Recursion Operators for $N \times N$ Matrix Nonlinear Evolution Equations

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In the Ablowitz-Kaup-Newell-Segur theory for $N \times N$ matrix system, the associated general nonlinear evolution equation is obtained, in which the recursion operator is directly derived from the integrability condition. Discussed are also the nonlinear superposition law and symmetries associated with the evolution equations.

§ 1. Introduction

The class of nonlinear evolution equations related to the general first order $N \times N$ matrix differential equation was extensively studied in connection with the inverse scattering method.[1],[2] In this paper, we study the evolution equations obtained as the integrability condition of the system of the differential equations for N-vector $\phi(x, t, \lambda)$

$$D\phi - Q\phi + i\lambda\tau\phi = 0, \tag{1·1}$$

$$\phi_t + A\phi = 0, \tag{1·2}$$

where $x \in \mathbf{R}$ and $t \in \mathbf{R}$ are independent variables, $Q(x, t)$ and $A(Q, \lambda)$ are $N \times N$ matrices, $\tau = \mathrm{diag}(\tau_1, \cdots, \tau_N)$ for non zero constants $\tau_1 > \cdots > \tau_N$, $\lambda \in \mathbf{C}$ a spectral parameter and $D := \partial/\partial x$, $\phi_t := \partial\phi/\partial t$. Without loss of generality, we assume that Q is off-diagonal, i.e., $Q_{ii} = 0$. Integrability condition of ϕ in (1·1) and (1·2) yields the λ dependent equation for Q as

$$Q_t + DA + i\lambda[\tau, A] - [Q, A] = 0, \tag{1·3}$$

where $[\alpha, \beta] := \alpha\beta - \beta\alpha$. We can eliminate λ from (1·3) to obtain λ independent evolution equation either by expanding A in powers of λ and equating coefficients of the power of λ to zero or by introducing "squared eigenfunction" to invert λ into recursion operator in a Hilbert space. By each of these two methods, we derive in § 2 the general evolution equation

$$Q_t + \sum_{l=1}^{N-1} f_l(L)\rho_l Q = 0 \tag{1·4}$$

from (1·3), here L is an integro-differential matrix operator[3] defined by (2·9), $f_l(L)$ ($l = 1, 2, \cdots, N-1$) arbitrary function of L and $\rho_l Q := [P_l, Q]$ with matrix unit $P_l = \{\delta_{li}\delta_{ij}\}$. It is noted that L (recursion operator) is derived by simple algebraic manipulations. The eigenfunctions of the adjoint of L which are usually named as "squared eigenfunctions" are assumed to consititute a complete set in the Hilbert space.

In § 3, proving the hereditary property of L,[4],[5] we show that L is in fact the recursion operator which transforms the symmetry[6] of (1·4) into the symmetry of (1·4). Here, the

symmetry of (1·4) is defined by the $N \times N$ matrix solution s of

$$s_t + K's = 0, \tag{1·5}$$

where $K's$ is the Gâteaux differentiation of $K[Q] = \sum_{l=1}^{N-1} f_l(L) \rho_l Q$ toward s, for the solution Q of (1·4).

Some structural properties of (1·4) is discussed in §5. First, we note that the solution of (1·4) is expressible as the linear superposition of the eigenfunction Ψ_λ of L, since there is a solution ρQ for (1·5) with the invertible matrix ρ and Ψ_λ also satisfies (1·5);

$$Q = \rho^{-1} \int_{\lambda \in \sigma(L)} d\lambda C_\lambda \Psi_\lambda. \tag{1·6}$$

Here $\sigma(L)$ denotes spectrum of L and C_λ is time-independent coefficient. We may call the property (1·6) nonlinear superposition law, which was previously proposed for $N = 2$.[4] Finally, from the invariance of (1·1) and (1·2) under the gauge transformation, we derive the symmetries of (1·3) δQ and (1·4) s

$$\delta Q = (L - \lambda) g + \sum_{l=1}^{N-1} g_l(\lambda) \rho_l Q, \tag{1·7}$$

$$s = \sum_{l=1}^{N-1} g_l(L) \rho_l Q, \tag{1·8}$$

where g is arbitrary off-diagonal matrix and $g_l(\lambda)$ arbitrary function of λ. Relations among (1·4), (1·7) and (1·8) are also discussed.

