Recursive Biorthogonal Interpolating Wavelets and Signal-Adapted Interpolating Filter Banks

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***Abstract*—In this paper, by combining the ideas of the recursive wavelets with second-generation wavelets, a family of recursive biorthogonal interpolating wavelets (RBIWs) is developed. RBIW's have simple shape parameter vectors on each level, which allows a multichannel decomposition algorithm and provides a flexible structure for designing signal-adapted interpolating filter banks. In the single-level case, an efficient approach to design an optimum two-channel biorthogonal interpolating filter bank is proposed, which maximizes the coding gain under the traditional quantization noise assumption. Furthermore, in the multilevel case, using level-wise optimization of the shape parameter vectors, signal-adapted tree-structured recursive biorthogonal interpolating filter banks (RBIFBs) are designed, which are efficient in computation and can remarkedly improve coding gain. Finally, numerical results demonstrate the effectiveness of the proposed methods.**

I. Introduction

RECENTLY, combining the interpolation theory, subdivision scheme with the wavelet and multiresolution analysis, Deslauriers and Dubuc [1], [2], Donoho [3], Lounsbery [4], Sweldens [5], [6], and Harten [7], [8], [25] had established the fundamental theory of interpolating wavelets, recursive wavelets, and second-generation wavelets. Due to the allowance of the use of a diverse filter pair on each level, the new framework provides a flexible structure for designing signal-adapted data representations and filter banks.

For tree-structured filter banks or wavelet transforms, it is more efficient to level-wise select the filters (or scaling function and wavelet) to fit into the properties of the input signal than the standard wavelet/filter banks. The general framework has been established in [6]–[8], [21]–[23], [25]. In this framework, the pivotal tool is the lifting scheme by which one can freely select the decimation operator/dual lifting filter and the prediction operator/lifting filter on each level and, yet, not influence the biorthogonality. These dramatically simplify the design of biorthogonal filter banks and extend the standard wavelets to interval domains, surfaces, and irregular samples.

Additionally, the signal-adapted subband coding systems/filter banks have been extensively discussed by many researchers, for example, [9]–[19]. The basic idea is to decompose a complicated signal into "more compact" and "simpler" low-resolution parts using some adaptive techniques. In data and image compression applications, there are the two typical adaptive approaches: the entropy-based adaptive wavelet packet decompositions (AWPDs) [9]–[14] and the signal-adapted filter banks (or principal component filter banks) [15]–[19]. The former uses the same pair of filters in all nodes, and an optimal time-frequency tiling or tree-structured decomposition is achieved by the minimal entropy criterion. The associated coding scheme often uses the zero-tree coding that creatively exploits the interlevel correlation presented in the transform domain. In the signal-adapted filter banks or the principal component filter banks, we can adaptively select the coefficients of filters by the subband coding gain. In this way, an optimal subband coding system is obtained with the maximal subband coding gain in which the quantizer error processes are assumed to be zero-mean and white and interchannel uncorrelated. The design of optimum orthogonal uniform FIR filter banks and their unconstrained counterparts have been completely solved [15]–[18]. In theory, the biorthogonal ones and tree-structured ones can achieve better coding performance than orthogonal ones. Nevertheless, the design in FIR case remains unresolved [19].

The main interest in this paper is to exploit the lifting scheme to design signal-adapted biorthogonal interpolating subband coding systems. First, we develop RBIWs and the multichannel decomposition algorithm, which is a flexible decomposition structure based on the lifting scheme. Second, the optimal two-channel biorthogonal interpolating filter banks and signal-adapted tree-structured interpolating filter banks are designed. This paper starts with a brief review on the basic concepts and properties of the lifting Donoho wavelets and the general interpolating filters. In Section III, combining the ideas of recursive wavelets, second-generation wavelets, and interpolating wavelets, we develop RBIWs with explicit shape parameter vectors. RBIWs provide a more flexible structure in which the free selection of the shape parameter vectors preserves not only the biorthogonality but the regular order and vanishing moments of filter banks as well. Next, a flexible multichannel decomposition algorithm is given, which is the foundation of our paper's work. In Section IV, two adaptive algorithms that fit into the high bit rate data compression are given. The first is used to design the optimum two-channel FIR interpolating filter banks, which maximize the subband coding gain under the traditional quantization noise assumption. The second is used to design signal-adapted tree-structured interpolating filter banks by level-wise optimizing the shape parameter vectors. Due to the level-wise optimization, the algorithm is efficient in computation and can achieve a larger coding gain, although it does not guarantee the global optimal

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one. In Section V, some numerical results are reported that demonstrate the effectiveness of our methods. Finally, we conclude the paper.

II. Lifting Donoho Wavelets and General Interpolating Filters

A. Lifting Donoho Wavelets

In 1992, Donoho [3] suggested the idea of wavelets built from Deslauriers–Dubuc fundamental interpolating functions [1], [2] and constructed a family of interpolating wavelets later referred to as Donoho wavelets. They can be described as follows:

$$\begin{aligned}\varphi(x) &= \varphi(2x) + \sum_k \boldsymbol{h}(k)\varphi(2x-2k+1)\\ \psi(x) &= 2\varphi(2x-1)\end{aligned} \tag{1}$$

with the duals

$$\begin{aligned}\tilde{\varphi}(x) &= \delta(x)\\ \tilde{\psi}(x) &= \tilde{\varphi}(2x-1) - \sum_k \boldsymbol{h}(-k)\tilde{\varphi}(2x-2k-2)\end{aligned}$$

where $\delta(x)$ denotes the Dirac impulse, and $\boldsymbol{h}$ is an interpolating filter.

