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Recursions for Distribution Functions and Stop-Loss Transforms

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Abstract

For any function f on the non-negative integers, we can evaluate the cumulative function Γf given by $\Gamma f(s) = \sum_{x=0}^{s} f(x)$ from the values of f by the recursion $\Gamma f(s) = \Gamma f(s-1) + f(s)$. Analogously we can use this procedure t times to evaluate the t-th order cumulative function $\Gamma^t f$. As an alternative, in the present paper we shall derive recursions for direct evaluation of $\Gamma^t f$ when f itself satisfies a certain sort of recursion. We shall also derive recursions for the t-th order tails $\Lambda^t f$ where $\Lambda f(s) = \sum_{x=s+1}^{\infty} f(x)$. The recursions can be applied for exact and approximate evaluation of distribution functions and stop-loss transforms of probability distributions. The class of recursions for f includes the classes discussed by Sundt (1992), incorporating the class studied by Panjer (1981). We discuss in particular convolutions and compound functions.

Keywords: aggregate claims, probability function, distribution function, stop-loss transform, recursive evaluation.

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1 Introduction

Since the publication of Panjer's (1981) paper there has grown up an extensive literature on recursive evaluation of the probability function of discrete compound distributions with severity distributions on the non-negative integers. Panjer assumed that the probability function p of the counting distribution satisfies

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1) \qquad (n = 1, 2, ...) \tag{1}$$

for some a and b.

In Sundt (1992) the following generalisation of Panjer's class of counting distributions is considered:

$$p(n) = \sum_{x=1}^{k} \left(a(x) + \frac{b(x)}{n}\right) p(n-x) \qquad (n = 1, 2, ...) \tag{2}$$

for some positive integer k and functions a and b on $\{1, 2, ..., k\}$ with $p(n) = 0$ for $n < 0$.

Almost the whole literature on recursive evaluation of probability distributions is restricted to the derivation of recursions for the probability functions. There are only a few references where recursions are considered for the distribution function and/or the stop-loss transform. A recursive algorithm for the distribution function of a convolution of discrete uniform distributions can be found in Sundt (1988). In Sundt (1992) recursions are derived for the distribution function and the stop-loss transform of a compound distribution whose counting distribution has a probability function satisfying the recursion (2) with b identical to zero. The compound geometric case is considered in Sundt (1982). Waldmann (1995) considers a recursion for the distribution function of compound distributions having a counting distribution satisying the recursion (1).

In this paper we shall derive recursions for distribution functions and stop-loss transforms within a general class of discrete probability distributions.

Compound distributions with counting distributions satisfying the recursion (2) appear as a special case. For the Panjer class of counting distributions our recursion for the distribution function seems to be an efficient reformulation of Waldmann's recursion.

We propose to use the recursions for distribution functions and stop-loss transforms, rather than using the known recursions for the probability function and then making the appropriate summations. Although these new recursions will not always give rise to time-reduction, there is an advantage in that the distribution function and the stop-loss transform are monotonic functions which will give some stability advantages for the recursions for evaluating these values, cf. Waldmann (1995). As an application, we shall use a result of Panjer & Wang (1993) to derive conditions under which the recursion for the distribution function of the number of claims in an insurance portfolio (individual model) is strongly stable.

To allow for application of our results not only to proper probability distributions, but also to approximations which are not necessarily probability distributions themselves, we shall derive our results for more general functions.

2 Main result

Let $\mathcal{F}_0$ denote the class of functions g on the non-negative integers with $g(0) > 0$. In the remainder of this paper, for any $g \in \mathcal{F}_0$, we will set $g(s) = 0$ if $s < 0$. For functions f on the non-negative integers the summation operator Γ is defined by

$$\Gamma f(s) = \sum_{x=0}^{s} f(x) \qquad (s = 0, 1, ...)$$

Further, let

$$\Gamma^0 f = f$$

$$\Gamma^{t+1} = \Gamma\left(\Gamma^t\right) \qquad (t = 0, 1, ...)$$

Our main result is stated in the following theorem.

