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RECURRENCE RELATIONS FOR HIGHER MOMENTS OF ORDER STATISTICS FROM DOUBLY TRUNCATED BURR DISTRIBUTION

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1. INTRODUCTION

The Burr distribution was first introduced in the literature by Burr (1942). Burr and Cislak (1968) and Burr (1968) have shown that if one chooses the parameters appropriately, the Burr distribution covers a large proportion of curve shape characteristic of type I, IV and VI in the Pearson family of distributions. Thus the use of Burr distribution as a failure model is appropriate and useful in applied statistics, specially in survival analysis and actuarial studies. Burr distribution has been widely studied by Tadikamalla (1980) among others. He has established its relationship with some other distributions. The Burr distribution is very important in modeling of finance and insurance data. Experience has shown that, there is sometimes a need to find heavy tailed distributions which offer greater flexibility than the Pareto law. It may be mentioned that the Burr distribution has been used extensively to model franchise deductible premium, fixed amount deductible premium, proportional deductible premium, limited proportional deductible premium and disappearing deductible premium (see Burnecki et al. (2004)).

The probability density function (pdf) of Burr type XII distribution is given by (see Figure 1)

$$f(x) = mp\theta x^{p-1}\left(1+\theta x^{p}\right)^{-(m+1)},\ 0 \le x < \infty, m > 0,\ p > 0,\ \theta > 0, \tag{1.1}$$

and the cumulative distribution function (cdf) is given by

$$F(x) = 1-\left(1+\theta x^{p}\right)^{-m},\ 0 \le x < \infty, m, p, \theta > 0. \tag{1.2}$$

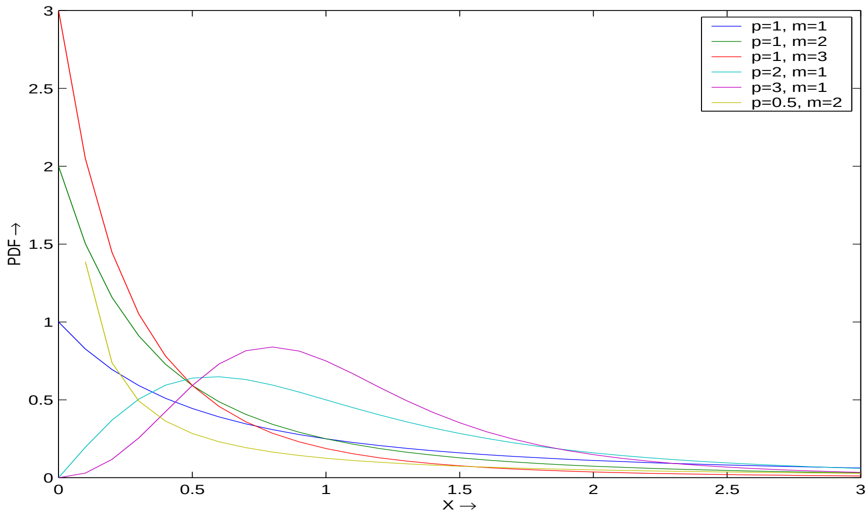

Figure 1 – Burr type XII Distribution (Non-Truncated case).

Let the random variable X have a doubly truncated Burr type XII distribution with probability density function (pdf)

$$f(x) = \frac{mp\theta x^{p-1}\left(1+\theta x^{p}\right)^{-(m+1)}}{P-Q}, \quad Q_1 \le x \le P_1,\ m>0,\ p>0,\ \theta>0, \tag{1.3}$$

and cumulative distribution function (cdf)

$$F(x) = \begin{cases} 0 & \text{for } x < Q_1 \\ \dfrac{1-Q-\left(1+\theta x^{p}\right)^{-m}}{P-Q} & \text{for } Q_1 \le x \le P_1,\ \ m, p, \theta > 0 \\ 1 & \text{for } x > P_1 \end{cases} \tag{1.4}$$

where Q and $(1-P)$, $(Q<P)$ are the proportions of truncation on the left and the right of the distribution, respectively, and