Such wavelets suffer from the following disadvantages: Wavelets do not have any vanishing moments and thus do not form a Riesz basis for $\boldsymbol{L}^2(\boldsymbol{R})$, whereas the duals do not even belong to $\boldsymbol{L}^2(\boldsymbol{R})$, and the dual scaling filter is fullpass without frequency localization. To overcome these disadvantages, Sweldens uses the lifting scheme to Donoho wavelets and obtains a novel family of interpolating wavelets, which is termed here "lifting Donoho wavelets" [5]. In this way, the system satisfies the following two-scale relation

$$\begin{aligned}\varphi(x) &= \varphi(2x) + \sum_k \boldsymbol{h}(k)\varphi(2x-2k+1)\\ \psi(x) &= 2\varphi(2x-1) - \sum_k \boldsymbol{g}(k)\varphi(x-k)\end{aligned} \tag{2}$$

with the duals

$$\begin{aligned}\tilde{\varphi}(x) &= 2\tilde{\varphi}(2x) + \sum_k \boldsymbol{g}(-k)\tilde{\psi}(x-k)\\ \tilde{\psi}(x) &= \tilde{\varphi}(2x-1) - \sum_k \boldsymbol{h}(-k)\tilde{\varphi}(2x-2k-2)\end{aligned} \tag{3}$$

where $\boldsymbol{h}$, $\boldsymbol{g}$ are a pair of real FIR general interpolating filters. Such a wavelet system can be designed by dual lifting and lifting the "lazy wavelet" [5] in turn, and thus, the filters $\boldsymbol{h}$, $\boldsymbol{g}$ are also referred as the dual lifting filter and the lifting filter, respectively. Obviously, $\boldsymbol{h} = \boldsymbol{g} = \boldsymbol{0}$ results in the "lazy wavelet," and $\boldsymbol{g} = \boldsymbol{0}$ in the Donoho wavelet. Due to compact support, such interpolating wavelets are more attractive. Upon the vanishing moments, there exists the following conclusion.

Proposition 1: φ, $\tilde{\varphi}$, ψ and $\tilde{\psi}$ in (2) and (3) have $D+1$ vanishing moments iff the general interpolating filters $\boldsymbol{h}$, $\boldsymbol{g}$ satisfy [5, th. 12]

$$\sum_k k^d \boldsymbol{h}(k) = \sum_k k^d \boldsymbol{g}(k) = \left(\tfrac{1}{2}\right)^d, \; d = 0, 1, \cdots, D. \tag{4}$$

A scaling function has $D+1$ vanishing moments (or $D+1$ regular) if it satisfies $\int x^d \varphi(x)\, dx = \delta(d)$, $d = 0, 1, 2, \cdots, D$.

The regular order of a scaling function is closely related to the approximation power of the associated multiresolution analysis to smooth signals or functions.

B. General Interpolating Filters

The subdivision scheme and the interpolating filters have been successfully used to design wavelets on real line and interval, two-dimensional (2-D) inseparable wavelets [20] and wavelets on manifolds [4].

Definition 1: An interpolating filter is of $D+1$ order if the interpolator derived from it is accurate for all polynomials less than $D+1$ degrees but not accurate for $D+1$-degree polynomials.

An equivalent condition is that the interpolating filter satisfies the constraints in (4) and $\sum_k k^{D+1}\boldsymbol{h}(k) \neq (1/2)^{D+1}$. Deslauriers–Dubuc filters are a family of special interpolating filters, and a $D+1$ order Deslauriers–Dubuc filter can be calculated by the Largrange interpolation formula

$$\boldsymbol{h}_D^{(n)}(k) = \prod_{l=n,\, l\neq k}^{n+D} \frac{2l-1}{2(l-k)}, \qquad k = n, n+1, \cdots, n+D. \tag{5}$$

Each n corresponds to a $D+1$-order interpolating filter, and such diversity is advantageous to boundary processing [3], [7], [8]. They are shortest among $D+1$-order interpolating filters, and it is more important that using them as bases, the general interpolating filters can be parameterized. We have the following proposition.

Proposition 2: Let $\boldsymbol{h}$ be at least a $D+1$-order interpolating filter with the support set $\{n, n+1, \cdots, n+N\} (N \geqslant D)$; then, there exists an unique parameter vector $\boldsymbol{a}$ satisfying $\sum_k \boldsymbol{a}(k) = 1$ such that

$$\boldsymbol{h}(k) = \sum_{p=n}^{n+N-D} \boldsymbol{a}(p)\boldsymbol{h}_D^{(p)}(k). \tag{6}$$

By optimization of the filters $\boldsymbol{h}$ and $\boldsymbol{g}$, one can design the interpolating wavelets with good properties, e.g., near semi-orthogonal interpolating wavelets [4] and near orthogonal interpolating wavelets [24]. In the lifting scheme, the free choice of the lifting filter does not influence the biorthogonality, which dramatically simplifies the design of biorthogonal wavelets/filter banks. Following (6), the free choice of the parameter vector $\boldsymbol{a}$ preserves both the biorthogonality and the regular order/vanishing moments of the wavelet system. Moreover, from the data approximation view, Deslauriers–Dubuc filters originate from the piecewise polynomial interpolation, whereas the general interpolating filters originate from the piecewise polynomial fitting. The latter is more flexible and efficient in many applications.

III. RBIWs and Multichannel Decomposition

A biorthogonal recursive interpolating wavelets is the generalization of the lifting Donoho wavelets and a particular case of second-generation wavelets. Its basic idea is not to use the cascade *ad infinitum* to construct a scaling function but, instead, to fix the interpolating scaling function on an arbitrary finest level and then define the interpolating scaling functions and wavelets on the coarser level through recursive use of the two-scale relation. In this section, we give the concept and properties of RBIW's and their multichannel decomposition algorithm.

A. RBIWs

Without loss of generality, let the initial scaling and dual scaling functions $\varphi_0(x)$ and $\tilde{\varphi}_0(x)$ be determined by the lifting Donoho wavelet in (2) and (3) with $\boldsymbol{h} = \boldsymbol{g} = \boldsymbol{h}_{2n+1}^{(-n)}$. Naturally, their shifts by integer constitute a Riesz basis of $\boldsymbol{V}_0 = Span\{\varphi_0(x-k),\, k \in \boldsymbol{Z}\}$ and $\tilde{\boldsymbol{V}}_0 = Span\{\tilde{\varphi}_0(x-k),\, k \in \boldsymbol{Z}\}$ and satisfy the biorthogonality [3], [5]

$$\langle \varphi_0(x-k),\, \tilde{\varphi}_0(x-l)\rangle = \delta(k-l)$$

where $\langle\rangle$, $\delta(k)$, and $\boldsymbol{Z}$ denote the inner product in $\boldsymbol{L}^2(\boldsymbol{R})$, the Kronecker's symbol, and the integer set, respectively.

