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RECURSIVE DERIVATION OF REFLECTION COEFFICIENTS
FROM NOISY SEISMIC DATA

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ABSTRACT

We consider plane-wave motion at normal inci-
dence in a horizontally layered system. The sys-
tem is assumed lossless, and only the compres-
sional waves are treated. A procedure is introduced
for determining the reflection coefficients of the
layered system when the observed seismic data
may contain random noise. No deconvolution of the
measured seismic data is required by the proce-
dure when the input is a narrow wavelet.

1. INTRODUCTION

In recent years much attention has been given to
the problem of determining reflection coefficients
for a layered media from the observed seismic
data [1-4]. In line with the customary assumptions
and restrictions, we also limit our attention to a
horizontally stratified nonabsorptive earth with
vertically traveling plane compressional waves.
Such a system is completely described by a set of
reflection coefficients and travel times within lay-
ers.

A fundamental procedure described in detail in
the above references for deriving values of the re-
flection coefficients can be summarized by the fol-
lowing assumptions and steps.

Standard Assumptions:

(A1) The input wavelet is assumed known.
(A2) The data is assumed noise free.
(A3) The layered system is assumed to have uni-
form travel times between layers where a number
of the layers are hypothetical, i. e., they may not
correspond to an actual interface of the substruc-
ture and are associated with zero reflection and
unity transmission coefficients.

Standard Steps:

(S1) The observed seismic data is deconvolved us-
ing the input waveform to produce the system re-
sponse to a unit spike input.
(S2) The number of layers is chosen high enough to
result in travel times short compared with the in-
verse of the bandwidth of the observed seismic
data.
(S3) The deconvolved data is sampled with samp-
ling interval equal to the chosen one-way travel
time between layers.*
(S4) The system structure is used to arrive at a
set of normal equations (linear simultaneous equa-
tions) in terms of reflection coefficients and the

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*Of course the deconvolution may be performed in
discrete time using the same sampling interval.

discretized and deconvolved observed data.
(S5) The normal equations of the preceding step
have the Toeplitz structure which makes it possible
to utilize the very efficient Levinson algorithm to
recursively solve for the reflection coefficients.

In this paper the method of solution to the inverse
problem stated above is fundamentally modified to
cope with the existence of the noise in the measure-
ment data, often without need for any deconvolution.
More specifically, although again a uniform layered
system is assumed, the choice of number of layers
can now be made independent of the sampling rate
requirement of the data (step (S2) above) often re-
sulting in the need for far fewer layers. No decon-
volution is necessary (step (S1)) for wavelets of dur-
ation of the order of twice the layer travel times.
The exact deconvolution of step (S1) is either not
possible in practice or, at the least, will further ag-
gravate the harmful effects of the noise in the ob-
servation [5]. Furthermore, the deconvolution is a
time consuming operation. Finally, the procedure is
very simple to derive and does not need the con-
cepts of z-transforms, minimum phase, forward and
backward polynomials, spectral factorization, etc.
The results reduce to the existing solution of the in-
verse problem in the absence of noise and with a
spike input signal (wavelet) [1].

2. STATEMENT OF THE PROBLEM

We are considering a uniform K layered system
and normal incident compressional waves. Figure 1
represents such a system where $d_j(t)$ is the down-
going wave at the bottom of the jth layer and $u_j(t)$ is
the up-going wave at the top of the layer. The re-
flection, downward transmission and upward trans-
mission coefficients associated with the interface at
the bottom of jth layer are denoted r_j, t_j and t'_j re-
spectively where $t_j = 1 + r_j$, $t'_j = 1 - r_j$. The one way trav-
el time between layers is denoted by τ.

