Recursive Prediction Error Methods for Adaptive Estimation

JOHN B. MOORE AND HAIM WEISS

Abstract—**Convenient recursive prediction error algorithms for identification and adaptive state estimation are proposed, and the convergence of these algorithms to achieve off-line prediction error minimization solutions is studied. To set the recursive prediction error algorithms in another perspective, specializations are derived from significant simplifications to a class of extended Kalman filters. The latter are designed for linear state space models with the unknown parameters augmenting the state vector and in such a way as to yield good convergence properties. Also, specializations to approximate maximum likelihood recursions, Kalman filters with adaptive gains, and connections to the extended least squares algorithms are noted.**

I. Introduction

An important class of nonlinear filtering and control problems arises when there is a finite dimensional linear stochastic signal model with unknown parameters, and a simultaneous state and parameter estimation is required. An optimal solution to this adaptive filtering problem is usually not feasible, and schemes based on approximations must be investigated.

The extended Kalman filter (EKF) approach to adaptive filtering is an obvious possibility. In this approach the state space model, including noise covariances, is parameterized by an unknown parameter vector (or matrix) θ, and then the elements of θ are treated as additional states. Experience indicates that this approach may lead to parameter estimation bias [1] or divergence [2] in the case that the initial estimates are not sufficiently good. As pointed out by Ljung [3], the estimates will, in general, be biased unless the noise characteristics of the signal model are known. However, very natural modifications introduced in [3] ensure good convergence of the parameter estimate to the true parameter and thus good performance. The modifications which require the computation of $\partial K/\partial\theta$ where K is the Kalman gain, in general, add considerably to the computational effort. Of course when K is independent of θ or is explicitly expressed in terms of θ, as can frequently be arranged, then the additional effort required is negligible.

For a restricted class of linear signal models parameterized by θ, alternative approaches via extended least squares (ELS) [3]–[5] and via approximate maximum likelihood recursions (RML) [3], [6], [7] are also attractive. The restricted models are those for which the measurements z_k must satisfy $z_k = \theta^T H_k x_k + v_k$ with H_k measurable, x_k the model state vector, and v_k a white noise process. Though not obvious, this includes the case when autoregressive moving average (ARMA) models are assumed with θ comprising the autoregressive and moving average parameters. (This and other examples are given in [4], [5].) The approximate maximum likelihood recursions for these models are claimed in [3] to be asymptotically equivalent to specializations of the extended Kalman filter scheme of [3] and have the same attractive convergence properties. The ELS schemes are also widely used, being mildly simpler to implement and having good convergence properties. However, these require additional restrictions on the signal generating system to achieve convergence.

The ideas of [3], [6], and [7] can in fact be exploited to yield a recursive maximum likelihood method for simultaneous state and parameter estimation in a linear state space system with Gaussian disturbances. In Section II, such a theory is built upon using results in [13] to yield, first, an off-line prediction error scheme and, then, a recursive prediction error algorithm in convenient form for state and parameter estimation from vector measurements. In prediction error methods [8], [9], the index to be minimized $J_{t|\theta}$ is expressed in terms of the prediction error $\tilde{z}_{k|\theta} = z_k - \hat{z}_{k|k-1,\theta}$; $k = 1, 2, \cdots, t$ for some one step ahead prediction estimate $\hat{z}_{k|k-1,\theta}$, and takes the form

$$J_{t|\theta} = \frac{1}{2}\sum_{k=1}^{t} [\tilde{z}_{k|\theta}^T \Lambda_{k|\theta}^{-1} \tilde{z}_{k|\theta} + \log \det \Lambda_{k|\theta}]$$

where $\Lambda_{k|\theta}$ is some positive definite weighing matrix. The derived recursive prediction error (RPE) algorithm is based on approximated minimization of $J_{t|\theta}$ for each t to achieve a recursive update of a parameter estimate $\hat{\theta}_t$.