Then, the scaling function, wavelet, and their duals on the coarser level are defined by recursive two-scale equations as follows:

$$\begin{aligned}
\varphi_m(x) &= \varphi_{m-1}(2x) + \sum_k \boldsymbol{h}_m(k)\varphi_{m-1}(2x-2k+1)\\
\psi_m(x) &= 2\varphi_{m-1}(2x-1) - \sum_k \boldsymbol{g}_m(k)\varphi_m(x-k)\\
\tilde{\varphi}_m(x) &= 2\tilde{\varphi}_{m-1}(2x) + \sum_k \boldsymbol{g}_m(-k)\tilde{\psi}_m(x-k)\\
\tilde{\psi}_m(x) &= \tilde{\varphi}_{m-1}(2x-1) - \sum_k \boldsymbol{h}_m(-k)\tilde{\varphi}_{m-1}(2x-2k-2)
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\boldsymbol{h}_m(k) &= \sum_{p=-n}^{n+1-I} \Gamma_m(p)\boldsymbol{h}_I^{(p)}(k), \quad \sum_{p=-n}^{n+1-I} \Gamma_m(p) = 1\\
\boldsymbol{g}_m(k) &= \sum_{p=-n}^{n+1-I} \Lambda_m(p)\boldsymbol{h}_I^{(p)}(k), \quad \sum_{p=-n}^{n+1-I} \Lambda_m(p) = 1.
\end{aligned} \tag{8}$$

According to Propositions 1 and 2, the general interpolating filters $\boldsymbol{h}_m, \boldsymbol{g}_m$ are at least of the order $I+1$ and have the same support set $\{-n, -n+1, \cdots, n+1\}$. Moreover, from the lifting scheme, the associated filter bank is biorthogonal, and all scaling functions and wavelets belong to $\boldsymbol{L}^2(\boldsymbol{R})$. The family of the vector pair $\{\Gamma_m, \Lambda_m\}$ completely determines the filter bank, which is referred as the shape parameter vector of the system.

Let

$$\begin{aligned}
\Pi_m(l) &= \langle \varphi_m(x),\, \varphi_m(x-l)\rangle\\
\Omega_m(l) &= \langle \psi_m(x),\, \psi_m(x-l)\rangle\,.
\end{aligned} \tag{9}$$

Then, from the recursive relation (7), one can easily derive a recursive relation of Π_m, Ω_m, which can be represented by the shape parameter vector and Π_0 as well. Through Fourier transforms of Π_m, Ω_m, it can be judged whether $\{2^{-(m/2)}\varphi_m(2^{-m}x-k),\, k \in \boldsymbol{Z}\}$ and $\{2^{-(m/2)}\psi_m(2^{-m}x-k),\, k \in \boldsymbol{Z}\}$ constitute the Riesz bases of the subspace $\boldsymbol{V}_m$ and $\boldsymbol{W}_m$, where

$$\begin{aligned}
\boldsymbol{V}_m &= Span\{2^{-(m/2)}\varphi_m(2^{-m}x-k),\, k \in \boldsymbol{Z}\}\\
\boldsymbol{W}_m &= Span\{2^{-(m/2)}\psi_m(2^{-m}x-k),\, k \in \boldsymbol{Z}\}.
\end{aligned}$$

In fact, due to finite level decomposition, the Riesz condition is not essential, and therefore, in the adaptive approaches below, the Riesz condition is loosened. The RBIW defined in (7) has the following properties.

i) Every scaling function has an interpolation property.
ii) The RBIW has biorthogonality.
iii) The number of vanishing moments and regular orders lie between $I+1$ and $2n+2$.

B. Multichannel Decomposition

For a given signal $f(x) \in \boldsymbol{L}^2(\boldsymbol{R})$, without loss of generality, assume that $f(x) \in \boldsymbol{V}_0$. Following the interpolation property, we have

$$\boldsymbol{c}_0(l) = \int f(x)\varphi_0(x-l)dx = f(l),\, l \in \boldsymbol{Z}.$$

Even though $f(x) \notin \boldsymbol{V}_0$, the right band of the above equation provides a more efficient approximation of $\boldsymbol{c}_0$ than the noninterpolating one with the same regular order [3]. For a discrete-time signal, the initial approximation coefficients $\boldsymbol{c}_0$ can substitute for the samples of the signal. An M-level decomposition of the RBIW is as follows. For $m > 0$

$$\begin{aligned}
\boldsymbol{c}_m(l) &= 2^{-(m/2)} \int f(x)\tilde{\varphi}_m(2^{-m}x-l)\,dx\\
\boldsymbol{d}_m(l) &= 2^{-(m/2)} \int f(x)\tilde{\psi}_m\left(2^{-m}x-l\right)\,dx\\
& l \in \boldsymbol{Z},\, m = 1, 2, \cdots, M.
\end{aligned} \tag{10}$$

In this way, the continuous-time signal $f(x)$ is expanded as a wavelet series

$$\begin{aligned}
f(x) = &\sum_{m=1}^{M} 2^{-(m/2)} \sum_l \boldsymbol{d}_m(l)\psi_m\left(2^{-m}x-l\right)\\
&+ 2^{-(M/2)} \sum_l \boldsymbol{c}_M(l)\varphi_M\left(2^{-M}x-l\right).
\end{aligned}$$

This decomposition process is level-wise implemented with a multichannel decomposition algorithm. The algorithm is described as follows. Assume that the approximation coefficients $\boldsymbol{c}_{m-1}(l)$ have been obtained, and following (8) and (10), the detail coefficients $\boldsymbol{d}_m(l)$ on the next coarse level are

$$\boldsymbol{d}_m(l) = \sum_{p=-n}^{n+1-I} \Gamma_m(p)\boldsymbol{d}_m^{(p)}(l) \tag{11}$$

where for $p = -n, -n+1, \cdots, n+1-I$

$$\boldsymbol{d}_m^{(p)}(l) = \frac{1}{\sqrt{2}}\left[\boldsymbol{c}_{m-1}(2l+1) - \sum_k \boldsymbol{h}_I^{(p)}\boldsymbol{c}_{m-1}(2l+2k+2)\right]. \tag{12}$$

