-(1+x)D(G^1+\{u,\cdot\},x)+ \\ & (1-x)D(G^1-u+\{v,\cdot\},x)-(1+x)D(G^1+\{v,\cdot\},x)+ \\ & (1+x)^2D(G^1,x)+2D(G^1+\{u,\cdot\}+\{v,\cdot\},x) \\ & -2xD(G^1-u-v,x)\Big]. \end{aligned}$$

Substituting $G^1 = G - e$ gives the required answer. □

It is interesting that this formula, like Theorem 2.3, is a linear combination of exactly nine terms. Unfortunately, the formula is not as useful as Theorem 2.3, since we have to compute domination polynomials of graphs that are emerging from G by adding additional edges.

6 A recurrence relation using derivatives

Lemma 6.1. *Let $G = (V, E)$ be a graph and $u \in V$. Then $D_{u\in W}(G, x) = D(G, x) - D(G - u, x) + p_u(G, x)$.*

Proof. From Theorem 2.1 we have

$$D_{u\notin W}(G,x) = D(G-u,x) - p_u(G,x)\,. \tag{24}$$

Adding $D_{u\in W}(G, x)$ to both sides of Equation (24) and rearranging the terms, we get the lemma. □

We denote by $A(G,x)$ the following sum:

$$A(G,x) = \sum_{v\in V} D(G-v,x) - D(G/v,x) - D(G-N[v],x)\,.$$

$D^{(i)}(G,x)$ denotes the ith derivative of the domination polynomial $D(G,x)$ with respect to its indeterminate x. Similarly, $A^{(i)}(G,x)$ denotes the ith derivative of $A(G,x)$.

Theorem 6.2. *Let $G=(V,E)$ be a graph. For every $i \geqslant 0$,*

$$D^{(i)}(G,x) = \frac{1+x}{|V|-i} D^{(i+1)}(G,x) + \frac{1}{|V|-i} A^{(i)}(G,x)\,.$$

Proof. The proof is by induction. First we prove the case for $i=0$.

$$\begin{aligned} D(G,x) &= \sum_{W\subseteq V:N[W]=V} x^{|W|} \\ &= \sum_{W\subseteq V:N[W]=V} \sum_{w\in W} \frac{1}{|W|} x^{|W|} \\ &= \sum_{w\in V} \sum_{\{w\}\subseteq W\subseteq V:N[W]=V} \frac{1}{|W|} x^{|W|}\,. \end{aligned}$$

Taking the derivative of $D(G,x)$ with respect to x and using Lemma 6.1 we get

$$\begin{aligned} D^{(1)}(G,x) &= \sum_{w\in V} \sum_{\{w\}\subseteq W\subseteq V:N[W]=V} x^{|W|-1} \\ &= \sum_{w\in V} \frac{1}{x} D_{w\in W}(G,x) \\ &= \frac{1}{x} \sum_{w\in V} D(G,x) - D(G-w,x) + p_w(G,x)\,. \end{aligned}$$

Now, using Theorem 2.1,

$$\begin{aligned} D^{(1)}(G,x) &= \frac{1}{1+x} \sum_{w\in V} \big(D(G,x) - D(G-w,x) + D(G/w,x) \\ & \quad + D(G-N[w],x)\big) = \frac{|V|}{1+x} D(G,x) - \frac{1}{1+x} A(G,x)\,, \end{aligned}$$

and the case $i=0$ follows.

Assuming that we have $D^{(i)}(G,x)$ as stated in the theorem we can multiply by $|V|-i$ and take the derivative of both sides to get

$$(|V|-i)D^{(i+1)}(G,x) = (1+x)D^{(i+2)}(G,x) + D^{(i+1)}(G,x) + A^{(i+1)}(G,x)\,.$$

We can then re-arrange this equation to get

$$(|V|-(i+1))D^{(i+1)}(G,x) = (1+x)D^{(i+2)}(G,x) + A^{(i+1)}(G,x)$$

which establishes the induction on division by $|V|-(i+1)$. □

We can obtain a representation of $D(G,x)$ in terms of $A(G,x)$.

