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RECURRENT EVALUATION OF INTEGRALS OF THE TYPE $\int_0^1 x^\alpha \sin 2\pi px \, dx$, $\int_0^1 x^\alpha \cos 2\pi px \, dx$ WITH RESPECT TO NUMERICAL STABILITY

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1. INTRODUCTION

In connection with the evaluation of Fourier coefficients and Fourier transform of a function, the following problem often occurs [1], [2]: Given a real $\mu > -1$ and positive integers m_0, p_0, compute the values of $V_m(\mu; p)$ and $W_m(\mu; p)$ where

$$(1) \qquad V_m(\mu; p) = \int_0^1 x^{m+\mu} \sin 2\pi px \, dx \quad (m = 0, 1, \ldots, m_0;\ p = 1, 2, \ldots, p_0),$$

$$(2) \qquad W_m(\mu; p) = \int_0^1 x^{m+\mu} \cos 2\pi px \, dx \quad (m = 0, 1, \ldots, m_0;\ p = 1, 2, \ldots, p_0).$$

Throughout this paper we shall deal only with $W_m(\mu; p)$. The considerations concerning $V_m(\mu; p)$ are analogous.

The most effective method for solving the problem seems to be the application of a recurrence relation between $W_m(\mu; p)$ and $W_{m+2}(\mu; p)$. If we know the values of $W_0(\mu; p)$ and $W_1(\mu; p)$, we can consecutively compute $W_m(\mu; p)$ for $m = 2, 3, \ldots, m_0$; or, conversely, from the values of $W_{m_0}(\mu; p)$ and $W_{m_0-1}(\mu; p)$ we can obtain $W_m(\mu; p)$ for $m = m_0 - 2, m_0 - 3, \ldots, 0$. For some recurrence relations it happens that if the application in forward direction is numerically unstable, then the application in backward direction is numerically stable and vice versa [3]. In our case, however, there appears a substantial difficulty at practical realization of any of the two mentioned procedures on computer due to round-off errors accumulation, i.e., none of the procedures by itself can satisfactorily solve the problem.

The aim of this paper is to suggest a method for evaluating $W_m(\mu; p)$ based on a combination of the recurrence processes both in forward and backward directions in such a way to prevent numerical instability. Note that the method is applicable to some other recursive computations.

Let μ and p be given. For the sake of simplicity of notation let us denote $m' = m + \mu$, $W_m = W_m(\mu; p)$, $V_m = V_m(\mu; p)$ and for any real a and positive integer k,

$$(a)_k = (a + k - 1)(a + k - 2) \ldots a \,, \quad (a)_0 = 1 \,.$$

Integrating (2) by parts we find

$$W_m = -\frac{m'}{2\pi p} V_{m-1} \quad (m = 1, 2, \ldots) \tag{3}$$

and the recurrence relation (for convenience written in two forms)

$$W_m = \frac{m'}{(2\pi p)^2} (1 - (m' - 1) W_{m-2}) \quad (m = 2, 3, \ldots) \,, \tag{4}$$

$$W_m = \frac{1}{m' + 1} \left(1 - \frac{(2\pi p)^2}{m' + 2} W_{m+2}\right) \quad (m = 0, 1, \ldots) \,. \tag{5}$$

The following properties of W_m can be readily shown:

$$W_m > 0 \quad (m = 2, 3, \ldots) \,, \tag{6}$$

$$W_m < \frac{1}{m' + 1} \quad (m = 0, 1, \ldots) \,. \tag{7}$$

Without loss of generality we can evidently restrict ourselves to the values of μ from the interval $(-1, 0]$. This restriction will be tacitly assumed.

We shall demonstrate the numerical execution of the recurrence processes defined by formulae (4) and (5) for $\mu = 0$. It holds

$$W_0(0; p) = 0 = W_1(0; p) \,. \tag{8}$$

The computations were carried out for $p = 1, 5, 10$ by means of formulae (4) and (5) for $m = 2, 3, \ldots, 120$ and $m = 118, 117, \ldots, 0$, respectively. In the latter case the starting values W_{120}, W_{119} were evaluated by the method to be presented in the next section. Some representative values are shown in Tables 1 and 2, respectively. (An integer in parentheses indicates the power of ten by which the preceding mantissa must be multiplied.) For sufficiently large numbers m in Tab. 1 and sufficiently small numbers m in Tab. 2 the results obtained are absurd, being in contradiction to (6), (7) and (8).