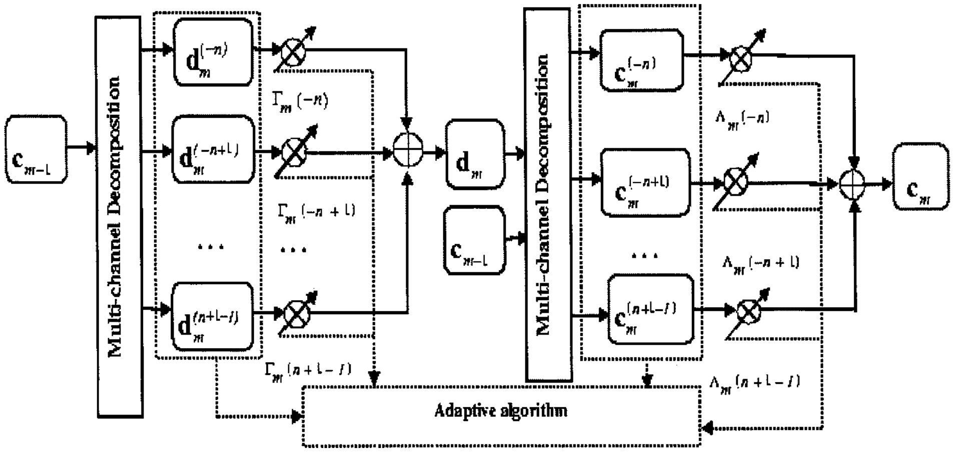

Fig. 1. Flow diagram of adaptive decomposition.

The detail coefficients are the weigh sum of the detail coefficients of the output of $2n+2-I$ subchannels and the decomposition of every subchannel fits into the forward transform in the lifting scheme [6].

Having obtained the detail coefficients, the approximation coefficients $\boldsymbol{c}_m$ on the level are calculated by an analogous algorithm, which equals the weight sum of the outputs of $2n+2-I$ subchannels or $(2n+2-I)^2$ subchannels.

$$\boldsymbol{c}_m(l) = \sum_{p=-n}^{n+1-I} \Lambda_m(p)\boldsymbol{c}_m^{(p)}(l) = \sum_{p=-n}^{n+1-I}\sum_{p'=-n}^{n+1-I} \Lambda_m(p)\Gamma_m(p')\,\boldsymbol{c}_m^{(p,p')}(l) \tag{13}$$

where $\boldsymbol{c}_m^{(p)}$, $\boldsymbol{c}_m^{(p,p')}$ are the approximation coefficients of subchannels

$$\boldsymbol{c}_m^{(p)}(l) = \sqrt{2}\boldsymbol{c}_{m-1}(2l) + \sum_k \boldsymbol{h}_I^{(p)}(k)\boldsymbol{d}_m(l+k)$$
$$\boldsymbol{c}_m^{(p,p')}(l) = \sqrt{2}\boldsymbol{c}_{m-1}(2l) + \sum_k \boldsymbol{h}_I^{(p)}(k)\boldsymbol{d}_m^{(p')}(l+k). \tag{14}$$

Using vector and matrix notations, their more compact forms are obtained. Let $\boldsymbol{x}_m$, $\boldsymbol{a}_m$ and $\boldsymbol{b}_m$ be a column and two column vector sequences

$$\boldsymbol{x}_m((2n+2-I)p+p') = \Gamma_m(p)\Lambda_m(p')$$
$$\boldsymbol{a}_m(p,l) = \boldsymbol{d}_m^{(p)}(l)$$
$$\boldsymbol{b}_m((2n+2-I)p+p') = \boldsymbol{c}_m^{(p,p')}$$

and

$$\boldsymbol{d}_m = \Gamma_m^T\boldsymbol{a}_m = \Gamma_m^T\left(\tilde{\boldsymbol{G}} * \boldsymbol{c}_{m-1}\downarrow 2\right) = \left(\left(\Gamma_m^T\tilde{\boldsymbol{G}}\right) * \boldsymbol{c}_{m-1}\right)\downarrow 2$$
$$\boldsymbol{c}_m = \boldsymbol{x}_m^T\boldsymbol{b}_m = \boldsymbol{x}_m^T\left(\tilde{\boldsymbol{H}} * \boldsymbol{c}_{m-1}\downarrow 2\right) = \left(\left(\boldsymbol{x}_m^T\tilde{\boldsymbol{H}}\right) * \boldsymbol{c}_{m-1}\right)\downarrow 2 \tag{15}$$

where $*$ and $\downarrow 2$ denote the discrete convolution operator and the downsampling by two operator, respectively, and each row of $\tilde{\boldsymbol{G}}$ is a highpass filter with the z-transform $(1/\sqrt{2})z^{-1}(1 - z^{-1}\sum_k \boldsymbol{h}_I^{(p)}(-k)z^{-2k})$, $-n \le p \le n+1-I$, and each row of $\tilde{\boldsymbol{H}}$ is a lowpass filter that equals to the lowpass analysis filter in the lifting Donoho wavelet with $\boldsymbol{h} = \boldsymbol{h}_I^{(p)}$ and $\boldsymbol{g} = \boldsymbol{h}_I^{(p')}$.

The synthesis process fits into the inverse transform in the lifting scheme [6]

$$\boldsymbol{c}_{m-1}(2l) = \frac{1}{\sqrt{2}}\left[\boldsymbol{c}_m(l) - \sum_k \boldsymbol{g}_m(-k)\boldsymbol{d}_m(l+k)\right.$$
$$\boldsymbol{c}_{m-1}(2l+1) = \sqrt{2}\boldsymbol{d}_m(l) + \sum_k \boldsymbol{h}_m(-k)\boldsymbol{c}_{m-1}(2l+2k+2)$$
$$f(x) = \sum_l \boldsymbol{c}_0(l)\varphi_0(x-l).$$

In terms of (7), the synthesis filters are the weight sum of some basic filters, and the lowpass and highpass filters can be written as $\Gamma_m^T\boldsymbol{H}$ and $\boldsymbol{x}_m^T\boldsymbol{G}$, respectively, in which each row of $\boldsymbol{H}$ is a filter with the z-transform $(1/\sqrt{2})(1 + z^{-1}\sum_k \boldsymbol{h}_I^{(p)}(k)z^{2k})$, and each $\boldsymbol{G}$ is the highpass synthesis filter in the lifting Donoho wavelet with $\boldsymbol{h} = \boldsymbol{h}_I^{(p)}$, $\boldsymbol{g} = \boldsymbol{h}_I^{(p')}$. $\tilde{\boldsymbol{G}}$, $\boldsymbol{H}$ are two $(2n+2-I)\times(4n+3)$ matrices, and $\tilde{\boldsymbol{H}}$, $\boldsymbol{G}$ are two $(2n+2-I)^2\times(8n+5)$ matrices. The four matrices are completely determined by the systemic parameter n and I or the Deslauriers–Dubuc filters in (8) and independent of the level. In other words, the synthesis filters are divided into the product of the shape parameter vectors and the fixed matrices, which brings the great facility into the design of signal-adapted filter banks.