$$\theta Q_1^{p} = \left[(1-Q)^{-1/m} - 1\right], \tag{1.5}$$

$$\theta P_1^{p} = \left[\left(1-P\right)^{-1/m} - 1 \right]. \tag{1.6}$$

It may be noted that by letting $Q \to 0$ and $P \to 1$ (or, equivalently, $Q_1 \to 0$ and $P_1 \to \infty$), the distribution in (1.3) reduces to the non-truncated Burr distribution given in (1.1). The following figure (Figure 2) shows the graphs of doubly truncated Burr type XII distribution with $Q = 0.25$ and $P = 0.75$ for different values of p and m.

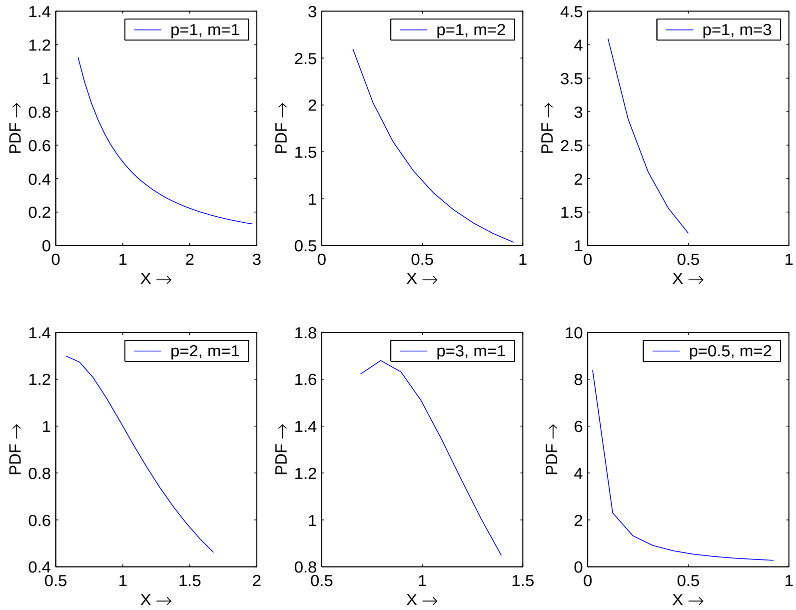

Figure 2 – Doubly Truncated Burr type XII Distribution with $Q = 0.25$ and $P = 0.75$.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the doubly truncated Burr distribution given in (1.3), and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics, and $X_{r_1:n} \le X_{r_2:n} \le \ldots \le X_{r_k:n}$ be the corresponding sub-sample order statistics of size $k \le n$. Let us denote the single moments $E\left(X_{r:n}^{k}\right)$ by $\mu_{r:n}^{(k)}$ $(1 \le r \le n)$ and the product moments of k order statistics, viz. $E\left(X_{r_1:n}^{i_1} X_{r_2:n}^{i_2} \ldots X_{r_k:n}^{i_k}\right)$ by

$$\mu_{r_1, r_2, \ldots, r_k:n}^{(i_1, i_2, \ldots, i_k)} \ (1 \le r_1 < r_2 < \ldots < r_k \le n \text{ and } i_1, i_2, \ldots, i_k = 0, 1, 2, \ldots).$$

From (1.3) and (1.4) we observe that the characterizing differential equation for the doubly truncated Burr distribution is

$$\left(1+\theta x^{p}\right)f(x)=mp\theta x^{p-1}\left(P_2+1-F(x)\right) \tag{1.7}$$

or, equivalently,

$$\left(1+\theta x^{p}\right)f(x)=mp\theta x^{p-1}\left(Q_2-F(x)\right), \tag{1.8}$$

where $Q_2=(1-Q)/(P-Q)$ and $P_2=(1-P)/(P-Q)$.

We shall use equations (1.7) and (1.8) to establish several recurrence relations satisfied by higher moments of order statistics from doubly truncated Burr distribution defined in (1.3), thus generalizing the earlier work due to Khan and Khan (1987) and Pushkarna, Saran and Tiwari (2012).