Tab. 1. Evaluation of $W_m(0; p)$ by (4)

m	p		
	1	5	10
0	0·000000 (0)	0·000000 (0)	0·000000 (0)
30	−0·261142 (3)	0·161804 (−1)	0·627666 (−2)
60	0·929062 (28)	0·131863 (−1)	0·801904 (−2)
90	−0·188108 (61)	−0·289808 (8)	0·748730 (−2)
120		0·158880 (24)	−0·176598 (1)

Tab. 2. Evaluation of $W_m(0; p)$ by (5)

m	p		
	1	5	10
120	0·824277 (−2)	0·775349 (−2)	0·653386 (−2)
90	0·109385 (−1)	0·984802 (−2)	0·748696 (−2)
60	0·162293 (−1)	0·130530 (−1)	0·801904 (−2)
30	0·310909 (−1)	0·161804 (−1)	0·624322 (−2)
0	0·107288 (−5)	0·742928 (4)	0·111203 (18)

We shall analyze this phenomenon using the idea of [4]. Let real r_{m-2} and s_{m+2} be given. Evaluating the expression on the right in (4), having replaced W_{m-2} by $W_{m-2} + r_{m-2}$, we compute $W_m + r_m$ where

$$(9) \qquad r_m = -\frac{(m' - 1)\, m'}{(2\pi p)^2}\, r_{m-2} + \varrho_m \quad (m = 2, 3, \ldots),$$

if ϱ_m denotes the round-off error of the computation. In a similar fashion, substituting $W_{m+2} + s_{m+2}$ instead of W_{m+2} into (5) we compute $W_m + s_m$ where

$$(10) \qquad s_m = -\frac{(2\pi p)^2}{(m' + 1)(m' + 2)}\, s_{m+2} + \sigma_m \quad (m = 0, 1, \ldots),$$

if the round-off error is denoted by σ_m. Hence r_m and s_m may be regarded as total errors of W_m arising in evaluations by means of (4) and (5) provided the total errors corresponding to W_{m-2} and W_{m+2} at the preceding step were r_{m-2} and s_{m+2}, respectively. We shall suppose r_m and s_m to have this meaning. If $(m' - 1)\, m' \gg (2\pi p)^2$, then $|r_m| \gg |r_{m-2}|$, i.e., with increasing m, $|r_m|$ highly increases, hence, the evaluation of W_m according to scheme (4) leads to a worthless result. An analogous assertion holds for W_m computed by formula (5) for m such that $(2\pi p)^2 \gg (m' + 1)(m' + 2)$ because the error s_m has the property $|s_m| \gg |s_{m+2}|$.

For given μ and p there exists exactly one positive integer $m_c = m_c(\mu; p)$ such that

$$m_c + \mu \leqq 2\pi p < m_c + \mu + 1 .$$

This number m_c will be called the critical index of the sequence $\{W_m\}$. Clearly,

$$\frac{(m' - 1)\, m'}{(2\pi p)^2} < 1 \quad (m = 2, 3, \ldots, m_c)\,, \tag{11}$$

$$\frac{(2\pi p)^2}{(m' + 1)(m' + 2)} < 1 \quad (m = m_c, m_c + 1, \ldots)\,. \tag{12}$$

Taking absolute values in (9) and (10) we have

$$|r_m| \leqq \frac{(m' - 1)\, m'}{(2\pi p)^2}\, |r_{m-2}| + |\varrho_m| \quad (m = 2, 3, \ldots)\,, \tag{13}$$

$$|s_m| \leqq \frac{(2\pi p)^2}{(m' + 1)(m' + 2)}\, |s_{m+2}| + |\sigma_m| \quad (m = 0, 1, \ldots)\,. \tag{14}$$

If we compute the values of W_m using the recurrence formula (4) for $m = 2, 3, \ldots, m_c$ and the recurrence formula (5) for $m = m_0 - 2, m_0 - 3, \ldots, m_c$, it follows from (11) and (13), (12) and (14) that the total errors of W_m grow in absolute values at most linearly.

In other words, the set of the required values of W_m is divided into two subsets $\{W_m;\ m = 2, 3, \ldots, m_c\}$ and $\{W_m;\ m = m_c, m_c + 1, \ldots, m_0 - 2\}$, the former to be computed by realization of the recurrence process in forward, the latter in backward directions; then the undesirable destruction of the values W_m due to round-off errors accumulation does not occur. The problem of computing the starting values W_0, W_1 and W_{m_0}, $W_{m_0 - 1}$ is postponed to the next sections.