The input to the system, $d_0(t)$, is assumed known
(the wavelet) and the output may be either $u_1(t)$ (in
the marine environment) or $u_0(t)$. The measured
seismic data, $y(t)$, consists of the output and an addi-
tive noise component $n(t)$. The source of this noise
may be the instrument measurement noise, the un-
certainty in the knowledge of the input wavelet or
response to unwanted inputs (ambient noise). It is
desired to process $y(t)$, $t \geq 0$ and derive values for
the reflection coefficients r_j, $j = 1, \ldots, K$ (r_0 may be
assumed known in cases such as the marine envi-
ronment).

3. STATE EQUATIONS

Using the notation of Fig. 1, for a general j^{th} layer

we have [6, 7].

$$u_j(t+\tau) = t_j' u_{j+1}(t) + r_j d_j(t) \tag{1}$$

$$d_{j+1}(t+\tau) = -r_j u_{j+1}(t) + t_j d_j(t) . \tag{2}$$

These equations are valid for $j = 1, \ldots, K-1$. The They should be augmented at the surface with

$$u_0(t) = t_0' u_1(t) + r_0 d_0(t) \tag{3}$$

$$d_1(t+\tau) = -r_0 u_1(t) + t_0 d_0(t) \tag{4}$$

and at the basement with

$$u_K(t+\tau) = t_K' u_{K+1}(t) + r_K d_K(t) = r_K d_K(t) \tag{5}$$

$$d_{K+1}(t) = -r_{K-1} u_{K+1}(t) + t_K d_K(t) = t_K d_K(t) . \tag{6}$$

Equations (3, 4) and (5, 6) can be derived from (1) and (2) [letting $j = 0, 1, \ldots, K$] by noting that $u_0(t)$ is taken at the <u>bottom</u> of layer 0 and $d_{K+1}(t)$ represents the down-going wave leaving the last interface and is not reflected by any other interface; hence $u_{K+1}(t) \equiv 0$. These equations, called causal functional, are not difference equations since t is the continuous time variable [6].

Using the state equations given above, it can be shown [Appen. B, 9] that the function d_{K+1} satisfies the equation

$$d_{K+1}[t+K\tau] + a_1 d_{K+1}[t+(K-2)\tau] + \cdots + a_{K-1} d_{K+1}[t-(K-2)\tau]$$

$$+ r_0 r_K d_{K+1}[t-K\tau] = \prod_0^K t_i d_0(t) . \tag{7}$$

Note that the coefficient of the highest term of the left hand side is unity and that of the lowest term is $r_0 r_K$. The precise form of the other coefficients is not important. Only the structural form of (7) will be utilized in the sequel. In this equation, the unknowns are the reflections coefficients which are embedded in the coefficients $a_1, \ldots, a_{K-1}, r_0 r_K$ and $\prod_0^K t_i$. The input $d_0(t)$ is assumed known. Equation (7) is the starting point for our inverse procedure: however, it is in terms of a signal which, in general, is not measurable. In the following two sections we relate $d_{K+1}(t)$ to measured seismic data. We do this so that we will be able to extract the reflection coefficients from measured data.

4. A GENERALIZED ENERGY TRANSFER (KUNETZ) RELATION

Consider ϵ to be a non-negative continuous or discrete variable with dimension of time. Equations (1) and (2) where $j = 0, 1, \ldots, K$ are multiplied by $u_j(t+\tau+\epsilon)$ and $d_{j+1}(t+\tau+\epsilon)$ respectively resulting in

$$u_j(t+\tau) u_j(t+\tau+\epsilon) = t_j'^2 u_{j+1}(t) u_{j+1}(t+\epsilon) + r_j^2 d_j(t) d_j(t+\epsilon)$$

$$+ r_j t_j'[u_{j+1}(t) d_j(t+\epsilon) + u_{j+1}(t+\epsilon) d_j(t)] \tag{8}$$

$$d_{j+1}(t+\tau) d_{j+1}(t+\tau+\epsilon) = r_j^2 u_{j+1}(t) u_{j+1}(t+\epsilon) + t_j^2 d_j(t) d_j(t+\epsilon)$$

