Recursive Decoding and Its Performance for Low-Rate Reed–Muller Codes

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***Abstract*—Recursive decoding techniques are considered for Reed–Muller (RM) codes of growing length n and fixed order r. An algorithm is designed that has complexity of order $n \log n$ and corrects most error patterns of weight up to $n(1/2 - \varepsilon)$ given that ε exceeds $n^{-1/2^r}$. This improves the asymptotic bounds known for decoding RM codes with nonexponential complexity.**

To evaluate decoding capability, we develop a probabilistic technique that disintegrates decoding into a sequence of recursive steps. Although dependent, subsequent outputs can be tightly evaluated under the assumption that all preceding decodings are correct. In turn, this allows us to employ second-order analysis and find the error weights for which the decoding error probability vanishes on the entire sequence of decoding steps as the code length n grows.

***Index Terms*—Decoding threshold, Plotkin construction, recursive decoding, Reed–Muller (RM) codes.**

I. Introduction

In this paper, our goal is to design new decoding algorithms that can enhance techniques known to date for Reed–Muller (RM) codes. In general, RM codes can be designed from the set $\{f_r^m\}$ of all m-variate Boolean polynomials of degree r or less. Here each polynomial $f \in \{f_r^m\}$ is defined on the m-dimensional space E_2^m. For any f, we consider the sequence of binary values $f(x)$ obtained as argument x runs through E_2^m. These sequences—codewords $\boldsymbol{c}(f)$—form an RM code, which is below denoted $\left\{ {m \atop r} \right\}$ and has length n, dimension k, and distance d as follows:

$$n = 2^m, \quad k = \sum_{i=0}^{r} \binom{m}{i}, \quad d = 2^{m-r}.$$

The decoding algorithms discussed in this paper (including the new algorithms) can be applied to any RM code. However, we will mostly focus on their asymptotic performance obtained for long RM codes of *fixed order* r. To define their error-correcting performance, we use the following definition. Given an infinite sequence of codes $A_i(n_i, d_i)$, we say that a decoding algorithm Ψ has a threshold sequence δ_i and a residual sequence $\epsilon_i \to 0$ if for $n_i \to \infty$

- Ψ correctly decodes all but a vanishing fraction of error patterns of weight $\delta_i(1 - \epsilon_i)$ or less;
- Ψ fails to decode a nonvanishing fraction of error patterns of weight δ_i or less.[1]

Nonexponential decoding algorithms known for RM codes can be loosely separated into three groups. First, majority decoding was developed in the seminal paper [1]. The algorithm requires complexity of order nk or less. For RM codes of fixed order r, it was proven in [6] that majority decoding achieves maximum possible threshold $\delta = n/2$ (here and subsequently we omit the index i) with a residual

$$\varepsilon_r^{\mathrm{maj}} = (cm/d)^{1/2^{r+1}}, \qquad m \to \infty \tag{1}$$

where c is a constant that does not depend on m and r.

The second type of decoding algorithms makes use of the symmetry group of RM codes. One very efficient algorithm is presented in [7]. For long RM codes $\left\{ {m \atop 2} \right\}$, this algorithm reduces the residual term $\varepsilon_2^{\mathrm{maj}}$ from (1) to its square $(cm/d)^{1/4}$, where $c > \ln 4$. On the other hand, the complexity order of nm^2 of majority decoding is also increased in the algorithm [7] to almost its square $n^2 m$. The corresponding thresholds for higher orders $r \geq 3$ are yet unknown.

Another result of [7] concerns maximum-likelihood (ML) decoding. It is shown that ML decoding of RM codes of fixed-order r yields a substantially lower residual

$$\varepsilon_r^{\min} = m^{r/2} n^{-1/2} (c(2^r - 1)/r!)^{1/2}, \qquad m \to \infty \tag{2}$$

where $c > \ln 4$. However, even the best known algorithm of ML decoding designed by the multilevel trellis structure in [8] has yet complexity that is exponential in n.

Finally, various recursive techniques were introduced in [2]–[4], and [10]. All these algorithms use different recalculation rules but rely on the same code design based on the *Plotkin construction* $(\boldsymbol{u}, \boldsymbol{u} + \boldsymbol{v})$. The construction allows to decompose RM codes $\left\{ {m \atop r} \right\}$ onto shorter codes, by taking subblocks $\boldsymbol{u}$ and $\boldsymbol{v}$ from codes $\left\{ {m-1 \atop r} \right\}$ and $\left\{ {m-1 \atop r-1} \right\}$. The results from [2], [4], and [10] show that this recursive structure enables both encoding and bounded distance decoding with the lowest complexity order of $n \min(r, m - r)$ known for RM codes of an arbitrary order r.

In the same vein, below we also employ Plotkin construction. The basic recursive procedure will split RM code $\left\{ {m \atop r} \right\}$ of length n into two RM codes of length $n/2$. Decoding is then relegated further to the shorter codes of length $n/4$ and so on, until we reach basic codes of order $r \leq 1$ or $r = m$. At these points, we use ML decoding or the variants derived therefrom. By contrast, in all intermediate steps, we shall only recalculate

[1]Note that multiple sequences with $\epsilon_i \to 0$ can satisfy the same definition.

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the newly defined symbols. Here our goal is to find efficient *recalculation rules* that can *provably* improve the performance of RM codes. Our results presented in Theorems 1 and 2 show that recursive techniques indeed outperform other polynomial algorithms known for RM codes. These results also show how decoding complexity can be traded for a higher threshold.

Theorem 1: Long RM codes $\left\{ {m \atop r} \right\}$ of fixed order r can be decoded with linear complexity $O(n)$ and decoding threshold

$$\delta = n/2, \quad \varepsilon_r = ((2r\ln m)/d)^{1/2^{r+1}}, \qquad m \to \infty.$$

Theorem 2: Long RM codes $\left\{ {m \atop r} \right\}$ of fixed order r can be decoded with quasi-linear complexity $O(n\log n)$ and decoding threshold

$$\delta = n/2, \quad \tilde{\varepsilon}_r = (cm/d)^{1/2^r}, \qquad c > \ln 4, \quad m \to \infty.$$

Rephrasing Theorems 1 and 2, we obtain the following.

Corollary 3: Long RM codes $\left\{ {m \atop r} \right\}$ of fixed order r can be decoded with vanishing output error probability and linear complexity $O(n)$ (or quasi-linear complexity $O(n\log n)$) on a binary channel with crossover error probability $(1-\varepsilon_r)/2$ (correspondingly, $(1-\tilde{\varepsilon}_r)/2$) as $n \to \infty$.

Note that Theorem 1 increases decoding threshold of the recursive techniques introduced in [2] and [4] from the order of $d/2$ to $n/2$ while keeping linear decoding complexity. Theorem 2 improves both the complexity and residual of majority decoding of RM codes. When compared with the algorithm of [7], this theorem reduces the quadratic complexity $O(n^2\log n)$ to a quasi-linear complexity $O(n\log n)$ and also extends this algorithm to an arbitrary order $r \geq 2$ of RM codes.

The algorithms designed in what follows differ from the earier algorithms of [2], [4], and [10] in both the intermediate recalculations and the stopping rules. First, we employ new *intermediate recalculations*, which yield the exact decoding thresholds, as opposed to the bounded distance threshold $d/2$ established in [4] and [10]. This leads us to Theorem 1. Second, by analyzing the results of Theorem 1, we also change the former *stopping rules*, all of which terminate decoding at the repetition codes. Now we terminate decoding earlier, once we achieve the biorthogonal codes. This change yields Theorem 2 and substantially improves decoding performance (this is discussed in Section VII). Finally, we employ a new probabilistic analysis of recursive algorithms. In Section VII, we will see that this analysis not only gives the actual thresholds but also shows how the algorithms can be advanced further.

In Section II, we consider recursive structure of RM codes in more detail. In Section III, we proceed with decoding techniques and design two different recursive algorithms Ψ_r^m and Φ_r^m. These algorithms are analyzed in Sections IV–VI, which are concluded with Theorems 1 and 2. In Section VII, we briefly discuss extensions that include decoding lists, subcodes of RM codes, and soft-decision channels. For the latter case, we will relate the noise power to the quantity ε^{-2}. Thus, the residual ε will serve as a measure of the highest noise power that can be withstood by a specific low-rate code.

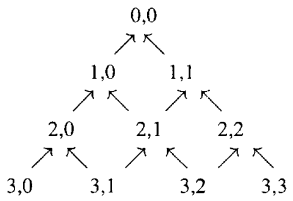

Fig. 1. Decomposition of RM codes of length 8.

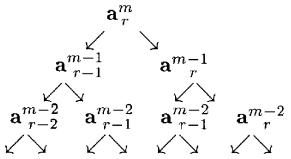

Fig. 2. Decomposition of information paths.

II. Recursive Encoding of RM Codes

Consider any m-variate Boolean polynomial $f = f_r^m$ and the corresponding codeword $\boldsymbol{c}(f)$ with symbols $f(x)$. Below we assume that positions $x = (x_1, \ldots, x_m)$ are ordered lexicographically, with x_1 being the senior digit. Note that any polynomial f can be split as

$$f_r^m(x_1, \ldots, x_m) = f_r^{m-1}(x_2, \ldots, x_m) + x_1 f_{r-1}^{m-1}(x_2, \ldots, x_m) \quad (3)$$

where we use the new polynomials f_r^{m-1} and f_{r-1}^{m-1}. These polynomials are defined over $m-1$ variables and have degrees at most r and $r-1$, respectively. Correspondingly, one can consider two codewords $\boldsymbol{u} = \boldsymbol{c}(f_r^{m-1})$ and $\boldsymbol{v} = \boldsymbol{c}(f_{r-1}^{m-1})$ that belong to the codes $\left\{ {m-1 \atop r} \right\}$ and $\left\{ {m-1 \atop r-1} \right\}$. Then representation (3) converts any codeword $\boldsymbol{c}(f) \in \left\{ {m \atop r} \right\}$ to the form $(\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v})$. This is the well-known Plotkin construction.

By continuing this process on codes $\left\{ {m-1 \atop r} \right\}$ and $\left\{ {m-1 \atop r-1} \right\}$, we obtain RM codes of length 2^{m-2}, and so on. Finally, we arrive at the end nodes, which are repetition codes $\left\{ {g \atop 0} \right\}$ for any $g = 1, \ldots, m-r$ and full spaces $\left\{ {h \atop h} \right\}$ for any $h = 1, \ldots, r$. This is schematically shown in Fig. 1 for RM codes of length 8.