It is worth mentioning here that the recurrence relations for the single, double (product) and higher moments of order statistics from various doubly truncated distributions, viz. Lomax, Weibull, Weibull-gamma, Weibull-exponential, log logistic, exponential, Rayleigh, generalized Rayleigh and generalized Pareto distributions can easily be deduced from the results of this paper as special cases.

For similar type of work, one can also see Afify (2008), Saran and Pandey (2004, 2008) and Saran and Pushkarna (2000).

2. Recurrence relations

The joint density function of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n} (1\le r_1<r_2<\ldots<r_k\le n)$ is given by

$$\begin{aligned} &f_{r_1,r_2,\ldots,r_k:n}(x_1,\ldots,x_k)=C_{r_1,r_2,\ldots,r_k:n}\left[F(x_1)\right]^{r_1-1}\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1} \\ &\ldots\left[F(x_k)-F(x_{k-1})\right]^{r_k-r_{k-1}-1}\left[1-F(x_k)\right]^{n-r_k} f(x_1)f(x_2)\ldots f(x_k), \\ &Q_1\le x_1<\ldots<x_k\le P_1 \end{aligned} \tag{2.1}$$

where

$$C_{r_1,r_2,\ldots,r_k:n}=\frac{n!}{(r_1-1)!(r_2-r_1-1)!\ldots(r_k-r_{k-1}-1)!(n-r_k)!}$$

(cf. David and Nagaraja (2003), p.12), and $f(x)$ and $F(x)$ are as given in equations (1.3) and (1.4), respectively. Then by making use of the characterizing differential equations in (1.7) and (1.8), we establish recurrence relations for the product moments of k order statistics.

THEOREM 2.1. For $1 \le r_1 < r_2 < \ldots < r_k \le n$, $r_1 = 1$, $r_2 = 2$ *and* $i_1, i_2, \ldots, i_k \ge 0$,

$$\mu_{1,2,r_3,\ldots,r_k:n}^{(i_1+p,i_2,\ldots,i_k)} = \frac{i_1+p}{mp\theta}\Big\{\mu_{1,2,r_3,\ldots,r_k:n}^{(i_1,i_2,\ldots,i_k)} + \theta\mu_{1,2,r_3,\ldots,r_k:n}^{(i_1+p,i_2,\ldots,i_k)}\Big\}$$
$$-nQ_2\Big\{\mu_{1,r_3-1,\ldots,r_k-1:n-1}^{(i_1+i_2+p,i_3,\ldots,i_k)} - Q_1^{i_1+p}\mu_{1,r_3-1,\ldots,r_k-1:n-1}^{(i_2,i_3,\ldots,i_k)}\Big\}$$
$$+\mu_{2,r_3,\ldots,r_k:n}^{(i_1+i_2+p,i_3,\ldots,i_k)} \tag{2.2}$$

and, for $1 \le r_1 < r_2 < \ldots < r_k \le n$, $r_1 = 1$, $r_2 \ge 3$ *and* $i_1, i_2, \ldots, i_k \ge 0$,

$$\mu_{1,r_2,\ldots,r_k:n}^{(i_1+p,i_2,i_3,\ldots,i_k)} = \frac{i_1+p}{mp\theta}\Big\{\mu_{1,r_2,r_3,\ldots,r_k:n}^{(i_1,i_2,i_3,\ldots,i_k)} + \theta\mu_{1,r_2,r_3,\ldots,r_k:n}^{(i_1+p,i_2,i_3,\ldots,i_k)}\Big\}$$
$$-nQ_2\Big\{\mu_{1,r_2-1,r_3-1,\ldots,r_k-1:n-1}^{(i_1+p,i_2,i_3,\ldots,i_k)} - Q_1^{i_1+p}\mu_{r_2-1,r_3-1,\ldots,r_k-1:n-1}^{(i_2,i_3,\ldots,i_k)}\Big\}$$
$$+\mu_{2,r_2,r_3,\ldots,r_k:n}^{(i_1+p,i_2,i_3,\ldots,i_k)}. \tag{2.3}$$

PROOF. Relations in (2.2) and (2.3) may be proved by following exactly the same steps as those in proving Theorem 2.2, which is presented here.

