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RECURRENT JACOBI OPERATOR OF REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

Imsoon Jeong, Juan de Dios Pérez, and Young Jin Suh

Abstract. In this paper we give a non-existence theorem for Hopf hypersurfaces in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator $\bar{R}_N$.

1. Introduction

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold. The Jacobi operator $\bar{R}_X$, for any tangent vector field X at $x \in \bar{M}$, is defined by

$$(\bar{R}_X Y)(x) = (\bar{R}(Y,X)X)(x)$$

for any $Y \in T_x\bar{M}$. It becomes a self adjoint endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$, where $\bar{R}$ denotes the curvature tensor of $(\bar{M}, \bar{g})$. That is, the Jacobi operator satisfies $\bar{R}_X \in \mathrm{End}(T_x\bar{M})$ and is symmetric in the sense of $\bar{g}(\bar{R}_X Y, Z) = \bar{g}(\bar{R}_X Z, Y)$ for any vector fields Y and Z on $\bar{M}$.

Let M be a hypersurface in a Riemannian manifold $\bar{M}$. Now by putting a unit normal vector N into the curvature tensor $\bar{R}$ of $\bar{M}$, the normal Jacobi operator $\bar{R}_N$ is defined by

$$\bar{R}_N X = \bar{R}(X,N)N$$

for any tangent vector field X on M in $\bar{M}$.

Related to the commuting problem with the shape operator for real hypersurfaces M in quaternionic projective space $\mathbb{H}P^m$ or in quaternionic hyperbolic space $\mathbb{H}H^m$, Berndt [1] has introduced the notion of normal Jacobi operator $\bar{R}_N \in \mathrm{End}(T_xM)$, $x \in M$, where $\bar{R}$ denotes the curvature tensor of the ambient spaces $\mathbb{H}P^m$ or $\mathbb{H}H^m$. He [1] showed that the *curvature adaptedness*, when

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the normal Jacobi operator commutes with the shape operator A, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$. Here, $\{J_\nu \mid \nu = 1, 2, 3\}$ is a canonical local basis of quaternionic Kähler structure $\mathfrak{J}$ and $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Moreover, he gave a complete classification of curvature adapted real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$ and in quaternionic hyperbolic space $\mathbb{H}H^m$, respectively.

We say that the normal Jacobi operator $\bar{R}_N$ is *parallel* on M if the covariant derivative of the normal Jacobi operator $\bar{R}_N$ identically vanishes, that is, $\nabla_X \bar{R}_N = 0$ for any vector field X on M.

Parallelness of the normal Jacobi operator means that the normal Jacobi operator $\bar{R}_N$ is parallel on a real hypersurface M in ambient space $\bar{M}$. This means that the eigenspaces of the normal Jacobi operator are parallel along any curve γ in M. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be *parallel* along any curve γ if they are *invariant* with respect to any *parallel displacement* along the curve γ.

The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$ has a remarkable geometric structure. It is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (See [2]). From these two structures J and $\mathfrak{J}$, we have geometric conditions naturally induced on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$: That $[\xi] = \text{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. From these two conditions, Berndt and Suh [3] have proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The structure vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Reeb* vector field. If the *Reeb* vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator, M is said to be a *Hopf hypersurface*. In such a case the integral curves of the *Reeb* vector field ξ are geodesics (See [4]). Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow.*

In paper [9], Jeong, Kim and Suh considered the notion of *parallel* normal Jacobi operator, that is, $\nabla_X \bar{R}_N = 0$ for any vector field X on M in $G_2(\mathbb{C}^{m+2})$, where ∇ denotes the induced connection from the Levi-Civita connection $\bar{\nabla}$ of $G_2(\mathbb{C}^{m+2})$. They proved a non-existence theorem for Hopf hypersurfaces in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator as follows:

Theorem B. *There does not exist any connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator.*

On the other hand, in [10] Jeong, Lee and Suh have considered a Lie parallelness of the normal Jacobi operator, that is, $\mathcal{L}_X \bar{R}_N = 0$, where $\mathcal{L}_X$ denotes the Lie derivative along any direction X on M in $G_2(\mathbb{C}^{m+2})$, and asserted the following:

Theorem C. *There does not exist any Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator if the integral curves of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components od the Reeb vector field are totally geodesics.*

The purpose of this paper is to study a generalized condition weaker than parallel normal Jacobi operator for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Let T be a tensor field of type (1,1) on M. T is said to be recurrent if there exists a certain 1-form ω on M such that for any vector fields X, Y tangent to M, $(\nabla_X T)(Y) = \omega(X)T(Y)$. This notion generalizes the fact of T being parallel (see [13]).