$$- r_j t_j[u_{j+1}(t) d_j(t+\epsilon) + u_{j+1}(t+\epsilon) d_j(t)] . \tag{9}$$

Multiplying (8) by t_j/t_j' and adding the resulting expression to (9) yields

$$d_{j+1}(t+\tau) d_{j+1}(t+\tau+\epsilon) + (t_j/t_j') u_j(t+\tau) u_j(t+\tau+\epsilon)$$

$$= (t_j/t_j') d_j(t) d_j(t+\epsilon) + u_{j+1}(t) u_{j+1}(t+\epsilon) . \tag{10}$$

Let us define the following correlation-type functions

$$D_j(\epsilon) \triangleq \int_{-\infty}^{+\infty} d_j(t) d_j(t+\epsilon) dt \tag{11}$$

$$U_j(\epsilon) \triangleq \int_{-\infty}^{+\infty} u_j(t) u_j(t+\epsilon) dt . \tag{12}$$

Integrating both sides of Eq. (10) from $-\infty$ to $+\infty$, and using Eqs. (11) and (12), we find that

$$D_{j+1}(\epsilon) - U_j(\epsilon) = t_j/t_j'[D_j(\epsilon) - U_j(\epsilon)] \tag{13}$$

where $j = 0, 1, 2, \ldots, K$. This is a generalization of the well-known [1] energy transfer (Kunetz) relation. Note that in our derivation, input $d_0(t)$ is not assumed to be an impulse and the seismic data is not discretized.

Iterating (13), starting with $j = \ell$ and ending with $j = K$, we obtain

$$D_{K+1}(\epsilon) = \frac{\Pi}{\Pi'}[D_\ell(\epsilon) - U_\ell(\epsilon)] \tag{14}$$

where ℓ can take on the values of $0, 1, \ldots,$ or K, $\Pi \triangleq \prod_0^K t_i$ and $\Pi' \triangleq \prod_0^K t_i'$. In the marine case this relationship is used with $\ell = 1$. In the non-marine case it is used with $\ell = 0$.

5. APPLICATION TO MARINE ENVIRONMENT

In this section we will direct our attention to the marine case and shall express $D_{K+1}(\epsilon)$ in terms of measured signals. To do this, we set $r_0 = 1$ and we see from (14) that we must express $D_1(\epsilon) - U_1(\epsilon)$ in terms of the measured signals. The first layer can be depicted as in Fig. 2. Observe that (4) becomes

$$d_1(t+\tau) = -u_1(t) + 2 d_0(t) . \tag{15}$$

From (11, 12, 13), we can evaluate the difference* term $D_1(\epsilon) - U_1(\epsilon)$,

$$D_1(\epsilon) - U_1(\epsilon) \triangleq P(\epsilon) = \int_{-\infty}^{+\infty} \{[2 d_0(t) - u_1(t)]$$

$$\cdot [2 d_0(t+\epsilon) - u_1(t+\epsilon)] - u_1(t) u_1(t+\epsilon)\} dt \tag{16}$$

or

$$P(\epsilon) = \int_{-\infty}^{+\infty} 4 d_0(t) d_0(t+\epsilon) dt - \int_{-\infty}^{+\infty} 2 d_0(t) u_1(t+\epsilon) dt$$

$$- \int_{-\infty}^{+\infty} 2 u_1(t) d_0(t+\epsilon) dt ; \tag{17}$$

hence, $P(\epsilon)$ can be evaluated from a knowledge of $d_0(t)$ and $u_1(t)$ for any desired ϵ. Observe, also, that (14) with $\ell = 1$ can be written in terms of $P(\epsilon)$, using (16), as

$$D_{K+1}(\epsilon) = \frac{\Pi}{\Pi'} P(\epsilon) . \tag{18}$$

We should point out at this stage that the quantity $u_1(t)$ needed in (17) is only available through the observation