Now let $\boldsymbol{a}_r^m = \{a_j | j = 1, k\}$ be a block of k information bits a_j that encode a vector $(\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v})$. By decomposing this vector into $\boldsymbol{u}$ and $\boldsymbol{v}$, we also split $\boldsymbol{a}_r^m$ into two information subblocks $\boldsymbol{a}_r^{m-1}$ and $\boldsymbol{a}_{r-1}^{m-1}$ that encode vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. In the following steps, information subblocks are split further, until we arrive at the end nodes $\left\{ {g \atop 0} \right\}$ or $\left\{ {h \atop h} \right\}$. This is shown in Fig. 2. Note that only one information bit is assigned to the left-end (repetition) code $\left\{ {g \atop 0} \right\}$, while the right-end code $\left\{ {h \atop h} \right\}$ includes 2^h bits. Below, these 2^h bits will be encoded using the unit generator matrix. Summarizing, we see that any codeword can be encoded from the information strings assigned to the end nodes $\left\{ {g \atop 0} \right\}$ or $\left\{ {h \atop h} \right\}$ by repeatedly combining codewords $\boldsymbol{u}$ and $\boldsymbol{v}$ in the $(\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v})$-construction.

Given any algorithm ψ, in the sequel we use the notation $|\psi|$ for its complexity. Let ψ_r^m denote the encoding described above for the code $\left\{ {m \atop r} \right\}$. Taking a complexity estimate from [2] and its enhancement from [4], we arrive at the following lemma.

Lemma 4: RM codes $\left\{ {m \atop r} \right\}$ can be recursively encoded with complexity

$$|\psi_r^m| \le n \min(r, m-r). \tag{4}$$

Proof: First, note that the end nodes $\left\{ {g \atop 0} \right\}$ and $\left\{ {h \atop h} \right\}$ require no encoding and therefore satisfy the complexity bound (4). Second, we verify that code $\left\{ {m \atop r} \right\}$ satisfies (4) if the two constituent codes do. Let the codewords $\boldsymbol{u} \in \left\{ {m-1 \atop r} \right\}$ and $\boldsymbol{v} \in \left\{ {m-1 \atop r-1} \right\}$ have encoding complexity $\left|\psi_r^{m-1}\right|$ and $\left|\psi_{r-1}^{m-1}\right|$ that satisfies (4). Then their $(\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v})$ combination requires complexity

$$|\psi_r^m| \le \left|\psi_{r-1}^{m-1}\right| + \left|\psi_r^{m-1}\right| + n/2$$

where $n/2$ extra additions $(\mathrm{mod}\, 2)$ were included to find the right half $\boldsymbol{u}+\boldsymbol{v}$. Now we substitute estimates (4) for quantities $\left|\psi_{r-1}^{m-1}\right|$ and $\left|\psi_r^{m-1}\right|$. If $r < m-r$, then

$$|\psi_r^m| \le n(r-1)/2 + nr/2 + n/2 = nr.$$

The two other cases, namely, $r > m-r$ and $r = m-r$, can be treated similarly. □

Now consider an information bit a_j associated with a left node $\left\{ {g \atop 0} \right\}$, where $g \in [1, m-r]$. We will map a_j onto a specific "binary path"

$$\xi \stackrel{\mathrm{def}}{=} (\xi_1, \ldots, \xi_m)$$

of length m leading from the origin $\left\{ {m \atop r} \right\}$ to the end node $\left\{ {0 \atop 0} \right\}$. To do so, we first define the senior bit

$$\xi_1 = \begin{cases} 0, & \text{if } a_j \in \boldsymbol{a}_{r-1}^{m-1} \\ 1, & \text{if } a_j \in \boldsymbol{a}_r^{m-1}. \end{cases}$$

Next, we take $\xi_2 = 0$ if a_j encodes the left descendant subcode on the following step. Otherwise, $\xi_2 = 1$. Similar procedures are then repeated at the steps $t = 3, \ldots, m-g$ and give some end subpath $\underline{\xi}$ that arrives at the node $\left\{ {g \atop 0} \right\}$. We then add g right-hand steps and obtain a full path ξ of length m that arrives at the node $\left\{ {0 \atop 0} \right\}$. Using notation 1^g for the sequence of g ones, we write

$$\xi = (\underline{\xi}, 1^g).$$

Now consider any right-end node $\left\{ {h \atop h} \right\}$, where $h \in [1, r]$ and let $\underline{\xi}$ be any right-end path that ends at this node. Then $\underline{\xi}$ is associated with 2^h information bits. Therefore, we extend $\underline{\xi}$ to the full length m by adding any binary suffix $(\xi_{m-h+1}, \ldots, \xi_m)$. This allows us to consider separately all 2^h information bits and use common notation $a(\xi)$.

When all left- and right-end paths are considered together, we obtain all paths of length m and binary weight $m-r$ or more. This gives one-to-one mapping between k information bits and extended paths ξ. Below all ξ are ordered lexicographically, as m-digital binary numbers.

III. Recursive Decoding

Now we turn to recursive decoding algorithms. We map any binary symbol a onto $(-1)^a$ and assume that all code vectors belong to $\{1, -1\}^n$. Obviously, the sum $a+b$ of two binary symbols is being mapped onto the product of their images. Then we consider any codeword

$$\boldsymbol{c} = (\boldsymbol{u}, \boldsymbol{uv})$$

transmitted over a binary symmetric channel with crossover probability $p < 1/2$. The received block $\boldsymbol{y} \in \{1, -1\}^n$ consists of two halves $\boldsymbol{y'}$ and $\boldsymbol{y''}$, which are the corrupted images of vectors $\boldsymbol{u}$ and $\boldsymbol{uv}$. We start with a basic algorithm $\Psi_{\mathrm{rec}}(\boldsymbol{y})$ that will be later used in recursive decoding. In our decoding, vector $\boldsymbol{y}$ will be replaced by the vectors whose components take on real values from the interval $[-1, +1]$. Therefore, we take a more general approach and assume that $\boldsymbol{y} \in \mathbb{R}^n$.

Step 1. We first try to find the codeword $\boldsymbol{v}$ from $\left\{ {m-1 \atop r-1} \right\}$. In the absence of noise, we have the equality $\boldsymbol{v} = \boldsymbol{y'y''}$ (which gives the binary sum of vectors $\boldsymbol{y'}$ and $\boldsymbol{y''}$ in the former notation). On a noisy channel, we first find the "channel estimate"

$$\boldsymbol{y}^v = \boldsymbol{y'y''} \tag{5}$$

of $\boldsymbol{v}$. Next, we employ (any) decoding $\Psi(\boldsymbol{y}^v)$, which will be specified later. The output is some vector $\hat{\boldsymbol{v}} \in \left\{ {m-1 \atop r-1} \right\}$ and its information block $\hat{\boldsymbol{a}}^v$.

Step 2. We try to find the block $\boldsymbol{u} \in \left\{ {m-1 \atop r} \right\}$ given $\hat{\boldsymbol{v}}$ from Step 1. Here we take two corrupted versions of vector $\boldsymbol{u}$, namely, $\boldsymbol{y'}$ in the left half and $\boldsymbol{y''}\hat{\boldsymbol{v}}$ in the right half. These two *real* vectors are added and combined in their "midpoint"

$$\boldsymbol{y}^u = (\boldsymbol{y'} + \boldsymbol{y''}\hat{\boldsymbol{v}})/2. \tag{6}$$

Then we use some decoding $\Psi(\boldsymbol{y}^u)$, which is also specified later. The output is some vector $\hat{\boldsymbol{u}} \in \left\{ {m-1 \atop r} \right\}$ and its information block $\hat{\boldsymbol{a}}^u$. So, decoding $\Psi_{\mathrm{rec}}(\boldsymbol{y})$ is performed as follows.

Algorithm $\Psi_{\mathrm{rec}}(\boldsymbol{y})$

1. Calculate vector $\boldsymbol{y}^v = \boldsymbol{y'y''}$.
Find $\hat{\boldsymbol{v}} = \Psi(\boldsymbol{y}^v)$ and $\hat{\boldsymbol{a}}^v$.

2. Calculate vector $\boldsymbol{y}^u = (\boldsymbol{y'} + \boldsymbol{y''}\hat{\boldsymbol{v}})/2$.
Find $\hat{\boldsymbol{u}} = \Psi(\boldsymbol{y}^u)$ and $\hat{\boldsymbol{a}}^u$.

3. Output decoded components:
$\hat{\boldsymbol{a}} := (\hat{\boldsymbol{a}}^v \mid \hat{\boldsymbol{a}}^u); \hat{\boldsymbol{c}} := (\hat{\boldsymbol{u}} \mid \hat{\boldsymbol{u}}\hat{\boldsymbol{v}})$.

In a more general scheme Ψ_r^m, we repeat this recursion by decomposing subblocks $\boldsymbol{y}^v$ and $\boldsymbol{y}^u$ further. On each intermediate step, we only recalculate the newly defined vectors $\boldsymbol{y}^v$ and $\boldsymbol{y}^u$ using (5) when decoder moves left and (6) when it goes right. Finally, vectors $\boldsymbol{y}^v$ and $\boldsymbol{y}^u$ are decoded, once we reach the end nodes $\left\{ {g \atop 0} \right\}$ and $\left\{ {h \atop h} \right\}$. Given any end code C of length l and any estimate $\boldsymbol{z} \in \mathbb{R}^l$, we employ the (soft decision) minimum-distance (MD) decoding $\Psi(\boldsymbol{z}) = \hat{\boldsymbol{c}}$ that outputs a codeword $\hat{\boldsymbol{c}}$ closest to $\boldsymbol{z}$ in the Euclidean metric. Equivalently, $\hat{\boldsymbol{c}}$

maximizes the inner product $(\boldsymbol{c}, \boldsymbol{z})$. The algorithm is described below.