In Section III, a convergence analysis along the lines given in [10] and [11] and exploitation of ergodicity results in [8] and [9] lead to the result that under very reasonable conditions, the recursive prediction error algorithm leads to the same parameter estimates (asymptotically) as an off-line prediction error scheme. The attractive feature of off-line prediction error index minimization schemes is that not only do they achieve a minimum index which has strong appeal but also the best predictor in the set of permitted predictors in terms of spectrum closeness and Kullback information function measures. See [9] for details.

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The authors are with the Department of Electrical Engineering, University of Newcastle, New South Wales, 2308, Australia.

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In Section IV specializations of the recursions to the case of known linear system innovations model structures with unknown parameters are derived by introducing significant simplifications to the extended Kalman filter designed with the unknown parameters augmenting the state vector and modified to yield good convergence properties. Thus it is possible to demonstrate that the recursive prediction error schemes are simpler than their nearest competitor, the extended Kalman filter. The convergence analysis is also more direct. The special case of a Kalman filter with an adaptive Kalman gain is highlighted.

Manipulations of the RPE algorithm in Section V involving approximations allow significant computational simplifications for the multivariable models discussed in [4] and [5]. Such algorithms are considerably simpler than the corresponding EKF algorithms. Moreover, additional manipulations are possible in the scalar measurement case to achieve further computational simplifications, and the resulting class of algorithms contains the RML2 algorithm of [6] and [7]. When specialized to single-input/single-output systems described by $x_{k+1} = Fx_k + K\varepsilon_k$; $z_k = \theta^T x_k + \varepsilon_k$, with $\{\varepsilon_k\}$ as an innovations sequence, the above scheme permits an interesting interpretation in terms of an ELS scheme for a nonminimal model.

The fact that the RPE schemes of this paper have the extensively studied and attractive RML2 algorithm of [6] and [7] as an important subclass, we believe, is sufficient support of the RPE schemes of this paper at this stage—at least as far as simulation studies are concerned.

In a paper under preparation, refinements of RML2 and more general RPE algorithms to avoid stability tests are proposed, and simulation studies are given which further support the approach developed here. In cases such as a Kalman filter with an adaptive gain where the algorithms of this paper are novel, the supportive simulation studies are of particular interest.

II. The Recursive Prediction Error Algorithm

In this section, we consider in turn the prediction error index, its minimization using off-line calculations, and then its minimization using on-line techniques.

A. The Prediction Error Index

Consider the index

$$J_{t|\theta} = \frac{1}{2}\sum_{k=1}^{t} [\tilde{z}_{k|\theta}^T \Lambda_{k|\theta}^{-1} \tilde{z}_{k|\theta} + \log\det \Lambda_{k|\theta}]. \tag{1}$$

Here θ is the unknown parameter r-vector, $\tilde{z}_{k|\theta}$ is the one step ahead prediction error, and $\Lambda_{k|\theta}$ is some positive definite weighting matrix.

In the case of linear state space signal models with Gaussian disturbances and unknown parameters θ, the one step ahead prediction estimate which is most widely used is given from a Kalman filter as $\hat{z}_{k|k-1,\theta} = E\{z_k | z_0, z_1, \cdots, z_{k-1}, \theta\}$. The prediction error $\tilde{z}_{k|\theta}$ is the innovations sequence, and if in (1) $\Lambda_{k|\theta}$ is the innovations covariance matrix, then (1) represents the negative log-likelihood function within a constant.

For the special case when $\Lambda_{k|\theta}$ is independent of θ and is denoted as Λ_k, the functional to be minimized simplifies as

$$\bar{J}_{t|\theta} = \frac{1}{2}\sum_{k=1}^{t} [\tilde{z}_{k|\theta}^T \Lambda_k^{-1} \tilde{z}_{k|\theta}].$$

This functional is coincident with the loss function described in [6] for the case of scalar $\tilde{z}_{k|\theta}$ with $\Lambda_k = 1$.