$$y(t) = u_1(t) + n(t) \tag{19}$$

where $n(t)$ is the additive noise. Consequently, $P(\epsilon)$ is not physically available; however, we can define $\bar{P}(\epsilon)$ by replacing $y(t)$ for $u_1(t)$ in (17),

$$\bar{P}(\epsilon) = \int_{-\infty}^{+\infty} 4 d_0(t) d_0(t+\epsilon) dt - \int_{-\infty}^{+\infty} 2 d_0(t) y(t+\epsilon) dt$$

$$- \int_{-\infty}^{+\infty} 2 y(t) d_0(t+\epsilon) dt \tag{20}$$

---

*Because of the range of integration in (11), we can also express $D_1(\epsilon)$ as $D_1(\epsilon) = \int_{-\infty}^{+\infty} d_1(t+\tau) d_1(t+\tau+\epsilon) dt$. We use this form of (11) in our development of $D_1(\epsilon) - U_1(\epsilon)$.

which can also be written as

$$\overline{P}(\epsilon) = P(\epsilon) + N(\epsilon),\qquad(21)$$

where

$$N(\epsilon) = -\int_{-\infty}^{+\infty} 2d_0(t)n(t+\epsilon)dt - \int_{-\infty}^{+\infty} 2n(t)d_0(t+\epsilon)dt.\qquad(22)$$

The statistics of noise term $N(\epsilon)$ can be determined in terms of those of $n(t)$. Using $\overline{P}(\epsilon)$ in (18) yields

$$D_{\kappa+1}(\epsilon) = \frac{\Pi}{\Pi} \overline{P}(\epsilon)\qquad(23)$$

where $\overline{P}(\epsilon)$ is a known quantity. Equation (23) is a fundamental relationship which will be used in the derivation of the inverse procedure.

6. DERIVATION OF THE NORMAL EQUATIONS

Equation (7) is the main relation which will be used to derive the reflection coefficients. Dividing both sides by $\prod_0^\kappa t_i$ and identifying the resulting coefficients by $g_0, \ldots, g_\kappa$ we get

$$g_0 d_{\kappa+1}[t+K\tau] + g_1 d_{\kappa+1}[t+(K-2)\tau] + \cdots$$
$$+ g_{\kappa-1} d_{\kappa+1}[t-(K-2)\tau] + g_\kappa d_{\kappa+1}[t-K\tau] = d_0(t).\qquad(24)$$

We compute the coefficients of this equation by means of the following "least squares" criterion:

$$\min_{g_0,\ldots,g_\kappa} \int_{-\infty}^{+\infty} \{ g_0 d_{\kappa+1}[t+K\tau] + \cdots + g_\kappa d_{\kappa+1}[t-K\tau] - d_0(t) \}^2 dt.\qquad(25)$$

The result of this minimization is equivalent to multiplying both sides of (7) by $d_{\kappa+1}[t+(K-2i)\tau]$ and integrating from $-\infty$ to ∞ for $i=0,1,\ldots,K$. By either approach, we obtain $K+1$ simultaneous equations, which, using (11), become

$$\begin{bmatrix} D_{\kappa+1}(0) & D_{\kappa+1}(2\tau) & \cdots & D_{\kappa+1}(2K\tau) \\ D_{\kappa+1}(2\tau) & D_{\kappa+1}(0) & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ D_{\kappa+1}(2K\tau) & \cdots & \cdots & D_{\kappa+1}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ r_\kappa \end{bmatrix} = 2\prod_0^K t_i \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\kappa+1} \end{bmatrix}\qquad(26)$$

where we have substituted $t_0 = 1 + r_0 = 2$ to represent the marine environment and

$$\alpha_{i+1} = \int_{-\infty}^{+\infty} d_0(t) d_{\kappa+1}(t+K\tau-2i\tau)dt,\quad i=0,1,2,\ldots,K.\qquad(27)$$