Hamada [5], [6] investigated real hypersurfaces M in complex projective space $\mathbb{C}P^m$ with recurrent shape operator. This means that the eigenspaces of the shape operator are parallel along any curve γ in M. That is, they are invariant with respect to parallel translation along γ. He proved that there does not exist real hypersurface in complex projective space $\mathbb{C}P^m$ with recurrent shape operator. In [7] he also proved that there does not exist any real hypersurface M in complex projective space $\mathbb{C}P^m$ with recurrent Ricci tensor if the structure vector field ξ is principal.

For a real hypersurface in complex projective space $\mathbb{C}P^m$, Pérez and Santos [15] introduced a new notion of $\mathfrak{D}$-recurrent, which is weaker than the structure Jacobi operator being recurrent. Here, the structure Jacobi operator R_ξ is said to be $\mathfrak{D}$-recurrent if it satisfies

$$(\nabla_X R_\xi)(Y) = \omega(X)R_\xi(Y),$$

where ω and $\mathfrak{D}$ respectively denote an 1-form on M and the orthogonal complement of the Reeb vector field ξ in TM, and any vector fields $X \in \mathfrak{D}$, $Y \in TM$. Namely, they proved the following:

Theorem D. *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 3$. Then its structure Jacobi operator is $\mathfrak{D}$-recurrent if and only if it is a minimal ruled real hypersurface.*

Related to the shape operator, in paper due to [12], first they have applied Hamada's results to hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and next obtained a non-existence property for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator.

Motivated by such a recurrent shape operator, and in order to make a generalization of Theorem B, in this paper we introduce a new notion of *recurrent Jacobi operator*, that is, the *recurrent normal Jacobi operator* for a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. A hypersurface M in $G_2(\mathbb{C}^{m+2})$ with *recurrent normal Jacobi operator* is defined by

$$(\nabla_X \bar{R}_N)(Y) = \omega(X)\bar{R}_N(Y),$$

where ω denotes an 1-form on M and any vector fields X, Y tangent to M (see Kobayashi and Nomizu [13] page 305). Consequently, we prove the following:

Theorem 1.1. *There does not exist any Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator.*

2. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details refer to [2], [3] and [4].

By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. The space $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$.

We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, negative B restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$.

In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximum sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$.

If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{2.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned} \bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y \\ & - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\} \end{aligned} \tag{2.2}$$

for any vector fields X, Y and Z on $G_2(\mathbb{C}^{m+2})$, where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$ [2].

3. Some fundamental formulas

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (See [2], [3] and [4]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression for $\bar{R}$, the Codazzi equation becomes

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ & + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\big\} \\ & + \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\} \\ & + \sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu\ . \end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned} &\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ &\phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}. \end{aligned} \tag{3.1}$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.2}$$

for any vector field X tangent to a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (2.1) and (3.1) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.3}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.4}$$

$$\begin{aligned} (\nabla_X\phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX,Y)\xi_\nu. \end{aligned} \tag{3.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{3.6}$$

4. Recurrent normal Jacobi operator

From (2.2) the normal Jacobi operator $\bar{R}_N$ of M is given by

$$\begin{aligned} \bar{R}_N(X) &= \bar{R}(X,N)N \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^3 \Big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\Big\}. \end{aligned}$$

Now let us consider the covariant derivative of the normal Jacobi operator $\bar{R}_N$ along the direction X (see [9]). It is given by

$$\begin{aligned} (\nabla_X\bar{R}_N)Y = &\, 3g(\phi AX,Y)\xi + 3\eta(Y)\phi \\ &+ 3\sum_{\nu=1}^3 \Big\{g(\phi_\nu AX,Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\Big\} \\ &- \sum_{\nu=1}^3 \Bigg[2\eta_\nu(\phi AX)(\phi_\nu\phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu AX,\phi Y)\phi_\nu\xi \\ &- \eta(Y)\eta_\nu(AX)\phi_\nu\xi - \eta_\nu(\phi Y)(\phi_\nu\phi AX - g(AX,\xi)\xi_\nu)\Bigg]. \end{aligned}$$