B. An Off-Line Prediction Error Method

The scheme of [13, p. 92] is here restated and generalized to handle the case when $\Lambda_{k|\theta}$ is a known function of θ.

The basic iteration is a gradient-type nonlinear programming algorithm

$$\hat{\theta}_{i+1} = \hat{\theta}_i - \rho_i M_i^{-1} g_i \tag{2}$$

where $\hat{\theta}_i$ is the parameter vector estimate at the ith iteration, $g_i = (J'_{t|\hat{\theta}_i})^T$ is the gradient vector,[1] ρ_i is a step size parameter to accelerate the algorithm (e.g., initially $\rho_i > 1$, but $\rho_i \to 1$ as $i \to \infty$), and M_i (short for $M_{t|\hat{\theta}_i}$) can here be chosen to have the following attractive properties:

i) $M_{t|\hat{\theta}_i} > 0$, for all i

ii) $E[M_{t|\hat{\theta}_i}] = E[J''_{t|\hat{\theta}_i}]$

iii) $\frac{1}{t} M_{t|\hat{\theta}_i} \to \frac{1}{t} J''_{t|\hat{\theta}_i}$, as $t \to \infty$ (if possible).

We do not select $M_i = J''_{t|\hat{\theta}_i}$ as in the Newton–Raphson method since its calculation is cumbersome and condition (i) above is not satisfied in general. The convenient expressions for g_i and an M_i to satisfy the above properties are

$$g_i^T = J'_{t|\hat{\theta}_i} = \sum_{k=1}^{t} [\tilde{z}_{k|\hat{\theta}_i}^T \Lambda_{k|\hat{\theta}_i}^{-1} \tilde{z}'_{k|\hat{\theta}_i} + \tfrac{1}{2}\phi_{k|\hat{\theta}_i}^T - \tfrac{1}{2}\mu_{k|\hat{\theta}_i}] \tag{3}$$

$$M_i = M_{t|\hat{\theta}_i} = \sum_{k=1}^{t} [(\tilde{z}'_{k|\hat{\theta}_i})^T \Lambda_{k|\hat{\theta}_i}^{-1} (\tilde{z}'_{k|\hat{\theta}_i}) + \tfrac{1}{2}\phi_{k|\hat{\theta}_i} \phi_{k|\hat{\theta}_i}^T] \tag{4}$$

where

$$\phi_{k|\theta}^T = \left[\operatorname{tr}\left(\Lambda_{k|\theta}^{-1} \frac{\partial \Lambda_{k|\theta}}{\partial \theta^{(1)}}\right) \cdots \operatorname{tr}\left(\Lambda_{k|\theta}^{-1} \frac{\partial \Lambda_{k|\theta}}{\partial \theta^{(r)}}\right)\right] \tag{5a}$$

$$\mu_{k|\theta} = \tilde{z}_{k|\theta}^T \Lambda_{k|\theta}^{-1} \left[\frac{\partial \Lambda_{k|\theta}}{\partial \theta^{(1)}} \Lambda_{k|\theta}^{-1} \tilde{z}_{k|\theta} \cdots \frac{\partial \Lambda_{k|\theta}}{\partial \theta^{(r)}} \Lambda_{k|\theta}^{-1} \tilde{z}_{k|\theta}\right]. \tag{5b}$$

C. Motivation and Verification for M_i Selection

To motivate the above selections of M_i first consider the Gaussian case when it is reasonable to replace $J''_{t|\hat{\theta}_i}$ by the Fisher information matrix $E[(J'_{t|\hat{\theta}_i})^T (J'_{t|\hat{\theta}_i})]$ as in the Gauss–Newton method [12]. It is immediate that this selection satisfies (i) above, and in fact

$$E[J''_{t|\hat{\theta}_i}] = E[(J'_{t|\hat{\theta}_i})^T (J'_{t|\hat{\theta}_i})].$$

Thus (ii) holds, as can be shown by direct manipulation. In fact, the direct manipulations do not exploit the Gaussian assumption, and thus (i) and (ii) hold with this selection for

[1] The following convention is adopted throughout the paper: $\partial\beta/\partial\theta$, where β and θ are vectors, denotes a matrix with ijth element $\partial\beta^{(i)}/\partial\theta^{(j)}$ where $\beta^{(i)}$ and $\theta^{(j)}$ are the ith and jth components of β and θ, respectively.