§2. General form of the evolution equation

Throughout the paper, we assume that Q and Q_t for fixed t are differentiable to any order and decrease as $|x| \to \infty$ faster than any inverse power of $|x|$. Then, the spectral problem of (1·1) can be solved for a set of generic potentials Q.[7]~[9] We rewrite (1·1) as the eigenvalue problem,

$$l\Phi_\lambda = \lambda \Phi_\lambda,$$
$$l := i\tau^{-1}(D - Q), \tag{2·1}$$

where Φ_λ is the holomorphic function in the upper or lower half of the complex λ plane under the Jost boundary condition $\Phi_\lambda \propto e^{-i\lambda \tau x}$ as $|x| \to \infty$. The inverse matrix Φ_λ^{-1} satisfies

$$\Phi_\lambda^{-1} \tau \overleftarrow{l} = \lambda \Phi_\lambda^{-1} \tau,$$
$$\overleftarrow{l} := -i\tau^{-1}(\overleftarrow{D} + Q). \tag{2·2}$$

Let $\phi_\lambda = \{\phi_{\lambda j}\}$ and $\bar{\phi}_\lambda = \{\bar{\phi}_{\lambda j}\}$ $(j = 1, 2, \cdots, N)$ be arbitrary column and row vectors in Φ_λ and Φ_λ^{-1} respectively. Then it is easy to verify that the off-diagonal part of the matrix Ψ, defined as $\Psi_n = \{\psi_{ij} | \psi_{ij} = \phi_{\lambda i} \bar{\phi}_{\lambda j}, i, j = 1, 2, \cdots, N, i \neq j\}$ satisfies

$$\tilde{L} \Psi_n = \lambda \Psi_n, \tag{2·3}$$

$$\tilde{L} := ix(D \cdot - [Q, \cdot]_n - [Q, D^{-1}[Q, \cdot]_d]), \tag{2·4}$$

where x is the inverse of $ad(\tau) \cdot = [\tau, \cdot]$, i.e., $x\Psi = \{x_{ij}\phi_{ij}\}, x_{ij} = (\tau_i - \tau_j)^{-1}$, suffices n and d mean off-diagonal and diagonal parts of the matrices respectively and $D^{-1} := \int_{-\infty}^{x} dx \cdot$. In (2·4) and the following D^{-1} may be replaced by the integral $-\int_{x}^{\infty} dx \cdot$. We introduce a Hilbert space H of off-diagonal matrix functions with the trace inner product

$$(\phi, \psi) := \int_{-\infty}^{\infty} dx \ \mathrm{tr}(\phi^{\dagger}\psi) .$$

Here $\phi^{\dagger} = \phi^{T*}$ is the complex conjugate of the transposed matrix ϕ^T of ϕ. For any operator U in H, we introduce an operator U^A through the relation $(\phi^{\dagger}, U\psi) = (\phi^{\dagger}, U^A\phi)$.

We rewrite (1·3) as

$$Q_t + ixDa - \lambda a - i[Q, xa]_n - [Q, A_d]_n = 0 , \tag{2·5}$$

$$DA_d - i[Q, xa]_d = 0 , \tag{2·6}$$

where off-diagonal matrix a is defined by $A_n = ixa$. Let the integration constant to A_d in (2·6) be $\sum_{l=1}^{N} C_l(\lambda) P_l$ with the matrix unit $P_l = \{\delta_{li}\delta_{lj}\}$ and arbitrary functions of λ, $C_l(\lambda)$. Then, (2·6) yields