Error estimates for the computed values of W_m will be derived under the assumption: the initial errors r_0, r_1, s_{m_0} and $s_{m_0 - 1}$ are introduced in the approximations of W_0, W_1, W_{m_0} and $W_{m_0 - 1}$, respectively, and

$$\max\left(|r_0|, |r_1|, |s_{m_0 - 1}|, |s_{m_0}|\right) < \varepsilon\,, \tag{15}$$

$$|\varrho_m| < \delta\,, \quad |\sigma_m| < \delta \quad (m = 2, 3, \ldots, m_0 - 2)\,.$$

Repeated application of inequalities (13) and (14) yields

$$|r_m| \leqq \sum_{\nu=0}^{k-1} \frac{(m' - 2\nu + 1)_{2\nu}}{(2\pi p)^{2\nu}}\, |\varrho_{m-2\nu}| + \frac{(m' - 2k + 1)_{2k}}{(2\pi p)^{2k}}\, |r_{m-2k}|$$

$$(m = 2k \quad \text{or} \quad m = 2k + 1)\,,$$

$$|s_m| \leqq \sum_{\nu=0}^{k-1} \frac{(2\pi p)^{2\nu}}{(n' - 2k + 1)_{2\nu}}\, |\sigma_{m+2\nu}| + \frac{(2\pi p)^{2k}}{(n' - 2k + 1)_{2k}}\, |s_{m+2k}|$$

$$(n = m_0 \quad \text{or} \quad n = m_0 - 1\,; \quad m = n - 2k)\,.$$

Using the assumption (15),

$$|r_m| < \delta \sum_{\nu=0}^{k-1} \left[\frac{(m'-1)\,m'}{(2\pi p)^2}\right]^\nu + \varepsilon \frac{(m'-2k+1)_{2k}}{(2\pi p)^{2k}}$$

$$(m = 2k \quad \text{or} \quad m = 2k+1),$$

$$|s_m| < \delta \sum_{\nu=0}^{k-1} \left[\frac{(2\pi p)^2}{(m'+1)(m'+2)}\right]^\nu + \varepsilon \frac{(2\pi p)^{n-m}}{(n'-2k+1)_{n-m}}$$

$$(n = m_0 \quad \text{or} \quad n = m_0 - 1\,; \quad m = n - 2k)\,.$$

Therefore,

$$|r_m| < \delta \frac{1 - R_m^k}{1 - R_m} + \varepsilon R_m^k \quad (m = 2k \quad \text{or} \quad m = 2k+1)$$

and

$$|s_m| < \delta \frac{1 - S_m^k}{1 - S_m} + \varepsilon S_m^k \quad (n = m_0 \quad \text{or} \quad n = m_0 - 1\,; \quad m = n - 2k)$$

with the abbreavitions

$$R_m = \frac{(m'-1)\,m'}{(2\pi p)^2}, \quad S_m = \frac{(2\pi p)^2}{(m'+1)(m'+2)}.$$

3. POWER SERIES REPRESENTATIONS

In the case

(16) $$m_0 - 1 > m_c(\mu; p)$$

the starting values $W_{m_0}(\mu; p)$, $W_{m_0-1}(\mu; p)$ for the backward recurrence (5) are required. Following (3) we could choose an integer $N > m_0$ sufficiently large, set

$$W_{N-1} = W_N = \lim_{m\to\infty} W_m \ (=0)$$

and succesively evaluate W_m for $m = N-2, N-3, \ldots, m_0, m_0 - 1$ by (5). However, it would not be possible to judge of the influence of the choice of N upon the accuracy achieved in the computation of W_{m_0} and W_{m_0-1}.

In the present section some power series representations of W_m are derived. One of them gives a new computational technique for evaluating W_{m_0} and W_{m_0-1} not possessing the disadvantage mentioned above, another will be made use of in the next section.

Let a nonnegative integer n be given.