IV. Signal-Adapted Recursive Interpolating Filter Banks

Using multichannel decomposition, one can signal-adapted choose the parameter vectors Γ_m, Λ_m or the pair of interpolating filters on each level. The adaptive process is illustrated in Fig. 1, where Γ_m, Λ_m can be optimized simultaneously or in turn. For special applications, the adaptive criterion and optimal strategy are the pivotal issues. The high bit rate data compression [15]–[19] and the low bit rate data (or image) compression [9]–[14] are frequently encountered in applications. The former requires that signal energy concentrates on as few subbands as possible in order to achieve a large subband coding gains, and the latter requires that signal energy concentrates on as a small number of its decomposition coefficients as possible.

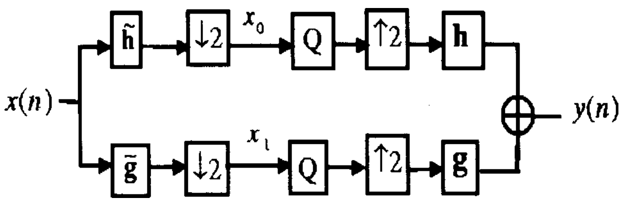

Fig. 2. Two-channel biorthogonal filter bank.

In this paper, we mainly focus our methods on the high bit rate case, although this flexible adaptive framework may be applied to other aspects.

The subband coder (SBC) is commonly used in data compression for lossy encoding of audio and image signals. When the subband quantizers are present in an orthogonal filter bank and the average bit rate is fixed, the optimal orthogonal filter banks were independently developed in [15]–[18]. For nonuniform or tree-structured filter banks, the asymptotic performance analysis has been reported in [27]. In theory, the biorthogonal ones can achieve better coding performance than orthogonal ones. For biorthogonal case, there are some important developments [19]. Nevertheless, the design of optimal FIR biorthogonal filter banks remains unsolved, including M-band and tree-structured ones.

A. Optimum Two-Band Biorthogonal Interpolating Filter Banks

Consider a two-band biorthogonal filter bank, as shown in Fig. 2, where $\{\boldsymbol{h}, \boldsymbol{g}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{g}}\}$ is a biorthogonal filter bank, the block Q denotes quantizer, $\uparrow 2$ and $\downarrow 2$ the upsampling and downsampling by two, and the input $x(n)$ is a zero-mean wide sense stationary (WSS) process or a deterministic signal.

Using the mean-square error as the criterion, with high bit rate assumption on the quantization noise sources and with optimal bit allocation, one can write the subband coding gain as [19]

$$G_{SBC} = \frac{\sigma_x^2}{\sigma_0 \sigma_1 ||\boldsymbol{h}||_2 ||\boldsymbol{g}||_2} \tag{16}$$

where σ_x^2, σ_0^2, and σ_1^2 are the variances of x, x_0, and x_1, respectively, and $||\boldsymbol{h}||_2$ and $||\boldsymbol{g}||_2$ are the l^2-norms of the synthesis filters $\boldsymbol{h}$, $\boldsymbol{g}$.

The biorthogonal filter bank that maximizes the subband coding gain in (16) is referred to as the optimum filter bank or biorthogonal PCFB. Moreover, some other constraints can be imposed on the filters, e.g., the order of filters, vanishing moments, and the special structure. In the orthonormal case, due to energy conservation, i.e., $\sigma_x^2 = \sigma_0^2 + \sigma_1^2$, the design is transferred into that of optimum compaction filter, which can be solved by a standard LSIP [16], [17]. However, biorthogonal filter banks lack the above precondition, and the design becomes difficult. Using the above flexible structure of biorthogonal interpolating filter banks, the optimal biorthogonal interpolating filter banks can be developed.

Following (16), the maximization of the SBC gain is equivalent to the minimization of $\sigma_0\sigma_1||\boldsymbol{h}||_2||\boldsymbol{g}||_2$. In addition, the minimization can be implemented by the optimization of the shape parameter vectors Γ_1 and Λ_1. Assume the input signal $\boldsymbol{c}_0$ with Toeplitz covariance $R_0(k, l) = E\{\boldsymbol{c}_0(k)\boldsymbol{c}_0(l)\}$, where $E\{\cdot\}$ denotes expectation. According to (15), we have

$$\begin{aligned} \sigma_0^2 &= \boldsymbol{x}_1^T \boldsymbol{U}_1 \boldsymbol{x}_1 \\ \sigma_1^2 &= \Gamma_1^T \boldsymbol{S}_1 \Gamma_1 \end{aligned}$$

in which $\boldsymbol{U}_1$ and $\boldsymbol{S}_1$ are the covariances of the column vectors $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$, respectively

$$\begin{aligned} \boldsymbol{U}_1 &= \tilde{\boldsymbol{H}}\boldsymbol{R}_0(0\colon 8n+4,\, 0\colon 8n+4)\tilde{\boldsymbol{H}}^T \\ \boldsymbol{S}_1 &= \tilde{\boldsymbol{G}}\boldsymbol{R}_0(0\colon 4n+2,\, 0\colon 4n+2)\tilde{\boldsymbol{H}}^T \end{aligned}$$

where $\boldsymbol{R}_0(k_1\colon k_2, l_1\colon l_2)$ denotes the submatrix formed from k_1 to k_2 rows and l_1 to l_2 columns of $\boldsymbol{R}_0$. The synthesis filters satisfy

$$||\boldsymbol{h}||_2 = \Gamma_1^T \boldsymbol{H}\boldsymbol{H}^T \Gamma_1,\ ||\boldsymbol{g}||_2 = \boldsymbol{x}_1^T \boldsymbol{G}\boldsymbol{G}^T \boldsymbol{x}_1.$$

