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RECURRENT Z FORMS ON RIEMANNIAN AND KAEHLER MANIFOLDS

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In this paper, we introduce a new kind of Riemannian manifold that generalize the concept of weakly Z-symmetric and pseudo-Z-symmetric manifolds. First a Z form associated to the Z tensor is defined. Then the notion of Z recurrent form is introduced. We take into consideration Riemannian manifolds in which the Z form is recurrent. This kind of manifold is named $(\mathrm{ZRF})_n$. The main result of the paper is that the closedness property of the associated covector is achieved also for $\mathrm{rank}(Z_{kl}) > 2$. Thus the existence of a proper concircular vector in the conformally harmonic case and the form of the Ricci tensor are confirmed for$(\mathrm{ZRF})_n$ manifolds with $\mathrm{rank}(Z_{kl}) > 2$. This includes and enlarges the corresponding results already proven for pseudo-Z-symmetric $(\mathrm{PZS})_n$ and weakly Z-symmetric manifolds $(\mathrm{WZS})_n$ in the case of non-singular Z tensor. In the last sections we study special conformally flat $(\mathrm{ZRF})_n$ and give a brief account of Z recurrent forms on Kaehler manifolds.

Keywords: Pseudo-Z-symmetric manifolds; weakly Z-symmetric manifolds; weakly-Ricci symmetric manifolds; pseudo-projective Ricci symmetric; conformal curvature tensor; quasi-conformal curvature tensor; conformally symmetric; conformally recurrent; Riemannian manifolds.

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1. Introduction

Weakly symmetric Riemannian manifolds are generalizations of the important and often investigated locally symmetric Riemannian spaces. In the last decades many kinds of weakly symmetric spaces were created and investigated. We give here a brief account of the most important results. Tamassy and Binh [36] introduced and studied a type of non-flat Riemannian manifold whose Ricci tensor is not identically zero and satisfies the following equation:

$$\nabla_k R_{jl} = A_k R_{jl} + B_j R_{kl} + D_l R_{kj}. \tag{1.1}$$

Such a manifold is called *weakly Ricci symmetric*, A_k, B_k, D_k are non-null covectors called associated one-forms, ∇ is the operator of covariant differentiation with respect to the (positive definite) metric g_{kl} and the manifold is denoted by $(\mathrm{WRS})_n$. Here we have defined the Ricci tensor to be $R_{kl} = -R_{mkl}^{m}$ [42] and the scalar curvature $R = g^{ij}R_{ij}$. If in (1.1) the one-form A_k is replaced by $2A_k$ and B_k and D_k are replaced by A_k, then the manifold is called pseudo-Ricci symmetric manifold introduced by Chaki [4] and denoted by $(\mathrm{PRS})_n$. This notion of pseudo-Ricci symmetric is different from that of Deszcz [18]. If in (1.1) the one-form A_k is replaced by $2A_k$, then the manifold is called generalized pseudo-Ricci symmetric manifold introduced by Chaki and Koley [6]. Later some authors studied these kind of manifolds [14, 21, 12]. In [14] some global properties of $(\mathrm{WRS})_n$ were pointed out and the form of the Ricci tensor was found. In [12] the authors considered conformally flat generalized pseudo-Ricci symmetric manifolds, where the conformal curvature tensor

$$\begin{aligned} C_{jkl}^{m} = R_{jkl}^{m} &+ \frac{1}{n-2}(\delta_j^m R_{kl} - \delta_k^m R_{jl} + R_j^m g_{kl} - R_k^m g_{jl}) \\ &- \frac{R}{(n-1)(n-2)}(\delta_j^m g_{kl} - \delta_k^m g_{jl}) \end{aligned} \tag{1.2}$$

vanishes, i.e. $C_{jkl}^{m} = 0$: it may be scrutinized that the conformal curvature tensor vanishes identically for $n = 3$ [31]. They also pointed out the existence of a proper concircular vector for such a manifold. In [2] a $(\mathrm{PRS})_n$ with harmonic curvature tensor i.e. $\nabla_m R_{jkl}^{m} = 0$ and with harmonic conformal curvature tensor i.e. $\nabla_m C_{jkl}^{m} = 0$ was taken into consideration. In [21] a quasi-conformally flat $(\mathrm{WRS})_n$ was studied. They also pointed out the existence of a proper concircular vector for such a manifold.

Later a generalization of $(\mathrm{PRS})_n$ manifolds was introduced in a paper by Chaki and Saha [8]. They considered the so-called Projective Ricci tensor P_{kl} obtained by a suitable contraction of the Projective Curvature tensor P_{jklm} [19]:

$$P_{jl} = \frac{n}{n-1}\left(R_{jl} - \frac{R}{n}g_{jl}\right), \tag{1.3}$$

where $g^{jl}P_{jl} = 0$. The generalization defined in [8] is thus written as:

$$\nabla_k P_{jl} = 2A_k P_{jl} + A_j P_{kl} + A_l P_{kj}. \tag{1.4}$$

This kind of manifold is called *pseudo-projective Ricci symmetric* and denoted by $(\mathrm{PPRS})_n$. Recently in [5] and [13] a further generalization of the condition of a $(\mathrm{PRS})_n$ manifold was considered. More precisely a manifold whose Ricci tensor satisfies the condition

$$\nabla_k R_{jl} = (A_k + B_k)R_{jl} + A_j R_{kl} + A_l R_{kj} \tag{1.5}$$

is defined. Such a manifold is called *almost pseudo-Ricci symmetric* and denoted by $A(\mathrm{PRS})_n$: A_k and B_k are non-null covectors. In [13] the properties of conformally flat $A(\mathrm{PRS})_n$ are studied. In the same paper the authors pointed out the importance of pseudo-Ricci symmetric manifolds in the theory of General Relativity.

More recently in [27] and subsequently in [26] a new generalized $(0, 2)$ symmetric tensor was introduced and studied; precisely the new tensor was defined as:

$$Z_{kl} = R_{kl} + \phi g_{kl}, \tag{1.6}$$

where ϕ is an arbitrary scalar function and was named *generalized Z tensor* (the *classical Z* tensor is defined as $Z_{kl} = R_{kl} - \frac{R}{n} g_{kl}$). These authors pointed out several interesting properties of such tensor that we recall here.

First of all the Z tensor allows to reinterpret many well-known structures on Riemannian manifolds. (a) *A Z* flat Riemannian manifold is simply an Einstein space $R_{kl} = \frac{R}{n} g_{kl}$ [3].

(b) If $\nabla_i Z_{kl} = 0$ (Z symmetric manifold) then $\nabla_i R_{kl} + \nabla_i \phi g_{kl} = 0$: transvecting with g^{ik} and g^{kl} we obtain $\nabla_k \phi = 0 = \nabla_k R$ and a Ricci symmetric manifold is recovered.

(c) If a Z recurrent Riemannian manifold is considered i.e. a space satisfying the condition $\nabla_i Z_{kl} = \lambda_i Z_{kl}$ one can easily find that this condition is equivalent to $\nabla_i R_{kl} = \lambda_i R_{kl} + (n-1)\mu_i g_{kl}$ with the choice $(n-1)\mu_i = \lambda_i \phi - \nabla_i \phi$. So the manifold reduces to a *Generalized Ricci Recurrent manifold* [11] and if $0 = \lambda_i \phi - \nabla_i \phi$, a Ricci recurrent manifold is recovered.

(d) Let us consider a Riemannian manifold with Codazzi Z tensor [17], i.e. with the property:

$$\nabla_k Z_{jl} = \nabla_j Z_{kl}. \tag{1.7}$$

One can easily find that this condition is equivalent to:

$$\nabla_k R_{jl} - \nabla_j R_{kl} = (\nabla_j \phi) g_{kl} - (\nabla_k \phi) g_{jl}. \tag{1.8}$$

Transvecting the previous relation with g^{jl} we get $\nabla_k (R + 2(n-1)\phi) = 0$ and finally:

$$\nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)} [(\nabla_k R) g_{jl} - (\nabla_j R) g_{kl}]. \tag{1.9}$$

A manifold with Codazzi Z tensor is thus a Nearly Conformally Symmetric manifold $(\mathrm{NCS})_n$: this condition was introduced and studied by Roter [32] and generalized in [35]. We note also that this is equivalent to have an harmonic conformal curvature tensor $\nabla_m C^m_{jkl} = 0$. Conversely a $(\mathrm{NCS})_n$ manifold has a Codazzi type Z tensor if the condition $\nabla_k (R + 2(n-1)\phi) = 0$ is satisfied. Moreover if $\nabla_j \phi = 0$, a manifold with harmonic curvature tensor is recovered.

(e) If the condition $\nabla_k Z_{jl} + \nabla_j Z_{lk} + \nabla_l Z_{kj} = 0$ is considered, then transvecting with g^{jl} we get $\nabla_k(2R + (n+2)\phi) = 0$ and thus:

$$\nabla_k \left(R_{jl} - \frac{2}{n+2} R g_{jl} \right) + \nabla_j \left(R_{lk} - \frac{2}{n+2} R g_{lk} \right) + \nabla_l \left(R_{kj} - \frac{2}{n+2} R g_{kj} \right) = 0. \tag{1.10}$$

These conditions on the Z tensor thus agree with the generalizations of the Einstein metrics, that is, a few classes of Riemannian manifolds characterized by tensorial conditions, which are consequences of the Einstein metric equation (see Besse [3]).