Algorithm $\Psi_r^m(\boldsymbol{y})$

1. If $0 < r < m$, perform $\Psi_{\rm rec}(\boldsymbol{y})$ using $\Psi(\boldsymbol{y}^v) = \Psi_{r-1}^{m-1}$ and $\Psi(\boldsymbol{y}^u) = \Psi_r^{m-1}$.
2. If $r = 0$, perform MD decoding $\Psi(\boldsymbol{y}^v)$ for code $\left\{ {r \atop 0} \right\}$.
3. If $r = m$, perform MD decoding $\Psi(\boldsymbol{y}^u)$ for code $\left\{ {r \atop r} \right\}$.

In the following algorithm Φ_r^m, we refine algorithm $\Psi_r^m(\boldsymbol{y})$ by terminating decoding Ψ at the biorthogonal codes $\left\{ {g \atop 1} \right\}$.

Algorithm $\Phi_r^m(\boldsymbol{y})$

1. If $1 < r < m$, perform $\Psi_{\rm rec}(\boldsymbol{y})$ using $\Psi(\boldsymbol{y}^v) = \Phi_{r-1}^{m-1}$ and $\Psi(\boldsymbol{y}^u) = \Phi_r^{m-1}$.
2. If $r = 1$, perform MD decoding $\Phi(\boldsymbol{y}^v)$ for code $\left\{ {r \atop 1} \right\}$.
3. If $r = m$, perform MD decoding $\Phi(\boldsymbol{y}^u)$ for code $\left\{ {r \atop r} \right\}$.

Thus, procedures Ψ_r^m and Φ_r^m have a recursive structure that calls itself until MD decoding is applied on the end nodes. Now the complexity estimate follows.

Lemma 5: For any RM code $\left\{ {m \atop r} \right\}$ algorithms Ψ_r^m and Φ_r^m have decoding complexity

$$|\Psi_r^m| \le 4n \min(r, m-r) + n \tag{7}$$
$$|\Phi_r^m| \le 3n \min(r, m-r) + n(m-r) + n. \tag{8}$$

Proof: First, note that for trivial codes $\left\{ {r \atop 0} \right\}$ and $\left\{ {r \atop r} \right\}$, MD decoding can be executed in n operations and satisfies the bound (7) (here we assume that finding the sign of a real value requires one operation). For biorthogonal codes, their MD decoding Φ_1^m can be executed in $n \log_2 n + n + 3$ operations using the Green machine or $n \log_2 n + 2n$ operations using fast Hadamard transform (see [18] or [5, Sec. 14.4]). Obviously, this decoding satisfies the upper bound (8).

Second, for both algorithms Ψ and Φ, vector $\boldsymbol{y}^v$ in (5) can be calculated in $n/2$ operations while vector $\boldsymbol{y}^u$ in (6) requires $3n/2$ operations. Therefore, our decoding complexity satisfies the same recursion

$$|\Psi_r^m| \le \left|\Psi_{r-1}^{m-1}\right| + \left|\Psi_r^{m-1}\right| + 2n$$
$$|\Phi_r^m| \le \left|\Phi_{r-1}^{m-1}\right| + \left|\Phi_r^{m-1}\right| + 2n.$$

Finally, we verify that (7) and (8) satisfy the above recursion, similarly to the derivation of (4). □

Discussion.

Both algorithms Ψ_r^m and Φ_r^m admit bounded distance decoding. This fact can be derived by adjusting the arguments of [4] for our recalculation rules (5) and (6). Algorithm Ψ_r^m is also similar to decoding algorithms of [2] and [10]. However, our *recalculation rules* are different from those used in the above papers. For example, the algorithm of [10] performs the so-called "min-sum" recalculation

$$\boldsymbol{y}^v = \operatorname{sign}(\boldsymbol{y}'\boldsymbol{y}'') \ \min\{|\boldsymbol{y}'|, |\boldsymbol{y}''|\} \tag{9}$$

instead of (5). This (simpler) recalculation (5) will allow us to substantially expand the "*provable*" decoding domain versus the bounded-distance domain established in [4] and [10]. We then further extend this domain in Theorem 2, also using the new *stopping rule* that replaces $r = 0$ in Ψ_r^m with $r = 1$ in Φ_r^m. However, it is still an open problem to find the decoding domain using any other recalculation rule, say those from [2], [4], [10], or [11].

Finally, note that the scaling factor $1/2$ in recalculation rule (6) brings any component y^u back to the interval $[-1, +1]$ used before this recalculation. This scaling will also allow us to simplify some proofs, in particular that of Lemma 10. However, replacing (6) by the simpler rule

$$\boldsymbol{y}^u = \boldsymbol{y}' + \boldsymbol{y}''\hat{\boldsymbol{v}}$$

does not change any decoding results. Though being equivalent, the new rule also reduces complexity estimates (7) and (8) to

$$|\Psi_r^m| \le 3n \min(r, m-r) + n \tag{10}$$
$$|\Phi_r^m| \le 2n \min(r, m-r) + n(m-r) + n. \tag{11}$$

These reductions in complexity notwithstanding, in the sequel we still use the original recalculation rule (6) for the only reason to simplify our proofs.

IV. Analysis of Recursive Algorithms

A. Intermediate Outputs

We begin with the algorithm Ψ_r^m and later will use a similar analysis for the algorithm Φ_r^m. Note that Ψ_r^m enters each end node multiple times, by taking all paths leading to this node. It turns out that the output bit-error rate (BER) significantly varies on different nodes and even on different paths leading to the same node. Therefore, our first problem is to fix a path ξ and estimate the output BER for the corresponding information symbol $a(\xi)$. In particular, we will define the most error-prone paths.

Consider any (sub)path of some length s. Let $\underline{\xi}$ be its prefix of length $s-1$, so that $\xi = (\underline{\xi}, \xi_s)$, where

$$\xi = (\xi_1, \ldots, \xi_s), \quad \underline{\xi} = (\xi_1, \ldots, \xi_{s-1}), \qquad s \in [1, m]. \tag{12}$$

First, note that algorithm $\Psi_r^m(\boldsymbol{y})$ repeatedly recalculates its input $\boldsymbol{y}$, by taking either an estimate $\boldsymbol{y}^v$ from (5) when a path ξ turns left or $\boldsymbol{y}^u$ from (6) otherwise. The following lemma shows that recursive decoding follows lexicographic order of our paths $\xi \in \Gamma$.

Lemma 6: For two paths γ and ξ, the bit $a(\xi)$ is decoded after $a(\gamma)$ if $\xi > \gamma$.

Proof: Given two paths ξ and γ, let s be the first (senior) position where they disagree. If $\xi > \gamma$, then $\xi_s = 1$ and $\gamma_s = 0$. Thus, after s steps, γ moves left while ξ moves right. Correspondingly, γ proceeds first. □

On any subpath ξ of length s, algorithm $\Psi_r^m(\boldsymbol{y})$ outputs some vector $\boldsymbol{y}(\xi)$ of length 2^{m-s}. Next, we derive a recursive expres-

sion for $\boldsymbol{y}(\xi)$ using formulas (5) and (6). In any step s, the algorithm first splits $\boldsymbol{y}(\underline{\xi})$ into halves $\boldsymbol{y}'(\underline{\xi})$ and $\boldsymbol{y}''(\underline{\xi})$. For $\xi_s = 0$, $\boldsymbol{y}(\xi)$ is given by recursion (5) and is rewritten later in the upper line of (13).

If $\xi_s = 1$, then $\boldsymbol{y}(\xi)$ is obtained from (6). Here we also need the vector $\hat{\boldsymbol{v}}(\xi)$ decoded on the preceding subpath $(\underline{\xi}, 0)$. The corresponding output is written in the second line of (13)

$$\boldsymbol{y}(\xi) = \begin{cases} \boldsymbol{y}'(\underline{\xi})\boldsymbol{y}''(\underline{\xi}), & \text{if } \xi_s = 0 \\ \boldsymbol{y}'(\underline{\xi})/2 + \hat{\boldsymbol{v}}(\xi)\boldsymbol{y}''(\underline{\xi})/2, & \text{if } \xi_s = 1. \end{cases} \tag{13}$$

Finally, consider any left-end path $\xi = (\underline{\xi}, 1^g)$ that passes some repetition code $\left\{ \begin{smallmatrix} g \\ 0 \end{smallmatrix} \right\}$. Note that no preceding decodings are used after ξ reaches the repetition code $\left\{ \begin{smallmatrix} g \\ 0 \end{smallmatrix} \right\}$. Here we define the end result on the path ξ as

$$y(\xi) \stackrel{\text{def}}{=} \sum_{i=1}^{2^g} y_i(\underline{\xi})/2^g \tag{14}$$

by taking $\hat{\boldsymbol{v}}(\xi) = 1$ in the last g steps of recursion (13). Note that MD decoding also makes its decision on the entire sum of symbols $y_i(\underline{\xi})$ and outputs the symbol[2]

$$\hat{a}(\xi) = \operatorname{sign}(y(\xi)). \tag{15}$$

For any right-end code $\left\{ \begin{smallmatrix} h \\ h \end{smallmatrix} \right\}$, the output is some vector $\boldsymbol{y}(\underline{\xi})$ of length 2^h. Again, MD decoding takes every symbol $y(\xi)$ on the full path ξ and converts it into the information bit $\hat{a}(\xi)$, making bit-by-bit decision (15). This is summarized as follows.

Lemma 7: For any end path ξ, the algorithm Ψ_r^m decodes the outputs $y(\xi)$ into the information bits $\hat{a}(\xi)$ using the rule (15).

B. Conditional Error Probabilities

Next, we consider the decoding error probability $P(\xi)$ for any information bit $a(\xi)$. On an additive binary symmetric channel, $P(\xi)$ does not depend on the transmitted codeword $\boldsymbol{c}$ and we can assume that $\boldsymbol{c} = \boldsymbol{1}$. According to our decoding rule (15), an error event $\{\hat{a}(\xi) = -1\}$ has probability

$$P(\xi) = \Pr\{y(\xi) < 0\}.$$

(Here we follow footnote 2 and assume that $y(\xi) < 0$ with probability $1/2$ if $y(\xi) = 0$.)