Theorem 1 *If* $f \in \mathcal{F}_0$ *satisfies the recursion*

$$f(s) = \frac{g(s)}{s} + \sum_{x=1}^{s} \left(a(x) + \frac{b(x)}{s}\right) f(s-x) \qquad (s = 1, 2, ...) \qquad (3)$$

then for $t = 0, 1, 2, ...$, $\Gamma^t f$ *satisfies the recursion*

$$\Gamma^t f(s) = \frac{\Gamma^t g(s)}{s} + \sum_{x=1}^{s} \left(a(x) + \frac{b_t(x)}{s}\right) \Gamma^t f(s-x) \qquad (s = 1, 2, ...) \quad (4)$$

with

$$b_t(x) = b(x) + t\,(1 - \Gamma a(x - 1))$$

and $a(0) = g(0) = 0$.

Proof. We first prove (4) for $t = 1$, that is

$$s\,\Gamma f(s) = \Gamma g(s) + \sum_{x=1}^{s} [s\,a(x) + b(x) + 1 - \Gamma a(x - 1)]\,\Gamma f(s-x) \qquad (s = 1, 2, ...) \qquad (5)$$

by induction on s .

It is easily shown that (5) holds for $s = 1$.

Let us now assume that (5) holds for $s = r$. By application of (5) and (3) we obtain

$$\begin{aligned}
&(r + 1)\Gamma f(r + 1) \\
&\quad = (r + 1)\,(\Gamma f(r) + f(r + 1)) \\
&\quad = r\Gamma f(r) + (r + 1)f(r + 1) + \Gamma f(r) \\
&\quad = \Gamma g(r) + \sum_{x=1}^{r} [ra(x) + b(x) + 1 - \Gamma a(x - 1)]\,\Gamma f(r - x) \\
&\qquad + g(r + 1) + \sum_{x=1}^{r+1} [(r + 1)a(x) + b(x)]\,f(r + 1 - x) + \Gamma f(r) \\
&\quad = \Gamma g(r + 1) + \sum_{x=1}^{r+1} [(r + 1)a(x) + b(x) + 1 - \Gamma a(x - 1)]\,\Gamma f(r + 1 - x) + I
\end{aligned}$$

with

$$
\begin{aligned}
I &= \Gamma a(r) f(0) + \sum_{x=1}^{r} [\Gamma a(x-1) f(r+1-x) - a(x)\Gamma f(r-x)] \\
&= \Gamma a(r) f(0) + \sum_{x=1}^{r} [\Gamma a(x-1) f(r+1-x) + \Gamma a(x-1)\Gamma f(r-x) \\
&\quad - \Gamma a(x-1)\Gamma f(r-x) - a(x)\Gamma f(r-x)] \\
&= \Gamma a(r) f(0) + \sum_{x=1}^{r} [\Gamma a(x-1)\Gamma f(r+1-x) - \Gamma a(x)\Gamma f(r-x)] \\
&= \Gamma a(0)\Gamma f(r) \\
&= 0
\end{aligned}
$$

This proves that (5) holds for $s = r + 1$. Induction now gives that (5) holds for all positive integers s, that is, the theorem holds for $t = 1$.

It remains to show that (4) also holds for $t > 1$. We once more apply induction. We assume that (4) holds for t equal to a positive integer r. By applying the case $t = 1$ to the function $\Gamma^r f$ we now easily obtain that (4) holds for $t = r + 1$, and by induction we obtain that (4) holds for all non-negative integers t.

This completes the proof of the theorem.

$\square$

Let us now assume that $f \epsilon \mathcal{F}_0$ is the probability function of a random variable S with a positive probability in zero, and satisfies the recursion (3). A recursion for the distribution function Γf is given by (4) with $t = 1$. The quantity $\Gamma^{t+1} f(s)$ $(t = 0, 1)$ can be interpreted as the expectation of a function of S. Indeed, one can prove that

$$\Gamma^{t+1} f(s-1) = E\Big[(s-S)_+^t\Big] \qquad (t = 0, 1; \quad s = 1, 2 \ldots)$$

As we have that

$$E[(S-s)_+] = E[(s-S)_+] + E[(S-s)]$$

we find that

$$E[(S-s)_+] = \Gamma^2 f(s-1) + E(S) - s \tag{6}$$

so that the stop-loss transform $\overline{f}$ of f defined by

$$\overline{f}(s) = E[(S - s)_+] \qquad (s = 0, 1, ...)$$

can be evaluated recursively by (4) and (6).