Moreover in [26] and [27] the authors pointed out also the importance of the Z tensor in mathematical physics. For example an Einstein-like equation $Z_{kl} = kT_{kl}$ may be written in n dimensions, being T_{kl} the energy-momentum tensor and k a suitable gravitational constant. The physical condition $\nabla^l T_{kl} = 0$ and the second contracted Bianchi identity that reads $\nabla_k(\frac{R}{2} + \phi) = 0$ give $\phi = -\frac{R}{2} + \Lambda$. The choice $\Lambda \neq 0$ describes some cosmological models (see [16, 41]): in this case the vacuum solution $Z_{kl} = 0$ implies $\Lambda = R\frac{n-2}{2n}$ and thus an Einstien space $R_{kl} = \frac{R}{n} g_{kl}$. The choice $\Lambda = 0$ agrees with observations over regions of space of galactic dimension: in this case the vacuum solution is reduced to $R_{kl} = 0$. Moreover it can be shown (see [24, p. 321]) that a linear combination of $R_{kl} - \frac{1}{2} R g_{kl}$ and g_{kl} is the most general two index symmetric tensor which is divergence-free and can be constructed locally from the metric and its derivatives up to second order. So the generalized Z tensor may be as though a generalized Einstein Gravitational tensor with arbitrary scalar function ϕ.

In [26] and [27] the Z tensor was used to introduce the new differential structures of *pseudo-Z-symmetric* and *weakly Z-symmetric* Riemannian manifolds. It turned out that these manifolds generalize the notions of $(\mathrm{PRS})_n$, $(\mathrm{WRS})_n$ and the $(\mathrm{PWRS})_n$ manifolds. *A pseudo-Z-symmetric* manifold $(\mathrm{PZS})_n$ is defined by the condition [27]:

$$\nabla_k Z_{jl} = 2A_k Z_{jl} + A_j Z_{kl} + A_l Z_{kj}. \tag{1.11}$$

In [27] the authors studied the fundamental properties of such manifolds: they focused the case with harmonic curvature tensors giving the conditions of closedness of the associated one-form when the Z tensor is non-singular. In the conformally flat case they gave the form of the Ricci and the metric tensors. Moreover they provided sufficient conditions for a $(\mathrm{PZS})_n$ manifold to be Ricci pseudo-symmetric in the sense of Deszcz [18]. Finally they studied $(\mathrm{PZS})_n$ space-time manifolds focusing the properties of perfect fluid space-times.

In [26] the authors studied the properties of a manifold on which the Z tensor is subjected to the condition:

$$\nabla_k Z_{jl} = A_k Z_{jl} + B_j Z_{kl} + D_l Z_{kj}. \tag{1.12}$$

Such a manifold is called weakly Z-symmetric and denoted by $(\mathrm{WZS})_n$. In particular they presented a great number of relations to the already known weakly symmetric spaces. Also in this case the authors found sufficient conditions for the existence of a proper concircular vector, and determined the Ricci and the metric tensors in the conformally harmonic and conformally flat cases. They described the impact of the closedness of the one-form $\omega_k = A_k - B_k$ when the Z tensor is non-singular.

In this paper, we present a generalization of the above-mentioned manifolds. First a Z form associated to the Z tensor is defined. Then the notion of Z *recurrent form* with its associated covector is introduced. We consider Riemannian manifolds on which the Z form is recurrent and study its properties in depth.

These manifolds are named $(\mathrm{RZF})_n$. The notion of Z recurrent form incorporates both *pseudo-Z-symmetric* and *weakly Z-symmetric* manifolds. The main result of this paper is that the closedness property of the associated covector is achieved also in the case of $\mathrm{rank}(Z_{kl}) > 2$. Thus the existence of a proper concircular vector in the conformally harmonic case is confirmed in this more general situation. This includes and enlarges already known results for *pseudo-Z-symmetric* and *weakly Z-symmetric* manifolds with non-singular Z tensor developed in [26] and [27]. In Sec. 5, we study a special conformally flat $(\mathrm{RZF})_n$. In the last section we give a brief survey of Z recurrent forms on Kaehlerian manifolds.

2. Recurrent *Z* Forms

In this section, we consider the recurrence of Z forms originating from Z tensors. In [29] the authors extended the recurrence notion from curvature tensors to the associated 2-forms. First we recall some basic definitions about generalized curvature tensors and associated forms. Consider a class of curvature tensors K_{jkl}^{m} with the usual symmetries of the Riemann tensor satisfying the first Bianchi identity. Specifically we admit a generalized curvature tensor satisfying the following relations [23, 25, 35]:

$$\begin{aligned} &\text{(a)} \quad K_{jkl}^{m} + K_{klj}^{m} + K_{ljk}^{m} = 0, \quad K_{jkl}^{m} = -K_{kjl}^{m}, \\ &\text{(b)} \quad \nabla_i K_{jkl}^{m} + \nabla_j K_{kil}^{m} + \nabla_k K_{ijl}^{m} = B_{ijkl}^{m}, \end{aligned} \tag{2.1}$$

where B_{ijkl}^{m} is a tensor source in the second Bianchi identity. Moreover we may define also an associated completely covariant $(0,4)$ tensor $K_{jklm} = g_{mn} K_{jkl}^{n}$ with the following further properties [23]:

$$\begin{aligned} K_{jklm} &= -K_{kjlm} = -K_{jkml}, \\ K_{jklm} &= K_{lmjk}. \end{aligned} \tag{2.2}$$

In this way the contraction $K_{kl} = -K^m_{mkl}$ defines a symmetric generalized Ricci tensor [28]. An n-dimensional Riemannian manifold is said to be K-flat if $K^m_{jkl} = 0$ and K-symmetric if $\nabla_i K^m_{jkl} = 0$, K-harmonic if $\nabla_m K^m_{jkl} = 0$ and K-recurrent if $\nabla_i K^m_{jkl} = \alpha_i K^m_{jkl}$ [25]. Now the curvature two-form associated to this tensor is defined in the following manner:

$$\Omega^m_{(K)l} = K^m_{jkl} dx^j \wedge dx^k. \tag{2.3}$$

If we consider the symmetric contraction $K_{kl} = -K^m_{mkl}$ a generalized *Ricci one-form* may be defined [28] as:

$$\Lambda_{(K)l} = K_{kl} dx^k. \tag{2.4}$$

Hereafter we consider n-dimensional non-K-flat Riemannian manifolds. An exterior covariant derivative D (see [3] and [24]) acting on the forms (2.3) and (2.4) can be defined as follows:

$$D\Omega^m_{(K)l} = \nabla_i K^m_{jkl} dx^i \wedge dx^j \wedge dx^k,$$
$$D\Lambda_{(K)l} = \nabla_i K_{kl} dx^i \wedge dx^k.$$

The forms $\Omega^m_{(K)l}, \Lambda_{(K)l}$ are said to be closed if $D\Omega^m_{(K)l} = 0$, $D\Lambda_{(K)l} = 0$. This implies respectively $\nabla_i K^m_{jkl} + \nabla_j K^m_{kil} + \nabla_k K^m_{ijl} = 0 = B^m_{ijkl}$ and $\nabla_i K_{kl} - \nabla_k K_{il} = 0$ for the previously defined forms [27]. The notion of recurrent curvature form introduced in [29] enlarges the closedness condition and the ordinary recurrence of curvature tensors. In [29] the following definition was stated:

Definition 2.1. Let M be an n-dimensional Riemannian manifold. The curvature two-form $\Omega^m_{(K)l} = K^m_{jkl} dx^j \wedge dx^k$ is said to be recurrent if there exist a nonzero scalar one-form α for which:

$$D\Omega^m_{(K)l} = \alpha \wedge \Omega^m_{(K)l}, \tag{2.5}$$

being $\alpha = \alpha_i dx^i$ the associated one-form.

It is easy to see that the previous condition is a generalization of the notion of K-recurrency. In fact if we write Eq. (2.5) in local components we have:

$$(\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl}) dx^i \wedge dx^j \wedge dx^k = 0. \tag{2.6}$$

If $\alpha = 0$ we recover the closedness of $\Omega^m_{(K)l}$. The following theorem stated in [29] explains the meaning of this kind of recurrence.

Theorem 2.2. *Let M be an n-dimensional Riemannian manifold. The curvature two-form $\Omega^m_{(K)l} = K^m_{jkl} dx^j \wedge dx^k$ satisfies condition* (2.5) *if and only if*:

$$B^m_{ijkl} = \nabla_i K^m_{jkl} + \nabla_j K^m_{kil} + \nabla_k K^m_{ijl} = \alpha_i K^m_{jkl} + \alpha_j K^m_{kil} + \alpha_k K^m_{ijl}. \tag{2.7}$$

Proof. From Eq. (2.6) we easily obtain the following expression:

$$
\begin{aligned}
&(\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl}) dx^i \wedge dx^j \wedge dx^k \\
&\quad = \frac{1}{3!}(\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl})\delta^{ijk}_{rst} dx^r \wedge dx^s \wedge dx^t \\
&\quad = \sum_{r<s<t} (\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl})\delta^{ijk}_{rst} dx^r \wedge dx^s \wedge dx^t = 0. \qquad (2.8)
\end{aligned}
$$

The above condition is fulfilled if and only if $(\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl})\delta^{ijk}_{rst} = 0$ from which Eq. (2.7) follows immediately. Obviously if the manifold is K-recurrent $\nabla_i K^m_{jkl} = \alpha_i K^m_{jkl}$, then condition (2.7) is satisfied. □

Now we focus on the notion of recurrence for the generalized Ricci one-form $\Lambda_{(K)l} = K_{kl} dx^k$ where $K_{kl} = -K^m_{jkl}$. Again in [29] the following definition was stated:

Definition 2.3. Let M be an n-dimensional Riemannian manifold. The generalized Ricci one-form $\Lambda_{(K)l} = K_{kl} dx^k$ is said to be recurrent if there exist a nonzero scalar one-form β for which:

$$D\Lambda_{(K)l} = \beta \wedge \Lambda_{(K)l}, \qquad (2.9)$$

being $\beta = \beta_i dx^i$ the associated one-form.