From this, together with formulas given in Section 3, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator satisfies the following

$$\begin{aligned}
&3g(\phi AX,Y)\xi+3\eta(Y)\phi AX+3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu AX,Y)\xi_\nu+\eta_\nu(Y)\phi_\nu AX\Big\}\\
&-\sum_{\nu=1}^{3}\Bigg[2\eta_\nu(\phi AX)(\phi_\nu\phi Y-\eta(Y)\xi_\nu)-g(\phi_\nu AX,\phi Y)\phi_\nu\xi\\
&-\eta(Y)\eta_\nu(AX)\phi_\nu\xi-\eta_\nu(\phi Y)(\phi_\nu\phi AX-g(AX,\xi)\xi_\nu)\Bigg]\\
&=\omega(X)\Bigg[Y+3\eta(Y)\xi+3\sum_{\nu=1}^{3}\eta_\nu(Y)\xi_\nu\\
&-\sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)(\phi_\nu\phi Y-\eta(Y)\xi_\nu)-\eta_\nu(\phi Y)\phi_\nu\xi\Big\}\Bigg].
\end{aligned}\tag{4.1}$$

In order to prove our Main Theorem in the introduction, we give the following.

Lemma 4.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator. Then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. From (4.1), we take $X=Y=\xi$ and suppose that M is Hopf, that is, $A\xi=\alpha\xi$ for a certain function α. Then this yields

$$\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi=\omega(\xi)(\xi+\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu).\tag{4.2}$$

Taking its scalar product with ξ we get $\omega(\xi)=0$. As $\omega(\xi)=0$, (4.2) yields

$$\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi=0.\tag{4.3}$$

From this, we consider the case that the function α is non-vanishing. Now let us put $\xi=\eta(X_0)X_0+\eta(\xi_1)\xi_1$ for some unit $X_0\in\mathfrak{D}$ and non-vanishing functions $\eta(X_0)$ and $\eta(\xi_1)$.

Then (4.3) yields

$$0=\eta(\xi_1)\phi_1\xi=\eta(X_0)\eta(\xi_1)\phi_1X_0.$$

This gives a contradiction with $\eta(\xi_1)\neq 0$ and $\eta(X_0)\neq 0$. So we get $\eta(\xi_1)=0$ or $\eta(X_0)=0$, which means $\xi\in\mathfrak{D}$ or $\xi\in\mathfrak{D}^\perp$.

When the function α vanishes, we can differentiate $A\xi=0$. Then by a theorem due to Berndt and Suh [4] we know that

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi=0.$$

This also gives $\xi\in\mathfrak{D}$ or $\xi\in\mathfrak{D}^\perp$. □

5. Recurrent normal Jacobi operator with $\xi \in \mathfrak{D}$

In paper [14], Lee and Suh gave a characterization of real hypersurfaces of type B in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field $\xi \in \mathfrak{D}$. Now we introduce the following:

Proposition 5.1. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$, then the distribution $\mathfrak{D}$ is invariant under the shape operator A of M, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Then by Proposition 5.1 and Theorem A in the introduction, we know naturally that a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator and $\xi \in \mathfrak{D}$ is a tube over a totally geodesic quaternionic projective space $\mathbb{H}P^n$, $m = 2n$.

Now let us check if a real hypersurface of type (B) in $G_2(\mathbb{C}^{m+2})$, that is, a tube over a totally geodesic $\mathbb{H}P^n$, satisfies the notion of recurrent normal Jacobi operator. Corresponding to such a real hypersurface of type (B), we introduce a proposition due to Berndt and Suh [3] as follows:

Proposition 5.2. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 3 = m(\gamma),\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi,\ T_\beta = \mathfrak{J}J\xi,\ T_\gamma = \mathfrak{J}\xi,\ T_\lambda,\ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp,\ \mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu,\ JT_\lambda = T_\mu.$$

Now let us suppose M is of type (B) with recurrent normal Jacobi operator $\bar{R}_N$. From (4.1), by putting $X = \xi_2$ and $Y = \xi$ we have

$$\omega(\xi_2)\xi - \beta\phi_2\xi = 0.$$

Then it follows that $\omega(\xi_2) = 0$ and $\beta = 0$. Since β is not zero, this makes a contradiction. Thus we conclude the following:

Theorem 5.1. *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with recurrent normal Jacobi operator if the Reeb vector ξ belongs to the distribution $\mathfrak{D}$.*

6. Recurrent normal Jacobi operator with $\xi \in \mathfrak{D}^{\perp}$

In this section, we consider the case that $\xi \in \mathfrak{D}^{\perp}$. Then the unit normal vector field N is a singular tangent vector of $G_2(\mathbb{C}^{m+2})$ of type $JN \in \mathfrak{J}N$. So there exists an almost Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1 N$. Then we have

$$\xi = \xi_1,\ \phi\xi_2 = -\xi_3,\ \phi\xi_3 = \xi_2,\ \phi\mathfrak{D} \subset \mathfrak{D}.$$