Instead of using (4) and (6) one could also start with evaluating the probabilty function from (3) and then using

$$E[(S - s)_+] = 2E[(S - (s - 1))_+] - E[(S - (s - 2))_+] + f(s - 1)$$

for $s = 2, 3, ...$ in order to evaluate the stop-loss transform. It is clear that this way of evaluation will mostly give rise to a computation time of the same order of magnitude. Nevertheless, if f is a probability function, then the recursion (4) has, for $t \geq 1$, the nice property of producing increasing values which will influence the stability of the recursion. Further research has to be done on this matter. A first attempt is made in Subsection 5.2.

A function $p \in \mathcal{F}_0$ is said to be in the form $R_k[a, b]$ if it satisfies the recursion (2), cf. Dhaene & Sundt (1994). In this case we will always silently assume that $a(x) = b(x) = 0$ for $x > k$. From Theorem 1 we immediately obtain the following corollary.

Corollary 1 *If $p \in \mathcal{F}_0$ is in the form $R_k[a, b]$, then for $t = 0, 1, 2, ...$ $\Gamma^t p$ is in the form $R_\infty[a, b_t]$ with*

$$b_t(x) = b(x) + t(1 - \Gamma a(x - 1)) \qquad (x = 1, 2, ...)$$

Sundt (1992) considered functions $p \in \mathcal{F}_0$ that satisfy the recursion

$$p(s) = \sum_{x=1}^{k} \left(a(x) + \frac{b(x)}{s}\right) p(s - x) \qquad (s = m + 1, m + 2, ...) \tag{7}$$

which is more general than (2). We easily see that p satisfies the recursion (3) with

$$g(s) = s\left[p(s) - \sum_{x=1}^{k} \left(a(x) + \frac{b(x)}{s}\right) p(s - x)\right] \qquad (s = 1, 2, ..., m)$$

and $g(s) = 0$ for $s > m$. Thus we can apply Theorem 1 for recursive evaluation of $\Gamma^t p$.

3 Convolutions

The convolution of two functions f and g on the non-negative integers is defined by

$$(f * g)(x) = \sum_{y=0}^{x} f(y)g(x-y) \qquad (x = 0, 1, 2, ...)$$

and the n-fold convolution f^{*n} of f by

$$f^{*0}(x) = 1 \qquad (x = 0, 1, 2, ...)$$

$$f^{*n} = f * f^{*(n-1)} \qquad (n = 1, 2, ...)$$

For simplicity we restrict to probability functions for the rest of Section 4. However, the results also hold for more general functions.

If a probability function $f \in \mathcal{F}_0$ is in the form $R_k[a,b]$, then we say that f is $R_k[a,b]$; in this case f is uniquely determined by a and b. Sundt (1992) discussed convolutions of such probability functions. In particular he showed that the convolution $f = *_{j=1}^{m} f_j$, where f_j is $R_k[a,b_j]$ $(j = 1, ..., m)$, is $R_k[a,b]$ with

$$b(x) = (m-1)xa(x) + \sum_{j=1}^{m} b_j(x) \qquad (x = 1, 2, ..., k) \tag{8}$$

By combining (8) with Theorem 1 we can evaluate $\Gamma^t f$ recursively.

In particular we see from (8) and Corollary 1 that if f is $R_k[a,b]$, then $\Gamma^t f^{*m}$ is $R_\infty[a, b_{m,t}]$ with

$$b_{m,t}(x) = mb(x) + (m-1)xa(x) + t(1 - \Gamma a(x-1)) \qquad (x = 1, 2, ...)$$

Sundt (1992) showed that any probability function $f \in \mathcal{F}_0$ can be expressed in the form $R_\infty[a,b]$ with

$$a(x) = -\frac{f(x)}{f(0)} \qquad b(x) = 2x\frac{f(x)}{f(0)} \qquad (x = 1, 2, ...) \tag{9}$$

By combining (8) and (9) we find that $\Gamma^t f^{*m}$ is $R_\infty[a, b_{m,t}]$ with

$$b_{m,t}(x) = \frac{1}{f(0)}[(m+1)xf(x) + t\Gamma f(x-1)] \qquad (x = 1, 2, ...)$$

that is, for any probability function $f \in \mathcal{F}_0$, $\Gamma^t f^{*m}$ satisfies the recursion

$$\Gamma^t f^{*m}(s) = \frac{1}{f(0)} \sum_{x=1}^{s} \left[\left((m+1)\frac{x}{s} - 1 \right) f(x) + \frac{t}{s}\Gamma f(x-1) \right] \Gamma^t f^{*m}(s-x) \qquad (s = 1, 2, ...)$$

For $t = 0$, this recursion was deduced by De Pril (1985).