In local components the previous equation may be written in the form:

$$(\nabla_i K_{kl} - \beta_i K_{kl}) dx^i \wedge dx^k = 0. \qquad (2.10)$$

If $\beta = 0$ the closedness of the generalized Ricci one-form is recovered. The following theorem explains the meaning of this kind of recurrence.

Theorem 2.4. *Let M be an n-dimensional Riemannian manifold. The generalized Ricci one-form $\Lambda_{(K)l} = K_{kl} dx^l$ satisfies condition* (2.9) *if and only if*:

$$\nabla_i K_{kl} - \nabla_k K_{il} = \beta_i K_{kl} - \beta_k K_{il}. \qquad (2.11)$$

Proof. From Eq. (2.9) we easily obtain the following expression:

$$
\begin{aligned}
(\nabla_i K_{kl} - \beta_i K_{kl}) dx^i \wedge dx^k &= \frac{1}{2!}(\nabla_i K_{kl} - \beta_i K_{kl})\delta^{ik}_{rs} dx^r \wedge dx^s \\
&= \sum_{r<s} (\nabla_i K_{kl} - \beta_i K_{kl})\delta^{ik}_{rs} dx^r \wedge dx^s = 0. \qquad (2.12)
\end{aligned}
$$

The above condition is fulfilled if and only if $(\nabla_i K_{kl} - \alpha_i K_{kl})\delta^{ik}_{rs} = 0$ from which one concludes immediately. □

We apply the previous definition to the case of Z form $\Lambda_{(Z)l} = Z_{kl} dx^k$; specifically we have the following definition.

Definition 2.5. Let M be an n-dimensional Riemannian manifold. The Z form $\Lambda_{(Z)l} = Z_{kl} dx^k$ is said to be recurrent if there exist a nonzero scalar one-form ω for which:

$$D\Lambda_{(Z)l} = \omega \wedge \Lambda_{(Z)l}. \tag{2.13}$$

From the above-mentioned theorems the Z form is recurrent if and only if:

$$\nabla_k Z_{jl} - \nabla_j Z_{kl} = \omega_k Z_{jl} - \omega_j Z_{kl}. \tag{2.14}$$

In the previous expression ω_k is called the *associated covector*. A manifold satisfying condition (2.14) is called $(\mathrm{ZRF})_n$, (Z recurrent form). Obviously if the Z tensor is recurrent, $\nabla_k Z_{il} = \omega_k Z_{il}$ the previous equation is satisfied.

Remark 2.6. We notice also that the notion of Z recurrent form incorporates the properties of both pseudo-Z-symmetric manifolds and of weakly Z-symmetric manifolds. In fact Eqs. (1.11) and (1.12) imply $\nabla_k Z_{jl} - \nabla_j Z_{kl} = \omega_k Z_{jl} - \omega_j Z_{kl}$. where $\omega_k = A_k$ for a pseudo-Z-symmetric manifold and $\omega_k = A_k - B_k$ for a weakly Z-symmetric manifold. Moreover if an almost pseudo-Z-symmetric manifold is defined:

$$\nabla_k Z_{jl} = (A_k + B_k) Z_{jl} + A_j Z_{kl} + A_l Z_{kj}, \tag{2.15}$$

then such a manifold is a $(\mathrm{ZRF})_n$ with $\omega_k = B_k$. This manifold is named $(\mathrm{APZS})_n$. Therefore all the results stated in this paper for $(\mathrm{ZRF})_n$ manifolds are valid also for pseudo-Z-symmetric, weakly Z-symmetric and almost pseudo-Z-symmetric manifolds and thus for all differential structures originating from them for various choices of ϕ. If $\omega_k = 0$ the closedness of the Z form is recovered and thus a Codazzi Z tensor: we have a $(\mathrm{NCS})_n$ manifold. If $\nabla_k \phi = \omega_k \phi$, the Z recurrent form reduces to a Ricci recurrent form and ω_k is a closed one-form. This happens if and only if $\phi = Ce^{D\omega_k x^k}$ being C, D arbitrary constants, and includes the case $\phi = 0$.

Transvecting Eq. (2.14) with g^{jl} and recalling the definition of the Z tensor $Z_{kl} = R_{kl} + \phi g_{kl}$ one can obtain:

$$\frac{1}{2}\nabla_k Z + \frac{n-2}{2}\nabla_k \phi = \omega_k Z - \omega^l Z_{kl}, \tag{2.16}$$

being $Z = g^{kl} Z_{kl}$. Now if we consider $(\mathrm{ZRF})_n$ manifold on which the Z tensor is a Codazzi one [17]; we have simply from Eq. (2.14):

$$\omega_k Z_{jl} = \omega_j Z_{kl}. \tag{2.17}$$

Let us suppose that $\omega_k \neq 0$: then transvecting equation with g^{jl} we have $\omega_k Z = \omega^l Z_{kl}$ and consequently that $Z_{jl} = Z\frac{\omega_j \omega_l}{\omega^k \omega_k}$. Thus we have just proved that the Z tensor has rank one. We may state the following (see also [26]):

Theorem 2.7. *Let M be an n-dimensional* $(\mathrm{ZRF})_n$ *Riemannian manifold whose Z tensor is of Codazzi type* [17], *i.e.* $\nabla_k Z_{jl} = \nabla_j Z_{kl}$: *if* $\mathrm{rank}(Z_{kl}) > 1$, *then the one-form ω_k is null, that is,* $\omega_k = 0$.

Remark 2.8. If $\omega_k \neq 0$ and $\nabla_k Z_{jl} = \nabla_j Z_{kl}$ then the Z tensor has rank one that is a singular tensor, that is, $\det(Z_{kl}) = 0$.

3. Recurrent Z Forms: Conditions for the Closedness of the ω One-Form

In this section, we investigate the conditions for the closedness of the one-form ω_k on a $(\mathrm{RZF})_n$ $(n > 3)$ Riemannian manifold. We generalize some already known results about the closedness property of associated forms in $(\mathrm{PZS})_n$ and $(\mathrm{WZS})_n$ manifolds presented in [26] and [27] in the case of non-singular Z tensor. First we need the following lemma known as *Lovelock's differential identity* [24–26].

Lemma 3.1 (Lovelock's differential identity). *Let M be an n-dimensional Riemannian manifold. Then the following identity is fulfilled*:

$$\nabla_i \nabla_m R^m_{jkl} + \nabla_j \nabla_m R^m_{kil} + \nabla_k \nabla_m R^m_{ijl} = -R_{im} R^m_{jkl} - R_{jm} R^m_{kil} - R_{km} R^m_{ijl}. \tag{3.1}$$

Here we collect some useful preliminary formulas for the divergence of the Riemann tensor on a $(\mathrm{ZRF})_n$ manifold. From the contracted second Bianchi identity and from the definition of the Z tensor $Z_{kl} = R_{kl} + \phi g_{kl}$, the following equation can be written:

$$\nabla_m R^m_{jkl} = \nabla_k Z_{jl} - \nabla_j Z_{kl} + (\nabla_j \phi g_{kl} - \nabla_k \phi g_{jl}). \tag{3.2}$$

On the other hand, from the definition of a $(\mathrm{RZF})_n$ manifold one easily finds that:

$$\nabla_k Z_{jl} - \nabla_j Z_{kl} = \omega_k Z_{jl} - \omega_j Z_{kl}. \tag{3.3}$$

From this we get:

$$\begin{aligned}\nabla_m R^m_{jkl} &= \nabla_k Z_{jl} - \nabla_j Z_{kl} + (\nabla_j \phi g_{kl} - \nabla_k \phi g_{jl}) \\ &= \omega_k Z_{jl} - \omega_j Z_{kl} + (\nabla_j \phi g_{kl} - \nabla_k \phi g_{jl}).\end{aligned} \tag{3.4}$$

In [27] the following theorem for pseudo-Z-symmetric manifolds was proved.

Theorem 3.2. *Let M be an $n(n > 3)$-dimensional* $(\mathrm{PZS})_n$ *Riemannian manifold with non-singular Z_{kl} tensor. Then A_k is a closed one-form if and only if the* *following algebraic relation is fulfilled.*

$$R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl} = 0. \tag{3.5}$$

In [26] the previous theorem was generalized to weakly Z-symmetric manifolds: precisely the authors stated.

Theorem 3.3. *Let M be an $n(n > 3)$-dimensional $(\mathrm{WZS})_n$ Riemannian manifold with non-singular Z_{kl} tensor. Then ω_k is a closed one-form if and only if $R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl} = 0$.*

Following the same procedure of Theorems 3.2 and 3.3 we may state an equivalent version of them for the present case of Z recurrent form. Nevertheless the main result of this section is a generalization of the previous theorems. We are able to assert a fundamental result that is valid under the condition $\mathrm{rank}(Z_{kl}) > 2$. Precisely we have the following theorem.