Substituting for $D_{\kappa+1}(\epsilon)$ from (23), we find that (26) reduces to

$$\begin{bmatrix} \overline{P}(0) & \overline{P}(2\tau) & \cdots & \overline{P}(2K\tau) \\ \overline{P}(2\tau) & \overline{P}(0) & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ \overline{P}(2K\tau) & \cdots & \cdots & \overline{P}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ r_\kappa \end{bmatrix} = 2\prod_0^K t_i' \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\kappa+1} \end{bmatrix}.\qquad(28)$$

Note that the $(K+1) \times (K+1)$ matrix on the left has the Toeplitz structure. The terms $N(0), N(2\tau), \ldots$ which appear in $\overline{P}(0), \overline{P}(2\tau), \ldots,$ are random variables with known statistics; they will be zero if the seismic data is noise free (i.e., $n(t) \equiv 0$). Observe, also that the first and last elements of the vector on the left-hand side of (28) are unity and r_κ, respectively, by virtue of the property which we stated for the $d_{\kappa+1}$ causal functional equation.

Equation (28) provides the second point in our procedure for identifying the reflection coefficients.

Note that in general the α_i are functions of $d_{\kappa+1}(t)$, a signal which is not determinable; however, we will show in the following section that when the input wavelet is narrow enough (not necessarily a spike), (28) has a unique solution for the reflection coefficients in terms of observable data.

7. SPECIAL CASE OF NARROW WAVELET

Let us now consider the case where $d_0(t)$ does not extend* beyond 2τ, i.e.,

$$d_0(t) = 0 \qquad t < 0,\ t > 2\tau.\qquad(29)$$

Since the time of arrival at the Kth interface is $K\tau$ and the time of arrival of the first reflections is $(K+2)\tau$,

$$d_{\kappa+1}(t) = 0 \qquad\qquad\qquad t < K\tau \quad(30a)$$
$$= 2\prod_0^\kappa t_i d_0(t-K\tau) \qquad K\tau < t \le (K+2)\tau \quad(30b)$$
$$= \text{more complicated terms}\quad t > (K+2)\tau.\quad(30c)$$

From (27, 30a, 30b, and 29) we see that

$$\alpha_1 = \int_{-\infty}^{+\infty} d_0(t) d_{\kappa+1}(t+K\tau)dt = 2\prod_0^\kappa t_i \int_0^{2\tau} d_0^2(t)dt,\qquad(31)$$

and that

$$\alpha_{i+1} = \int_{-\infty}^{+\infty} d_0(t) d_{\kappa+1}[t+K\tau-2i\tau]dt = 0 \quad i=1,\ldots,K.\qquad(32)$$

Note now that (28) will have precisely $K+1$ unknowns, K of them in the vector multiplying the Toeplitz matrix and one on the right-hand side, α_1.

Finally, Normal Equation (28) can be written in a compact matrix form, as

$$\overline{P}_\kappa \underline{a}_\kappa = \underline{c}_\kappa\qquad(33)$$

where $\overline{P}_\kappa$ is a $(K+1) \times (K+1)$ Toeplitz matrix with the first row being[†] $[\overline{P}(0), \overline{P}(2\tau), \ldots, \overline{P}(2K\tau)]$; $\underline{a}_\kappa$ is a $K+1$ column vector with first and last elements 1 and r_κ, respectively; and, $\underline{c}_\kappa = col(\bar{g}_\kappa, 0, 0, \ldots, 0)$ and

$$\bar{g}_\kappa = 2 \prod_{i=1}^\kappa (1-r_i^2) \int_0^{2\tau} d_0^2(t)dt.$$

The Normal Equation (33) can be solved for $\underline{a}_\kappa$. This only produces one of the K reflection coefficients, namely r_κ. We will show, in the following, that in the case of the marine environment, nested within (33) are a set of normal equations, the solutions of which produce each one of the reflection coefficients. The absence of this useful property in the non-marine case renders the procedure of this paper inapplicable in that case.