Consequently, the design of the optimum filter bank is described as an optimal problem:

$$\min_{\Gamma_1, \Lambda_1} \left\{ \left(\Gamma_1^T \boldsymbol{S}_1 \Gamma_1\right) \left(\Gamma_1^T \boldsymbol{H}\boldsymbol{H}^T \Gamma_1\right) \left(\boldsymbol{x}_1^T \boldsymbol{U}_1 \boldsymbol{x}_1\right) \left(\boldsymbol{x}_1^T \boldsymbol{G}\boldsymbol{G}^T \boldsymbol{x}_1\right) \right\}$$
$$\text{s.t.,} \sum_{p=-n}^{n+1-I} \Gamma_1(p) = \sum_{p=-n}^{n+1-I} \Lambda_1(p) = 1. \tag{17}$$

Obviously, the objective function is a 12-degree multivariate polynomial on Γ_1 and Λ_1. The problem is easily simplified as an unconstrained counterpart and then solved by the gradient algorithms. In the gradient algorithms, the initial point of iteration uses the parameter vectors to which the Deslauriers–Dubuc filter $\boldsymbol{h}_{2n+1}^{(-n)}$ corresponds. When the input is a deterministic signal, the covariances are substituted by the sample estimates [16]. Additionally, optimizing Γ_1 and Λ_1, in turn, is simpler in algorithms but often unsatisfactory in results [31].

Unlike an optimal orthonormal FIR filter banks in which all filter have the same length, an optimal biorthogonal interpolating filter bank (BIFB) is composed of four filters of different lengths. According to (17), the constraint on vanishing moments is naturally imposed on the filter bank. A tradeoff between the subband coding gain and the vanishing moments can be realized by the selection of the parameters n and I. Our approach completely solves the design of optimum BIFB's, and using the lifting scheme, this approach can be extended to the design of signal-adapted two-band biorthogonal filter banks. Its drawback is that the iteration may sometimes plunge into the local minimum points.

B. Signal-Adapted Recursive Interpolating Filter Banks

In a wavelet-based coder, a binary tree-structured filter bank is often used, which is equivalent to a nonuniform filter bank with decimation ratios that are powers of two. An M-level wavelet-based decomposition is shown in Fig. 3(a). For simplicity, we used the plain line to represent the branches of the tree-structure, and each line represents a filtering and downsampling by two. An equivalent filter bank of Fig. 3(a) are illustrated in Fig. 3(b), where $\boldsymbol{e}\tilde{\boldsymbol{h}}_M$ and $\boldsymbol{e}\tilde{\boldsymbol{g}}_m$ denote the equivalent filters for analysis and $\boldsymbol{eh}_M$ and $\boldsymbol{eg}_m$ the equivalent filters for synthesis.

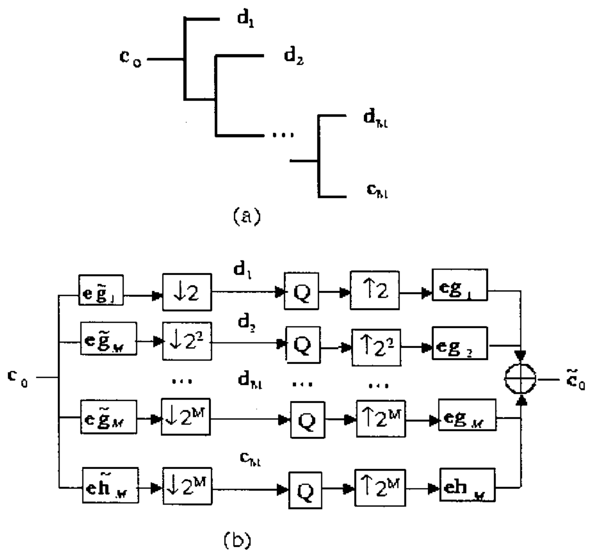

Fig. 3. M-level wavelet-based decomposition and its equivalent filter bank.

In this way, the global subband coding gain of the filter bank in Fig. 3(b) is written as [19]

$$G_{SBC} = \frac{\sigma_0^2}{\left(\sigma_{c_M}^2 \|\boldsymbol{eh}_M\|_2^2\right)^{(1/2^M)} \prod_{m=1}^{M} \left(\sigma_{d_m}^2 \|\boldsymbol{eg}_m\|_2^2\right)^{(1/2^m)}} \tag{18}$$

where σ_0^2, $\sigma_{c_M}^2$ and $\sigma_{d_m}^2$ denote the variances of $\boldsymbol{c}_0$, $\boldsymbol{c}_M$ and $\boldsymbol{d}_m$.

For an M-level recursive biorthogonal interpolating filter bank (RBIFB), if the global coding gain in (18) is directly taken as the objective function, it results in a complicated design. Therefore, using the above single-level algorithm, we use level-wise optimization of the parameter vector rather than the global optimization. In this way, a large global subband coding gain are achieved, although it maybe not the largest one. This is a cost-effective tradeoff between the global subband coding gain and the complexity of design.

Before starting the level-wise optimization, we give a brief review of the equivalent filters, which have been widely available in tree-structured filter banks. A cascaded filtering process including multiple filters and downsampling operators is shown in Fig. 4(a), and its equivalent process is shown in Fig. 4(b). The filter $\boldsymbol{eh}$ is referred as the equivalent filter of Fig. 4(a), satisfying

$$\boldsymbol{eh} = \boldsymbol{h}_1 * (\boldsymbol{h}_2 \uparrow 2) * \cdots * \left(\boldsymbol{h}_m \uparrow 2^M\right).$$

The level-wise optimization of the parameter vectors Γ_m and Λ_m is implemented by the following steps.

Step 1) For the first level, use the optimization (17) to obtain the optimal parameter vector $\Gamma_{1,\,opt}$ and $\Lambda_{1,\,opt}$. Then, calculate the optimal equivalent filters for synthesis $\boldsymbol{eh}_{1,\,opt}$ and $\boldsymbol{eg}_{1,\,opt}$ and the covariance matrix $\boldsymbol{R}_1$ of $\boldsymbol{c}_1$.