$$A_d = iD^{-1}[Q, xa]_d + \sum_{l=1}^{N} C_l(\lambda) P_l . \tag{2·7}$$

Substituting (2·7) into (2·5), we have

$$Q_t + (L - \lambda)a + \sum_{l=1}^{N} C_l(\lambda)\rho_l Q = 0 , \tag{2·8}$$

$$L := i(Dx \cdot - [Q, x \cdot]_n - [Q, D^{-1}[Q, x \cdot]_d]) . \tag{2·9}$$

As verified by the direct computation, L^A coincides with $\tilde{L}$ defined by (2·4). Hence, multiplying (2·8) by $\Psi_n \in H$ from the left and constructing the inner product, one has

$$\int_{-\infty}^{\infty} dx \, \mathrm{tr}[\Psi_n(Q_t + \sum_{l=1}^{N} C_l(L)\rho_l Q)] = 0 . \tag{2·10}$$

We assume the completeness of $\{\Psi_n\}$ in H,[10] then (2·10) gives the evolution equation

$$Q_t + \sum_{l=1}^{N} C_l(L)\rho_l Q = 0 . \tag{2·11}$$

Since there is a constraint $\sum_{l=1}^{N}\rho_l = 0$, (2·11) can be rewritten in terms of $N-1$ arbitrary functions $f_l(\lambda) := C_l(\lambda) - C_N(\lambda)$ $(l = 1, 2, \cdots, N-1)$ as given by (1·4).

It is possible to obtain (2·11) by the other method. We assume that a and C_l can be expanded as $a = \sum_{-\infty}^{\infty} a_j \lambda^j$ and $C_l = \sum_{-\infty}^{\infty} C_{lj}\lambda^j$. Substituting these expansions into (2·8) and equating coefficients of each power of λ to zero, we have

$$a_{j-1} = La_j + \sum_{l=1}^{N} C_{lj}\rho_l Q, \quad (j \neq 0) \tag{2·12}$$

$$Q_t + La_0 - a_{-1} + \sum_{l=1}^{N} C_{l0}\rho_l Q = 0 . \tag{2·13}$$

Assuming further, $L^n a_n \to 0$, $L^{-n} a_{-n-1} \to 0$ as $n \to \infty$ for L and the inverse L^{-1}, we have from (2·12) $a_0 = \sum_{j=1}^{\infty}\sum^{N} L^{j-1} C_{lj}\rho_l Q$ and $a_{-1} = -\sum_{-\infty}^{-1}\sum^{N} L^j C_{lj}\rho_l Q$ and hence, from (2·13), obtain (2·11).

It is noted that, for the case $C_l(\lambda) = i a_l \lambda$, $a_l =$ constant $(l = 1, 2, \cdots, N-1)$, the integral terms in (2·11) vanish identically due to the identity $[Q, x\rho_l Q]_d = 0$ and (2·11) or (1·4) reduces to

$$Q_t - \sum_{l=1}^{N-1} a_l (Dx\rho_l Q - [Q, x\rho_l Q]_n) = 0$$

which describes locally interacting $n(\leq N((N-1))$ waves. By a choice $C_l(\lambda) = -ih(\lambda)\lambda\tau_l$ $(l = 1, 2, \cdots, N-1)$ and the identity $\sum_{l=1}^{N-1} \tau_l x\rho_l Q = Q$, we obtain also, from (2·11)

$$Q_t + h(L)DQ = 0 , \tag{2·14}$$

where $h(\lambda)$ is an arbitrary function of λ.   Note that for $N = 2$, (2·11) is always expressible in the form (2·14).[4]

§ 3.   Hereditary property of L

In this section, we show that L is a recursion operator; for the solution Q of (2·11), L satisfies

$$\left[\frac{\partial}{\partial t} + K', L \right] = 0 , \tag{3·1}$$

where K' is the operator defined by

$$K' \phi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (K[Q + \epsilon\phi] - K[Q]) ,$$

$$K[Q] := \sum_{l=1}^{N} C_l(L) \rho_l Q . \tag{3·2}$$

The fact that L is a recursion operator is essentially a consequence of the hereditary property of L described in Lemma 3.3 below.   For any specified $K[Q]$, the pair K' and L satisfying (3·1) is called "recursion pair".   The recursion pair constitutes a new Lax pair for (2·11) and hence the eigenfunction ϕ of L, for the time independent eigenvalue, satisfies