4 Compound functions

4.1 The general class

Let $\mathcal{F}_+$ denote the class of functions on the positive integers. For $p \in \mathcal{F}_0$ and $h \in \mathcal{F}_+$ we define the compound function $p \vee h \in \mathcal{F}_0$ by

$$(p \vee h)(s) = \sum_{n=0}^{s} p(n) h^{*n}(s) \qquad (s = 0, 1, 2, ...)$$

If p and h are probability functions, then $p \vee h$ is the probability function of a compound distribution with counting probability function p and severity probability function h.

The following theorem is a trivial generalisation of Theorem 3.1 in Dhaene, Sundt & De Pril (1995).

Theorem 2 *If $h \in \mathcal{F}_+$ and $p \in \mathcal{F}_0$ satisfies the recursion*

$$p(n) = r(n) + \sum_{x=1}^{n} \left(a(x) + \frac{b(x)}{n} \right) p(n-x) \qquad (n = 1, 2, ...) \qquad (10)$$

then $p \vee h$ satisfies the recursion

$$(p \vee h)(s) = (r \vee h)(s) + \sum_{x=1}^{s} (p \vee h)((s-x) \sum_{y=1}^{x} \left(a(y) + \frac{b(y)}{y}\frac{x}{s} \right) h^{*y}(x) \qquad (s = 1, 2, ...) \qquad (11)$$

We see that the recursion (10) is equivalent with the recursion (3) with $g(s) = sr(s)$. Furthermore, we also have that the recursion (11) is in the form (3) as we can rewrite (11) as

$$(p \vee h)(s) = \frac{g(s)}{s} + \sum_{x=1}^{s} \left(c(x) + \frac{d(x)}{s} \right) p(s-x) \qquad (s = 1, 2, ...)$$

with

$$g(s) = s\,(r \vee h)(s) \qquad (s = 1, 2, ...) \tag{12}$$

$$c(x) = \sum_{y=1}^{x} a(y) h^{*y}(x) \qquad (x = 1, 2, ...)$$

$$d(x) = x \sum_{y=1}^{x} \frac{b(y)}{y} h^{*y}(x) \qquad (x = 1, 2, ...)$$

Combining this with Theorem 1 gives the following corollary.

Corollary 2 *If $h \in \mathcal{F}_+$ and $p \in \mathcal{F}_0$ satisfies the recursion* (10)*, then for $t = 0, 1, 2, ...$ $\Gamma^t(p \vee h)$ satisfies the recursion*

$$\begin{aligned} \Gamma^t(p \vee h)(s) \quad = \quad & \frac{\Gamma^t g(s)}{s} + \sum_{x=1}^{s} \left[\sum_{y=1}^{x} \left(a(y) + \frac{b(y)}{y} \frac{x}{s} \right) h^{*y}(x) \right. \\ & \left. + \frac{t}{s} \left(1 - \sum_{y=1}^{x} a(y) \Gamma(h^{*y})(x-1) \right) \right] \Gamma^t(p \vee h)(s-x) \qquad (s = 1, 2, ...) \end{aligned}$$

with g given by (12)*.*

4.2 Panjer's class

Let us now consider the special case where p is in the form $R_1[a, b]$. Then we find from Corollary 2 that $\Gamma^t(p \vee h)$ can be evaluated recursively by

$$\Gamma^t(p \vee h)(s) = \sum_{x=1}^{s} \left[\left(a + b\,\frac{x}{s} \right) h(x) + \frac{t}{s}(1 - a\,\Gamma h(x-1)) \right] \Gamma^t(p \vee h)(s-x) \qquad (s = 1, 2, ...) \tag{13}$$

Let us now assume that p and h are probability functions.