Theorem 3.4. *Let M be an $n(n > 3)$-dimensional $(\mathrm{RZF})_n$ Riemannian manifold with $\mathrm{rank}(Z_{kl}) > 2$. Then ω_k is a closed one-form if and only if $R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl} = 0$.*

Proof. By performing the covariant derivative of Eq. (3.2) recalling (3.2) one easily obtains:

$$\begin{aligned} \nabla_i\nabla_m R^m_{jkl} &= (\nabla_i\omega_k)Z_{jl} + \omega_k(\nabla_i Z_{jl}) - (\nabla_i\omega_j)Z_{kl} \\ &\quad - \omega_j(\nabla_i Z_{kl}) + (\nabla_i\nabla_j\phi g_{kl} - \nabla_i\nabla_k\phi g_{jl}). \end{aligned} \tag{3.6}$$

Now a cyclic permutation of the indices $i.j.k$ is performed and the resulting three equations are added to obtain:

$$\begin{aligned} &\nabla_i\nabla_m R^m_{jkl} + \nabla_j\nabla_m R^m_{kil} + \nabla_k\nabla_m R^m_{ijl} \\ &\quad = (\nabla_i\omega_k - \nabla_k\omega_i)Z_{jl} + (\nabla_j\omega_i - \nabla_i\omega_j)Z_{kl} + (\nabla_k\omega_j - \nabla_j\omega_k)Z_{il} \\ &\qquad + \omega_j(\nabla_k Z_{il} - \nabla_i Z_{kl}) + \omega_k(\nabla_i Z_{jl} - \nabla_j Z_{il}) + \omega_i(\nabla_j Z_{kl} - \nabla_k Z_{jl}). \end{aligned} \tag{3.7}$$

Inserting Eq. (3.3) in the previous result and recalling Lemma 3.1 one easily writes:

$$\begin{aligned} &(\nabla_i\omega_k - \nabla_k\omega_i)Z_{jl} + (\nabla_j\omega_i - \nabla_i\omega_j)Z_{kl} + (\nabla_k\omega_j - \nabla_j\omega_k)Z_{il} \\ &\quad = -R_{im}R^m_{jkl} - R_{jm}R^m_{kil} - R_{km}R^m_{ijl}. \end{aligned} \tag{3.8}$$

Now if ω_k is a closed one-form then Eq. (3.5) is fulfilled. Now suppose that Eq. (3.5) holds: Eq. (3.8) may be written in the algebraic form:

$$Z_{il}A_{jk} + Z_{jl}A_{ki} + Z_{kl}A_{ij} = 0, \tag{3.9}$$

with $A_{ij} = \nabla_i\omega_j - \nabla_j\omega_i$. We prove that $A_{ik} = 0$. □

First we introduce the following remark.

Remark 3.5 (Spectral theorem). Z is a symmetric (self-adjoint) endomorphism on a manifold with positive definite metric, thus: (1) It possesses n real eigenvalues $\lambda_1, \ldots, \lambda_n$ and n real eigenvectors $X_i^{(l)}, \ldots, X_i^{(n)}$.

(2) The n eigenvectors form an orthonormal base. Moreover Z is unitarily diagonalized, i.e. there exist a unitary matrix $U^+ = U^{-1}$ whose columns are the orthogonal Z eigenvectors such that $Z_D = U^+ZU$ being Z_D the diagonal form of Z. From this the spectral representation of Z is achieved:

$$Z_{ij} = \lambda_1 X_i^{(1)} X_j^{(1)} + \lambda_2 X_i^{(2)} X_j^{(2)} + \cdots + \lambda_n X_i^{(n)} X_j^{(n)},$$

where λ_i are the eigenvalues of Z. If $\mathrm{rank}(Z_{kl}) = n$, recalling that $\mathrm{rank}(Z_D) = \mathrm{rank}\ Z$, the eigenvalues are all different from zero, i.e. $\lambda_i \neq 0, i = 1, \ldots, n$ and they may be degenerate or not. If $\mathrm{rank}(Z_{kl}) = r < n$ the null space of Z has dimension $n - r$: thus $\lambda_i = 0, i = r + 1, \ldots, n$ and the corresponding eigenvectors $X^{(r+1)}, \ldots, X_i^n$ belong to the null space of Z. The spectral representation of Z is then:

$$Z_{ij} = \lambda_1 X_i^{(1)} X_j^{(1)} + \lambda_2 X_i^{(2)} X_j^{(2)} + \cdots + \lambda_r X_i^{(r)} X_j^{(r)},$$

where $X_i^{(p)}, X_i^{(q)} p \neq q$ are distinct eigenvectors in the range of Z and $\lambda_i \neq 0, i = 1 \cdots r$ are the relative eigenvalues: they may be still degenerate or not but the spectral theorem guarantees the eigenvectors to be orthogonal.

From Remark 3.5 if $\mathrm{rank}(Z_{kl}) > 2$ we may choose three distinct orthonormal eigenvectors $X_i^{(p)}, X_i^{(q)}, X_i^{(s)} : p \neq q \neq s$ in the range of Z such that

$$\begin{aligned} Z_{il} X^{l(p)} &= \lambda_p X_i^{(p)}, \\ Z_{il} X^{l(q)} &= \lambda_q X_i^{(q)}, \\ Z_{il} X^{l(s)} &= \lambda_s X_i^{(s)}, \end{aligned} \tag{3.10}$$

where $\lambda_p, \lambda_q, \lambda_s \neq 0$ may be equal or not and $X_m^{(h)} X^{m(k)} = \delta_{hk} : h, k = p, q, s$ (orthonormality condition). Transvecting (3.9) by $X^{(p)l}$ recalling that $\lambda_p \neq 0$ gives:

$$X_i^{(p)} A_{jk} + X_j^{(p)} A_{ki} + X_k^{(p)} A_{ij} = 0. \tag{3.11}$$

Now multiply the previous equation by $X^{j(q)} X^{k(s)}$ to get:

$$X_i^{(p)} X^{j(q)} X^{k(s)} A_{jk} + X_j^{(p)} X^{j(q)} X^{k(s)} A_{ki} + X_k^{(p)} X^{k(s)} X^{j(q)} A_{ij} = 0. \tag{3.12}$$

The second and the third terms vanish because of the orthogonality of distinct eigenvectors and we get:

$$X_i^{(p)} X^{j(q)} X^{k(s)} A_{jk} = 0. \tag{3.13}$$

Thus we have:

$$X^{j(q)}X^{k(s)}A_{jk} = 0. \tag{3.14}$$

Now Eq. (3.9) is transvected by $X^{j(q)}X^{k(s)}$ giving:

$$X^{j(q)}X^{k(s)}Z_{il}A_{jk} + X^{j(q)}X^{k(s)}Z_{jl}A_{ki} + X^{j(q)}X^{k(s)}Z_{kl}A_{ij} = 0. \tag{3.15}$$

The first term vanishes and the eigenvalues equation gives:

$$\lambda_q X_l^{(q)}X^{k(s)}A_{ki} + \lambda_s X_l^{(s)}X^{j(q)}A_{ij} = 0. \tag{3.16}$$

The previous condition is now multiplied respectively by $X^{l(q)}$ and by $X^{l(s)}$ giving (recalling the orthogonality condition):

$$\begin{aligned} \lambda_q X^{k(s)}A_{ki} &= 0, \\ \lambda_s X^{j(q)}A_{ij} &= 0. \end{aligned} \tag{3.17}$$

Thus the range eigenvectors of the tensor Z belong to the null space of A. Tranvection (3.9) by $X^{j(q)}$ we get:

$$X^{j(q)}Z_{il}A_{jk} + X^{j(q)}Z_{jl}A_{ki} + X^{j(q)}Z_{kl}A_{ij} = \lambda_q X_l^{(q)}A_{ki} = 0, \tag{3.18}$$

from which one concludes that $A_{ki} = 0$.

In [25] the authors proved that Lovelock's identity is left unchanged if the divergence of the Riemann tensor is replaced by the divergence of any curvature tensor satisfying certain properties. This property was discussed also in [26] and [28]. We state it here again for convenience.

Theorem 3.6. *Let M be an n-dimensional Riemannian manifold having a generalized curvature tensor K^m_{jkl} with the property*:

$$\nabla_m K^m_{jkl} = A\nabla_m R^m_{jkl} + B[(\nabla_j\psi)a_{kl} - (\nabla_k\psi)a_{jl}], \tag{3.19}$$

where A and B are non-null constants, ψ an arbitrary scalar function and a_{kl} a symmetric (0.2) *Codazzi tensor*, [17] *i.e. $\nabla_i a_{kl} = \nabla_k a_{il}$. Then the following relation is fulfilled*:

$$\begin{aligned} &\nabla_i\nabla_m K^m_{jkl} + \nabla_j\nabla_m K^m_{kil} + \nabla_k\nabla_m K^m_{ijl} \\ &\quad = -A(R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl}). \end{aligned} \tag{3.20}$$

In literature one always meets generalized curvature tensors whose divergence is given in the form (3.19) with trivial Codazzi tensor (i.e. constant multiple of the metric tensor) and $\psi = R$ (see [25, 26, 28]):

$$\nabla_m K^m_{jkl} = A\nabla_m R^m_{jkl} + B[(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}]. \tag{3.21}$$

If the generalized curvature tensor is harmonic, i.e. $\nabla_m K^m_{jkl} = 0$, we recall the following (see [26, 27]).

Theorem 3.7. *Let M be an n-dimensional Riemannian manifold having a generalized curvature tensor K^m_{jkl} with the property*:

$$\nabla_m K^m_{jkl} = A\nabla_m R^m_{jkl} + B[(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}], \tag{3.22}$$

where a and b are constants. If $\nabla_m K^m_{jkl} = 0$ and the condition $B \neq \frac{A}{2(n-1)}$ is satisfied then the scalar curvature is a covariant constant $\nabla_k R = 0$.