$$\phi_t + K' \phi = 0 . \tag{3·3}$$

In the following, we use the identities

$$x_{ki} - x_{kj} = \frac{x_{ki} x_{kj}}{x_{ij}}$$

and

$$x[A_n, xB_n]_n - x[B_n, xA_n]_n = [xA_n, xB_n]_n , \tag{3·4}$$

$$[A, [B_n, xC_n]_d] = [A, [C_n, xB_n]_d] \tag{3·5}$$

for matrices A, B and C, where $x[\ ,\ ]_n$ is the multiplication of x on the matrix $[\ ,\ ]_n$ as defined in § 2.   We also use symbol $K_l[Q] = \rho_l Q$ $(l = 1, 2, \cdots, N)$.

Lemma 3.1. K_l' $(l = 1, 2, \cdots, N)$ and L are recursion pair.

Proof   For $N = 1$ and $C_l(\lambda) = \delta_{l1}$, (2·11) is reduced to

$$Q_{t_1} + K_1 = 0 \,,$$

where we denote $t = t_1$ to distinguish each $K_l[Q] = \rho_l Q$ ($l = 1, 2, \cdots, N$). Since $K_1'\phi = \rho_1\phi$ and

$$L_{t_1}\phi = i\{[\rho_1 Q, \varkappa\phi]_n + [\rho_1 Q, \phi_d] + [Q, D^{-1}[\rho_1 Q, \varkappa\phi]_d]\},$$

$$[L, K']\phi = i\{D\varkappa\rho_1\phi - [Q, \varkappa\rho_1\phi]_n - [Q, D^{-1}[Q, \varkappa\rho_1\phi]_d]$$

$$- \rho_1 D\varkappa\phi + \rho_1[Q, \varkappa\phi]_n + \rho_1[Q, D^{-1}[Q, \varkappa\phi]_d]\} \,,$$

here $\phi_d := D^{-1}[Q, \varkappa\phi]_d$, the L. H. S. of (3·1) is reduced to

$$(L_{t_1} + [K', L])\phi = i\{[\rho_1 Q, \varkappa\phi]_n + [Q, \varkappa\rho_1\phi]_n - \rho_1[Q, \varkappa\phi]_n\}$$

$$+ i\{[\rho_1 Q, \phi_d] + [Q, D^{-1}[\rho_1 Q, \varkappa\phi]_d]$$

$$+ [Q, D^{-1}[Q, \varkappa\rho_1\phi]_d] - \rho_1[Q, \phi_d]\} \,. \tag{3·6}$$

The first bracket of the R. H. S. of (3·6) vanishes due to ρ_1 and the Jacobi identity. The second bracket vanishes because of the identities

$$-[\rho_1 Q, \phi_d] + \rho_1[Q, \phi_d] = -[[P_1, Q], \phi_d] + [P_1, [Q, \phi_d]]$$

$$= -[Q, [P_1, \phi_d]] = 0 \,,$$

$$[\rho_1 Q, \varkappa\phi]_d + [Q, \varkappa\rho_1\phi]_d = [[P_1, Q], \varkappa\phi]_d + [Q, [P_1, \varkappa\phi]]_d$$

$$= -[P_1, [\varkappa\phi, \phi]]_d = 0 \,.$$

For $l = 2, \cdots, N$, the proof is similar.

Lemma 3.2.   Let K_l' and L are recursion pair and $K' = \sum_{l=1}^{N} C_l K_l'$ ($C_l = $constant), then K' and L are recursion pair.

Proof   Let $L'(\phi)$ be Gâteaux derivative of L toward ϕ. It is to be noted that, since $L_t = L'(Q_t)$, the equation $L_{t_l} + [K_l', L] = 0$ is equivalent to the identity

$$-L'(K_l) + [K_l', L] \equiv 0 \tag{3·7}$$

independent of Q_{t_l}. Then, from the equation

$$Q_t + \sum_{l=1}^{N} C_l K_l[Q] = 0 \,,$$

we have

$$L_t + [K', L] = \sum_{l=1}^{N} C_l(-L'(K_l) + [K_l', L]) \,. \tag{3·8}$$

The R. H. S. of (3·8) vanishes by (3·7).