When $t = 0$, (13) reduces to Panjer's (1981) well-known recursion

$$(p \vee h)(s) = \sum_{x=1}^{s} \left(a + b\,\frac{x}{s}\right) h(x)(p \vee h)(s-x) \qquad (s = 1, 2, \ldots)$$

Let us now consider the case $t = 1$. Then we have for $s = 1, 2, \ldots$. that

$$\Gamma(p \vee h)(s) = \sum_{x=1}^{s} \left[\left(a + b\,\frac{x}{s}\right) h(x) + \frac{1}{s}(1 - a\,\Gamma h(x-1))\right] \Gamma(p \vee h)(s-x) \quad (14)$$

We obtain

$$\begin{aligned}
\sum_{x=1}^{s} \Gamma h(x-1)\Gamma(p \vee h)(s-x) &= \sum_{x=0}^{s-1} \Gamma(p \vee h)(x)\Gamma h(s-x-1) \\
= \sum_{x=0}^{s-1} \Gamma(p \vee h) \sum_{y=1}^{s-x-1} h(y) &= \sum_{y=1}^{s-1} \sum_{x=0}^{s-y-1} \Gamma(p \vee h)(x) \\
= \sum_{y=1}^{s} h(y)\Gamma^2(p \vee h)(s-y-1) &
\end{aligned}$$

Introduction of this expression in (14) gives

$$\begin{aligned}
s\,\Gamma(p \vee h)(s) = \sum_{x=1}^{s} (as + bx)h(x)\Gamma(p \vee h)(s-x) + \Gamma^2(p \vee h)(s-1) \\
-a \sum_{x=1}^{s} h(x)\Gamma^2(p \vee h)(s-x-1) \qquad (s = 1, 2, \ldots)
\end{aligned}$$

This recursion was derived by Waldmann (1995). The recursion (14) is an efficient reformulation of Waldmann's recursion.

5 The case $a \equiv 0$

5.1 A general result

It is easy to see that a function $f \in \mathcal{F}_0$ always satisfies a recursion of the form

$$f(s) = \frac{1}{s} \sum_{x=1}^{s} b(x) f(s-x) \qquad (s = 1, 2, \ldots) \tag{15}$$

with the function b uniquely determined by f. Recursions in the form (15) appear in many areas, in particular in actuarial science where relevant references include White & Greville (1959), De Pril (1989), Dhaene & De Pril (1994), Dhaene & Sundt (1996), Sundt (1995) and Sundt, Dhaene and De Pril (1996).

As $b(x)$ may alternate between positive and negative values when x varies, stability problems may arise, see e.g. Panjer & Wang (1993) who also state a definition of strong stability.

From Theorem 1 we find that $\Gamma^t f$ can be evaluated by

$$\Gamma^t f(s) = \frac{1}{s} \sum_{x=1}^{s} (b(x) + t)\Gamma^t f(s-x) \qquad (s = 1, 2, ...) \tag{16}$$

From (16) we see that if $b(x) > -t$ for all x, then the coefficients in the recursion for $\Gamma^t f$ are positive so that this recursion is strongly stable, see Panjer & Wang (1993). It is interesting to note that the greater t is, the more likely it is that the recursion for evaluating $\Gamma^t f$ is strongly stable. Moreover, if the recursion for $\Gamma^t f$ is strongly stable, then the recursion for $\Gamma^s f$, $s \geq t$ is strongly stable. Thus we see that if b is bounded, then it is always possible to obtain a stable recursion for $\Gamma^s f$ by choosing s sufficiently large. From $\Gamma^s f$ we can evaluate $\Gamma^t f$ for $t < s$ by taking differences. The evaluation of these differences will not accumulate errors, and thus one might feel tempted to conclude that we have also found a stable way of evaluating $\Gamma^t f$. However, this is not necessarily the case as the differences would be of a different order of magnitude than $\Gamma^s f$.

5.2 The number of claims in the individual risk theory model

As an example we will derive stability conditions for the recursions related to the number of claims in the individual risk theory model, see White & Greville (1959) and De Pril (1989).