Let us now hypothesize a j-layer system (i.e., the basement layer is the jth) consisting of the top j-layers of the above K-layer system ($K \ge j$). Clearly, from (33), we have

$$\overline{P}_j \underline{a}_j = \underline{c}_j\qquad(34)$$

---
* If this condition is not satisfied, we can always deconvolve the data to achieve this. Since the requirement here is not to deconvolve down to an impulse function (only (29) has to be satisfied), this results in a more practical solution.

[†] For a narrow wavelet and $\epsilon \ge 2$, the calculation of $\overline{P}(\epsilon)$ simplifies since (20) reduces to

$$\overline{P}(\epsilon) = -2 \int_0^{2\tau} d_0(t) y(t+\epsilon)dt.$$

where $\underline{a}_j$ will again have 1 and r_j as first and last elements. We shall now show that, in the case of the marine environment, $\bar{P}_j$ is a $(j+1) \times (j+1)$ Toeplitz matrix composed of the top left corner of $\bar{P}_\varepsilon$; i.e., its first row is given by $[\bar{P}(0), \bar{P}(2\tau), \ldots, \bar{P}(2j\tau)]$.

For the moment let us ignore the additive noise term in (20). Let us denote by $u_1^j(t)$ the response of the j-layer system (i.e., the term $u_1(t)$ in Fig. 2 is replaced by $u_1^j(t)$). In (17), due to the fact that $d_0(t)=0$ for $t > 2\tau$, the last value of $u_1(t)$ contributing to $P(\varepsilon)$ is $u_1(2\tau+\varepsilon)$. In determination of $\bar{P}_j$, with elements $\bar{P}(\varepsilon)$, $\varepsilon = 0, \ldots, 2j\tau$, for a j-layered system, therefore the last value of $u_1^j(t)$ contributing to $\bar{P}_j$ is $u_1^j[2(j+1)\tau]$. On the other hand, $u_1(t)$ is the response of the K-layer system, and

$$u_1(t) = u_1^j(t) \qquad 0 \le t \le 2(j+1)\tau \qquad (35)$$

since the first return from the interfaces below the jth will not appear earlier than $t=2(j+1)\tau$. Hence, the elements of $\bar{P}_j$, which are functions of $u_1^j(t)$ and $\bar{P}_\varepsilon$ which are functions of $u_1(t)$, are identical when (35) is satisfied. In other words, the numerical values of $\bar{P}(0), \ldots, \bar{P}(2j\tau)$ will be identical to those of the K-layer system for all $j \le K$. Furthermore, the additive noise term in (21) is independent of the number of layers, as is evident from (22).

The set of normal equations given by (34), for $j= 1, \ldots, K$, can now be solved for the vectors $\underline{a}_j$, and hence, their last elements, r_j, $j=1, \ldots, K$. The matrix $\bar{P}_\varepsilon$ is Toeplitz and consequently, the Levinson algorithm [1] can be used to solve for the vectors $\underline{a}_j$, $j=1, \ldots, K$ recursively.

Since the r_j's are reflection coefficients, for the solution to this problem to be acceptable, each r must be less than unity in magnitude. It is shown in [9] that any solution of (34) with $\bar{s}_j > 0$ for all j yields a set of r_j's which satisfy this condition. Moreover if $\bar{P}_\varepsilon$ is positive definite, a compatible solution with $\bar{s}_j > 0$ is guaranteed. We see therefore, that the requirement that $|r_j| < 1$ has nothing to do with a specific method of solution of the normal equations (i.e., the Levinson procedure). This result is different from the comparable result in [1-3], where one is left with the impression that a specific method of solution leads to $|r_j| < 1$.

8. EXPERIMENTAL RESULTS

A seven layer system was chosen with the following reflection coefficients: $r_1=1$; $r_2=0.1$; $r_3= 0.15$; $r_4=-0.3$; $r_5=0.25$; $r_6=0.12$; $r_7=0.05$; $r_8=0.2$. We used a one-way layer travel time and data sampling rate of 20 msec and 2 msec, respectively.