Lemma 3.3.   Bilinear operator $\Gamma(\cdot, \cdot)$

$$\Gamma(\phi, \psi) = LL'(\phi)\psi + L'(L\psi)\phi \tag{3·9}$$

is symmetric, i.e., $\Gamma(\phi, \psi) = \Gamma(\phi, \psi)$ for $\phi, \psi \in H$.

Proof   According to the order of the power of Q, we divide L and Γ as $L = \sum_{i=0}^{2} L_{(i)}$ and $\Gamma = \sum_{i=0}^{3} \Gamma_{(i)}$, where

$$L_{(0)}\phi := iDx\phi, \qquad L_{(1)}\phi := -i[Q, x\phi]_n,$$

$$L_{(2)}\phi := -i[Q, D^{-1}\tilde{\phi}_d]. \quad (\tilde{\phi}_d = [Q, x\phi]_d)$$

As easily shown from (3·9), $\Gamma_{(0)}$ takes the form

$$\Gamma_{(0)}(\phi, \psi) = L_{(0)}L_{(1)}'(\phi)\psi + L_{(1)}'(L_{(0)}\psi)\phi$$

$$= xD[\phi, x\psi]_n + [xD\psi, x\phi]_n,$$

which is symmetric due to (3·4). The first order component $\Gamma_{(1)}$ is given by

$$\Gamma_{(1)} = L_{(0)}(L_{(2)}'\phi)\psi + L_{(1)}(L_{(1)}'\phi)\psi + L_{(2)}'(L_{(0)}\psi)\phi + L_{(1)}'(L_{(1)}\psi)\phi$$

$$= xD[\phi, D^{-1}\tilde{\phi}_d] + xD[Q, D^{-1}[\phi, x\psi]_d] - [Q, x[\phi, x\psi]_n]_n$$

$$+ [xD\psi, D^{-1}\tilde{\phi}_d] + [Q, D^{-1}[xD\psi, x\phi]_d] - [[Q, x\psi]_n, x\phi]_n.$$

From (3·4), (3·5) and the Jacobi identity, we have

$$\Gamma_{(1)}(\phi, \psi) - \Gamma_{(1)}(\psi, \phi) = [x\phi, \tilde{\phi}_d] - [x\psi, \tilde{\phi}_d] + D[x\phi, D^{-1}([\phi, x\psi]_d - [\psi, x\phi]_d)]$$

$$- [Q, x[\phi, x\psi]_n - x[\psi, x\phi]_n]_n + [Q, [x\psi, x\phi]_d]$$

$$- [[Q, x\psi]_n, x\phi]_n + [[Q, x\phi]_n, x\psi]_n$$

$$= [x\phi, [Q, x\psi]]_n + [x\psi, [x\phi, Q]]_n + [Q, [x\psi, x\phi]]_n$$

$$= 0.$$

$\Gamma_{(2)}$ is constituted from four terms

$$L_1L_2'(\phi)\psi = [Q, x[\phi, D^{-1}[Q, x\psi]_d]]_n - [Q, x[Q, D^{-1}[\phi, x\psi]_d]]_n,$$

$$L_2L_1'(\phi)\psi = -[Q, D^{-1}[Q, x[\phi, x\psi]_n]_d],$$

$$L_1'(L_2\psi)\phi = -[[Q, D^{-1}[Q, x\psi]_d], x\phi]_n$$

$$= [Q, [x\phi, D^{-1}[Q, x\psi]_d]]_n - [[Q, x\phi], D^{-1}[Q, x\psi]_d]_n,$$

$$L_2'(L_1\psi)\phi = -[[Q, x\psi]_n, D^{-1}[Q, x\phi]_d] - [Q, D^{-1}[[Q, x\psi]_n, x\phi]_d].$$