Step 2) For the mth level, assume that the optimal equivalent filters $\boldsymbol{eh}_{m-1,\,opt}$ and $\boldsymbol{eg}_{m-1,\,opt}$ for synthesis and the covariance matrix $\boldsymbol{R}_{m-1}$ of $\boldsymbol{c}_{m-1}$ have been calculated, and optimize the parameter vectors Γ_m and Λ_m on the mth level such that the product $\sigma_{c_m}^2 \sigma_{d_m}^2 \|\boldsymbol{eh}_m\|_2^2 \|\boldsymbol{eg}_m\|_2^2$ is minimal. Like the single-level case, set

$$\begin{aligned}
\boldsymbol{U}_m &= \tilde{\boldsymbol{H}}\boldsymbol{R}_{m-1}(0{:}\,8n+4,\, 0{:}\,8n+4)\tilde{\boldsymbol{H}}^T \\
\boldsymbol{S}_m &= \tilde{\boldsymbol{G}}\boldsymbol{R}_{m-1}(0{:}\,4n+2,\, 0{:}\,4n+2)\tilde{\boldsymbol{H}}^T \\
\boldsymbol{\sigma}_{c_m}^2 &= \boldsymbol{x}_m^T\boldsymbol{U}_m\boldsymbol{x}_m,\ \boldsymbol{\sigma}_{d_m}^2 = \boldsymbol{\Gamma}_m^T\boldsymbol{S}_m\boldsymbol{\Gamma}_m \\
\|\boldsymbol{eh}_m\|_2^2 &= \Gamma_m^T\boldsymbol{H}_{m,\,e}\boldsymbol{H}_{m,\,e}^T\Gamma_m \\
\|\boldsymbol{eg}_m\|_2^2 &= \boldsymbol{x}_m^T\boldsymbol{G}_{m,\,e}\boldsymbol{G}_{m,\,e}^T\boldsymbol{x}_m
\end{aligned}$$

where $\boldsymbol{H}_{m,\,e}(p,\,:)$ is the lowpass equivalent filter on the mth level when the interpolating filter $\boldsymbol{h}_m = \boldsymbol{h}_I^{(p)}$ and $\boldsymbol{G}_{m,\,e}((2n+2-I)p+p',\,:)$ is the highpass equivalent filter on the mth level when $\boldsymbol{h}_m = \boldsymbol{h}_I^{(p)}$ and $\boldsymbol{g}_m = \boldsymbol{h}_I^{(p')}$. [$\boldsymbol{A}(k,\,:)$ denotes the kth row of $\boldsymbol{A}$, $-n \le p,\, p' \le n+1-I$].

In conclusion, the adaptive selection of $\Gamma_m,\,\Lambda_m$ boils down to the following optimal problem:

$$\begin{aligned}
&\min_{\Gamma_m,\,\Lambda_m} \left\{\left(\Gamma_m^T\boldsymbol{S}_m\Gamma_m\right)\left(\Gamma_m^T\boldsymbol{H}_{m,\,e}\boldsymbol{H}_{m,\,e}^T\Gamma_m\right)\right. \\
&\qquad \left.\times \left(\boldsymbol{x}_m^T\boldsymbol{U}_{m,\,e}\boldsymbol{x}_m\right)\left(\boldsymbol{x}_m^T\boldsymbol{G}_{m,\,e}\boldsymbol{G}_{m,\,e}^T\boldsymbol{x}_m\right)\right\} \\
&\text{s.t.,} \sum_{p=-n}^{n+1-I} \Gamma_m(p) = \sum_{p=-n}^{n+1-I} \Lambda_m(p) = 1.
\end{aligned} \tag{19}$$

Solve the optimal problem, and obtain the optimal parameter vectors $\Gamma_m,\,\Lambda_m$.

Step 3) Calculate the optimal equivalent filters and the covariance matrix of $\boldsymbol{c}_m$.

$$\begin{aligned}
\boldsymbol{eh}_{m,\,opt} &= \Gamma_{m,\,opt}^T\boldsymbol{H}_{m,\,e},\ \boldsymbol{eg}_{m,\,opt} = \boldsymbol{x}_{m,\,opt}^T\boldsymbol{G}_{m,\,e} \\
\boldsymbol{R}_m(k,\,l+k) &= E\left[\boldsymbol{c}_m(k)\boldsymbol{c}_m(l+k)\right] = E\left[\boldsymbol{c}_m(0)\boldsymbol{c}_m(l)\right] \\
&= \boldsymbol{x}_{m,\,opt}^T\tilde{\boldsymbol{H}}\boldsymbol{R}_{m-1}(0{:}\,8n+4,\,2l{:}\,2l+8n+4)) \\
&\quad \cdot \tilde{\boldsymbol{H}}^T\boldsymbol{x}_{m,\,opt}.
\end{aligned}$$

Step 4) If $m = M$, the process ends; otherwise, go to Step 2.

In the above design, the limitation that each level uses the same subchannel interpolating filters can be relaxed, namely, the different number of the interpolating filter of subchannel in each level can be employed. In this way, there is more flexibility in design. In orthogonal case, with the number of level M increasing, the global subband coding gain is nondecreasing, but in biorthogonal case, this conclusion is invalid for absence of energy conservation. As a result, the level number of decomposition will influence the subband coding performance.

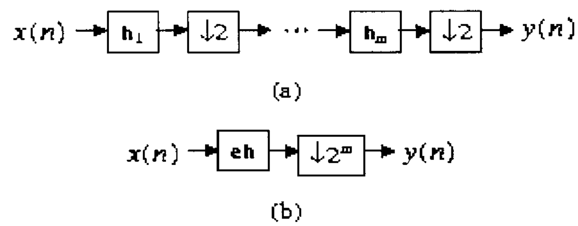

Fig. 4. Cascaded filtering process and its equivalent filter.

C. Relation to Other Work

The flexible parameter structure reported above allows us to level-wise optimize the interpolating filters according to a predetermined criterion, which can be applied to several other aspects in filter bank design. First, for lifting Donoho wavelets, using the optimization of parameter vectors, one can design the interpolating wavelets with good properties, such as the more regularity ones and nearly orthogonal ones [24]. Second, combining the lifting scheme and our method, it is possible to design the optimum two-band biorthogonal filter banks. Finally, it is an interesting problem to exploit RBIFB's application in the low bit rate compression.