THEOREM 2.2. For $1 \le r_1 < r_2 < \ldots < r_k \le n$, $i_1, i_2, \ldots, i_k \ge 0$ *and* $r_2 = r_1 + 1$,

$$\mu_{r_1,r_1+1,r_3,\ldots,r_k:n}^{(i_1+p,i_2,\ldots,i_k)} = \frac{1}{r_1}\Bigg[\frac{i_1+p}{mp\theta}\Big\{\mu_{r_1,r_1+1,r_3,\ldots,r_k:n}^{(i_1,i_2,\ldots,i_k)} + \theta\mu_{r_1,r_1+1,r_3,\ldots,r_k:n}^{(i_1+p,i_2,\ldots,i_k)}\Big\}$$
$$-nQ_2\Big\{\mu_{r_1,r_3-1,\ldots,r_k-1:n-1}^{(i_1+i_2+p,i_3,\ldots,i_k)} - \mu_{r_1-1,r_1,r_3-1,\ldots,r_k-1:n-1}^{(i_1+p,i_2,\ldots,i_k)}\Big\}\Bigg]$$
$$+\mu_{r_1+1,r_3,\ldots,r_k:n}^{(i_1+i_2+p,i_3,\ldots,i_k)} \tag{2.4}$$

and, for $r_2 - r_1 \ge 2$,

$$\mu_{r_1+1,r_2,\ldots,r_k:n}^{(i_1+p,i_2,\ldots,i_k)} = \frac{1}{r_1}\left[\frac{i_1+p}{mp\theta}\left\{\mu_{r_1,r_2,\ldots,r_k:n}^{(i_1,i_2,\ldots,i_k)} + \theta\mu_{r_1,r_2,\ldots,r_k:n}^{(i_1+p,i_2,\ldots,i_k)}\right\}\right.$$
$$\left.-nQ_2\left\{\mu_{r_1,r_2-1,\ldots,r_k-1:n-1}^{(i_1+p,i_2\ldots,i_k)} - \mu_{r_1-1,r_2-1,\ldots,r_k-1:n-1}^{(i_1+p,i_2,\ldots,i_k)}\right\}\right] \quad (2.5)$$
$$+\mu_{r_1,r_2,\ldots,r_k:n}^{(i_1+p,i_2\ldots,i_k)}.$$

PROOF. From equation (2.1), we have for $1 \le r_1 < r_2 < \ldots < r_k \le n$,

$$\mu_{r_1,r_2,\ldots,r_k:n}^{(i_1,i_2,\ldots,i_k)} + \theta\mu_{r_1,r_2,\ldots,r_k:n}^{(i_1+p,i_2,\ldots,i_k)} = C_{r_1,r_2,\ldots,r_k:n}\int_{Q_1}^{P_1}\int_{Q_1}^{x_k}\ldots\int_{Q_1}^{x_2} x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k}$$
$$\left[F(x_1)\right]^{r_1-1}\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1}\ldots$$
$$\ldots\left[1-F(x_k)\right]^{n-r_k}\left[1+\theta x_1^p\right]f(x_1)f(x_2)\ldots$$
$$\ldots f(x_k)dx_1dx_2\ldots dx_k \quad (2.6)$$
$$= C_{r_1,r_2,\ldots,r_k:n}\int_{Q_1}^{P_1}\int_{Q_1}^{x_k}\ldots\int_{Q_1}^{x_2} x_2^{i_2}x_3^{i_3}\ldots. x_k^{i_k}$$
$$\left[F(x_3)-F(x_2)\right]^{r_3-r_2-1}\ldots$$
$$\ldots\left[1-F(x_k)\right]^{n-r_k} I(x_2)f(x_2)\ldots$$
$$\ldots f(x_k)dx_2\ldots dx_k,$$

where

$$I(x_2) = \int_{Q_1}^{x_2} x_1^{i_1}\left[F(x_1)\right]^{r_1-1}\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1}\left[1+\theta x_1^p\right]f(x_1)dx_1. \quad (2.7)$$