Note, however, that the recursive output $y(\xi)$ depends on the outputs $\boldsymbol{v}(\gamma)$ obtained on all preceding paths $\gamma < \xi$. To simplify our calculations, we wish to consider the above event $\{y(\xi) < 0\}$ on the condition that all preceding decodings are correct. This implies that any path $\gamma < \xi$ gives an information bit $\hat{a}(\gamma)$ and a codeword $\boldsymbol{v}(\gamma)$ as follows:

$$\hat{a}(\gamma) = 1, \quad \boldsymbol{v}(\gamma) = \boldsymbol{1}.$$

This assumption also allows us to simplify our recalculations (5), (6), and (13) by removing all vectors $\boldsymbol{v}(\gamma)$

$$\boldsymbol{y}^v = \boldsymbol{y}'\boldsymbol{y}'', \quad \boldsymbol{y}^u = (\boldsymbol{y}' + \boldsymbol{y}'')/2 \tag{16}$$

$$\boldsymbol{y}(\xi) = \begin{cases} \boldsymbol{y}'(\underline{\xi})\boldsymbol{y}''(\underline{\xi}), & \text{if } \xi_s = 0 \\ \boldsymbol{y}'(\underline{\xi})/2 + \boldsymbol{y}''(\underline{\xi})/2, & \text{if } \xi_s = 1. \end{cases} \tag{17}$$

[2]Below we assume that $\operatorname{sign}(0)$ takes values $+1$ and -1 with probability $1/2$.

Therefore, our first goal is to find how much unconditional probabilities $P(\xi)$ change given that the preceding decodings are correct. First, let

$$\xi_* = (0^r, 1^{m-r}) \tag{18}$$

be the leftmost path that begins with r zeros followed by $m - r$ ones. For any path ξ, let $A(\xi)$ and $B(\xi)$ denote the events

$$\begin{aligned} A(\xi) &= \cap_{\gamma \le \xi} \{\hat{a}(\gamma) = 1\} \\ B(\xi) &= \cap_{\gamma < \xi} \{\hat{a}(\gamma) = 1\} \end{aligned} \tag{19}$$

which include all error vectors that are correctly decoded on the paths $\gamma \le \xi$ or $\gamma < \xi$, respectively. We define the complete ensemble of all error vectors by $B(\xi_*)$. In the sequel, we replace each probability $P(\xi)$ by the probability

$$p(\xi) \stackrel{\text{def}}{=} \Pr\{y(\xi) < 0 | B(\xi)\} = \Pr\{\overline{A}(\xi) | B(\xi)\} \tag{20}$$

on the condition that all previous decodings are correct. The following upper bound (21) conservatively assumes that an information symbol $\hat{a}(\xi)$ is *always* incorrect whenever a failure occurs in any step $\gamma \le \xi$. Similarly, the upper bound in (22) uses the formula of total probability and adds up probabilities $p(\xi)$ over all paths ξ. By contrast, the lower bound takes into account that the block is always incorrect given the decoding failure on the first step ξ_*.

Lemma 8: For any path $\xi \in \Gamma$, its BER $P(\xi)$ satisfies the inequality

$$P(\xi) \le \sum_{\gamma \le \xi} p(\gamma). \tag{21}$$

Block error probability P satisfies inequalities

$$p(\xi_*) \le P \le \sum_{\xi \in \Gamma} p(\xi). \tag{22}$$

Proof: The probability $P(\xi)$ can be estimated as

$$\begin{aligned} P(\xi) &\le \Pr\{\overline{A}(\xi)\} = \sum_{\gamma \le \xi} \Pr\{\overline{A}(\gamma) \cap B(\gamma)\} \\ &\le \sum_{\gamma \le \xi} \Pr\{\overline{A}(\gamma) | B(\gamma)\} = \sum_{\gamma \le \xi} p(\gamma). \end{aligned}$$

Similarly, the total probability P is bounded as

$$\Pr\{\overline{A}(\xi_*)\} \le P \le \sum_{\xi \in \Gamma} \Pr\{\overline{A}(\xi) | B(\xi)\} = \sum_{\xi \in \Gamma} p(\xi). \quad \square$$

C. Asymptotic Setting

Given any path ξ, we now assume that the decoder gives correct solutions $\hat{a}(\gamma) = 1$ on all previous paths $\gamma < \xi$. Our next goal is to estimate the decoding error probability

$$p(\xi) = \Pr\{y(\xi) < 0\} \tag{23}$$

where $y(\xi)$ is a random variable (rv), which satisfies simplified recalculations (17). Here we begin with the original probability distribution

$$\Pr\{y_i\} = \begin{cases} 1 - p, & \text{if } y_i = +1 \\ p, & \text{if } y_i = -1 \end{cases} \tag{24}$$

where y_i are n independent and identically distributed (i.i.d.) rvs that form the received vector $\boldsymbol{y}$.

Remark: Note that the above problem is somewhat similar to that of "probability density evolution" researched in iterative algorithms. Namely, in both algorithms the original rv y_i undergoes two different transformations, similar to (17). However, in our setting, these transformations can also be mixed in an arbitrary (irregular) order that only depends on a particular path ξ, in general, and on its current symbol ξ_s, in particular.

To simplify this problem, in the following we estimate $p(\xi)$ using only the first two moments of variables y_i and their descendants. This will be done as follows.

1) First, note that the blocks $\boldsymbol{y}'$ and $\boldsymbol{y}''$ used in (16) always include different channel bits. Consequently, their descendants $\boldsymbol{y}'(\underline{\xi})$ and $\boldsymbol{y}''(\underline{\xi})$ used in (17) are also obtained from different channel bits. These bits are combined in the same operations. Therefore, all symbols $y_i(\xi)$ of the vector $\boldsymbol{y}(\xi)$ are i.i.d. rvs. This allows us to use the common notation $y(\xi)$ for any rv $y_i(\xi)$ obtained on the subpath ξ.

2) Let $e(\xi) = \mathsf{E}y(\xi)$ denote the expectation of any rv $y(\xi)$. Below we study the normalized rvs

$$z(\xi) = y(\xi)/e(\xi) \tag{25}$$

all of which have expectation 1. Our goal is to estimate their variances

$$\mu(\xi) \stackrel{\text{def}}{=} \mathsf{E}(z(\xi) - 1)^2. \tag{26}$$

Then decoding error probability always satisfies Chebyshev's inequality

$$p(\xi) = \Pr\{z(\xi) < 0\} \le \mu(\xi). \tag{27}$$

3) To prove Theorem 1, we first consider those left-end paths ξ that pass through the nodes $\left\{ {g \atop 0} \right\}$ with growing $g \ge m^{1/2}$. For any such path, we show that the corresponding rv $y(\xi)$ satisfies the central limit theorem as $m \to \infty$. This will allow us to replace Chebyshev's inequality (27) by (a stronger) Gaussian approximation. We will also see that the variance $\mu(\xi)$ rapidly declines as decoding progresses over the new paths ξ. For this reason, we shall still use Chebyshev's inequality (27) on the remaining paths with $g < m^{1/2}$, which will only slightly increase the block error probability P defined in (22).

D. Recalculation of the Variances

Our next goal is to recalculate the variances $\mu(\xi)$ defined in (26). Let the channel residual $\varepsilon = 1 - 2p$ be fixed. According to (24), the original channel outputs y_i have the means $\mathsf{E}y_i = \varepsilon$, in which case rv $z_i = y_i/\varepsilon$ have the variance

$$\mu_0 = \varepsilon^{-2} - 1.$$

Lemma 9: For any path $\xi = (\underline{\xi}, \xi_s)$, the variance $\mu(\xi)$ satisfies the recursions

$$\mu(\xi) + 1 = (\mu(\underline{\xi}) + 1)^2, \qquad \text{if } \xi_s = 0 \tag{28}$$

$$\mu(\xi) = \mu(\underline{\xi})/2, \qquad \text{if } \xi_s = 1. \tag{29}$$

Proof: First, we need to find the means $e(\xi)$ of rv $y(\xi)$ to proceed with new variables $z(\xi)$. Here we simply replace all three rvs used in (17) by their expectations. Then for any $\xi = (\underline{\xi}, \xi_s)$, the means $e(\xi)$ satisfy the recursion

$$e(\xi) = \begin{cases} e^2(\underline{\xi}), & \text{if } \xi_s = 0 \\ e(\underline{\xi}), & \text{if } \xi_s = 1. \end{cases} \tag{30}$$

Here we also use the fact that vectors $\boldsymbol{y}'(\underline{\xi})$ and $\boldsymbol{y}''(\underline{\xi})$ are independent and have symbols with the same expectation $e(\underline{\xi})$. Now we see that the normalized rvs $z(\xi)$ satisfy the recursion

$$z(\xi) = \begin{cases} z'(\underline{\xi}) \cdot z''(\underline{\xi}), & \text{if } \xi_s = 0 \\ z'(\underline{\xi})/2 + z''(\underline{\xi})/2, & \text{if } \xi_s = 1 \end{cases} \tag{31}$$

similarly to (17). By taking $\mathsf{E}z^2(\xi)$ we immediately obtain (28) and (29). □

Discussion.

Note that the means $e(\xi)$ only depend on the Hamming weight $w(\xi)$ of a binary subpath ξ. Indeed, a subpath ξ has $s - w(\xi)$ zero symbols ξ_i. According to (30), the original expectation $\mathsf{E}(y)$ of rv y_i is squared $s - w(\xi)$ times and is left unchanged $w(\xi)$ times. Therefore,

$$e(\xi) = \varepsilon^{2^{s - w(\xi)}}. \tag{32}$$

By contrast, equalities (28) and (29) show that variance $\mu(\xi)$ depends on positions of all ones in vector ξ. Thus, direct (nonrecurrent) calculations of $\mu(\xi)$ become more involved. In Lemma 12, we will see that even the simplest paths give rather bulky expressions (37) for $\mu(\xi)$. For this reason, we use a different approach. Namely, in Section IV-E we find the paths ξ that maximize $\mu(\xi)$.

E. The Weakest Paths

Preliminary discussion. Consider a channel with crossover error probability $(1 - \varepsilon)/2$ and residual ε. Initially, rvs $z(\xi)$ have the variance $\mu_0 = \varepsilon^{-2} - 1$ and always satisfy inequality $\mu(\xi) > 0$, by definition (26). According to (28), $\mu(\underline{\xi}) + 1$ is always squared when a path $\underline{\xi}$ is appended by $\xi_s = 0$. Thus, moving from a code $\left\{ {m \atop r} \right\}$ to its left descendant $\left\{ {m-1 \atop r-1} \right\}$ is equivalent to the replacement of the original residual ε by its square ε^2. In other words, any left-hand movement makes the descendant channel noisier. For small $\mu(\underline{\xi}) \approx 0$ (very-high-quality channel), squaring $\mu(\underline{\xi})+1$ is almost insignificant. However, it becomes more substantial as $\mu(\underline{\xi})$ grows.