Proof. Transvecting Eq. (3.21) with g^{kl} and using the second contracted Bianchi identity one easily obtains $(\nabla_j R)[\frac{1}{2}A - (n-1)B] = 0$ from which one concludes immediately. □

Some curvature tensors K^m_{jkl} with the property (3.21) and trivial Codazzi tensor are well-known: the *Projective curvature tensor* P^m_{jkl} [19], *the conformal curvature tensor* C^m_{jkl} [31], *the Concircular tensor* $\tilde{C}^m_{jkl}$ [33, 37] and *the quasi-conformal tensor* W^m_{jkl} [30, 40]. See also [22].

Now considering $(\mathrm{RZF})_n$ manifolds from Eqs. (3.21), (3.2) and (3.3) one simply gets (see [26, 27]):

$$\begin{aligned}\nabla_m K^m_{jkl} &= A[\omega_k Z_{jl} - \omega_j Z_{kl}] + A[\nabla_j \phi g_{kl} - \nabla_k \phi g_{jl}] \\ &\quad + B[(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}].\end{aligned} \tag{3.23}$$

If $\nabla_m K^m_{jkl} = 0$ from the previous relation we have:

$$A(\omega_k Z_{jl} - \omega_j Z_{kl}) = [\nabla_k(A\phi + BR)g_{jl} - \nabla_j(A\phi + BR)g_{kl}]. \tag{3.24}$$

In [26] the authors stated the following lemma (see also [27]):

Lemma 3.8. *Let M be an n-dimensional Riemannian* $(\mathrm{WZS})_n$ *manifold with non-singular Z tensor having a generalized curvature tensor satisfying the property* (3.21). *Then if $\nabla_m K^m_{jkl} = 0$ the one-form ω_k vanishes if and only if $\nabla_k(A\phi + BR) = 0$. Moreover, if $\frac{B}{A} \neq \frac{1}{2(n-1)}$ then $\nabla_k R = 0$ and the one-form ω_k vanishes if and only if $\nabla_k \phi = 0$.*

Here we extend the previous lemma to $\mathrm{rank}(Z_{kl}) > 1$; precisely we have (see also Theorem 2.7).

Lemma 3.9. *Let M be an n-dimensional Riemannian* $(\mathrm{RZF})_n$ *manifold with* $\mathrm{rank}(Z_{kl}) > 1$ *having a generalized curvature tensor satisfying the property* (3.21). *Then if $\nabla_m K^m_{jkl} = 0$, the one-form ω_k vanishes if and only if $\nabla_k(A\phi + BR) = 0$. Moreover if $\frac{B}{A} \neq \frac{1}{2(n-1)}$, then $\nabla_k R = 0$ and the one-form ω_k vanishes if and only if $\nabla_k \phi = 0$.*

Proof. If $\omega_k = 0$ from Eq. (3.24) we have $\nabla_k(A\phi + BR)g_{jl} = \nabla_j(A\phi + BR)g_{kl}$; transvecting with g^{jl} we get the result. Now if $\nabla_k(A\phi + BR) = 0$ we have immediately $\omega_k Z_{jl} = \omega_j Z_{kl}$. If $\text{rank}(Z_{kl}) > 1$ we claim that $\omega_k = 0$; in fact let us suppose on the contrary that $\omega_k \neq 0$: transvecting with g^{jl} we have $\omega_k Z = \omega^l Z_{kl}$ and consequently that $Z_{jl} = Z\frac{\omega_j\omega_l}{\omega^k\omega_k}$. Thus we have obtained a Z tensor of rank one, that is a contradiction: consequently $\omega_k = 0$. The second statement is obvious. □

In the K-harmonic case, in order to have $\omega_k \neq 0$ the condition $\nabla_k(A\phi+BR) \neq 0$ must be satisfied. In particular using Theorems 3.4 and 3.6 the following statement is fulfilled:

Theorem 3.10. *Let M be an n-dimensional Riemannian* $(\text{RZF})_n$ *manifold with* $\text{rank}(Z_{kl}) > 2$ *having a generalized curvature tensor satisfying the property* (3.21). *Then if $\nabla_m K^m_{jkl} = 0$ and $\nabla_k(A\phi + BR) \neq 0$, ω_k is a closed one-form.*

Remark 3.11. If we have an n-dimensional Riemannian $(\text{RZF})_n$ manifold with $\text{rank}(Z_{kl}) > 2$ having $\nabla_m R^m_{jkl} = 0$ from Eqs. (3.1)–(3.3) we have $\omega_k Z_{jl} - \omega_j Z_{kl} = \nabla_k\phi g_{jl} - \nabla_j\phi g_{kl}$. Thus if $\nabla\phi \neq 0$, ω_k is a closed one-form.

Corollary 3.12. *Let M be an n-dimensional Riemannian* $(\text{RZF})_n$ *manifold with* $\text{rank}(Z_{kl}) > 2$ *having harmonic Conformal curvature tensor $\nabla_m C^m_{jkl} = 0$ and with* $\nabla_k(R + 2(n-1)\phi) \neq 0$: *then ω_k is a closed one-form.*

Proof. We are in the case $\frac{B}{A} = \frac{1}{2(n-1)}$ (valid only with $K = C$), thus Lemma 3.9 and Theorem 3.10 apply. □

Now we consider the case of harmonic quasi-conformal curvature tensor. In 1968 Yano and Sawaki [40] defined and studied a tensor W^m_{jkl} on a Riemannian manifold of dimension n, which includes both the conformal curvature tensor C^m_{jkl} and the concircular curvature tensor $\tilde{C}^m_{jkl}$ as particular cases. This tensor is known as quasi-conformal curvature tensor and its components are given by:

$$W^m_{jkl} = -(n-2)bC^m_{jkl} + [a + (n-2)b]\tilde{C}^m_{jkl}. \tag{3.25}$$

In the previous equation $a \neq 0, b \neq 0$ are constants and $n > 3$ since the conformal curvature tensor vanishes identically for $n = 3$. We recall that the components of the concircular tensor are given by:

$$\tilde{C}^m_{jkl} = R^m_{jkl} + \frac{R}{n(n-1)}(\delta^m_j g_{kl} - \delta^m_k g_{jl}). \tag{3.26}$$

A non-flat manifold has the harmonic quasi-conformal curvature tensor if $\nabla_m W^m_{jkl} = 0$. If the equations for $\nabla_m C^m_{jkl}$ and $\nabla_m \tilde{C}^m_{jkl} = 0$ are employed and the covariant derivative with respect to the index m is applied on the definition of quasi-conformal curvature tensor, one obtains straightforwardly:

$$\nabla_m W^m_{jkl} = [a+b]\nabla_m R^m_{jkl} + \frac{2a - b(n-1)(n-4)}{2n(n-1)}[(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}]. \quad (3.27)$$

Now if $\nabla_m W^m_{jkl} = 0$, transvecting the previous equation with g^{kl} after some calculations it follows that:

$$(n-2)\frac{a+b(n-2)}{n}(\nabla_j R) = 0. \quad (3.28)$$

This means that or $\nabla_j R = 0$ or $a + b(n-2) = 0$.

Corollary 3.13. *Let M be an $n(n > 3)$-dimensional* $(\mathrm{RZF})_n$ *Riemannian manifold with* $\mathrm{rank}(Z_{kl}) > 2$ *having harmonic quasi-conformal tensor* $\nabla_m W^m_{jkl} = 0$ *and with* $\nabla_k(R + 2(n-1)\phi) \neq 0, \nabla\phi \neq 0$: *then ω_k is a closed one-form.*

Proof. If $\nabla_m W^m_{jkl} = 0$ we have $(n-2)\frac{a+b(n-2)}{n}(\nabla_j R) = 0$. This means that $\nabla_j R = 0$ or $a + b(n-2) = 0$. If $\nabla_j R = 0$ we are in the case $\nabla_m R^m_{jkl} = 0$, thus Remark 3.11 applies. If $a + b(n-2) = 0$, we have $\nabla_m C^m_{jkl} = 0$, thus Corollary 3.12 applies. □

Now for other curvature tensors the following corollaries are easily proven.