Making use of characterizing differential equation (1.8), we have

$$I(x_2) = mp\theta\left[Q_2 \int_{Q_1}^{x_2} x_1^{i_1+p-1}\left[F(x_1)\right]^{r_1-1}\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1} dx_1 \right.$$
$$\left. - \int_{Q_1}^{x_2} x_1^{i_1+p-1}\left[F(x_1)\right]^{r_1}\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1} dx_1 \right] \tag{2.8}$$
$$= mp\theta\left[Q_2 E(x_2,1) - E(x_2,0)\right],$$

where

$$E(x_2,k) = \int_{Q_1}^{x_2} x_1^{i_1+p-1}\left[F(x_1)\right]^{r_1-k}\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1} dx_1 \ ; k = 0,1. \tag{2.9}$$

Integrating by parts, treating $x_1^{i_1+p-1}$ for integration and rest of the integrand for differentiation, we get for $r_2 - r_1 \geq 2$,

$$E(x_2,k) = -\int_{Q_1}^{x_2} \frac{x_1^{i_1+p}}{i_1+p}\Big\{(r_1-k)\left[F(x_1)\right]^{r_1-k-1}\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1} f(x_1)dx_1$$
$$-(r_2-r_1-1)\left[F(x_1)\right]^{r_1-k}\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1} f(x_1)dx_1\Big\}$$

and, for $r_2 = r_1 + 1$,

$$E(x_2,k) = \frac{1}{i_1+p}\left[x_2^{i_1+p}\left[F(x_2)\right]^{r_1-k} - (r_1-k)\int_{Q_1}^{x_2} x_1^{i_1}\left[F(x_1)\right]^{r_1-k-1} f(x_1)dx_1 \right].$$

Upon substituting for $E(x_2, 0)$ and $E(x_2, 1)$ in (2.8) and then substituting the resulting expression for $I(x_2)$ in equation (2.6) and simplifying, we derive the relations in (2.4) and (2.5).

Proceeding on similar lines, one can derive the following recurrence relation.

THEOREM 2.3. For $1 \leq r_1 < r_2 < \ldots < r_k \leq n, \quad r_k \geq r_{k-1} + 2,$

$$\mu_{r_1,r_2,\ldots,r_{k-1},r_k:n}^{(i_1,i_2,\ldots,i_{k-1},i_k+p)} = \frac{1}{(n-r_k+1)}\left[\frac{i_k+p}{mp\theta}\left\{\mu_{r_1,r_2,\ldots,r_{k-1},r_k:n}^{(i_1,i_2,\ldots,i_{k-1},i_k)} + \theta\mu_{r_1,r_2,\ldots,r_{k-1},r_k:n}^{(i_1,i_2,\ldots,i_{k-1},i_k+p)}\right\}\right.$$
$$-nP_2\left\{\mu_{r_1,r_2,\ldots,r_{k-1},r_k:n-1}^{(i_1,i_2\ldots,i_{k-1},i_k+p)} - \mu_{r_1,r_2,\ldots,r_{k-1},r_k-1:n-1}^{(i_1,i_2,\ldots,i_{k-1},i_k+p)}\right\} \tag{2.10}$$
$$\left.+\mu_{r_1,r_2,\ldots,r_{k-1},r_k-1:n}^{(i_1,i_2\ldots,i_{k-1},i_k+p)}\right].$$