M_i without the Gaussian assumption. Now it is not appropriate to work with M_i as the expected value of some quantity, and so some sample approximation ideas (not developed here) lead to the convenient expression for M_i as in (4). Via the tedious derivations in [13, p. 269] it is immediate that $E[M_{t|\theta_i}]$, where $M_{t|\theta_i}$ is given from (4), is identical to $E[(J'_{t|\theta_i})^T(J'_{t|\theta_i})]$, and so this selection for M_i also satisfies (i) and (ii) above.

As for (iii), under the reasonable stability assumption that for all k, l, some $\alpha > 0$, and $0 < \beta < 1$,

$$\operatorname{cov}[\tilde{z}^T_{k|\theta}\Lambda^{-1}_{k|\theta}\tilde{z}_{k|\theta},\ \tilde{z}^T_{l|\theta}\Lambda^{-1}_{l|\theta}\tilde{z}_{l|\theta}] \le \alpha\beta^{|k-l|}$$

then application of lemma 3.1 of [9] yields the asymptotic ergodicity conditions $(1/t)M_{t|\theta} \to E[(1/t)M_{t|\theta}]$, $(1/t)J''_{t|\theta} \to E[(1/t)J''_{t|\theta}]$ as $t \to \infty$, and (iii) follows from (ii). As noted in [9], when the signal generating system and one-step-ahead prediction conditioned on θ are linear systems with impulse responses $w(\cdot, \cdot)$ satisfying $\|w(k, l)\| \le \alpha\beta^{(k-l)}1(k-l)$ and with no uncontrollable or unobservable modes, and the input to the signal generating system comprises a sequence of independent random variables with bounded fourth moments, then this condition is satisfied, and (iii) holds.

D. Recursive Prediction-Error Algorithm

Here we work with t as a running variable rather than as a fixed constant and consider a sequence of estimates $\hat{\theta}_1$, $\hat{\theta}_2, \cdots, \hat{\theta}_t$ for each t. Let us introduce the notation $\tilde{z}_t$ to denote $\tilde{z}_{t|\hat{\theta}_1\hat{\theta}_2\cdots\hat{\theta}_{t-1}}$ and likewise for Λ, ϕ, μ, and J'. Also denote

$$\tilde{z}'_t = \left.\frac{\partial \tilde{z}_{t|\hat{\theta}_1\hat{\theta}_2\cdots\hat{\theta}_{t-2}\theta}}{\partial\theta}\right|_{\theta=\hat{\theta}_{t-1}}.$$

In order to achieve an on-line algorithm from the above off-line scheme, the following approximations are introduced. They are not justified in this section but are rather presented as heuristics.

Rewriting (3) as

$$J'_{t|\hat{\theta}_i} = J'_{t-1|\hat{\theta}_i} + \tilde{z}^T_{t|\hat{\theta}_i}\Lambda^{-1}_{t|\hat{\theta}_i}\tilde{z}'_{t|\theta_i} + \tfrac{1}{2}\phi^T_{t|\hat{\theta}_i} - \tfrac{1}{2}\mu_{t|\hat{\theta}_i},$$

set $J'_{t-1|\hat{\theta}_i}$ to zero with the implicit assumption that $\hat{\theta}_i$ is chosen to minimize $J_{t-1|\theta}$ over θ as in the Newton–Raphson method. Then an estimate $\hat{J}'_{t|\hat{\theta}_i}$ is given from

$$\hat{J}'_{t|\hat{\theta}_i} = \tilde{z}^T_{t|\hat{\theta}_i}\Lambda^{-1}_{t|\hat{\theta}_i}\tilde{z}'_{t|\hat{\theta}_i} + \tfrac{1}{2}\phi^T_{t|\hat{\theta}_i} - \tfrac{1}{2}\mu_{t|\hat{\theta}_i}.$$