Expanding $\cos 2\pi px$ into the Maclaurin series, multiplying by $x^{n+\mu}$ and termwise integrating between the limits 0 and 1 we have

$$W_n = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{(2\pi p)^{2\nu}}{(2\nu)!\,(2\nu + n' + 1)} . \tag{17}$$

Substituting $n = 0$ and $\mu = 0$ into (17) and using (8) we find

$$0 = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{(2\pi p)^{2\nu}}{(2\nu + 1)!}$$

and subtracting this identity from (17) we obtain

$$W_n = -n' \sum_{\nu=0}^{\infty} (-1)^\nu \frac{(2\pi p)^{2\nu}}{(2\nu + 1)!\,(2\nu + n' + 1)} . \tag{18}$$

Expressing W_n in terms of W_{n+2} according to the recurrence formula (5), simultaneously expressing W_{n+2} in terms of W_{n+4}, etc., to W_{n+2k}, we get

$$W_n = \sum_{\nu=0}^{k-1} (-1)^\nu \frac{(2\pi p)^{2\nu}}{(n' + 1)_{2\nu+1}} + (-1)^k \frac{(2\pi p)^{2k}}{(n' + 1)_{2k}} W_{n+2k} ;$$

thus, the limit process $k \to \infty$ yields

$$W_n = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{(2\pi p)^{2\nu}}{(n' + 1)_{2\nu+1}} . \tag{19}$$

As to the three numbered representations of W_n, it can be found that from the point of view of the computational efficiency the most convenient are: (18) for $n = 0, 1$, (19) for $n = 2, 3, \ldots$ Thus, under the assumption (16) the values of W_{m_0} and W_{m_0-1} are to be computed by means of (19); the series in (19) is formed by terms with alternating signs and decreasing in absolute value so that the evaluation of the series by its truncating after k-th term achieves the precise value with error less in absolute value than $(2\pi p)^{2k}/(n' + 1)_{2k+1}$.

4. COMPUTATION OF W_0 AND W_1

The problem of computing $W_0(\mu; p)$, $W_1(\mu; p)$ $(p = 1, 2, \ldots, p_0)$ has been solved exactly for $\mu = 0$, as is seen from (8). Further, in the case $\mu \in (-1, 0)$ we can use the power series (18), as immediately follows from the argument of the preceding section; however, to obtain results of greater accuracy a more sophisticated computational process which generates the values $W_0(\mu; p)$ and $W_1(\mu; p)$ for $p = 1, 2, \ldots, p_0$ and given $\mu \in (-1, 0)$ will be now described.

First we shall modify the idea of [5] in a suitable way. Let us denote $\alpha = n + \mu$ where n is either 0 or 1 so that $\alpha \in (-1, 1]$ and let $W(p)$ be given by

$$(20) \qquad W(p) = \int_0^1 x^\alpha \cos 2\pi px \, dx = \frac{1}{p^{\alpha+1}} \int_0^p x^\alpha \cos 2\pi x \, dx \quad (p = 1, 2, \ldots).$$

We can easily derive

$$W(p) = \left(1 - \frac{1}{p}\right)^{\alpha+1} W(p-1) + \frac{1}{p^{\alpha+1}} \int_{p-1}^p x^\alpha \cos 2\pi x \, dx \quad (p = 2, 3, \ldots).$$

By an obvious substitution in the integral this relation is brought into the form

$$(21) \quad W(p) = \left(1 - \frac{1}{p}\right)^{\alpha+1} W(p-1) + \frac{1}{p}\left(1 - \frac{1}{p}\right)^\alpha \int_0^1 \left(1 + \frac{x}{p-1}\right)^\alpha \cos 2\pi x \, dx$$
$$(p = 2, 3, \ldots).$$

The binomial expansion of $[1 + x/(p-1)]^\alpha$ for $x < p - 1$ yields

$$\int_0^1 \left(1 + \frac{x}{p-1}\right)^\alpha \cos 2\pi x \, dx = \int_0^1 \sum_{m=0}^\infty \binom{\alpha}{m} \left(\frac{x}{p-1}\right)^m \cos 2\pi x \, dx \quad (p = 3, 4, \ldots)$$

and termwise integration of the series leads to

$$\int_0^1 \left(1 + \frac{x}{p-1}\right)^\alpha \cos 2\pi x \, dx = \sum_{m=2}^\infty \binom{\alpha}{m} (p-1)^{-m} W_m(0; 1) \quad (p = 3, 4, \ldots).$$

Combining this identity with (21) we obtain

$$(22) \quad W(p) = \left(1 - \frac{1}{p}\right)^{\alpha+1} W(p-1) + \frac{1}{p}\left(1 - \frac{1}{p}\right)^\alpha \sum_{m=2}^\infty \binom{\alpha}{m} (p-1)^{-m} W_m(0; 1)$$
$$(p = 3, 4, \ldots).$$

The recurrence relation (22) gives a rule for the computation of values $W_0(\mu; p)$ and $W_1(\mu; p)$ when the values for $p = 1, 2$ are known.