Theorem 6.3. *Let $G=(V,E)$ be a graph. Then*

$$D(G,x) = (1+x)^{|V|} + \sum_{i=0}^{|V|-1} (1+x)^i \frac{(|V|-(i+1))!}{|V|!} A^{(i)}(G,x)\,.$$

Proof. Rearranging Theorem 6.2 to make $A^{(i)}(G,x)$ the subject we have that

$$A^{(i)}(G,x) = (|V|-i)D^{(i)}(G,x) - (1+x)D^{(i+1)}(G,x)\,.$$

All but two terms of this sum cancel by combining adjacent terms:

$$\begin{aligned}
\sum_{i=0}^{|V|-1} (1+x)^i \frac{(|V|-(i+1))!}{|V|!} \quad A^{(i)}(G,x) \quad &= \frac{|V|D(G,x) - (1+x)D^{(1)}(G,x)}{|V|} \\
&+ \quad (1+x)\frac{(|V|-1)D^{(1)}(G,x) - (1+x)D^{(2)}(G,x)}{|V|(|V|-1)} \\
&+ \quad \ldots \\
&+ \quad (1+x)^{|V|-1}\frac{D^{(|V|-1)}(G,x) - (1+x)D^{|V|}(G,x)}{|V|!} \\
&= \quad D(G,x) - (1+x)^{|V|}\,.
\end{aligned}$$

We use that $D(G,x)$ is a monic polynomial of degree $|V|$ giving $D^{|V|}(G,x) = |V|!$ □

7 Conclusion

The domination polynomial resembles such graph polynomials as the independence polynomial $I(G,x)$ and the vertex cover polynomial $VC(G,x)$ insofar as they are all defined as generating functions of certain subsets of vertices. However, we showed that the domination polynomial has quite a different behavior with respect to recurrence relations. Theorems 2.4 and 2.5 show that $D(G,x)$ cannot satisfy a linear recurrence relation analogous to $I(G,x)$, $VC(G,x)$ and other prominent graph polynomials.

We gave many recurrence relations and splitting formulas for the domination polynomial. Theorem 2.1 gives a reduction formula for $D(G,x)$ based on the related $p_u(G,x)$ polynomial. This theorem gives rise to recurrence steps in various special cases, as well as to a linear recurrence relation for arbitrary graphs which uses compositions of standard edge and vertex elimination operations, Theorem 2.3. We gave splitting formulas for $D(G,x)$ in the case that it is 1-connected. We then generalized this in Theorem 5.14 to a splitting formula which allows to compute the domination polynomial of a graph G, by separating the graph into two parts, so that we have to compute domination polynomials of the modifications of the resulting subgraphs G^1 and G^2. Finally, we gave a rather simple recurrence relation for $D(G,x)$ using derivatives of domination polynomials of smaller graphs in Theorem 6.2.

The domination polynomial is therefore established to have surprisingly diverse and unique decomposition formulas and warrants further research. The paper leaves some open problems, among them:

Open Problem 1. Are there simple graph operations which can be used to give a simpler recurrence relation for the domination polynomial of arbitrary graphs?

In Theorem 5.14 it is necessary to attach additional edges to the subgraphs G^1 and G^2 in order to state this formula.

Open Problem 2. Is it possible to give a splitting formula in the vein of Theorem 5.14 which avoids adding edges?

Another interesting observation is that some edges might be shifted to different positions in a graph, without changing the resulting domination polynomial.

Open Problem 3. Is there a criterion which characterizes the edges which can be shifted without changing $D(G, x)$?

References

[1] S. Akbari, S. Alikhani, and Y.H. Peng. Characterization of graphs using domination polynomials. *European Journal of Combinatorics*, 31(7):1714–1724, 2010.

[2] S. Akbari, S. Alikhani, M.R. Oboudi and Y.H. Peng. On the zeros of domination polynomial of a graph. *Contemporary Mathematics*, American Mathematical Society, 531:109-115, 2010.