By contrast, $\mu(\underline{\xi})$ is always cut in half when $\xi_s = 1$. In general, any right-hand movement makes the descendant channel less noisy. For example, we obtain $\mu_0 \approx \varepsilon^{-2}$ on (bad) channels with small residual ε. Then performing the right step, the recursion replaces this residual ε with the quantity almost equal to $\varepsilon\sqrt{2}$. Therefore, our first conclusion is that variance $\mu(\xi)$ increases if $\xi_s = 1$ is replaced by $\xi_s = 0$.

Neighboring paths. Our next step is to consider two "equally balanced" movements. Namely, in (33), we consider two subpaths ξ_- and ξ_+ of length s that have the same prefix $\underline{\xi}$ of length $s - 2$ but diverge in the last two positions as follows:

$$\begin{cases} \xi_- = (\underline{\xi}, 0, 1), \\ \xi_+ = (\underline{\xi}, 1, 0). \end{cases} \quad \begin{array}{c} 0 \diagup\!\diagdown 1 \\ \underline{\xi} = \xi_1, \ldots, \xi_{s-2}. \end{array} \tag{33}$$

We say that ξ_- and ξ_+ are left and right neighbors, correspondingly.

Lemma 10: Any two neighbors ξ_- and ξ_+ satisfy inequality

$$\mu(\xi_-) \geq \mu(\xi_+). \tag{34}$$

Proof: Let $\mu(\underline{\xi}) = \tau$. Then we use recursive equations (28) and (29), which give

$$\mu(\xi_-) = \tau^2/2 + \tau, \qquad \mu(\xi_+) = \tau^2/4 + \tau.$$

Therefore, (34) holds. □

The weakest paths. Now we see that any path ξ that includes two adjacent symbols $(1,0)$ increases its $\mu(\xi)$ after permutation $(1,0) \Rightarrow (0,1)$. In this case, we say that this path ξ becomes weaker. From now on, let Γ be the complete set of k extended paths ξ. Also, let Γ^g be the subset of all left-end paths ξ that enter the node $\left\{ {g \atop 0} \right\}$ and Γ_0 be the subset of the right-end paths. Given any subset $I \subseteq \Gamma$, we now say that $\xi_*(I)$ is the weakest path in I if

$$\mu(\xi_*(I)) = \max\{\mu(\xi) | \xi \in I\}.$$

Then we have the following.

Lemma 11: The weakest path on the full set Γ of all k paths is the leftmost path (18). More generally, for any $g \in [1, m-r]$, the weakest path on the subset Γ^g is its leftmost path

$$\xi_*^g = (0^{r-1}, 1^{m-r-g}, 0, 1^g). \tag{35}$$

Proof: First, note that on all left-end paths ξ, the variances $\mu(\xi)$ are calculated after m steps, at the same node $\left\{ {0 \atop 0} \right\}$. By contrast, all right-end paths ξ end at different nodes $\left\{ {h \atop h} \right\}$; therefore, their variances $\mu(\xi)$ are found after $m-h$ steps. To use Lemma 10, we consider an extended right-end path $\xi^e = (\xi, 0^h)$ obtained by adding h zeros. Then we have inequality $\mu(\xi^e) > \mu(\xi)$, since the variance μ increases after zeros are added. Despite this fact, below we prove that ξ_* from (18) and ξ_*^g from (35) still represent the weakest paths, even after this extension.

Indeed, now all paths have the same length m and the same weight $m-r$, so we can apply Lemma 10. Recall that each path $\xi \in \Gamma^g$ ends with the same suffix $0, 1^g$. In this case, ξ_*^g is the leftmost path on Γ^g. By Lemma 10, ξ_*^g maximizes the variance $\mu(\xi)$ over all $\xi \in \Gamma^g$. Finally, note that ξ_* is the leftmost path on the total set Γ since all r zeros form its prefix 0^r. Thus, ξ_* is the weakest path. □

V. Threshold of Algorithm Ψ_r^m

Now we find the variances $\mu(\xi_*)$ and $\mu(\xi_*^g)$ for the weakest paths ξ_* and ξ_*^g.

Lemma 12: For crossover error probability $(1-\varepsilon)/2$, the weakest paths ξ_* and ξ_*^g give the variances

$$\mu(\xi_*) = 2^{-(m-r)}(\varepsilon^{-2^{r+1}} - 1) \tag{36}$$

$$\mu(\xi_*^g) = 2^{-g}\left(\left(\left(\varepsilon^{-2^r} - 1\right)2^{r+g-m} + 1\right)^2 - 1\right). \tag{37}$$

Proof: Consider the weakest path ξ_* from (18). The recursion (28) begins with the original quantity $\mu(\xi) + 1 = \varepsilon^{-2}$. After completing r left steps $\underline{\xi} = 0^r$, the result is

$$\mu_r(\underline{\xi}) + 1 = \varepsilon^{-2^{r+1}}. \tag{38}$$

Then we proceed with $m-r$ right steps, each of which cuts $\mu_r(\underline{\xi})$ in half according to (29). Thus, we obtain equality (36). Formula (37) follows from representation (35) in a similar (though slightly longer) way. □

Lemma 12 allows us to use Chebyshev's inequality

$$p(\xi) \leq \mu(\xi) \leq \mu(\xi_*) \tag{39}$$

for any path ξ. However, this bound is rather loose and insufficient to prove Theorem 1. Therefore, we improve this estimate, separating all paths into two different sets. Namely, let Γ^* be the subset of all left-end paths that enter the node $\left\{ {g \atop 0} \right\}$ with $g \geq m^{1/2}$.

We will use the fact that any path $\xi \in \Gamma^*$ satisfies the central limit theorem as m grows. However, we still use Chebyshev's inequality on the complementary subset $\Gamma \setminus \Gamma^*$. In doing so, we take ε equal to the ε_r from Theorem 1

$$\varepsilon_r = (d^{-1} 2r \ln m)^{1/2^{r+1}}, \qquad m \to \infty. \tag{40}$$

Theorem 13: For RM codes with $m \to \infty$ and fixed order r used on a binary channel with crossover error probability $(1-\varepsilon_r)/2$, algorithm Ψ_r^m gives on a path ξ the asymptotic BER

$$p(\xi) \lesssim m^{-r}/\sqrt{4\pi r \ln m}, \qquad m \to \infty \tag{41}$$

with asymptotic equality on the weakest path ξ_*.

Proof: According to (14), any left-end path ξ gives the rv $y(\xi)$, which is the sum of $2^{g(\xi)}$ i.i.d. limited rv $y_i(\xi)$. For $\xi \in \Gamma^*$, this number grows as $2^{\sqrt{m}}$ or faster as $m \to \infty$. In this case, the normalized rv $z(\xi)$ satisfies the central limit theorem and its probability density function (pdf) tends to the Gaussian distribution $\mathcal{N}(1, \mu(\xi))$.

According to Lemmas 11 and 12, the weakest path ξ_* gives the maximum variance $\mu(\xi_*)$. In particular, for $\varepsilon = \varepsilon_r$ equality (36) gives

$$\mu(\xi_*) = (2r \ln m)^{-1} - 2^{r-m}. \tag{42}$$

Using Gaussian distribution $\mathcal{N}(1, \mu(\xi_*))$ to approximate $p(\xi_*)$, we take $\mu^{-1/2}(\xi_*)$ standard deviations and obtain (see also Remark 1 following the proof)

$$p(\xi_*) \sim Q(\mu^{-1/2}(\xi_*)), \qquad m \to \infty. \tag{43}$$

Here we also use the asymptotic

$$Q(x) \stackrel{\text{def}}{=} \int_x^\infty \exp\{-x^2/2\} dx/\sqrt{2\pi} \sim \exp\{-x^2/2\}/(x\sqrt{2\pi})$$

valid for large x. This yields asymptotic equality for $p(\xi_*)$ in (41). For any other path $\xi \in \Gamma^*$, $z(\xi)$ is approximated by the normal rv with a smaller variance $\mu(\xi) < \mu(\xi_*)$. Therefore, we use inequality in (41)

$$p(\xi) < Q(\mu^{-1/2}(\xi_*)). \tag{44}$$

Finally, consider any path ξ with $g < m^{1/2}$. In this case, we directly estimate the asymptotics of $\mu(\xi_*^g)$. Namely, we use the substitution $\varepsilon = \varepsilon_r$ in (37), which gives a useful estimate

$$\mu(\xi_*^g) \sim \begin{cases} 2^{-m-r+g}(2r \ln m)^{-1}, & \text{if } g > \frac{m-r}{2} + \ln m \\ 2^{-(m-r-2)/2}(2r \ln m)^{-1/2}, & \text{if } g < \frac{m-r}{2} - \ln m \end{cases} \tag{45}$$

Thus, we see that for all $g < m^{1/2}$, variances $\mu(\xi_*^g)$ have the same asymptotics and decline exponentially in m, as opposed to the weakest estimate (42). Then we have

$$p(\xi) \le \mu(\xi) \le \mu(\xi_*^g) < 2^{-(m-r)/2}$$

which also satisfies (41) as $m \to \infty$. □

Discussion.

1) Considering approximation (43) for a general path ξ, we arrive at the estimate

$$p(\xi) \sim Q(\mu^{-1/2}(\xi)), \qquad m \to \infty. \tag{46}$$

According to [9, Theorem XVI.7.1], this approximation is valid if the number of standard deviations $\mu^{-1/2}(\xi)$ is small relative to the number $2^{g(\xi)}$ of rvs $z_i(\xi)$ in the sum $z(\xi)$

$$\mu^{-1/2}(\xi) = o(2^{g(\xi)/6}). \tag{47}$$

In particular, we can use (43) for the path ξ_*, since (42) gives

$$\mu^{-1/2}(\xi_*) \sim (2r \ln m)^{1/2} = o(2^{\sqrt{m}/6}).$$

2) Note that for $\varepsilon = \varepsilon_r$, the variance $\mu(\xi_*^g)$ in (45) declines exponentially as g moves away from $m - r$. On the other hand, we can satisfy the asymptotic condition (47) for any path $\xi \in \Gamma^*$, if $\mu^{-1/2}(\xi)$ in (47) is replaced with a parameter[3]

$$\tau^{-1/2}(\xi) = \min\{\mu^{-1/2}(\xi), \mathsf{poly}(m)\}$$

as $m \to \infty$. We then use the inequality

$$p(\xi) \le Q\left(\tau^{-1/2}(\xi)\right)$$

valid for any $\xi \in \Gamma^*$ instead of the weaker inequality (44). Thus, the bounds on probabilities $p(\xi)$ rapidly decline as g moves away from $m - r$, and the total block error rate P also satisfies the same asymptotic bound (41).