PROOF. From equation (2.1), we have

$$\mu_{r_1,r_2,\ldots,r_k:n}^{(i_1,i_2,\ldots,i_k)} + \theta\mu_{r_1,r_2,\ldots,r_k:n}^{(i_1,i_2,\ldots,i_k+p)} = C_{r_1,r_2,\ldots,r_k:n}\int_{Q_1}^{P_1}\int_{x_1}^{P_1}\ldots\int_{x_{k-1}}^{P_1} x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k}\left[F(x_1)\right]^{r_1-1}$$
$$\left[F(x_2)-F(x_1)\right]^{r_2-r_1-1}\ldots\left[1-F(x_k)\right]^{n-r_k}$$
$$\left[1+\theta x_k^p\right]f(x_1)f(x_2)\ldots f(x_k)$$
$$dx_k dx_{k-1}\ldots dx_2 dx_1 \tag{2.11}$$
$$= C_{r_1,r_2,\ldots,r_k:n}\int_{Q_1}^{P_1}\int_{x_1}^{P_1}\ldots\int_{x_{k-2}}^{P_1} x_1^{i_1}x_2^{i_2}\ldots x_{k-1}^{i_{k-1}}$$
$$\left[F(x_1)\right]^{r_1-1}\ldots\left[F(x_{k-1})-F(x_{k-2})\right]^{r_{k-1}-r_{k-2}-1}$$
$$I(x_{k-1})f(x_1)f(x_2)\ldots f(x_{k-1})dx_{k-1}\ldots dx_1,$$

where

$$I(x_{k-1}) = \int_{x_{k-1}}^{P_1} x_k^{i_k}\left[F(x_k)-F(x_{k-1})\right]^{r_k-r_{k-1}-1}\left[1-F(x_k)\right]^{n-r_k}$$
$$\left[1+\theta x_k^p\right]f(x_k)dx_k. \tag{2.12}$$

Using the characterizing differential equation (1.7) in (2.12), we get

$$I\left(x_{k-1}\right)=mp\theta\left[P_2\int_{x_{k-1}}^{P_1}x_k^{i_k+p-1}\left[F\left(x_k\right)-F\left(x_{k-1}\right)\right]^{r_k-r_{k-1}-1}\left[1-F\left(x_k\right)\right]^{n-r_k}dx_k\right.$$

$$\left.+\int_{x_{k-1}}^{P_1}x_k^{i_k+p-1}\left[F\left(x_k\right)-F\left(x_{k-1}\right)\right]^{r_k-r_{k-1}-1}\left[1-F\left(x_k\right)\right]^{n-r_k+1}dx_k\right] \tag{2.13}$$

$$=mp\theta\left[P_2E\left(x_{k-1},0\right)+E\left(x_{k-1},1\right)\right],$$

where

$$E\left(x_{k-1},t\right)=\int_{x_{k-1}}^{P_1}x_k^{i_k+p-1}\left[F\left(x_k\right)-F\left(x_{k-1}\right)\right]^{r_k-r_{k-1}-1}\left[1-F\left(x_k\right)\right]^{n-r_k+t}dx_k\ ;\ t=0,1.$$

Integrating by parts, treating $x_k^{i_k+p-1}$ for integration and rest of the integrand for differentiation, we have

$$E\left(x_{k-1},t\right)=\frac{\left(n-r_k+t\right)}{i_k+p}\int_{x_{k-1}}^{P_1}x_k^{i_k+p}\left[F\left(x_k\right)-F\left(x_{k-1}\right)\right]^{r_k-r_{k-1}-1}\left[1-F\left(x_k\right)\right]^{n-r_k+t-1}$$

$$f\left(x_k\right)dx_k-\frac{\left(r_k-r_{k-1}-1\right)}{i_k+p}\int_{x_{k-1}}^{P_1}x_k^{i_k+p}\left[F\left(x_k\right)-F\left(x_{k-1}\right)\right]^{r_k-r_{k-1}-2}$$

$$\left[1-F\left(x_k\right)\right]^{n-r_k+t}f\left(x_k\right)dx_k$$

Upon substituting for $E\left(x_{k-1},0\right)$ and $E\left(x_{k-1},1\right)$ in (2.13) and then substituting the resulting expression of $I\left(x_{k-1}\right)$ in (2.11) and simplifying, we derive the relation in (2.10).

Likewise, one can easily derive the recurrence relations given in the following theorem.