[3] S. Alikhani. On the graphs with four distinct domination roots. *International Journal of Computer Mathematics*, 88(13):2717–2720, 2011.

[4] S. Alikhani. Dominating sets and domination polynomials of graphs: Domination polynomial: A new graph polynomial *LAMBERT Academic Publishing*, ISBN: 978-3847344827, 2012.

[5] S. Alikhani. Graphs which their certain polynomials have few distinct roots – a survey. Available as arXiv:1112.0838v1 [math.CO].

[6] S. Alikhani, Y. H. Peng. Introduction to domination polynomial of a graph. *Ars Combin.*, in press. Available as arXiv:0905.2251 [math.CO].

[7] S. Alikhani, and Y. H. Peng. Domination polynomials of cubic graphs of order 10. *Turk J Math* 35:355–366, 2011.

[8] S. Alikhani and Y.H. Peng. Dominating Sets and Domination Polynomials of Paths. *International Journal of Mathematics and Mathematical Sciences*, 10, 2009.

[9] Alikhani, S., Peng, Y.: Domination Sets and Domination Polynomials of Certain Graphs, II, *Opuscula Mathematica*, 30(1):37–51, 2010.

[10] Arocha, J.L. and Liano, B.. Mean value for the matching and dominating polynomial. *Discussiones Mathematicae Graph Theory*, 20(1):57–70, 2000.

[11] I. Averbouch, B. Godlin and J.A. Makowsky. An extension of the bivariate chromatic polynomial. *European Journal of Combinatorics*, 31(1):1–17, 2010.

[12] K. Dohmen and P. Tittmann. Domination Reliability. *The Electronic Journal of Combinatorics*, 19, 2012. #P15.

[13] R. Frucht and F. Harary. On the Corona of two Graphs. *Aequationes Mathematicae*, 4:322–325, 1970.

[14] C.D. Godsil and I. Gutman. On the theory of the matching polynomial. *Journal of Graph Theory*, 5(2):137–144, 1981.

[15] V.E. Levit and E. Madrescu. The independence polynomial of a graph - a survey. In *Proceedings of the 1st International Conference on Algebraic Informatics*, 233–254, 2005.

[16] M. Marcus and H. Minc. *A survey of matrix theory and matrix inequalities*, volume 14. Dover Publications, 1992.

[17] J.A. Makowsky. From a zoo to a zoology: Towards a general theory of graph polynomials. *Theory of Computing Systems*, 43(3-4):542–562, 2008.