3) Note that the same minimum residual ε_r can also be used for majority decoding. Indeed, both the majority and the recursive algorithms are identical on the weakest path ξ_*. Namely, both algorithms first estimate the product of 2^r channel symbols and then combine 2^{m-r} different estimates in (14). However, a substantial difference between the two algorithms is that recursive decoding uses the previous estimates to process any other path ξ. Because of this, the algorithm outperforms majority decoding in both the complexity and BER $p(\xi)$ for any $\xi \ne \xi_*$.

4) Theorem 13 almost entirely carries over to any $\varepsilon > \varepsilon_r$. Namely, we use the normal pdf $\mathcal{N}(1, \mu(\xi))$ for any $\xi \in \Gamma^*$. Here any variance $\mu(\xi)$ declines as ε grows. Therefore, we can always employ inequality (44) by taking the *maximum possible* variance $\mu(\xi_*)$ obtained in (42). On the other hand, asymptotic equality (43) becomes invalid as ε grows.

In this case, tighter bounds (say, the Chernoff bound) must be used on ξ_*. However, in this case, we also need to extend the second-order analysis of Lemma 10 to exponential moments. Such an approach can also give the asymptotic error probability $p(\xi)$ for any $\varepsilon > \varepsilon_r$. However, finding the bounds on $p(\xi)$ is an important issue still open to date.

5) It can be readily proven that, for sufficiently large $\varepsilon > 2^{-(m-r-g)/2^r}$, the variance $\mu(\xi_*^g)$ becomes independent of g, similar to the estimates obtained in the second line of (45). More generally, more and more paths yield almost equal contributions to the block error rate as ε grows. This is due to the fact that the neighboring paths exhibit similar performance on sufficiently good channels.

Now Theorem 1 directly follows from Theorem 13.

Proof of Theorem 1: Consider a channel with crossover probability $p = (1 - \varepsilon_r)/2$ for $m \to \infty$. The output block error probability P of the algorithm Ψ_r^m has the order at most $kp(\xi_*)$, where k is the number of information symbols. This number has polynomial order of $\binom{m}{r}$. On the other hand, formula (41) shows that $p(\xi_*)$ declines faster than k^{-1} for any $\varepsilon \ge \varepsilon_r$. As a result, $P \to 0$.

Next, we note that the error patterns of weight pn or less occur with a probability

$$\sum_{i=0}^{pn} \binom{n}{i} p^i (1-p)^{n-i} \to Q(0) = 1/2.$$

Since $P \to 0$, the above argument shows that the decoder fails to decode only a vanishing fraction of error patterns of weight pn or less.

Next, we need to prove that Ψ_r^m fails to correct a nonvanishing fraction of errors of weight $n/2$ or less. In proving this, consider a higher crossover probability $p_1 = (1 - \varepsilon)/2$, where

$$\varepsilon = \varepsilon_r (\ln m)^{-1/2^r}.$$

For this ε, our estimates (36) and (43) show that $\mu(\xi_*) \to 0$ and $p(\xi_*) \to 1/2$. Also, according to (22), $P > p(\xi_*)$. On the other hand, the central limit theorem shows that the errors of weight $n/2$ or more still occur with a vanishing probability

$$\sum_{i=n/2}^{n} \binom{n}{i} p_1^i (1-p_1)^{n-i} \to 0.$$

Thus, we see that Ψ_r^m necessarily fails on the weights $n/2$ or less, since the weights over $n/2$ still give a vanishing contribution to the nonvanishing error rate $p(\xi_*)$. □

VI. Threshold of Algorithm Φ_r^m

Before proceeding with a proof of Theorem 2, we summarize three important points that will be used below to evaluate the threshold of the algorithm Φ_r^m.

1) The received rv y_i, all intermediate recalculations (13), and end decodings on the right-end paths ξ are identical in both algorithms Φ_r^m and Ψ_r^m.

   By contrast, any left-end path $\xi = \underline{\xi}, 1^g$ first arrives at some biorthogonal code $\left\{ {g+1 \atop 1} \right\}$ of length $l = 2^{g+1}$ and is then followed by the suffix 1^g. Also, let c_s^g denote the sth codeword of $\left\{ {g+1 \atop 1} \right\}$ where $s = 1, \ldots, 2l$. Here we also assume that the first two codewords form the repetition code

$$c_1^g = 1^l, \quad c_2^g = -c_1^g.$$

   For each c_s^g, define its support I_s^g as the subset of positions that have symbols -1. Here $2l - 2$ codewords with $s > 2$ have support of the same size $|I_s^g| = 2^g$, whereas $|I_2^g| = 2^{g+1}$. Also, below $a(\xi)$ denotes any information symbol associated with a path ξ.

[3]Here we take any positive polynomial $\mathsf{poly}(m)$ of a fixed degree as $m \to \infty$.

2) Let the all-one codeword 1^n be transmitted and $\boldsymbol{y}$ be received. Consider the vector $\boldsymbol{y}(\underline{\xi})$ obtained on some left-end path $\underline{\xi}$ that ends at the node $\left\{ {g+1 \atop 1} \right\}$. By definition of MD decoding, $\boldsymbol{y}(\underline{\xi})$ is incorrectly decoded into any $c \neq c_1^g$ with probability

$$P(\xi) = \Pr\left\{\cup_{s=2}^{2l} \Omega_s(\xi)\right\}$$

where

$$\Omega_s(\xi) = \left\{ \boldsymbol{y} : \sum_{i \in I_s^g} y_i(\underline{\xi}) < 0 \right\}. \tag{48}$$

In our probabilistic setting, each event $\Omega_s(\xi)$ is completely defined by the symbols $y_i(\underline{\xi})$, which are i.i.d. rvs.

3) Recall that Lemma 8 is "algorithm-independent" and therefore is left intact in Φ_r^m. Namely, we again consider the events $A(\xi)$ and $B(\xi)$ from (19). Similarly to (20), we assume that all preceding decodings are correct and replace the unconditional error probability $P(\xi)$ with its conditional counterpart

$$p(\xi) = \Pr\{\overline{A}(\xi) | B(\xi)\}.$$

This probability satisfies the bounds

$$\Pr \Omega_s(\xi) \leq p(\xi) \leq \sum_{s=2}^{2l} \Pr \Omega_s(\xi). \tag{49}$$

Here we take the probability of incorrect decoding into any single codeword c_s^g as a lower bound (in fact, below we choose $s > 2$), and the union bound as its upper counterpart.

Now we take parameters

$$\tilde{\varepsilon}_r = (cm2^{r-m})^{1/2^r}, \quad c > \ln 4, \quad c' = c/2 - \ln 2.$$

Theorem 14: For RM codes with $m \to \infty$ and fixed order r used on a binary channel with crossover error probability $(1 - \tilde{\varepsilon}_r)/2$, algorithm Φ_r^m gives for any path ξ a vanishing BER

$$p(\xi) < \max\{e^{-c'm}, 2^{-(m-r)/2 + m^{1/2}}\}. \tag{50}$$

Proof: Consider any left-end path $\underline{\xi}$ that ends at the node $\left\{ {g+1 \atop 1} \right\}$. For any vector $\boldsymbol{y}(\underline{\xi})$ and any subset I of 2^g positions, define the sum

$$y_I(\xi) \stackrel{\text{def}}{=} \sum_{i \in I} y_i(\underline{\xi}).$$

Here $y_i(\underline{\xi})$ form 2^g i.i.d. rvs. Thus, the sum $y_I(\xi)$ has the same pdf for any I. In turn, this allows us to remove index I from $y_I(\xi)$ and use common notation $y(\xi) = y_I(\xi)$. Then we rewrite bounds (49) as

$$\Pr\{y(\xi) \leq 0\} < p(\xi) \leq (2l - 1) \Pr\{y(\xi) \leq 0\}.$$

Equivalently, we use the normalized rv $z(\xi) = y(\xi)/\mathsf{E}y(\xi)$ with expectation 1 and rewrite the latter bounds as

$$\Pr\{z(\xi) \leq 0\} < p(\xi) \leq (2l - 1) \Pr\{z(\xi) \leq 0\}. \tag{51}$$

Similarly to the proof of Theorem 13, note that the sum $z(\xi)$ also satisfies the central limit theorem for any $g \geq m^{1/2}$ and has pdf that tends to $\mathcal{N}(1, \mu(\xi))$ as $m \to \infty$. Thus, we see that $p(\xi)$ depends only on the variance $\mu(\xi)$ obtained on the sum $z(\xi)$ of i.i.d. rv. This variance can be found using calculations identical to those performed in Lemmas 10 to 12. In particular, for any g, we can invoke the proof of Lemma 11, which shows that $\mu(\xi)$ achieves its maximum on the leftmost path

$$\xi_* = (0^{r-1}, 1^{m-r}).$$

Similarly to (36), we then find

$$\mu(\xi_*) = 2^{-(m-r)}(\varepsilon^{-2^r} - 1). \tag{52}$$

Direct substitution $\varepsilon = \tilde{\varepsilon}_r$ in (52) gives

$$\mu(\xi_*) = (cm)^{-1} - 2^{r-m}.$$

Now we almost repeat the proof of Theorem 13. For the first path ξ_*, we employ Gaussian approximation

$$\Pr\{z(\xi_*) \leq 0\} \sim Q(\mu^{-1/2}(\xi_*)) \sim e^{-cm/2}(2\pi cm)^{-1/2}$$

as $m \to \infty$. For maximum $l = 2^{m-r+1}$, the latter inequality and (51) give the upper bound

$$\begin{aligned} p(\xi_*) &\leq 2^{m-r+2} e^{-cm/2} (2\pi cm)^{-1/2} \\ &< e^{-c'm}. \end{aligned} \tag{53}$$

Also,

$$p(\xi_*) \geq Q(\mu^{-1/2}(\xi_*)). \tag{54}$$

For any other path ξ with $g \geq m^{1/2}$, we can use the same estimates in (51) due to the inequalities $\mu(\xi) < \mu(\xi_*)$ and $l \leq 2^{m-r}$.