THEOREM 2.4. For $1\le r_1<r_2<\ldots<r_k\le n,\quad r_k=r_{k-1}+1,$

$$\mu_{r_1,r_2,\ldots,r_{k-1},r_{k-1}+1:n}^{(i_1,i_2,\ldots,i_{k-1},i_k+p)}=\frac{1}{\left(n-r_k+1\right)}\left[\frac{i_k+p}{mp\theta}\left\{\mu_{r_1,r_2,\ldots,r_{k-1},r_{k-1}+1:n}^{(i_1,i_2,\ldots,i_{k-1},i_k)}+\theta\mu_{r_1,r_2,\ldots,r_{k-1},r_{k-1}+1:n}^{(i_1,i_2,\ldots,i_{k-1},i_k+p)}\right\}\right.$$

$$-nP_2\left\{\mu_{r_1,r_2,\ldots,r_{k-1},r_{k-1}+1:n-1}^{(i_1,i_2\ldots,i_{k-1},i_k+p)}-\mu_{r_1,r_2,\ldots,r_{k-1}:n-1}^{(i_1,i_2,\ldots,i_{k-1}+i_k+p)}\right\} \tag{2.14}$$

$$\left.+\mu_{r_1,r_2,\ldots,r_{k-1}:n}^{(i_1,i_2\ldots,i_{k-1}+i_k+p)}\right]$$

and, for $r_k=n$,

$$\mu_{r_1,r_2,\ldots,r_{k-1},n:n}^{(i_1,i_2,\ldots,i_{k-1},i_k+p)} = \frac{i_k+p}{mp\theta}\left\{\mu_{r_1,r_2,\ldots,r_{k-1},n:n}^{(i_1,i_2,\ldots,i_{k-1},i_k)} + \theta\mu_{r_1,r_2,\ldots,r_{k-1},n:n}^{(i_1,i_2,\ldots,i_{k-1},i_k+p)}\right\}$$
$$-nP_2\left\{P_1^{i_k+p}\mu_{r_1,r_2,\ldots,r_{k-1}:n-1}^{(i_1,i_2\ldots,i_{k-1})} - \mu_{r_1,r_2,\ldots,r_{k-1},n-1:n-1}^{(i_1,i_2,\ldots,i_{k-1},i_k+p)}\right\} \quad (2.15)$$
$$+\mu_{r_1,r_2,\ldots,r_{k-1},n-1:n}^{(i_1,i_2\ldots,i_{k-1},i_k+p)}.$$

REMARK 1. By using recurrence relations established in Section 2 one can easily compute all the single, double (product) and higher moments of order statistics from doubly truncated Burr distribution for all sample sizes in a simple recursive way as demonstrated in Section 3.

REMARK 2. By letting both the proportions of truncation Q *and* $1-P \to 0(\Rightarrow Q_2 \to 1, P_2 \to 0)$ *in Theorems 2.1–2.4, we deduce the corresponding relations for the non-truncated Burr distribution (1.1).*

REMARK 3. Let us consider several other distributions:
Lomax distribution:

$$F(x) = 1 - \left[1 + \frac{x}{a}\right]^{-m} \quad (2.16)$$

Weibull distribution:

$$F(x) = 1 - e^{-x^p} \quad (2.17)$$

Compound Weibull or Weibull-gamma distribution:

$$F(x) = 1 - \left[1 + \frac{x^p}{\delta}\right]^{-m} \quad (2.18)$$

Weibull-exponential distribution:

$$F(x) = \frac{\left(x^p / \delta\right)}{1 + x^p / \delta} \quad (2.19)$$

Log logistic distribution:

$$F(x) = \frac{(x/a)^p}{1+(x/a)^p} \tag{2.20}$$

Exponential distribution:

$$F(x) = 1 - e^{-x} \tag{2.21}$$

Rayleigh distribution:

$$F(x) = 1 - e^{-x^2/2} \tag{2.22}$$

Generalized Rayleigh distribution:

$$F(x) = 1 - \left[1 + \alpha x^2 / 2\right]^{-(1/\alpha)} \tag{2.23}$$

Generalized Pareto distribution:

$$F(x) = 1 - [1 + \beta x]^{-(1/\beta)}. \tag{2.24}$$

It can be readily seen that (2.16) to (2.24) are special cases of (1.2). Thus, the recurrence relations obtained in Section 2 for higher order moments of order statistics from doubly truncated Burr distribution are also true for the following doubly truncated distributions such as Lomax ($p = 1,\ \theta = 1/\alpha$), *Weibull* ($m = 1/\theta,\ \theta \to 0$), *Weibull-gamma* ($\theta = 1/\delta$), *Weibull-exponential* ($m = 1/\theta = 1/\delta$), *Log logistic* ($m = 1,\ \theta = a^{-p}$), *exponential* ($m = 1/\theta,\ p = 1,\ \theta \to 0$), *Rayleigh* $\left(p = 2,\ m = 1/(2\theta),\ \theta \to 0\right)$, *Generalized Rayleigh* $\left(p = 2,\ m = 1/\alpha,\ \theta = \alpha/2\right)$ *and Generalized Pareto* $\left(p = 1,\ m = 1/\beta,\ \theta = \beta\right)$ *distributions. Hence, the results obtained in Section 2 unify all the results derived by different authors for the single and product moments of order statistics from the above mentioned specific distributions [cf. Balakrishnan, Malik and Ahmed (1988)].*

REMARK 4. It may be mentioned that the results obtained by Pushkarna, Saran and Tiwari (2012) for higher moments of order statistics from doubly truncated exponential distribution follow as special cases of the results of Section 2 by taking therein ($m = 1/\theta,\ p = 1,\ \theta \to 0$).

3. RECURSIVE ALGORITHM

It may be noted that the single and the product moments of order statistics from doubly truncated Burr type XII distribution have already been calculated by Khan and Khan (1987). Utilizing this prior knowledge of single and product moments and the recurrence relations derived in Section 2, one can easily calculate the higher order moments of order statistics from the doubly truncated Burr distribution. As an illustration, the procedure for calculating the higher order moments of order 3 (i.e., for $k=3$) for the doubly truncated Burr distribution (with $p=1$) will be as follows:

Setting $i_1=0,1,2,\ldots$ in (2.2), we get $\mu_{1,2,r_3:n}^{(1,i_2,i_3)}$, $\mu_{1,2,r_3:n}^{(2,i_2,i_3)}$, $\mu_{1,2,r_3:n}^{(3,i_2,i_3)}$ and so on. Similarly, setting $i_1=0,1,2,\ldots$ in (2.4) with $r_1=2$, we will get $\mu_{2,3,r_3:n}^{(1,i_2,i_3)}$, $\mu_{2,3,r_3:n}^{(2,i_2,i_3)}$, $\mu_{2,3,r_3:n}^{(3,i_2,i_3)},\ldots$. Further, utilizing the above product moments along with Eq. (2.5) with $r_1=1, r_2=3$ and $i_1=0,1,2,\ldots$ will give the values of $\mu_{1,3,r_3:n}^{(1,i_2,i_3)}$, $\mu_{1,3,r_3:n}^{(2,i_2,i_3)}$, $\mu_{1,3,r_3:n}^{(3,i_2,i_3)},\ldots$. Likewise, utilizing the above obtained moments along with Eq. (2.3) with $r_2=4$ and $i_1=0,1,2,\ldots$ will give the values $\mu_{1,4,r_3:n}^{(1,i_2,i_3)}$, $\mu_{1,4,r_3:n}^{(2,i_2,i_3)}$, $\mu_{1,4,r_3:n}^{(3,i_2,i_3)},\ldots$. Proceeding in a similar manner, one can obtain all other triple moments for all sample sizes in a simple recursive manner.

Utilizing the above obtained triple moments, one can obtain quadruple moments for all sample sizes. In the similar manner higher order moments can be obtained for doubly truncated Burr type XII distribution.

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SUMMARY

Recurrence relations for higher moments of order statistics from doubly truncated Burr distribution

In this paper, we have obtained recurrence relations for higher moments of order statistics from doubly truncated Burr distribution, which enable one to obtain all the single, double (product) and higher moments of any order of all order statistics for any sample size from doubly truncated Burr distribution in a simple recursive manner, thus generalizing the earlier work done by Khan and Khan (1987) and also by Pushkarna, Saran and Tiwari (2012).