To derive the error estimates, let t_p be the total error of $W(p)$, ϑ_{p-1} the total error of the evaluation of the series in (22) and τ_p the round-off error of the evaluation of the right-hand side in (22). Since $p^{-1}(1 - 1/p)^\alpha < (1 - 1/p)^{\alpha+1}$, it holds

$$|t_p| \leqq \left(1 - \frac{1}{p}\right)^{\alpha+1} \left(|t_{p-1}| + |\vartheta_{p-1}|\right) + \tau_p \quad (p = 3, 4, \ldots).$$

This inequality, by repeated use, becomes

$$|t_p| \leqq \left(|t_2| + |\vartheta_2|\right) \prod_{\nu=3}^p \left(1 - \frac{1}{\nu}\right)^{\alpha+1} + \sum_{k=3}^{p-1} \left(|\vartheta_k| + |\tau_k|\right) \prod_{\nu=k+1}^p \left(1 - \frac{1}{\nu}\right)^{\alpha+1} + |\tau_p|$$
$$(p = 3, 4, \ldots).$$

Under the assumption

$$|t_2| + |\vartheta_2| < \varepsilon\,, \quad |\vartheta_k| + |\tau_k| < \varepsilon \quad (k = 3, 4, \ldots)$$

we obtain the following estimate:

$$|t_p| < \varepsilon \sum_{k=2}^{p} \prod_{v=k+1}^{p} \left(1 - \frac{1}{v}\right)^{\alpha+1} < \varepsilon \sum_{k=2}^{p} \left(1 - \frac{1}{p}\right)^{(\alpha+1)(p-k)} = \varepsilon \sum_{k=0}^{p-2} \left(1 - \frac{1}{p}\right)^{(\alpha+1)k}$$

$$(p = 3, 4, \ldots)\,.$$

Consequently,

$$|t_p| < \varepsilon \frac{1 - T_p^{p-1}}{1 - T_p} \quad (p = 3, 4, \ldots) \tag{23}$$

where

$$T_p = \left(1 - \frac{1}{p}\right)^{\alpha+1}\,.$$

Another method suggested in [2] will be treated. $W(p)$ is now supposed to be defined for the real variable $p \in [1, p_0]$. Differentiating the expression $W(p) = \int_0^1 x^\alpha \cos 2\pi p x \, \mathrm{d}x$ we can derive a differential equation for $W(p)$, viz.

$$\frac{\mathrm{d}}{\mathrm{d}p} W + \frac{\alpha + 1}{p} W = \frac{1}{p} \cos 2\pi p \quad (p \in [1, p_0])\,. \tag{24}$$

Since the function $f(p, y) = [\cos 2\pi p - (\alpha + 1)\, y]/p$ is continuous in $[1, p_0] \times (-\infty, +\infty)$ and satisfies the Lipschitz condition with respect to y there, equation (24) has exactly one solution in $[1, p_0]$ for any initial condition at $p = 1$. Thus, the required values of $W(p)$ can be obtained by solving numerically the differential equation (24) provided the value of $W(1)$ is known.

For numerical illustration, three types of computation were carried out:

A. $W(1)$, $W(2)$ by (18),
$W(p)$ $(p = 3, 4, \ldots, 10)$ by (22).

B. $W(1)$ by (18),
$W(p)$ $(p = 2, 3, \ldots, 10)$ by (24).

C. $W(1)$ by (18),
$W(2)$ by (24),
$W(p)$ $(p = 3, 4, \ldots, 10)$ by (22).

To evaluate the series in (18) and (22), numbers of terms necessary for the truncating errors not to influence the first six significant digits of the results were summed. As the computations verified, the scheme A is not suitable when α is close to 1 – because of relatively low accuracy in evaluating the initial value $W(2)$ for the recurrence formula (22). Applying the extrapolation method [6] for the solution of the differential equation (24) we found that the scheme B may lead to more accurate results than the scheme A; however, much more computational time was needed. The undesirable phenomena appearing in the processes following the schemes A and B are avoided when the scheme C is used. Hence, it is the scheme C that is recommended.

The relative errors corresponding to some results obtained by the numerical processes described above are presented in Tab. 3. As the exact values of $W(p)$ we took those given by the computation in double word-length number representation: we found that they, when restricted to six significant digits, did not depend on which of the processes A, B, C generated them.