Appendix A: $\mathbf{M}_X$ with $X = \{u, v\}$

$((\varnothing,\{u\}),(\varnothing,\{v\}))$	$D(P_0)D(P_0)$	$D(P_1)D(P_0)$	$D(P_2)D(P_0)$	$D(P_0)D(P_1)$	$D(P_1)D(P_1)$	$D(P_2)D(P_1)$	$D(P_0)D(P_2)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$
$((\varnothing,\varnothing),(\varnothing,\{v\}))$	$D(P_1)D(P_0)$	$D(P_2)D(P_0)$	$D(P_1)D(P_0)$	$D(P_1)D(P_1)$	$D(P_2)D(P_1)$	$D(P_3)D(P_1)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$
$((\{u\},\{u\}),(\varnothing,\{v\}))$	$D(P_2)D(P_0)$	$D(P_1)D(P_0)$	$D(P_4)D(P_0)$	$D(P_2)D(P_1)$	$D(P_3)D(P_1)$	$D(P_4)D(P_1)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_4)D(P_2)$
$((\varnothing,\{u\}),(\varnothing,\varnothing))$	$D(P_0)D(P_1)$	$D(P_1)D(P_1)$	$D(P_2)D(P_1)$	$D(P_0)D(P_2)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_0)D(P_3)$	$D(P_1)D(P_3)$	$D(P_2)D(P_3)$
$((\varnothing,\varnothing),(\varnothing,\varnothing))$	$D(P_1)D(P_1)$	$D(P_2)D(P_1)$	$D(P_3)D(P_1)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_1)D(P_3)$	$D(P_2)D(P_3)$	$D(P_3)D(P_3)$
$((\{u\},\{u\}),(\varnothing,\varnothing))$	$D(P_2)D(P_1)$	$D(P_3)D(P_1)$	$D(P_4)D(P_1)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_4)D(P_2)$	$D(P_2)D(P_3)$	$D(P_3)D(P_3)$	$D(P_4)D(P_3)$
$((\varnothing,\{u\}),(\{v\},\{v\}))$	$D(P_0)D(P_2)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_0)D(P_3)$	$D(P_1)D(P_3)$	$D(P_2)D(P_3)$	$D(P_0)D(P_4)$	$D(P_1)D(P_4)$	$D(P_2)D(P_4)$
$((\varnothing,\varnothing),(\{v\},\{v\}))$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_1)D(P_3)$	$D(P_2)D(P_3)$	$D(P_3)D(P_3)$	$D(P_1)D(P_4)$	$D(P_2)D(P_4)$	$D(P_3)D(P_4)$
$((\{u\},\{u\}),(\{v\},\{v\}))$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_4)D(P_2)$	$D(P_2)D(P_3)$	$D(P_3)D(P_3)$	$D(P_4)D(P_3)$	$D(P_2)D(P_4)$	$D(P_3)D(P_4)$	$D(P_4)D(P_4)$

	$(\varnothing,\{u,v\})$	$(\varnothing,\{v\})$	$(\{u\},\{u,v\})$	$(\varnothing,\{u\})$	$(\varnothing,\varnothing)$	$(\{u\},\{u\})$	$(\{v\},\{u,v\})$	$(\{v\},\{v\})$	$(\{u,v\},\{u,v\})$
$(\varnothing,\{u,v\})$	$D(P_0)D(P_0)$	$D(P_1)D(P_0)$	$D(P_2)D(P_0)$	$D(P_0)D(P_1)$	$D(P_1)D(P_1)$	$D(P_2)D(P_1)$	$D(P_0)D(P_2)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$
$(\varnothing,\{v\})$	$D(P_1)D(P_0)$	$D(P_2)D(P_0)$	$D(P_3)D(P_0)$	$D(P_1)D(P_1)$	$D(P_2)D(P_1)$	$D(P_3)D(P_1)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$
$(\{u\},\{u,v\})$	$D(P_2)D(P_0)$	$D(P_3)D(P_0)$	$D(P_4)D(P_0)$	$D(P_2)D(P_1)$	$D(P_3)D(P_1)$	$D(P_4)D(P_1)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_4)D(P_2)$
$(\varnothing,\{u\})$	$D(P_0)D(P_1)$	$D(P_1)D(P_1)$	$D(P_2)D(P_1)$	$D(P_0)D(P_2)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_0)D(P_3)$	$D(P_1)D(P_3)$	$D(P_2)D(P_3)$
$(\varnothing,\varnothing)$	$D(P_1)D(P_1)$	$D(P_2)D(P_1)$	$D(P_3)D(P_1)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_1)D(P_3)$	$D(P_2)D(P_3)$	$D(P_3)D(P_3)$
$(\{u\},\{u\})$	$D(P_2)D(P_1)$	$D(P_3)D(P_1)$	$D(P_4)D(P_1)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_4)D(P_2)$	$D(P_2)D(P_3)$	$D(P_3)D(P_3)$	$D(P_4)D(P_3)$
$(\{v\},\{u,v\})$	$D(P_0)D(P_2)$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_0)D(P_3)$	$D(P_1)D(P_3)$	$D(P_2)D(P_3)$	$D(P_0)D(P_4)$	$D(P_1)D(P_4)$	$D(P_2)D(P_4)$
$(\{v\},\{v\})$	$D(P_1)D(P_2)$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_1)D(P_3)$	$D(P_2)D(P_3)$	$D(P_3)D(P_3)$	$D(P_1)D(P_4)$	$D(P_2)D(P_4)$	$D(P_3)D(P_4)$
$(\{u,v\},\{u,v\})$	$D(P_2)D(P_2)$	$D(P_3)D(P_2)$	$D(P_4)D(P_2)$	$D(P_2)D(P_3)$	$D(P_3)D(P_3)$	$D(P_4)D(P_3)$	$D(P_2)D(P_4)$	$D(P_3)D(P_4)$	$D(P_4)D(P_4)$