Corollary 3.14. *Let M be an n-dimensional Riemannian* $(\mathrm{RZF})_n$ *manifold with* $\mathrm{rank}(Z_{kl}) > 2$ *having harmonic Projective curvature tensor* $\nabla_m P^m_{jkl} = 0$ *and* $\nabla\phi \neq 0$: *then ω_k is a closed one-form.*

Proof. The components of the Projective curvature tensor are defined as [19, 34]:

$$P^m_{jkl} = R^m_{jkl} + \frac{1}{n-1}(\delta^m_j R_{kl} - \delta^m_k R_{jl}). \quad (3.29)$$

Applying the operator of covariant derivative to the previous equation and recalling the second contracted Bianchi identity one obtains:

$$\nabla_m P^m_{jkl} = \frac{n-2}{n-1}\nabla_m R^m_{jkl}. \quad (3.30)$$

Thus applying Theorem 3.6 we are in the conditions of Theorem 3.10. □

Corollary 3.15. *Let M be an n-dimensional Riemannian* $(\mathrm{RZF})_n$ *manifold with* $\mathrm{rank}(Z_{kl}) > 2$ *having harmonic Concircular curvature tensor that* $\nabla_m \tilde{C}^m_{jkl} = 0$ *and* $\nabla\phi \neq 0$: *then ω_k is a closed one-form.*

Proof. The components of the Concircular [30, 34] curvature tensor are defined as:

$$\tilde{C}^m_{jkl} = R^m_{jkl} + \frac{R}{n(n-1)}(\delta^m_j g_{kl} - \delta^m_k g_{ji}). \tag{3.31}$$

Applying the operator of covariant derivative to the previous equation and considering the second contracted Bianchi identity one obtains:

$$\nabla_m \tilde{C}^m_{jkl} = \nabla_m R^m_{jkl} + \frac{1}{n(n-1)}[(\nabla_j R)g_{jl} - (\nabla_k R)g_{jl}]. \tag{3.32}$$

Thus applying Theorem 3.6 we are in the conditions of Theorem 3.10. □

Corollary 3.16. *Let M be an n-dimensional Riemannian* $(\mathrm{RZF})_n$ *manifold with* $\mathrm{rank}(Z_{kl}) > 2$ *having harmonic Conharmonic curvature tensor* $\nabla_m N^m_{jkl} = 0$ *and* $\nabla\phi \neq 0$: *then ω_k is a closed one-form.*

Proof. The components of the Conharmonic curvature tensor are defined as [30, 34]:

$$N^m_{jkl} = R^m_{jkl} + \frac{1}{n-2}(\delta^m_j R_{kl} - \delta^m_k R_{jl} + R^m_j g_{kl} - R^m_k g_{jl}). \tag{3.33}$$

Applying the operator of covariant derivative to the previous equation and considering the second contracted Bianchi identity one obtains:

$$\nabla_m N^m_{jkl} = \frac{n-3}{n-2}\nabla_m R^m_{jkl} + \frac{1}{2(n-2)}[(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}]. \tag{3.34}$$

Thus applying Theorem 3.6 we are in the conditions of Theorem 3.10. □

Finally we can state the following remark.

Remark 3.17. Since manifolds on which Z forms are recurrent incorporate the cases of $(\mathrm{PZS})_n$, $(\mathrm{WZS})_n$ and $(\mathrm{APZS})_n$ manifolds, we get: (a) Theorem 3.10 and Corollaries from (3.1) to (3.5) are valid for $(\mathrm{PZS})_n$, $(\mathrm{WZS})_n$ and $(\mathrm{APZS})_n$ with $\mathrm{rank}(Z_{kl}) > 2$.

(b) If $\phi = 0$, Theorem 3.10 and Corollaries from (3.1) to (3.5) are valid for $(\mathrm{PRS})_n$, $A(\mathrm{PRS})_n$ and $(\mathrm{PWRS})_n$ with $\mathrm{rank}(R_{kl}) > 2$.

4. Conformally Harmonic Recurrent Z Forms: Local Form of the Ricci and Metric Tensors

In this section, we study in depth conformally harmonic (i.e. $\nabla_m C^m_{jkl} = 0$) recurrent Z forms; in particular we point out the existence of a proper concircular vector in such a manifold and give the local form of the metric tensor in the conformally flat case. It is worth to notice that the proof in the present paper is based only on the request of $\mathrm{rank}(Z_{kl}) > 2$. Similar results were obtained in [26] and [27] for $(\mathrm{PZS})_n$

and $(\mathrm{WZS})_n$ manifolds with the stronger condition of non-singular Z tensor. It is well-known that the divergence of the conformal tensor satisfies the relation:

$$\nabla_m C^m_{jkl} = \frac{n-3}{n-2}\left[\nabla_m R^m_{jkl} + \frac{1}{2(n-1)}(\nabla_j R g_{kl} - \nabla_k R g_{jl})\right]. \tag{4.1}$$

So if we consider $\nabla_m C^m_{jkl} = 0$ one immediately obtains:

$$\nabla_m R^m_{jkl} = \nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)}(\nabla_k R g_{jl} - \nabla_j R g_{kl}). \tag{4.2}$$

Now considering Eqs. (3.3), (3.2) and (4.2) one can see that the following relation holds for a $(\mathrm{RZF})_n$ manifold with harmonic conformal curvature tensor:

$$\omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{2(n-1)}[\nabla_k(R + 2(n-1)\phi)g_{jl} - \nabla_j(R + 2(n-1)\phi)g_{kl}]. \tag{4.3}$$

This is the starting point for the proofs of the most important properties of a $(\mathrm{RZF})_n$ manifold having harmonic conformal curvature tensor.

Remark 4.1. We note that the condition $\nabla_m C^m_{jkl} = 0$ implies that the manifold is a $(\mathrm{NCS})_n$ one. Thus if the condition $\nabla_k(R + 2(n-1)\phi) = 0$ is satisfied, the Z tensor becomes a Codazzi tensor [32]. From Theorem 2.7 if $\mathrm{rank}(Z_{kl}) > 1$ then the one-form ω_k is null i.e. $\omega_k = 0$ in this condition. To avoid Z_{kl} to be of rank one we suppose in this section that $\nabla_k(R + 2(n-1)\phi) \neq 0$.

Now we follow the procedure explained in [13] to point out other properties of a $(\mathrm{RZF})_n$ manifold. Transvecting Eq. (4.3) with g^{kl} gives:

$$\omega_j Z - \omega^m Z_{jm} = \frac{1}{2}\nabla_j(R + 2(n-1)\phi). \tag{4.4}$$

Inserting this result in (4.3) one can write the following relation:

$$\omega_j Z_{jl} - \omega_j Z_{kl} = \frac{1}{(n-1)}[(\omega_k Z - \omega^m Z_{km})g_{jl} - (\omega_j Z - \omega^m Z_{jm})g_{kl}]. \tag{4.5}$$

Transvecting the previous equation with ω^l one straightforwardly obtains:

$$\omega_k \omega^l Z_{jl} = \omega_j \omega^l Z_{kl}. \tag{4.6}$$

Again we multiply the previous equation by ω^j to obtain:

$$\omega_k \omega^j \omega^l Z_{jl} = \omega_j \omega^j \omega^l Z_{kl}. \tag{4.7}$$

This last can be rewritten as:

$$\omega^l Z_{kl} = \frac{\omega_k \omega^j \omega^l Z_{jl}}{\omega_j \omega^j} = t\omega_k, \tag{4.8}$$

where $t = \frac{\omega^j \omega^l Z_{jl}}{\omega_j \omega^j}$ is a scalar function. We have just proved the following theorem that generalizes a similar result in [13] for an $A(\mathrm{PRS})_n$ Riemannian manifold (see also [26] and [27]).

Theorem 4.2. *Let M be an $n(n > 3)$-dimensional* $(\mathrm{RZF})_n$ *Riemannian manifold with the property $\nabla_m C^m_{jkl} = 0$. Then the vector ω^l is an eigenvector of the Z_{kl} tensor with eigenvalue t.*

Inserting (4.8) in Eq. (4.4) one easily obtains:

$$\omega_k(t - Z) = -\frac{1}{2}\nabla_k(R + 2(n-1)\phi). \tag{4.9}$$

This result is again a natural generalization of a similar equation given in [13] for an $A(\mathrm{PRS})_n$ Riemannian manifold.

Now transvecting Eq. (4.5) with ω^j and using the result (4.8) one straightforwardly shows that the following equation holds:

$$R_{kl} = \frac{\omega_k\omega_l}{\omega_j\omega^j}\left[\frac{nt - Z}{n-1}\right] + g_{kl}\left[\frac{Z - t}{n-1} - \phi\right]. \tag{4.10}$$

Again we have a quasi-Einstein manifold [7]. This result can be written in the more compact form:

$$R_{kl} = \alpha g_{kl} + \beta T_k T_l, \tag{4.11}$$

where $\alpha = \frac{Z-t}{n-1} - \phi, \beta = \frac{nt-Z}{n-1}$ are the associated scalars and $T_k = \frac{\omega_k}{\sqrt{\omega_j\omega^j}}$ is naturally a unit covector. We have just proved the following.

Theorem 4.3. *Let M be an $n(n > 3)$-dimensional* $(\mathrm{RZF})_n$ *Riemannian manifold with the property $\nabla_m C^m_{jkl} = 0$: then the manifold is quasi-Einstein.*

Remark 4.4. In the case of $(\mathrm{RZF})_n$ with K-harmonic curvature tensor (i.e. $\nabla_m K^m_{jkl} = 0$) Eq. (4.3) takes the form:

$$A(\omega_k Z_{jl} - \omega_j Z_{kl}) = \nabla_k(A\phi + bR)g_{jl} - \nabla_j(A\phi + BR)g_{kl}. \tag{4.12}$$

Transvecting this with g^{kl} we obtain:

$$\omega_j Z - \omega^m Z_{jm} = \frac{n-1}{A}\nabla_j(A\phi + BR). \tag{4.13}$$

Inserting (4.13) in (4.12) we get again the result (4.5). Following the same procedure used for the conformally harmonic case we get that a K-harmonic $(\mathrm{RZF})_n$ manifold is quasi-Einstein [7].

The notion of manifold of *quasi-constant curvature* was introduced by Chen and Yano [9] and generalizes a space of constant curvature. A Riemannian manifold $(n > 3)$ is said to be a manifold of quasi-constant curvature if it is conformally flat and the Riemann curvature tensor may be written in the form:

$$\begin{aligned} R_{jklm} &= p[g_{mj}g_{kl} - g_{mk}g_{jl}] \\ &\quad + q[g_{mj}T_kT_l - g_{mk}T_jT_l + g_{kl}T_mT_l + g_{jl}T_mT_k], \end{aligned} \tag{4.14}$$

where p and q are scalars (with $q \neq 0$) and T_i is a unit covector.