The input wavelet was chosen as depicted in Figure 3. Note that this is a non-minimum phase function. It was specifically chosen as such to indicate that the method introduced in this paper is not limited to minimum phase wavelets. Figure 4 is the synthetic seismogram response of the system. Figure 5 is obtained by adding white Gaussian noise with variance 1 to the sampled seismogram. Similar results were obtained for variances of 0.1 and 10. These responses were then utilized to produce estimates of reflection coefficient values. The results of a Monte-Carlo simulation for 100 different samples of noise appear in Tables 1 and 2. Table 1 presents the mean value of the estimates and Table 2 presents the estima-

tion error variance. As seen, the results are exact for zero noise variance (as expected) and are quite good for variances of 0.1 and 1. For noise variance of 10, although the average is not poor, the error variance indicates that the estimates are not very reliable.

A comparison between the procedure of this paper and the standard procedures described in [1-4] is warranted. Let us consider the noise term $n(t)$ to be white. Clearly, (22) indicates that the random variables $N(\cdot)$ have finite variances. For this case $[n(t)$ white], had we performed the necessary deconvolution and sampling required by the classical approach to the inverse problem, the resulting $N(\cdot)$ random variables would have infinite variance, clearly rendering the approach meaningless.* Of course, "approximate" deconvolution will eliminate this problem but at a great sacrifice in the information available within the seismic data. It should also be noted that, for the narrow wavelets, no deconvolution is required by the procedure outlined in this paper.

9. CONCLUSIONS

We have developed a procedure for extracting reflection coefficients from noisy data which we feel is a substantial generalization of similar procedures which have been reported in the literature. Associated with these earlier procedures are Standard Assumptions and Steps (see Introduction, p. 1) which include requirements that the data be noise free and that the observed seismic data be deconvolved. The procedure of our paper avoids these restrictive requirements. Furthermore, our procedure totally avoids the concepts of z-transforms, minimum phase, spectral factorization, forward and reverse polynomial manipulations, etc., which appear in the literature on this subject. Finally, since our derivation is so straightforward, it suggests a number of extensions, including the following, which are presently under study: (1) nonstandard locations of source and sensors (e.g., both in the first layer); (2) minimum mean-square estimation in the non-marine environment; and (3) optimal prefiltering of noisy data.

REFERENCES

1. J. F. Claerbout, Fundamentals of Geophysical Data Processing, McGraw-Hill, N. Y., 1976.

2. E. A. Robinson & S. Treitel, "The Spectral Function of a Layered System and the Determination of the Waveforms at Depth," Geophysical Prospecting, to be published, 1977.

3. E. A. Robinson, Multichannel Time Series Analysis with Digital Computer Programs, Holden-Day, San Francisco, 1967 (Chapt. 3).

*If $n(t)$ is not white, then the variance of $N(\cdot)$ may not be infinite, but will be very large.

4. E. A. Robinson, "Dynamic Predictive Deconvolution," _Geophysical Prospecting_, v. 23, Dec. 1975.

5. K. B. Theriault & A. B. Baggeroer, "Structure Estimation from Acoustic Reflection Measurements," presented at unknown conference.

6. N. E. Nahi & J. M. Mendel, "A Time-Domain Approach to Seismogram Synthesis for Layered Media," presented at the 46th Ann. Int'l. Mtg. of Society of Exploration Geophysicists, Houston, Texas, Oct. 24-28, 1976.

7. J. M. Mendel, N. E. Nahi, L. M. Silverman & H. D. Washburn, "State Space Models of Lossless Layered Media," presented at 1977 Joint Auto. Cont. Conf., San Francisco, Calif., June 1977.