Finally, consider any path ξ with $g \leq m^{1/2}$. In this case, we use Chebyshev's inequality instead of Gaussian approximation. Again, for any node $\left\{ {g+1 \atop 1} \right\}$ we can consider its leftmost path

$$\xi_1^g = (0^{r-2}, 1^{m-r-g}, 0, 1^g).$$

Similarly to our previous calculations in (36) and (37), it can be verified that

$$\mu(\xi_1^g) = 2^{-g} \left(\left(\left(\varepsilon^{-2^{r-1}} - 1 \right) 2^{r+g-m} + 1 \right)^2 - 1 \right).$$

Then for small $g \leq m^{1/2}$, substitution $\varepsilon = \tilde{\varepsilon}_r$ gives the equality

$$\mu(\xi_1^g) \sim 2^{-(m-r-2)/2} (cm)^{-1/2}, \qquad m \to \infty. \tag{55}$$

Thus, we obtain Chebyshev's inequality in the form

$$p(\xi_1^g) \leq \mu(\xi_1^g) \cdot 2 < 2^{-(m-r)/2 + m^{1/2}} \tag{56}$$

and complete the proof, since bound (50) combines both estimates (53) and (56). □

Proof of Theorem 2: We repeat the proof of Theorem 1 almost entirely. Consider a channel with crossover probability $(1 - \tilde{\varepsilon}_r)/2$ as $m \to \infty$. The output block error probability P of the algorithm Φ_r^m satisfies the estimate $P \leq k \max p(\xi)$, where the number k of different paths ξ is bounded by $\binom{m}{r}$. Formula (50) shows that all $p(\xi)$ decline exponentially in m. As a result, we obtain an asymptotic estimate $P \to 0$. On the other hand, the error patterns of weight pn or less occur with a total probability that tends to $1/2$. So decoder fails to decode only a vanishing fraction of these error patterns.

Now we take a smaller residual

$$\varepsilon = \tilde{\varepsilon}_r / m^{1/2^{r-1}}$$

and consider a channel with crossover probability $(1-\varepsilon)/2$. Then direct substitution of ε in (52) gives $\mu(\xi_*) \to \infty$. Then formula (54) shows that the decoding block error rate is

$$P \geq Q(\mu^{-1/2}(\xi_*)) \to 1/2.$$

Note also that errors of weight $n/2$ or more occur with vanishing probability. Thus, Ψ_r^m fails on errors of weight $n/2$ or less. □

Discussion.

The proofs of Theorems 1 and 2 also reveal the main shortcoming of our probabilistic technique, which employs rather loose estimates for probabilities $p(\xi)$. Indeed, the first two moments of the rvs $y(\xi)$ give tight approximation only for Gaussian rv. By contrast, error probabilities $p(\xi)$ slowly decline as $2^{-(m-r)/2}$, whenever Chebyshev's inequality (56) is applied for small parameters $g < m^{1/2}$. As a result, we can obtain a vanishing block error rate only if

$$k = o(2^{-(m-r)/2}).$$

This is the case of RM codes of fixed order r.

In contrast, the number of information symbols k is linear in n for RM codes of fixed rate $R \in (0,1)$. This fact does not allow us to extend Theorems 1 and 2 to nonvanishing code rates. More sophisticated arguments—that include the moments $\mathsf{E}z^s(\xi)$ of an arbitrary order s—can be developed in this case. The end result of this study is that recursive decoding of RM codes $\left\{ {m \atop r} \right\}$ of fixed rate R achieves the error-correcting threshold

$$\delta = (d \ln d)/2.$$

This increases $\ln d$ times the threshold of bounded distance decoding. However, the overall analysis becomes more involved and is beyond the scope of this paper.

VII. Further Enhancements and Open Problems

Now consider an infinite sequence of optimal binary codes of a low code rate R used on a channel with high crossover error probability $p = (1-\varepsilon)/2$. According to the Shannon coding theorem, ML decoding of such a sequence gives a vanishing block error probability if

$$p < H^{-1}(1-R)$$

where H^{-1} is the inverse (binary) entropy function. Note that for $R \to 0$

$$1 - 2H^{-1}(1-R) \sim \sqrt{R \ln 4}.$$

Correspondingly, a vanishing block error probability is obtained for any residual

$$\varepsilon_{\text{opt}} \sim \sqrt{cR}, \qquad c > \ln 4. \tag{57}$$

Next, recall that RM codes $\left\{ {m \atop r} \right\}$ of fixed order r have a code rate

$$R \sim m^r n^{-1} (r!)^{-1}.$$

For this rate, ML decoding of optimal codes gives

$$\varepsilon_{\text{opt}} \sim (cm^r)^{1/2} n^{-1/2} (r!)^{-1/2}, \qquad m \to \infty. \tag{58}$$

Thus, we see that optimal codes give approximately the same residual order (58) as the former order (2) derived in [7] for RM codes $\{ {m \atop r} \}$. In other words, RM codes of low-rate R can achieve nearly optimum performance for ML decoding. By contrast, low-complexity algorithm Φ_r^m has a substantially higher residual that has the order of $(m/n)^{1/2^r}$. This performance gap shows that further advances are needed for the algorithm Φ_r^m. The main problem here is whether possible improvements can be coupled with low complexity order of $n \log n$.

The performance gap becomes even more noticeable when a binary symmetric channel is considered as a "hard-decision" image of an additive white Gaussian noise (AWGN) channel. Indeed, let the input symbols ± 1 be transmitted over a channel with the AWGN $\mathcal{N}(0, \sigma^2)$. For code sequences of rate $R \to 0$, we wish to obtain a vanishing block error probability when $\sigma \to \infty$. In this case, the transmitted symbols ± 1 are interchanged with very high crossover probability $Q(1/\sigma)$, which gives the residual

$$\varepsilon \sim 1 - 2Q(1/\sigma) \sim \sigma^{-1} \sqrt{2/\pi}. \tag{59}$$

Thus,

$$\varepsilon^{-2} \sim \pi \sigma^2 / 2$$

serves (up to a small factor of $\pi/2$) as a measure of noise power σ^2. In particular, ML decoding operates at the above residual ε_{opt} from (58) and can withstand noise power σ^2 of order up to nm^{-r}.

By contrast, algorithm Φ_r^m can successfully operate only when noise power σ^2 has the lower order of $(n/m)^{1/2^{r-1}}$. Similarly, algorithm Ψ_r^m is efficient when σ^2 is further reduced to the order of $(n/m)^{1/2^r}$. Therefore, for long RM codes, algorithm Φ_r^m can increase $(n/m)^{1/2^r}$ times the noise power that can be sustained using the algorithm Ψ_r^m or majority decoding. However, performance of Φ_r^m also degrades for longer blocks when compared to optimum decoding, though this effect is slower in Φ_r^m than in other low-complexity algorithms known for RM codes.

For moderate lengths, this relative degradation is less pronounced, and algorithm Φ_r^m achieves better performance. In particular, some simulation results are presented in Figs. 3–5 for RM codes $\left\{ {7 \atop 2} \right\}$, $\left\{ {8 \atop 2} \right\}$, and $\left\{ {8 \atop 3} \right\}$, respectively. On the horizontal axis, we plot both input parameters—the signal-to-noise ratio $(2R\sigma^2)^{-1}$ of an AWGN channel and the crossover error probability $Q(1/\sigma)$ of the corresponding binary channel. The output code word-error rates (WER) of algorithms Ψ_r^m and Φ_r^m represent the first two (rightmost) curves. Decoding is performed on a binary channel, without using any soft-decision information.

These simulation results show that Φ_r^m gains about 1 dB over Ψ_r^m on the code $\left\{ {8 \atop 2} \right\}$ and about 0.5 dB on the code $\left\{ {8 \atop 3} \right\}$ even for high WER. A subsequent improvement can be obtained if we consider *soft-decision* decoding, which recursively recalculates the *posterior probabilities* of the new variables obtained in both Steps 1 and 2. These modifications of algorithms Ψ_r^m and Φ_r^m—called below $\tilde{\Psi}_r^m$ and $\tilde{\Phi}_r^m$—are designed along these lines in [11]. The simulation results for the algorithm $\tilde{\Phi}_r^m$ are also presented in Figs. 3–5, where these results are given by the third curve.

This extra gain can be further increased if a few most plausible code candidates are recursively retrieved and updated in all intermediate steps. We note that the list decoding algorithms

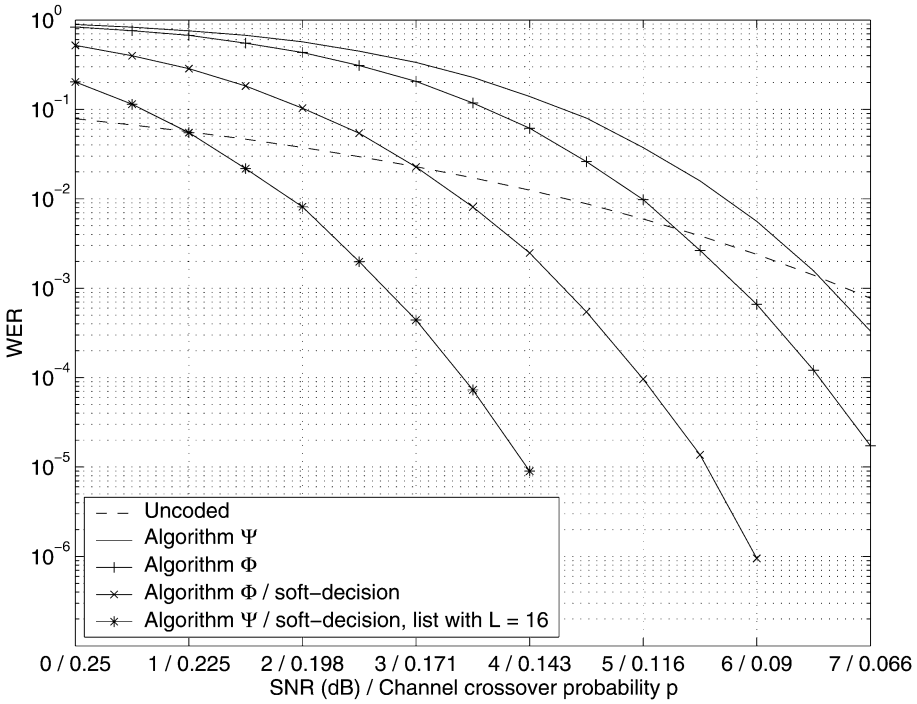

Fig. 3. $\left\{ {7 \atop 2} \right\}$ RM code, $n = 128, k = 29$. Code WER for hard-decision algorithms Ψ_r^m and Φ_r^m, and soft-decision algorithms $\bar{\Phi}_r^m$ and $\bar{\Psi}_r^m(L)$ (list of size $L = 16$).