The sum of the first terms of $L_1L_2'(\phi)\psi$ and of the second equation for $L_1'(L_2\psi)\phi$ vanishes. The second terms of $L_1L_2'(\phi)$ is symmetric with respect to ϕ and ψ. The sum of $L_2L_1'(\phi)\psi$ and the second term of $L_2'(L_1\psi)\phi$ is reduced to $-[Q, \tilde{\Gamma}(\phi, \psi) + \tilde{\Gamma}(\psi, \phi)]$, where $2\tilde{\Gamma}(\phi, \psi) = [Q, [x\psi, x\phi]_n + x[\phi, x\psi]_n] + [\phi, x[Q, x\psi]_n]_d$, while the second term of $L_1'(L_2\psi)$ is symmetric with the first term of $L_2'(L_1\psi)\phi$. It is shown that

$$\Gamma_{(3)} = -[Q, D^{-1}[Q, x[\phi, D^{-1}[Q, x\psi]_d]]_d] - [Q, D^{-1}[Q, x[Q, D^{-1}[\phi, x\psi]_d]]_d]$$

$$- [[Q, D^{-1}[Q, x\psi]_d], D^{-1}[Q, x\phi]_d] - [Q, D^{-1}[[Q, x\psi]_d, x\phi]_d].$$

The sum of the first and the fourth terms vanishes. Each of the second and the third term is symmetric with repect to ϕ and ψ. Thus, the proof is completed.

Lemma 3.4.   Let $K^j[Q] = L^j \rho_l Q$ $(j \in \mathbf{Z})$ for any fixed l.   If $(K^j)'$ and L are the recursion pair, then so $(K^{j\pm1})'$ and L.

Proof   Since $(K^{j+1})' \psi = L'(\psi) K^j + L(K^j)' \psi$, we have

$$\left[\frac{\partial}{\partial t} + (K^{j+1})', L\right]\psi = -L'(LK^j)\psi + L'(L\psi)K^j + L(K^j)'L\psi - LL'(\psi)K^j - L^2(K^j)'\psi$$

$$= -\Gamma(\psi, K^j) + \Gamma(K^j, \psi) + L[-L'(K^j) + (K^j)'L - L(K^j)']\psi .$$

From Lemma 3.3 the first two terms of the R. H. S. vanish and hence one can immediately obtain the statement for K^{j+1}.   The proof for K^{j-1} is similar.

Collecting Lemmas 3.1~3.4, we have a theorem.

THEOREM K' and L defined by (3·2) and (2·9) respectively for arbitrary $C_l(\lambda)$ $(l=1, 2, \cdots, N)$ are the recursion pair for the evolution equation (2·11).

§4.   Superposition law and symmetries

We discuss two characteristic structures of (1·4) or (2·11), i.e., superposition law for the solutions of (2·11) and infinitesimal symmetries.

First, we note that a solution of (2·11) is expressible, in the form of a linear superposition in term of the eigenfunction $\phi_\lambda(t)$ in H with the time independent eigenvalue λ.   To see this, first we write solutions of the linear equation (3·3) in H in terms of a linear operator $T(t)$ as

$$\phi(t) = T(t)\phi(0), \qquad T_t + K'(t)T(t) = 0 . \qquad (4·1)$$

Since by the Lax theory we have

$$T(t)L(0)T(t)^{-1} = L(t)$$

and the spectrum of $L(t)$ is invariant under the evolution by $T(t)$, we may write, under the assumption of the completeness of $\{\phi_\lambda | \lambda \in \sigma(L)\}$

$$\phi(t) = \int_{\lambda \in \sigma(L)} d\lambda\, C_\lambda \phi_\lambda(t) , \qquad (4·2)$$