Now in a conformally flat manifold the curvature tensor may be written as:

$$R_{jklm} = \frac{1}{(n-2)}[g_{mk}R_{jl} - g_{jm}R_{kl} - g_{kl}R_{jm} + g_{jl}R_{km}] + \frac{R}{(n-1)(n-2)}[g_{mj}g_{kl} - g_{mk}g_{jl}]. \tag{4.15}$$

Now if we consider an $n(n > 3)$-dimensional $(\mathrm{RZF})_n$ Riemannian manifold with the property $\nabla_m C^m_{jkl} = 0$ we get $R_{kl} = \alpha g_{kl} + \beta T_k T_l$, and inserting this in Eq. (4.15) we obtain a manifold of quasi-constant curvature with $q = -\frac{\beta}{(n-2)}, p = \frac{R-2(n-1)\alpha}{(n-1)(n-2)}$. We may assert the following.

Theorem 4.5. *Let M be an $n(n > 3)$-dimensional conformally flat* $(\mathrm{RZF})_n$ *Riemannian manifold*: *then the manifold is of quasi-constant curvature.*

We shall give now sufficient conditions for the existence of a proper concircular vector in a conformally harmonic $(\mathrm{RZF})_n$. We follow the same procedure already used in [26] and [27] for pseudo-Z-symmetric and weakly Z-symmetric manifolds (see also [15] and [12]).

Theorem 4.6. *Let M be an $n(n > 3)$-dimensional manifold whose Ricci tensor is given by $R_{kl} = \alpha g_{kl} + \beta T_k T_l$ where T_k is a unit vector*: *if the manifold is conformally harmonic and the condition $T_j(\nabla_k \beta) = T_k(\nabla_j \beta)$ is satisfied*, *then T_k is a proper concircular vector.*

Proof. If the manifold is conformally harmonic then Eq. (4.2) holds. The definition $R_{kl} = \alpha g_{kl} + \beta T_k T_l$ is then substituted in (4.2) and the operations of covariant differentiation are performed to give straightforwardly:

$$(\nabla_k \beta)T_j T_l + \beta(\nabla_k T_j)T_l + \beta T_j(\nabla_k T_l) - (\nabla_j \beta)T_k T_l - \beta(\nabla_j T_k)T_l - \beta T_k(\nabla_j T_l) = \frac{1}{2(n-1)}(\nabla_k \tilde{R} g_{jl} - \nabla_j \tilde{R} g_{kl}), \tag{4.16}$$

where $\tilde{R} = R - 2(n-1)\alpha$. Recalling that T_k is a unit vector and so $(\nabla_k T_l)T^l = 0$, Eq. (4.16) is then transvected with g^{jl} to obtain:

$$(\nabla_k \beta) - (\nabla^l \beta)T_k T_l - \beta(\nabla^l T_k)T_l - \beta T_k(\nabla^l T)_l = \frac{1}{2}(\nabla_k \tilde{R}). \tag{4.17}$$

Transvecting again Eq. (4.16) with $T^j T^l$ gives:

$$(\nabla_k \beta) - (\nabla_l \beta)T_k T^l - \beta T^l(\nabla_l T_k) = \frac{1}{2(n-1)}(\nabla_k \tilde{R}) - \frac{1}{2(n-1)}(\nabla_l \tilde{R})T_k T^l. \tag{4.18}$$

Comparing the last two equations gives immediately:

$$\beta T_k(\nabla^l T_l) = \frac{2-n}{2(n-1)}(\nabla_k \tilde{R}) - \frac{1}{2(n-1)}(\nabla_l \tilde{R})T_k T^l. \tag{4.19}$$

The last result is then transvected with T^k so that the following holds:

$$\beta(\nabla^l T_l) = -\frac{1}{2}(\nabla_l \tilde{R})T^l. \tag{4.20}$$

Now Eq. (4.20) is substituted in (4.19) to give:

$$(\nabla_l \tilde{R})T_k T^l = (\nabla_k \tilde{R}). \tag{4.21}$$

If the last result is substituted in (4.18) one can easily obtain:

$$(\nabla_k \beta) - (\nabla_l \beta)T_k T^l = \beta T^l(\nabla_l T_k). \tag{4.22}$$

It is worth to notice that, by (4.21) one easily has $(\nabla_k \tilde{R})T_j = (\nabla_j \tilde{R})T_k$. Thus transvecting Eq. (4.16) with T^l gives immediately:

$$\beta[\nabla_k T_j - \nabla_j T_k] + T_j \nabla_k \beta - T_k \nabla_j \beta = 0. \tag{4.23}$$

Following the hypothesis of the theorem we immediately conclude that T_k is a closed one-form, that is:

$$\nabla_k T_j - \nabla_j T_k = 0. \tag{4.24}$$

Now with this condition in mind we can transvect again Eq. (4.16) with T^j recalling that $(\nabla_j T_l)T^j = (\nabla_l T_j)T^j = 0$ being T_j closed to obtain the following relation:

$$\nabla_k T_l = \frac{T^m(\nabla_m \tilde{R})}{2\beta(n-1)}[T_l T_k - g_{kl}]. \tag{4.25}$$

So we conclude that T_k is a concircular vector. □

Now we can state the following remarks.

Remark 4.7. From $(\nabla_m \tilde{R})T^m T_k = (\nabla_k \tilde{R})$ we easily obtain by a covariant derivative that the following is true:

$$\nabla_j \nabla_k \tilde{R} = (\nabla_j T_k)(\nabla_m \tilde{R})T^m + T_k \nabla_j (\nabla_m \tilde{R})T^m. \tag{4.26}$$

A similar relation is written with indices k and j exchanged and the resulting equations are then subtracted recalling that T_k is a closed one-form to obtain finally:

$$\nabla_j (\nabla_m \tilde{R})T^m = T_j(T^k \nabla_k (T^m \nabla_m \tilde{R})). \tag{4.27}$$

Remark 4.8. From $(\nabla_k \beta) - (\nabla_l \beta)T_k T^l = \beta T^l(\nabla_l T_k)$ recalling that T_k is a closed one-form one easily writes:

$$(\nabla_k \beta) = (\nabla_l \beta)T_k T^l. \tag{4.28}$$

Now if the scalar function $f = \frac{T^m(\nabla_m \tilde{R})}{2\beta(n-1)}$ is considered by the previous remarks one can write $\nabla_j f = \mu T_j$ where μ is in another scalar function: thus the one-form $\omega_k = fT_k$ is closed and T_k is a proper concircular vector.

Now from Eqs. (2.16) and (4.8) it follows immediately that:

$$\frac{1}{2}\nabla_k Z + \frac{n-2}{2}\nabla_k \phi = \omega_k (Z - t). \tag{4.29}$$

Now a process of covariant differentiation is employed to obtain:

$$\frac{1}{2}\nabla_j \nabla_k Z + \frac{n-2}{2}\nabla_j \nabla_k \phi = \nabla_j \omega_k (Z - t) + \omega_k \nabla_j (Z - t). \tag{4.30}$$

Exchanging the indices k and j and subtracting the resulting equations one easily obtains:

$$(\nabla_j \omega_k - \nabla_k \omega_j)(Z - t) + \omega_k \nabla_j (Z - t) - \omega_j \nabla_k (Z - t) = 0. \tag{4.31}$$

Again from (4.29) one easily finds that the following is fulfilled:

$$\frac{1}{2}\omega_j \nabla_k Z + \frac{n-2}{2}\omega_j \nabla_k \phi = \omega_j \omega_k (Z - t). \tag{4.32}$$

Now the indices k and j are again exchanged and the resulting equation then subtracted to obtain:

$$\omega_j \nabla_k Z - \omega_k \nabla_j Z + (n-2)[\omega_j \nabla_k \phi - \omega_k \nabla_j \phi] = 0. \tag{4.33}$$

If ω_k is a closed one-form and the condition $\omega_j \nabla_k \phi - \omega_k \nabla_j \phi = 0$ is fulfilled, then one obtains the following equations:

$$\begin{aligned} \omega_k \nabla_j (Z - t) - \omega_j \nabla_k (Z - t) &= 0, \\ \omega_j \nabla_k Z - \omega_k \nabla_j Z &= 0. \end{aligned} \tag{4.34}$$

According to Theorem 4.3 a $(\mathrm{RZF})_n$ Riemannian manifold with the property $\nabla_m C^m_{jkl} = 0$ is quasi-Einstein, [7] that is, the Ricci tensor satisfies $R_{kl} = \alpha g_{kl} + \beta T_k T_l$. If $\mathrm{rank}(Z_{kl}) > 2$ the covector ω_k is closed and if the condition $\omega_j \nabla_k \phi - \omega_k \nabla_j \phi = 0$ is fulfilled by Eq. (4.34) we have $\omega_j(\nabla_k t) = \omega_k(\nabla_j t)$ and $\omega_j(\nabla_k Z) = \omega_k(\nabla_j Z)$. One easily obtains the following relation:

$$\omega_j \left(\nabla_k \frac{nt - Z}{n-1} \right) = \omega_k \left(\nabla_j \frac{nt - Z}{n-1} \right). \tag{4.35}$$

Thus multiplying the previous result by $\frac{1}{\sqrt{\omega_j \omega^j}}$ and considering Theorem 4.5 we can state the following.