8. F. A. Graybill, _An Introduction to Linear Statistical Models_, Vol. I, McGraw-Hill, N.Y., 1961.

9. N. E. Nahi, J. M. Mendel & L. M. Silverman, "Recursive Derivation of Reflection Coefficients from Noisy Seismic Data," USCEE Report 496, July 1977. Also, submitted for publication.
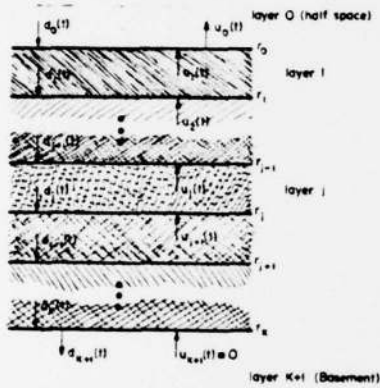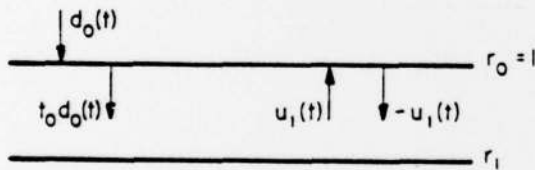
Figure 1. K Layered System
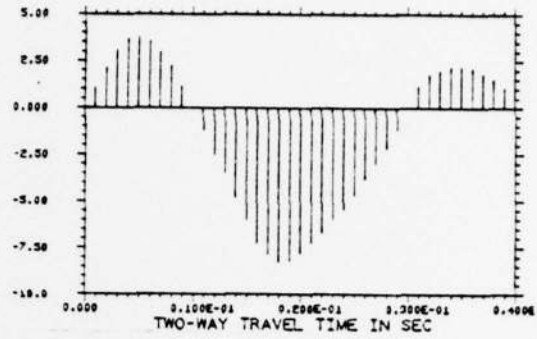

Figure 2. First Layer in the Marine Case

Figure 3. Source Wavelet

Figure 4. Synthetic Seismogram
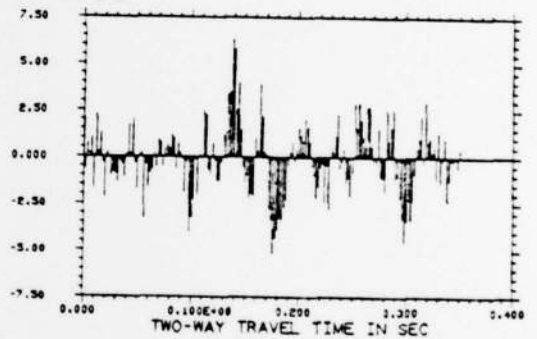
Figure 5. Noisy Sampled Seismogram ($\sigma_n^2 = 1.0$)

Table 1

Average of the Estimates of Each Reflection Coefficient for 100 Runs

σ^2	0	0.1	1.0	10.0
r_0	1.0	1.0	1.0	1.0
r_1	0.1	0.1001863	0.1008552	0.1060553
r_2	0.15	0.1507439	0.1538406	0.1816449
r_3	-0.3	-0.3002447	-0.3019902	-0.3243447
r_4	0.25	0.2515463	0.2578027	0.3565731
r_5	0.12	0.1198497	0.1170640	0.8808238
r_6	0.05	0.04983276	0.05660696	0.2625223
r_7	0.20	0.2013467	0.2024367	0.2065443

Table 2

The Estimation Error Variance for 100 Runs

σ^2	0	0.1	1.0	10.0
r_0	0	0	0	0
r_1	0	0.01308585	0.1339389	1.568616
r_2	0	0.00326687	0.03422876	0.5127926
r_3	0	0.001491613	0.01586429	0.2686259
r_4	0	0.003347942	0.02723132	2.344950
r_5	0	0.02002368	0.2376210	4623.396
r_6	0	0.09040674	1.071580	789.0087
r_7	0	0.004428605	0.05160568	32.63854