Appendix B: $x^2(1+x)^2\mathbf{M}_X^{-1}$ with $X=\{u,v\}$

$$\begin{pmatrix}
(x{+}3)^2x^4 & (x{+}1)(x{+}3)x^3 & -2\,(x{+}3)x^3 & (x{+}1)(x{+}3)x^3 & (x{+}1)^2x^2 & -2\,(x{+}1)x^2 & -2\,(x{+}3)x^3 & -2\,(x{+}1)x^2 & 4\,x^2 \\
(x{+}1)(x{+}3)x^3 & -(x{+}1)^2(x{+}3)x^3 & (x{+}1)(x{+}3)x^3 & (x{+}1)^2x^2 & -(x{+}1)^3x^2 & (x{+}1)^2x^2 & -2\,(x{+}1)x^2 & 2\,(x{+}1)^2x^2 & -2\,(x{+}1)x^2 \\
-2\,(x{+}3)x^3 & (x{+}1)(x{+}3)x^3 & -(x{-}1)(x{+}3)x^2 & -2\,(x{+}1)x^2 & (x{+}1)^2x^2 & -(x{-}1)(x{+}1)x & 4\,x^2 & -2\,(x{+}1)x^2 & 2\,(x{-}1)x \\
(x{+}1)(x{+}3)x^3 & (x{+}1)^2x^2 & -2\,(x{+}1)x^2 & -(x{+}1)^2(x{+}3)x^3 & -(x{+}1)^3x^2 & 2\,(x{+}1)^2x^2 & (x{+}1)(x{+}3)x^3 & (x{+}1)^2x^2 & -2\,(x{+}1)x^2 \\
(x{+}1)^2x^2 & -(x{+}1)^3x^2 & (x{+}1)^2x^2 & -(x{+}1)^3x^2 & (x{+}1)^4x^2 & -(x{+}1)^3x^2 & (x{+}1)^2x^2 & -(x{+}1)^3x^2 & (x{+}1)^2x^2 \\
-2\,(x{+}1)x^2 & (x{+}1)^2x^2 & -(x{-}1)(x{+}1)x & 2\,(x{+}1)^2x^2 & -(x{+}1)^3x^2 & (x{-}1)(x{+}1)^2x & -2\,(x{+}1)x^2 & (x{+}1)^2x^2 & -(x{-}1)(x{+}1)x \\
-2\,(x{+}3)x^3 & -2\,(x{+}1)x^2 & 4\,x^2 & (x{+}1)(x{+}3)x^3 & (x{+}1)^2x^2 & -2\,(x{+}1)x^2 & -(x{-}1)(x{+}3)x^2 & -(x{-}1)(x{+}1)x & 2\,(x{-}1)x \\
-2\,(x{+}1)x^2 & 2\,(x{+}1)^2x^2 & -2\,(x{+}1)x^2 & (x{+}1)^2x^2 & -(x{+}1)^3x^2 & (x{+}1)^2x^2 & -(x{-}1)(x{+}1)x & (x{-}1)(x{+}1)^2x & -(x{-}1)(x{+}1)x \\
4\,x^2 & -2\,(x{+}1)x^2 & 2\,(x{-}1)x & -2\,(x{+}1)x^2 & (x{+}1)^2x^2 & -(x{-}1)(x{+}1)x & 2\,(x{-}1)x & -(x{-}1)(x{+}1)x & (x{-}1)^2
\end{pmatrix}$$