where $\phi_\lambda(t)$ evolves by $T(t)$ whereas C_λ is time independent and is determined at $t=0$ from $\phi(0)$.   On the other hand, any symmetry s of (2·11) also satisfies (3·3) and hence its evolution is expressible in the form (4·2).   We show in the following that $\rho_l Q$ $(l=1, 2, \cdots, N)$ is the symmetry.   Thus, for such constants a_l that the matrix $\rho = \sum_{l=1}^N a_l \rho_l$ has the inverse ρ^{-1}, one can construct Q in the form

$$Q(t) = \rho^{-1} \int_{\lambda \in \sigma(L)} d\lambda\, C_\lambda \phi_\lambda(t) , \qquad (4·3)$$

where C_λ is determined from

$$Q(0) = \rho^{-1} \int_{\lambda \in \sigma(L)} d\lambda\, C_\lambda \phi_\lambda(0) . \qquad (4·4)$$

We can rewrite (4·3) as

$$Q(t) = \rho^{-1} \int_{\lambda \in \sigma(L)} d\lambda\, C_\lambda T(t) \phi_\lambda(0) . \qquad (4·5)$$

The set of $(4\cdot4)$ and $(4\cdot5)$ is a nonlinear analogue of the Fourier method for initial value problem for linear evolution equation. In this sense, we may call the formula $(4\cdot3)$ nonlinear superposition law in the solution space of $(2\cdot11)$.

Finally, we note the symmetry property of $(2\cdot11)$. The system $(1\cdot1)$ and $(1\cdot2)$ is invariant under the gauge transformation $\phi \to \tilde{\phi} = G\phi$ $(G \in GL(N))$, $Q \to \tilde{Q} = (GQ + DG + i\lambda[\tau, G])G^{-1}$ and $A \to \tilde{A} = (-G_t + GA)G^{-1}$. It is easy to see that these transformations constitute a group and have the infinitesimal form $G = 1 + \delta G$, $\tilde{Q} = Q + \delta Q$, and $\tilde{A} = A + \delta A$, where

$$\delta Q = [\delta G, Q] + D\delta G + i\lambda[\tau, \delta G], \qquad (4\cdot6)$$

$$\delta A = -\delta G_t + [\delta B, A]. \qquad (4\cdot7)$$

Since $(1\cdot3)$ is invariant under the variations $(4\cdot6)$ and $(4\cdot7)$, δQ defined by $(4\cdot6)$ is a symmetry of $(1\cdot3)$ and hence of $(2\cdot8)$ for arbitrary δG. We can rewrite $(4\cdot6)$ as

$$\delta Q_d = 0 = [ixg, Q]_d + D\delta G_d, \qquad (4\cdot8)$$

$$\delta Q_n = [\delta G, Q]_n + ixDg - \lambda[\tau, xg]_n, \qquad (4\cdot9)$$

where δG is decomposed into $\delta G = \delta G_d + ixg$ $(g \in H)$. Let the integration constant to δG_d obtained from $(4\cdot8)$ be $\sum_{l=1}^{N} g_l(\lambda)P_l$ where $g_l(\lambda)$ is arbitrary function, then $(4\cdot9)$ reduces to

$$\delta Q = \delta Q_n = (L - \lambda)g + \sum_{l=1}^{N} g_l(\lambda)\rho_l Q. \qquad (4\cdot10)$$

From $(4\cdot10)$, we have λ independent symmetry s of $(2\cdot11)$ by the same method used to obtain $(2\cdot11)$ from $(2\cdot8)$. Thus for instance, by expanding g in powers of λ and equating each power of λ equal to zero (c.f., $(2\cdot12)$, $(2\cdot13)$), one has

$$s = \sum_{l=1}^{N} g_l(L)\rho_l Q. \qquad (4\cdot11)$$

In particular, each $\rho_m Q$ $(m = 1, 2, \cdots, N-1)$ is symmetries corresponding to the choice $g_l(\lambda) = \delta_{lm}$ under the constraint $\sum_{l=1}^{N}\rho_l = 0$. Symmetry DQ associated with x translation is obtained by the choice $g_l(\lambda) = -i\lambda\tau_l$. Evolution equations $(2\cdot11)$ and $(2\cdot14)$ are interpreted by these symmetries.[4]

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