Let N be the number of claims occurred during a certain reference period in a portfolio consisting of m independent risks, labelled from 1 to m. Risk i either produces a claim (with probability q_i) or no claim (with probability $p_i = 1 - q_i$). We assume that

$$0 < q_i < 1/2 \qquad (i = 1, 2, ..., m) \tag{17}$$

From White & Greville (1959) we find that the probability function f of the number of claims produced during the reference period can be evaluated recursively by

$$f(s) = \frac{1}{s} \sum_{x=1}^{s} (-1)^{x+1} f(s - x) \sum_{i=1}^{m} \left(\frac{q_i}{p_i}\right)^x \qquad (s = 1, 2, ..., m)$$

We see that this recursion is of the form (15) with

$$b(x) = (-1)^{x+1} \sum_{i=1}^{m} \left(\frac{q_i}{p_i}\right)^x \qquad (x = 1, 2, ...) \tag{18}$$

From (16) we obtain

$$\Gamma^t f(s) = \frac{1}{s} \sum_{x=1}^{s} \left[t + (-1)^{x+1} \sum_{i=1}^{m} \left(\frac{q_i}{p_i}\right)^x\right] \Gamma^t f(s-x) \qquad (s = 1, 2, ..., m)$$

The alternating sign of b may cause stability problems. However, from (17) and (18) we see that

$$b(x) \geq b(2) = \sum_{i=1}^{m} \left(\frac{q_i}{p_i}\right)^2 \qquad (x = 1, 2, ...)$$

Hence, if $b(2) > -t$, or equivalently,

$$\sum_{i=1}^{m} \left(\frac{q_i}{p_i}\right)^2 < t$$

then $b(x) > -t$ for $x = 1, 2, ...$, and the recursion for $\Gamma^s f$ is strongly stable for all integers $s \geq t$.

6 Other classes of recursions

6.1 The general case

In the present subsection we shall deduce an alternative to Theorem 1 for recursive evaluation of $\Gamma^t f$.

For functions h on the integers we define the difference operator ∇ by

$$\nabla h(x) = h(x) - h(x-1)$$

We also introduce the notation

$$\nabla^0 h = h$$

$$\nabla^t h = \nabla(\nabla^{t-1} h) \qquad (t = 1, 2, ...)$$

Theorem 3 *If $f \in \mathcal{F}_0$ satisfies the recursion* (10)*, then for $t = 0, 1, 2, ...$ $\Gamma^t f$ satisfies the recursion*

$$\Gamma^t f(s) = r(s) + \sum_{x=1}^{s} \left(\nabla^t a(x) + \frac{\nabla^t b(x)}{s} \right) \Gamma^t f(s-x) \qquad (s = 1, 2, ...) \quad (19)$$

with $a(0) = -1$, $a(x) = 0$ for $x < 0$ and $b(x) = 0$ for $x \leq 0$.

Proof. From (10) we immediately see that (19) holds for $t = 0$.

Now let us assume that (19) holds for $t = u$. We shall prove that it also holds for $t = u + 1$. For $s = 1, 2, ...$ we have

$$\begin{aligned}
\Gamma^{u+1} f(s) &= \Gamma^{u+1} f(s-1) + \Gamma^u f(s) \\
&= \Gamma^{u+1} f(s-1) + r(s) + \sum_{x=1}^{s} \left(\nabla^u a(x) + \frac{\nabla^u b(x)}{s} \right) \Gamma^u f(s-x) \\
&= \Gamma^{u+1} f(s-1) + r(s) + \sum_{x=1}^{s} \left(\nabla^u a(x) + \frac{\nabla^u b(x)}{s} \right) (\Gamma^{u+1} f(s-x) - \\
&\quad \Gamma^{u+1} f(s-x-1)) \\
&= \Gamma^{u+1} f(s-1) + r(s) + \sum_{x=1}^{s} \left(\nabla^u a(x) + \frac{\nabla^u b(x)}{s} \right) \Gamma^{u+1} f(s-x) -
\end{aligned}$$

$$\sum_{x=2}^{s}\left(\nabla^u a(x-1)+\frac{\nabla^u b(x-1)}{s}\right)\Gamma^{u+1}f(s-x)$$

$$= r(s)+\sum_{x=1}^{s}\left(\nabla^{u+1}a(x)+\frac{\nabla^{u+1}b(x)}{s}\right)\Gamma^{u+1}f(s-x)$$

that is, (19) holds for $t = u+1$.