Tab. 3. Relative errors of $\int_0^1 x^\alpha \cos 2\pi px \, dx$ evaluated by A, B, C

α	Scheme	p				
		1	2	3	5	10
0·5	A	0·50 (−5)	−0·68 (−3)	−0·65 (−3)	−0·62 (−3)	−0·58 (−3)
	B	0·50 (−5)	0·78 (−4)	0·28 (−3)	−0·40 (−4)	−0·61 (−4)
	C	0·50 (−5)	0·78 (−4)	0·76 (−4)	0·75 (−4)	0·80 (−4)
0·6	A	0·24 (−4)	0·32 (−2)	0·30 (−2)	0·29 (−2)	0·27 (−2)
	B	0·24 (−4)	0·15 (−3)	0·41 (−3)	0·63 (−5)	−0·24 (−3)
	C	0·24 (−4)	0·15 (−3)	0·14 (−3)	0·13 (−3)	0·13 (−3)
0·7	A	0·00	−0·13 (−2)	−0·12 (−2)	−0·11 (−2)	−0·10 (−2)
	B	0·00	0·00	0·61 (−3)	0·99 (−4)	0·23 (−3)
	C	0·00	0·00	0·24 (−5)	0·63 (−5)	0·15 (−4)
0·8	A	0·11 (−3)	0·83 (−2)	0·76 (−2)	0·69 (−2)	0·62 (−2)
	B	0·11 (−3)	0·12 (−3)	0·11 (−2)	0·40 (−3)	0·69 (−3)
	C	0·11 (−3)	0·12 (−3)	0.11 (−3)	0·10 (−3)	0·10 (−3)
0·9	A	0·11 (−3)	0·75 (−2)	0·68 (−2)	0·61 (−2)	0·53 (−2)
	B	0·11 (−3)	0·14 (−3)	0·24 (−2)	0·32 (−2)	0·45 (−2)
	C	0·11 (−3)	0·14 (−3)	0·12 (−3)	0·12 (−3)	0·11 (−3)

5. REMARKS

At this stage the problem of computing the values of $W_m(\mu; p)$ stated in Sec. 1 has been completely solved. We close the paper pointing out explicitly the proposed algorithm under the assumption

$$m_0 - 1 > m_c(\mu; p) \quad (p = 1, 2, \ldots, p_0) . \tag{25}$$

1. If $\mu = 0$, set $W_0(0; p) = 0 = W_1(0; p)$ $(p = 1, 2, \ldots, p_0)$; otherwise, evaluate $W_0(\mu; p)$, $W_1(\mu; p)$ $(p = 1, 2, \ldots, p_0)$ according to scheme C of Sec. 4.
2. Compute the critical indices $m_c(\mu; p)$ $(p = 1, 2, \ldots, p_0)$.
3. For each $p = 1, 2, \ldots, p_0$ evaluate $W_m(\mu; p)$ $(m = 2, 3, \ldots, m_c(\mu; p))$ by means of the recurrence formula (4).

4. Evaluate $W_{m_0}(\mu; p)$, $W_{m_0-1}(\mu; p)$ $(p = 1, 2, \ldots, p_0)$ using the power series representation (19).
5. For each $p = 1, 2, \ldots, p_0$ evaluate $W_m(\mu; p)$ $(m = m_0 - 2, m_0 - 1, \ldots, m_c(\mu; p))$ by applying the recurrence formula (5).

Each of the values $W_{m_c}(\mu; p)$ $(p = 1, 2, \ldots, p_0)$ is thus computed in two ways, viz. sub 3 and 4; the coincidence of both results verifies the correctness of the computation. If (25) does not hold the algorithm is to be rather modified in an evident way.

The theory has been illustrated by numerical examples. All the calculations were carried out on a computer TESLA 270 in TESLA-FORTRAN language. A part of this paper was prepared during author's previous employment at ÚVT ČVUT, Horská 3, Praha 2 and some results were published in [7].

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Souhrn

REKURENTNÍ VÝPOČET INTEGRÁLŮ TYPU $\int_0^1 x^\alpha \sin 2\pi px \, dx$, $\int_0^1 x^\alpha \cos 2\pi px \, dx$ S OHLEDEM NA NUMERICKOU STABILITU

FRANTIŠEK CHVÁLA

Článek se zabývá rekurentními procesy pro výpočet integrálů vyskytujících se při numerické Fourierově analýze. Především je řešena otázka odstranění podstatného vlivu akumulace zaokrouhlovacích chyb v případě, že tohoto cíle nelze dosáhnout pouhým obrácením směru rekurentního výpočtu. Teorie je doplněna ilustrativními numerickými příklady.

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