Then, by putting $X = \xi_\mu$ and $Y = \xi$ in (4.1), we get

$$3\phi A\xi_\mu + 5\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\xi_\nu + 3\phi_1 A\xi_\mu + \sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\phi_\nu\xi = 8\omega(\xi_\mu)\xi.$$

From this, by taking its scalar product with Reeb vector field ξ we get $\omega(\xi_\mu) = 0$. As $\omega(\xi_\mu) = 0$, we have

$$\begin{aligned} 0 &= (\nabla_{\xi_\mu}\bar{R}_N)X \\ &= 3g(\phi A\xi_\mu, X)\xi + 3\eta(X)\phi A\xi_\mu \\ &\quad + 3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu A\xi_\mu, X)\xi_\nu + \eta_\nu(X)\phi_\nu A\xi_\mu\Big\} \\ &\quad - \sum_{\nu=1}^{3}\Bigg[2\eta_\nu(\phi A\xi_\mu)(\phi_\nu\phi X - \eta(X)\xi_\nu) - g(\phi_\nu A\xi_\mu, \phi X)\phi_\nu\xi \\ &\quad - \eta(X)\eta_\nu(A\xi_\mu)\phi_\nu\xi - \eta_\nu(\phi X)(\phi_\nu\phi A\xi_\mu - g(A\xi_\mu, \xi)\xi_\nu)\Bigg] \end{aligned}$$

for any $X \in TM$. From this, by putting $X = \xi$ and using $\xi = \xi_1$, we have

$$0 = 3\phi A\xi_\mu + 5\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\xi_\nu + 3\phi_1 A\xi_\mu + \sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\phi_\nu\xi.$$

From this, taking its inner product with $X \in \mathfrak{D}$ and using $g(\phi_\nu\xi, X) = 0$, we obtain

$$0 = 3g(\phi A\xi_\mu, X) + 3g(\phi_1 A\xi_\mu, X). \tag{6.1}$$

On the other hand, by using (3.4) we know that

$$\phi A\xi_\mu = \nabla_{\xi_\mu}\xi = \nabla_{\xi_\mu}\xi_1 = q_3(\xi_\mu)\xi_2 - q_2(\xi_\mu)\xi_3 + \phi_1 A\xi_\mu.$$

From this, taking its inner product with $X \in \mathfrak{D}$, we have

$$g(\phi A\xi_\mu, X) = g(\phi_1 A\xi_\mu, X).$$

Substituting this formula into (6.1) gives

$$0 = g(\phi A\xi_\mu, X).$$

From this, let us replace X by ϕX. Then it follows that

$$0 = g(\phi A\xi_\mu, \phi X) = -g(A\xi_\mu, \phi^2 X) = g(AX, \xi_\mu)$$

for any vector field $X \in \mathfrak{D}$, $\mu = 1, 2, 3$.

Then we obtain the following:

Lemma 6.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with recurrent normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

From this together with Theorem A in the introduction we know that any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator $\bar{R}_N$ and $\xi \in \mathfrak{D}^\perp$ is congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now let us check if real hypersurfaces of type (A) satisfy the condition of recurrent normal Jacobi operator. Then we recall a proposition given by Berndt and Suh [3] as follows:

Proposition 6.1. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = 2m-2 = m(\mu),$$

and as corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Now let us suppose M is of type (A) with recurrent normal Jacobi operator $\bar{R}_N$ and $\xi \in \mathfrak{D}^\perp$. From (4.1), by putting $X = \xi_2$ and $Y = \xi$ we have

$$8\omega(\xi_2)\xi - 6\beta\xi_3 = 0.$$

Then it follows that $\omega(\xi_2) = 0$ and $\beta = 0$. Since β is not zero, this gives a contradiction. Thus we conclude the following:

Theorem 6.1. *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with recurrent normal Jacobi operator if the Reeb vector ξ belongs to the distribution $\mathfrak{D}^\perp$.*

Accordingly, by Lemma 4.1 together with Theorems 5.1 and 6.1 we give a complete proof of our Theorem 1.1 mentioned in the introduction.

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Imsoon Jeong
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail address: imsoon.jeong@gmail.com

Juan de Dios Pérez
Departamento de Geometría y Topología
Facultad de Ciencias
Universidad de Granada
18071-Granada, Spain
E-mail address: jdperez@ugr.es

Young Jin Suh
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail address: yjsuh@knu.ac.kr