The theorem is now proved by induction. □

If $a(x) = b(x) = 0$ for all x greater than some finite k, then $\nabla^t a(x) = \nabla^t b(x) = 0$ for all $x > k+t$ so that we can rewrite (10) and (19) as

$$f(s) = r(s)+\sum_{x=1}^{k}\left(a(x)+\frac{b(x)}{s}\right)f(s-x) \qquad (s = 1,2,...) \tag{20}$$

$$\Gamma^t f(s) = r(s)+\sum_{x=1}^{k+t}\left(\nabla^t a(x)+\frac{\nabla^t b(x)}{s}\right)\Gamma^t f(s-x) \qquad (s = 1,2,...) \tag{21}$$

with $\Gamma^t f(s) = 0$ for all $s < 0$. On the other hand, a similar property does not hold for b_t defined in Theorem 1 for $t > 0$, and thus it seems that the recursion (21) will be more convenient than the recursion (4) if k is small.

With $r \equiv 0$ we see from (21) that if f is in the form $R_k[a,b]$, then $\Gamma^t f$ is in the form $R_{k+t}[\nabla^t a, \nabla^t b]$.

Let us now define the tail operator Λ for functions h on the integers by

$$\Lambda h(s) = \sum_{x=s+1}^{\infty} h(x)$$

assuming that this sum exists and is finite. Further, let

$$\Lambda^0 h = h$$

$$\Lambda^t h = \Lambda(\Lambda^{t-1}h) \qquad (t = 1,2,...)$$

If h is a probabilty function, then Λh is the tail of the distribution. The stop-loss transform $\overline{h}$ is easily found from $\Lambda^2 h$ as $\overline{h} = \Lambda^2 h(s-1)$ for $s = 0,1,\ldots$.

The following theorem gives a similar recursion to (21) for $\Lambda^t f$ and can be proved in the same way as Theorem 3.

Theorem 4 *If $f \in \mathcal{F}_0$ satisfies the recursion*

$$f(s) = r(s) + \sum_{x=1}^{k} \left(a(x) + \frac{b(x)}{s} \right) f(s-x) \qquad (s = 1, 2, ...)$$

with $k < \infty$, then for $t = 0, 1, 2, ...$ $\Lambda^t f$ satisfies the recursion

$$\Lambda^t f(s) = (-1)^t r(s) + \sum_{x=1}^{k+t} \left(\nabla^t a(x) + \frac{\nabla^t b(x)}{s} \right) \Lambda^t f(s-x) \qquad (s = 1, 2, ...) \tag{22}$$

with $a(0) = -1$, $a(x) = 0$ for $x < 0$ and $b(x) = 0$ for $x \leq 0$.

As $\Lambda^t f(s)$ is in general not equal to zero for $s < 0$, we cannot apply the recursion (21) to evaluate $\Lambda^t f$ when $k = \infty$. For the same reason, the assumption that f is in the form $R_k[a, b]$, does not imply that $\Lambda^t f$ is in the form $R_{k+t}[\Lambda^t a, \Lambda^t b]$.

6.2 The case $b \equiv 0$

Let us now assume that f satisfies the recursion (10) with $b \equiv 0$, that is,

$$f(s) = r(s) + \sum_{x=1}^{s} a(x) f(s-x) \qquad (s = 1, 2, ...) \tag{23}$$

We see that in this case the recursion given by Theorem 3 is also in the form (23). We shall now deduce an alternative recursion for $\Gamma^t f$.

Theorem 5 *If $f \in \mathcal{F}_0$ satisfies the recursion (23), then for $t = 0, 1, 2, ...$ $\Gamma^t f$ satisfies the recursion*

$$\Gamma^t f(s) = \Gamma^t r(s) + \sum_{x=1}^{s} a(x) \Gamma^t f(s-x) \qquad (s = 1, 2, ...) \tag{24}$$

with $r(0) = f(0)$.