Corollary 4.9. *Let M be an $n(n > 3)$-dimensional $(\mathrm{RZF})_n$ Riemannian manifold with the property $\nabla_m C^m_{jkl} = 0$. Then*: (A) *The manifold is quasi-Einstein and* (B) *If $\mathrm{rank}(Z_{kl}) > 2$ and the condition $\omega_j \nabla_k \phi - \omega_k \nabla_j \phi = 0$ is fulfilled, then the following is true*:

$$T_j(\nabla_k \beta) = T_k(\nabla_j \beta). \tag{4.36}$$

So if we recall Theorem 4.6 we can finally state the following theorem.

Theorem 4.10. *Let M be an $n(n > 3)$-dimensional conformally harmonic* $(\mathrm{RZF})_n$. *If* $\mathrm{rank}(Z_{kl}) > 2$ *and the condition* $\omega_j \nabla_k \phi - \omega_k \nabla_j \phi = 0$ *is fulfilled, then the manifold admits a proper concircular vector.*

Hereafter we specialize to the conformally flat case. It is well-known [1] that if a conformally flat space admits a proper concircular vector, then this space is subprojective in the sense of Kagan. In this way the following holds.

Theorem 4.11. *Let M be an $n(n > 3)$-dimensional conformally flat* $(\mathrm{RZF})_n$. *If* $\mathrm{rank}(Z_{kl}) > 2$ *and the condition* $\omega_j \nabla_k \phi - \omega_k \nabla_j \phi = 0$ *is fulfilled, then the manifold is a subprojective space.*

In [38] Yano proved that a necessary and sufficient condition for a Riemannian manifold to admit a concircular vector is that there is a coordinate system in which the first fundamental form may be written as:

$$ds^2 = (dx^1)^2 + e^{q(x^1)} g^*_{\alpha\beta} dx^\alpha dx^\beta, \tag{4.37}$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only $(\alpha, \beta, \gamma = 2, 3, \ldots, n)$ and q is a function of x^1 only. Since a conformally flat $(\mathrm{RZF})_n$ manifold with $\mathrm{rank}(Z_{kl}) > 2$ admits a proper concircular vector field, this space is the warped product $1 \times e^q M^*$ where (M^*, g^*) is an $(n-1)$-dimensional Riemannian manifold. Gebarosky [20] proved that the warped product $1 \times e^q M^*$ satisfies the condition (4.2) if and only if M^* is Einstein. Thus the following theorem holds:

Theorem 4.12. *Let M be an $n(n > 3)$-dimensional conformally flat* $(\mathrm{RZF})_n$. *If* $\mathrm{rank}(Z_{kl}) > 2$ *and the condition* $\omega_j \nabla_k \phi - \omega_k \nabla_j \phi = 0$ *is fulfilled, then the manifold is the warped product* $1 \times e^q M^*$, *where* M^* *is Einstein.*

Remark 4.13. The condition $\omega_j \nabla_k \phi - \omega_k \nabla_j \phi = 0$ is satisfied if $\nabla_k \phi = \omega_k \chi$, being χ an arbitrary scalar function. If $\phi = \chi$ from Sec. 1 we have that the Z recurrent form reduces to Ricci recurrent form.

5. Special Conformally Flat $(\mathrm{RZF})_n$ Manifolds

In this section, we prove that a conformally flat $(\mathrm{RZF})_n$ is a special conformally flat manifold. In [10] Chen and Yano introduced the notion of *special conformally flat manifold* that is a generalization of a subprojective space. A conformally flat Riemannian manifold is said to be special conformally flat if the $(0, 2)$ tensor defined as:

$$H_{ij} = -\frac{1}{n-2} R_{ij} + \frac{R}{2(n-1)(n-2)} g_{ij}, \tag{5.1}$$

may be written in the form:

$$H_{ij} = -\frac{\gamma^2}{2} g_{ij} + \delta(\nabla_i \gamma)(\nabla_j \gamma), \tag{5.2}$$

where γ, δ are scalar functions and $\gamma > 0$. In the previous sections we have proved that a conformally flat $(\mathrm{RZF})_n$ is quasi-Einstein [7] and the Ricci tensor is written in the form $R_{ij} = \alpha g_{ij} + \beta T_i T_j$; inserting this in (5.1) one easily gets:

$$H_{ij} = -\frac{\gamma^2}{2} g_{ij} - \frac{\beta T_i T_j}{(n-2)}, \tag{5.3}$$

where $\gamma^2 = -\frac{\tilde{R}}{(n-1)(n-2)}$ and $\tilde{R} = R - 2(n-1)\alpha$ as previously defined (provided that $\tilde{R} < 0$).

Now recalling (4.21) $\nabla_k \tilde{R} = T_k T^l (\nabla_l \tilde{R}) = \lambda T_k$ we have $T_i = -\frac{2(n-1)(n-2)}{\lambda} \gamma (\nabla_i \gamma)$, and consequently that:

$$T_i T_j = -\frac{4(n-1)(n-2)\tilde{R}}{\lambda^2} (\nabla_i \gamma)(\nabla_j \gamma). \tag{5.4}$$

We have thus proved the result in (5.2) with the choice $\delta = \frac{4\beta \tilde{R}(n-1)}{\lambda^2}$. Thus a conformally flat $(\mathrm{RZF})_n$ manifold is a special conformally flat manifold. We state the following theorem.

Theorem 5.1. *An n-dimensional* $(n > 3)$ *conformally flat* $(\mathrm{RZF})_n$ *manifold is a special conformally flat manifold.*

In [10] Chen and Yano proved that every simply connected special conformally flat manifold can be isometrically immersed in an Euclidean space E^{n+1} as a hypersurface. Thus from Theorem 5.1 we may assert the following theorem.

Theorem 5.2. *An n-dimensional simply connected* $(n > 3)$ *conformally flat* $(\mathrm{RZF})_n$ *manifold can be isometrically immersed in an Euclidean manifold* E^{n+1} *as a hypersurface.*

6. Recurrent *Z* Forms on Kaehlerian Manifolds

In this section, we give a brief account of the behavior of Z recurrent form on a Kaehlerian manifold. An $n(= 2m)$-dimensional Kaehlerian manifold is a Riemannian space equipped with a structure tensor field F_i^α (an affinor) satisfying the following relations [39]:

$$F_i^\alpha F_s^i = -\delta_s^\alpha; \quad F_i^\alpha g_{\alpha j} + F_j^\alpha g_{\alpha i} = 0; \quad \nabla_j F_i^\alpha = 0. \tag{6.1}$$

From these it is easily shown that:

$$g_{sj} = g_{\alpha i} F_j^\alpha F_s^i; \quad F_i^\alpha R_{\alpha j} + F_j^\alpha R_{\alpha i} = 0; \quad R_{sj} = R_{\alpha i} F_j^\alpha F_s^i. \tag{6.2}$$

Consequently also the Z tensor $Z_{kl} = R_{kl} + \phi g_{kl}$ obeys to the same relations that is:

$$F_i^{\alpha} Z_{\alpha j} + F_j^{\alpha} Z_{\alpha i} = 0; \quad Z_{sj} = Z_{\alpha i} F_j^{\alpha} F_s^{i}. \tag{6.3}$$

We now consider a recurrent Z form defined on a Kaehlerian manifold, that is by definition the following equation:

$$\nabla_k Z_{ij} - \nabla_j Z_{ik} = \omega_k Z_{ij} - \omega_j Z_{ik}. \tag{6.4}$$

The previous relation is thus transvected with $F_l^i F_h^j$ and a rearrangement of the indices is made to obtain:

$$\nabla_k Z_{\alpha\beta} F_i^{\alpha} F_j^{\beta} - \nabla_{\beta} Z_{\alpha k} F_i^{\alpha} F_j^{\beta} = \omega_k Z_{\alpha\beta} F_i^{\alpha} F_j^{\beta} - \omega_{\beta} Z_{\alpha k} F_i^{\alpha} F_j^{\beta}. \tag{6.5}$$

From Eq. (6.3) then we have simply:

$$\nabla_k Z_{ij} - \nabla_{\beta} Z_{\alpha k} F_i^{\alpha} F_j^{\beta} = \omega_k Z_{ij} - \omega_{\beta} Z_{\alpha k} F_i^{\alpha} F_j^{\beta}. \tag{6.6}$$

Now a similar equation with indices k and i exchanged is written and the two relations are added to obtain (recalling (6.1)):

$$\begin{aligned} &\nabla_k Z_{ij} + \nabla_i Z_{kj} - \nabla_{\beta}(Z_{\alpha k} F_i^{\alpha} F_j^{\beta} + Z_{\alpha i} F_k^{\alpha} F_j^{\beta}) \\ &\quad = \omega_k Z_{ij} + \omega_i Z_{kj} - \omega_{\beta}(Z_{\alpha k} F_i^{\alpha} F_j^{\beta} + Z_{\alpha i} F_k^{\alpha} F_j^{\beta}). \end{aligned} \tag{6.7}$$

Thus from (6.3) we get (after an indices rearrangement):

$$\nabla_k Z_{ij} + \nabla_j Z_{ki} = \omega_k Z_{ij} + \omega_j Z_{ki}. \tag{6.8}$$

Now Eqs. (6.8) and (6.4) are added to get the final result:

$$\nabla_k Z_{ij} = \omega_k Z_{ij}. \tag{6.9}$$

We are able to state the following.

Theorem 6.1. *On an $n(=2m)$-dimensional Kaehler manifold the notion of Z recurrent form is equivalent to the ordinary recurrency of the Z tensor.*

From Sec. 1 we recall that if the Z tensor is recurrent, then the Ricci tensor is generalized recurrent; we state the following result.

Theorem 6.2. *On an $n(=2m)$-dimensional Kaehler manifold with Z recurrent form: then the Ricci tensor is generalized recurrent.*

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