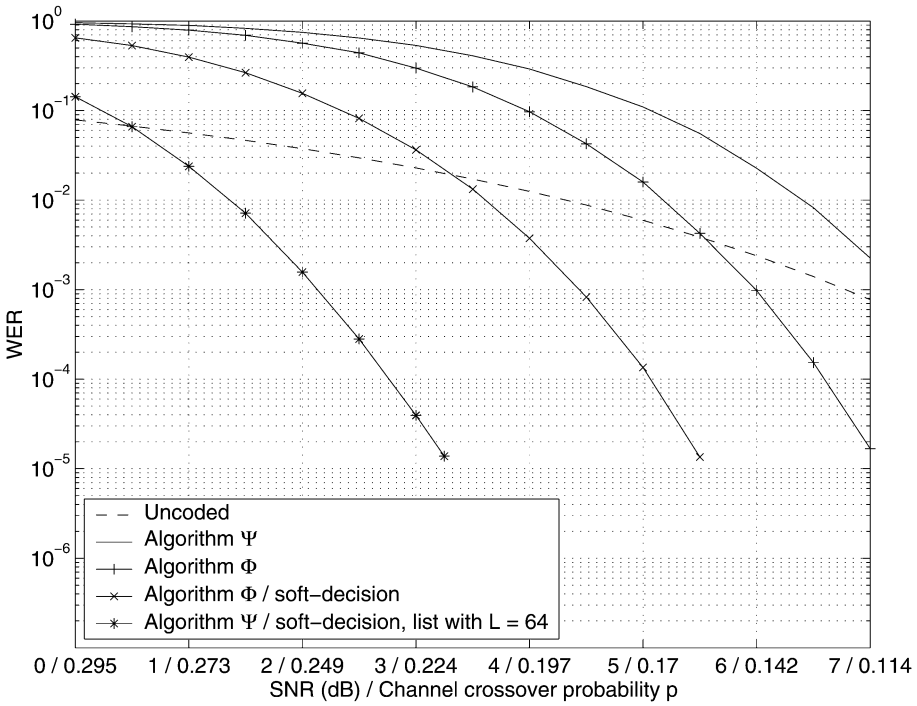

Fig. 4. $\left\{ {8 \atop 2} \right\}$ RM code, $n = 256, k = 37$. Code WER for hard-decision algorithms Ψ_r^m and Φ_r^m, and soft-decision algorithms $\bar{\Phi}_r^m$ and $\bar{\Psi}_r^m(L)$ (list of size $L = 64$).

have been of substantial interest not only in the area of error control but also in the learning theory. For long biorthogonal codes $\left\{ {m \atop 1} \right\}$, the pioneering randomized algorithm is presented in [16]. For $m \to \infty$ and any constants $\varepsilon > 0$, $s > 0$, this algorithm outputs a complete list of codewords located within the distance $n(1-\varepsilon)/2$ from any received vector, while taking only a polynomial time $\mathsf{poly}(ms/\varepsilon)$ to complete this task with high probability $1 - \exp\{-s\}$. Substantial further advances are obtained for some low-rate q-ary RM codes in [17] and the papers cited therein.

For binary RM codes of *any order* r, we mention three different soft-decision list decoding techniques, all of which reduce the output WER at the expense of higher complexity. The algorithm of [12] and [13] re-evaluates the most probable in-

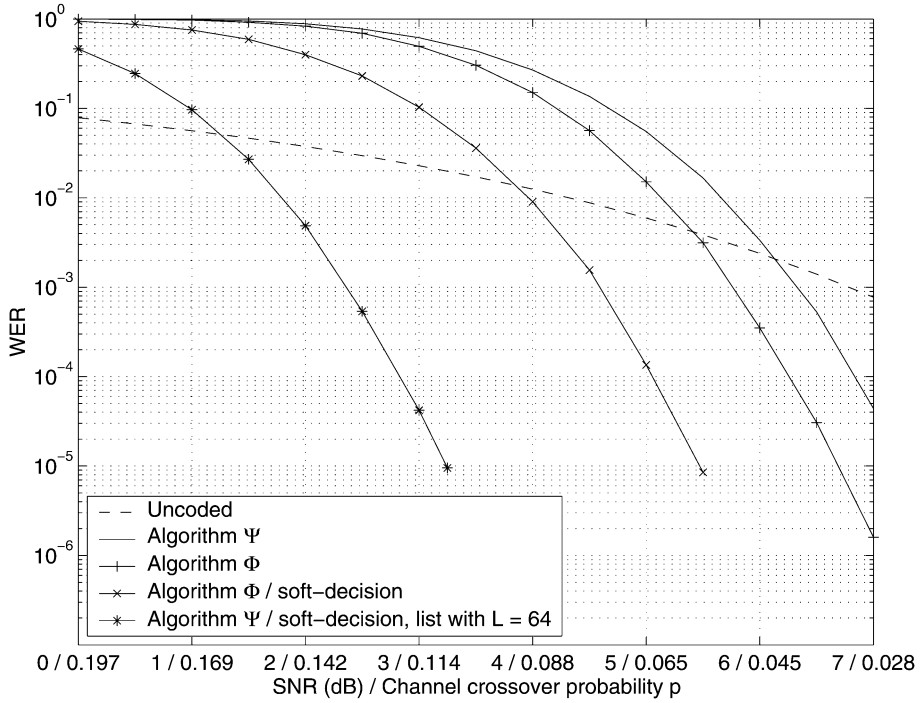

Fig. 5. $\{{}^{8}_{3}\}$ RM code, $n = 256, k = 93$. Code WER for hard-decision algorithms Ψ_r^m and Φ_r^m, and soft-decision algorithms $\tilde{\Phi}_r^m$ and $\tilde{\Psi}_r^m(L)$ (list of size $L = 64$).

formation subblocks on a *single run*. For each path ξ, the decoder—called below $\tilde{\Psi}_r^m(L)$—updates the list of L most probable information subblocks $\hat{\boldsymbol{a}}(\gamma)$ obtained on the previous paths γ. This algorithm has overall complexity of order $Ln \log n$. The technique of [14] proceeds recursively at any intermediate node, by choosing L codewords closest to the input vector processed at this node. These lists are updated in multiple recursive runs. Finally, the third novel technique [15] executes sequential decoding using the main stack, but also utilizes the complementary stack in this process. The idea here is to lower-bound the minimum distance between the received vector and the closest codewords that will be obtained in the *future steps*.

Computer simulations show that the algorithm of [13] achieves the best complexity–performance tradeoff known to date for RM codes of moderate lengths 128 to 512. In Figs. 3–5, this algorithm $\tilde{\Psi}_r^m(L)$ is represented by the fourth curve, which shows a gain of about 2 dB over $\tilde{\Phi}_r^m$. Here we take $L = 16$ in Fig. 3 and $L = 64$ in Figs. 4 and 5. Finally, complexity estimates (given by the overall number of floating-point operations) are presented for all three codes in Table I.

Recall also that different information bits—even those retrieved in consecutive steps—become much better protected as recursion progresses. This allows one to improve code performance by considering a subcode of the original code, obtained after a few least protected information bits are removed. The corresponding simulation results can be found in [12] and [13].

Summarizing this discussion, we outline a few important open problems. Recall that the above algorithms Ψ_r^m and Φ_r^m use two simple recalculation rules

$$\boldsymbol{y} \Longrightarrow \{\boldsymbol{y}'\boldsymbol{y}'', (\boldsymbol{y}' + \boldsymbol{y}'')/2\}. \tag{60}$$

Therefore the first important issue is to define whether any asymptotic gain can be obtained:

TABLE I
COMPLEXITY ESTIMATES FOR HARD-DECISION ALGORITHMS Ψ_r^m AND Φ_r^m AND SOFT-DECISION VERSIONS $\tilde{\Phi}_r^m$ AND $\tilde{\Psi}_r^m(L)$

Code	$\lvert\Psi_r^m\rvert$	$\lvert\Phi_r^m\rvert$	$\lvert\tilde{\Phi}_r^m\rvert$	$\lvert\tilde{\Psi}_r^m(L)\rvert$
$\{{}^{7}_{2}\}$	857	1264	6778	29602, $L = 16$
$\{{}^{8}_{2}\}$	1753	2800	16052	220285, $L = 64$
$\{{}^{8}_{3}\}$	2313	2944	12874	351657, $L = 64$

— by changing the recalculation rules (60);
— by using intermediate lists of small size L;
— by removing a few weakest information paths (bits).

The second important problem is to obtain tight bounds on the decoding error probability in addition to the decoding threshold derived above. This is an open problem even for the simplest recalculations (60) utilized in this paper, let alone other rules, such as (9) or those outlined in [11].

From practical perspective, recursive algorithms show substantial promise at the moderate lengths up to 256, on which they efficiently operate at signal-to-noise ratios below 3 dB. It is also interesting to extend these algorithms for low-rate subcodes of RM codes, such as the duals of the Bose–Chaudhuri–Hocquenghem (BCH) codes and other sequences with good autocorrelation.

In summary, the main result of the paper is a new probabilistic technique that allows one to derive exact asymptotic thresholds of recursive algorithms. First, we disintegrate the decoding process into a sequence of recursive steps. Second, these dependent steps are estimated by independent events, which occur when all preceding decodings are correct. Finally, we develop a

second-order analysis that defines a few weakest paths over the whole sequence of consecutive steps.

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