Now approximate $\tilde{z}_{t|\hat{\theta}_i}$ by $\tilde{z}_t$ (recall $\tilde{z}_t \triangleq \tilde{z}_{t|\hat{\theta}_1\hat{\theta}_2\cdots\hat{\theta}_{t-1}}$) and likewise for $\tilde{z}'$, Λ, ϕ, μ, and $\hat{J}'$, since on the assumption that $\hat{\theta}_t$ converges, the approximations are good ones as $t \to \infty$. Similarly, employing $\tilde{z}'_k$ and ϕ_k in (4) rather than $\tilde{z}'_{k|\hat{\theta}_i}$ and $\phi_{k|\hat{\theta}_i}$, approximate $M_{t|\hat{\theta}_i}$ and $M_{t-1|\hat{\theta}_i}$. Let us denote the approximations by M_t and M_{t-1}, respectively.

With the above approximations, the recursive prediction error algorithm is simply

$$\hat{\theta}_{t+1} = \hat{\theta}_t - \rho_t M_t^{-1}(\hat{J}'_t)^T \tag{6}$$

where with the definitions (5)

$$\hat{J}'_t = \tilde{z}^T_t\Lambda_t^{-1}\tilde{z}'_t + \tfrac{1}{2}\phi^T_t - \tfrac{1}{2}\mu_t \tag{7}$$

$$M_t = M_{t-1} + (\tilde{z}'_t)^T\Lambda_t^{-1}(\tilde{z}'_t) + \tfrac{1}{2}\phi_t\phi^T_t. \tag{8}$$

A further manipulation using the matrix inversion lemma and the definition $P_t = M_t^{-1}$ leads to the more convenient algorithm.

E. RPE Algorithm

$$\hat{\theta}_t = \hat{\theta}_{t-1} - \rho_t P_t[(\tilde{z}'_t)^T\Lambda_t^{-1}\tilde{z}_t + \tfrac{1}{2}\phi_t - \tfrac{1}{2}\mu^T_t] \tag{9}$$

where

$$P^*_t = P_{t-1} - P_{t-1}(\tilde{z}'_t)^T S_t^{-1}(\tilde{z}'_t)P_{t-1},$$

$$S_t = \Lambda_t + (\tilde{z}'_t)P_{t-1}(\tilde{z}'_t)^T \tag{10a}$$

$$P_t = P^*_t - P^*_t\phi_t(2 + \phi^T_t P_t\phi_t)^{-1}\phi^T_t P^*_t. \tag{10b}$$

[See also definition (5).] In the absence of *a priori* knowledge, the initial conditions of (9) and (10) are usually chosen as $\hat{\theta}_0 = 0$ and $P_0 = \alpha I$ where α is a large positive number.

F. Remarks

1) A *special case* of some importance is the one in which the weighting matrix $\Lambda_{k|\theta}$ is independent of θ. For this case the RPE algorithm simplifies as

$$\hat{\theta}_t = \hat{\theta}_{t-1} - \rho_t P_{t-1}(\tilde{z}'_t)^T S_t^{-1}\tilde{z}_{t/t-1}$$

$$P_t = P_{t-1} - P_{t-1}(\tilde{z}'_t)^T S_t^{-1}(\tilde{z}'_t)P_{t-1},$$

$$S_t = \Lambda_t + (\tilde{z}'_t)P_{t-1}(\tilde{z}'_t)^T.$$

Notice that the crucial intermediate results in the simplification procedure are

$$P_t(\tilde{z}'_t)^T\Lambda_t^{-1} = P_{t-1}(\tilde{z}'_t)^T S_t^{-1}, \qquad P_t = P^*_t.$$

2) In the case of single-input/single-output system represented by

$$x_{t+1} = Fx_t + Gy_t + Bu_t + K\varepsilon_t$$

$$y_t = \theta^T x_t,\ z_t = y_t + \varepsilon_t$$

where $\{\varepsilon_t\}$ is an innovations sequence with constant covariance, the RPE algorithm is coincident with the recursive approximate maximum likelihood algorithm RML2 of [6] and [7] for which extensive simulation results are available. There are similarities in the derivation, but the method of [6] can not be immediately generalized to the vector case given here.