Proof. The recursion (24) trivially holds for $t = 0$. Let us now assume that it holds for $t = u$. Then, for $s = 0, 1, 2, ...$, we have

$$\Gamma^{u+1} f(s) \;\; = \sum_{x=0}^{s} \Gamma^u f(x) = \Gamma^u r(0) + \sum_{x=1}^{s} \left(\Gamma^u r(x) + \sum_{y=1}^{x} a(y) \Gamma^u f(x-y) \right)$$

$$= \Gamma^{u+1} r(s) + \sum_{y=1}^{s} a(y) \Gamma^{u+1} f(s-y)$$

Thus (24) also holds for $t = u + 1$, and by induction, it follows that (24) holds for all t. $\square$

We shall now deduce a recursion for the tails $\Lambda^t f$.

Theorem 6 *If $f \in \mathcal{F}_0$ satisfies the recursion* (23), *then for* $t = 0, 1, 2, \ldots$ *$\Lambda^t f$ satisfies the recursion*

$$\Lambda^t f(s) = \Lambda^t r(s) + \Lambda f(-1) \sum_{j=1}^{t} \Lambda^j a(s) + \sum_{x=1}^{s} a(x) \Lambda^t f(s-x) \quad (s = 1, 2, \ldots) \quad (25)$$

Proof. We shall prove the special case $t = 1$; the general case follows easily by induction. For $s = 1, 2, \ldots$ we have

$$\begin{aligned}
\Lambda f(s) &= \sum_{x=s+1}^{\infty} f(x) = \sum_{x=s+1}^{\infty} [r(x) + \sum_{y=1}^{x} a(y) f(x-y)] \\
&= \Lambda r(s) + \sum_{y=1}^{\infty} a(y) \sum_{x=\max(y,s+1)}^{\infty} f(x-y) = \Lambda r(s) + \sum_{y=1}^{\infty} a(y) \Lambda f(\max(-1, s-y)) \\
&= \Lambda r(s) + \Lambda f(-1) \Lambda a(s) + \sum_{y=1}^{s} a(y) \Lambda f(s-y)
\end{aligned}$$

that is, (25) holds for $t = 1$. This completes the proof of Theorem 6. $\square$

If $f \in \mathcal{F}_0$ is a probability function, then $\Lambda f(-1) = 1$.

We now turn to compound functions. If $h \in \mathcal{F}_+$ and $p \in \mathcal{F}_0$ satisfies the recursion (23), then Theorem 2 gives

$$(p \vee h)(s) = (r \vee h)(s) + \sum_{x=1}^{s} (p \vee h)(s-x) \sum_{y=1}^{x} a(y) h^{*y}(x) \qquad (s = 1, 2, \ldots) \quad (26)$$

This recursion is also in the form (24), and thus we can evaluate $\Gamma^t(p \vee h)$ and $\Lambda^t(p \vee h)$ by respectively Theorems 5 and 6.

Let us consider the special case of compound geometric distributions, that is, h and p are probability functions and p is given by

$$p(n) = (1 - \pi)\pi^n \qquad (n = 0, 1, 2, , \ldots)$$

This counting probability function satisfies (1) with $a = \pi$ and $b = 0$. Thus (26) gives

$$(p \vee h)(s) = \pi \sum_{x=1}^{s} h(x)(p \vee h)(s-x) \qquad (s = 1, 2, ...)$$

As this recursion is in the form (23) with

$$r(x) = 0 \qquad a(x) = \pi h(x) \qquad (x = 1, 2, ...)$$

we can evaluate $\Gamma^t(p \vee h)$ recursively by Theorem 5. We obtain in particular

$$\Gamma(p \vee h)(s) = 1 - \pi + \pi \sum_{x=1}^{s} h(x)\Gamma(s-x) \qquad (s = 1, 2, ...)$$

$$\Gamma^2(p \vee h)(s) = (1-\pi)(s+1) + \pi \sum_{x=1}^{s} a(x)\Gamma^2(s-x) \qquad (s = 1, 2, ...)$$

For recursive evaluation of $\Lambda^t(p \vee h)$ Theorem 6 gives

$$\Lambda^t(p \vee h)(s) = \pi \left(\sum_{j=1}^{t} \Lambda^j h(s) + \sum_{x=1}^{s} h(x)\Lambda^t(p \vee h)(s-x) \right) \qquad (s = 1, 2, ...)$$

These recursions for compound geometric distributions can be applied to obtain upper and lower bounds for the probability of ultimate ruin in the classical ruin model, cf. e.g. Dickson (1995).

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