Under the high bit rate hypothesis, the subband coding gains efficiently measure the compression performance of a subband coder. However, current image transform coders operate below one bit per pixel; for such low bit rates, the subband coding gain yields an incorrect estimate of the distortion rate. In this case, the distortion rate depends mostly on the ability to precisely approximate a signal with a small number of larger coefficients [12]. In orthogonal case, duo to energy conservation, the minimal entropy criterion fits well into this case, and the best wavelet packet bases based on the minimal entropy criterion have been widely used [9]–[11]. However, for tree-structured RBIFB's, the energy conservation no longer holds; thus, for every level, both the entropy of decomposition coefficients and the interlevel redundancy should be considered together. Future work will develop an design criterion and an efficient algorithm corresponding to the low bit rate case. Additionally, a natural generalization of RBIFB's is the recursive biorthogonal interpolating wavelet packets, which has been reported in our work [32].

V. Numerical Results

Wavelet techniques have been successfully applied to data compression, and the adaptive subband coders markedly improve the performance in compression. Below, our method is tested on three input signals. These three WSS input signals are commonly used in [16]–[18], and they are typical narrowband processes.

1) $AR(1)$ process with correlation coefficient $\rho = 0.95$ (simple image model). In this case, the correlation function $r_n = \rho^n$.
2) $AR(2)$ process with poles at $z_{\pm} = \rho e^{\pm i\theta}$ in which $\rho = 0.975$ and $\theta = \pi/3$ (models certain types of image texture). The correlation function $r_n = 2\rho r_{n-1}\cos\theta - \rho^2 r_{n-2}$ with $r_0 = 1$, $r_1 = (2\rho\cos\theta/1+\rho^2)$.
3) Lowpass process with box spectrum $S(\omega) = (\pi/\omega_s)rect_{[-\omega_s,\omega_s]}(\omega)$ with $\omega_s = 0.45\pi$ and the correlation function $r_n = (\sin(\omega_s n)/\omega_s n)$.

At first, the experiments reported below are the subband coding gains of optimum two-band biorthogonal interpolating filter banks, maximal regular interpolating filter banks, and optimum two-band orthogonal filter banks where the length of filters is $4n+4$. In BIFB's, the lowpass and highpass filters are not equal in length; in order to contrast them with orthogonal ones, we use optimum orthogonal filter banks that provide the same number of free parameters as the lowpass filter in BIFB's. Below, take $n = 1, 2$, respectively, and $I = 1$. Here, the two maximal regular filters satisfy the interpolating filters $\boldsymbol{h} = \boldsymbol{g} = \boldsymbol{h}_{2n+1}^{(-n)}$, $n = 1, 2$, and the two optimum orthogonal filter banks are eight-tap and 12-tap, respectively. The experiment results are illustrated in Table I.

TABLE I
SUBBAND CODING GAINS OF OPTIMUM BIFBS, MAXIMAL REGULAR BIFBS, AND OPTIMUM ORTHOGONAL FILTER BANKS

Process		Biorthogonal case		Orthogonal
		adaptive	maximal regular	adaptive
AR(1)	n=1	4.2742	4.2082	3.8847
	n=2	4.4817	4.1682	3.9134
AR(2)	n=1	4.7237	2.4525	4.3327
	n=2	5.2845	3.2349	4.6303
Box	n=1	4.7882	3.1389	3.7972
spectrum	n=2	7.8809	3.8881	5.6127

TABLE II
GLOBAL SUBBAND CODING GAINS OF THREE-LEVEL SIGNAL-ADAPTED RBIFBS AND THREE-LEVEL MAXIMAL REGULAR BIFBS

Process		Adaptive	Maximal regular
AR(1)	n=1	12.5275	12.7197
	n=2	12.6480	12.8241
AR(2)	n=1	6.9099	2.3806
	n=2	8.0179	3.3647
Box	n=1	4.8192	2.9375
spectrum	n=2	8.0575	3.6416

The results show that compared with their maximal regular counterparts, the optimum BIFB's can markedly improve the subband coding gain, and they achieve larger subband coding gains than the optimum orthogonal ones. The biorthogonal subband coders have an advantages over orthogonal subband coders, which have been demonstrated in many papers [19], [28]–[30]. Without the order constraint, the optimal design of biorthogonal filter banks has recently had some important developments [19]. However, for the FIR case, the optimal design remains unsolved. Our method allows the design of the optimal two-band interpolating filter banks. We envision that the lifting scheme may make a fresh start of the optimal design of FIR biorthogonal filter banks.

Second, we consider three level signal-adapted recursive filter banks. Take $n = 1, 2$ and $I = 1$; the global subband coding gains for three tested signals are shown in Table II.

From experiment results, for $AR(2)$ and the box-spectrum, signal-adapted RBIFB's perform considerably better in global subband coding gains. However, for $AR(1)$, compared with the maximal regular ones, the global subband coding gains degrade a little. These results happen for two reasons: the $AR(1)$ is a typical lowpass process, and the maximal regular one is close to the optimal one. The level-wise optimization is not a global optimal algorithm.

VI. Conclusion

In this paper, a novel family of recursive biorthogonal interpolating wavelets is developed, and a flexible multichannel decomposition technique based on RBIW's is proposed. For signal-adapted RBIFB's, an efficient method to design optimum two-band FIR BIFB's is given first. Its underlying tool is the lifting scheme, and this new idea may make a breakthrough in the design of optimum two-band biorthogonal filter banks. Next, a level-wise optimization algorithm is proposed to design multilevel signal-adapted RBIFB's, and this algorithm can achieve a considerable improvement in global subband coding gain. However, due to level-wise optimization, it does not always achieve the maximal global coding gain. Finally, the experimental results are reported, which confirm the effectiveness of our approaches.

Future work will focus on incorporating the method into the design of the optimal FIR biorthogonal filter banks and signal-adapted tree-structured RBIWP's. Especially for low bit rate compression, it is an interesting issue to exploit new adaptive algorithm based on the framework of multichannel decomposition.

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