3) To avoid numerical difficulties in the RPE algorithm, square root filtering versions (or fast square-root versions [15]) are used in practice. Stochastic approximation versions are also readily defined [16].

4) Let us introduce the notation $R_t = \gamma_t P_t^{-1}$ for any decreasing scalar sequence γ_t satisfying the assumptions, $\sum_{t=1}^{\infty}\gamma_t = \infty$, $\sum_{t=1}^{\infty}\gamma^2(t) < \infty$ (or more generally, A.6–A.9 of [10]). Then (9) and (10) with $\rho_t = 1$ can be reorganized with minor variations as

$$\hat{\theta}_t = \hat{\theta}_{t-1} - \gamma_t R_t^{-1}[(\tilde{z}'_t)^T\Lambda_t^{-1}\tilde{z}_t + \tfrac{1}{2}\phi_t - \tfrac{1}{2}\mu^T_t] \tag{11a}$$

$$R_t = R_{t-1} + \gamma_t[(\tilde{z}'_t)^T\Lambda_t^{-1}(\tilde{z}'_t) + \tfrac{1}{2}\phi_t\phi^T_t + \delta I - R_{t-1}] \tag{11b}$$

where the introduction of δI ensures that $R_t \ge \delta I$ for some $\delta > 0$ and all t if $R_0 \ge \delta I$. Usually the new step size γ_t is chosen with $\gamma_t > 1/t$ initially and $\gamma_t \to 1/t$ as $t \to \infty$.

III. Asymptotic Equivalence to Off-Line Identification

The aim of this section is to apply the theory of [8]–[11] to show that under reasonable conditions, the RPE algorithm is asymptotically equivalent to an off-line prediction error minimization, and to thereby justify the RPE scheme.

Consider now the ordinary differential equation (ODE) associated with the recursions (11)

$$\frac{d}{d\tau}\theta(\tau, t) = R^{-1}(\tau, t) f[\theta(\tau, t), t] \tag{12a}$$

$$\frac{d}{d\tau}R(\tau, t) = G[\theta(\tau, t), t] + \delta I - R(\tau, t), \qquad R(0, t) \geq \delta I \tag{12b}$$

where t is here a fixed parameter and with $\theta(\tau, t)$ abbreviated as θ (not the true θ):

$$f(\theta, t) = -E\{(\tilde{z}'_{t|\theta})^T \Lambda_{t|\theta}^{-1} \tilde{z}_{t|\theta} + \tfrac{1}{2}\phi_{t|\theta} - \tfrac{1}{2}\mu_{t|\theta}^T\} \tag{13}$$

$$G(\theta, t) = E\{(\tilde{z}'_{t|\theta})^T \Lambda_{t|\theta}^{-1} (\tilde{z}'_{t|\theta}) + \tfrac{1}{2}\phi_{t|\theta}\phi_{t|\theta}^T\}. \tag{14}$$

Applying the theory of Ljung (Section V of [10]), which works with the limiting form of the ODE as $t \to \infty$ under conditions ensuring the existence of the various limits, gives immediately the following convergence result.

A. Convergence Result

The recursions (11) converge almost surely to the set $\bar{D}_c = \{\theta \mid \lim_{t\to\infty} f(\theta, t) = 0\}$ if the following hold.

1) $\hat{\theta}_t \in D_s$ for all t where D_s is the set of all θ such that the predictor generating $\tilde{z}_{t|\theta}$ and the equations generating the intermediate variables $\tilde{z}'_t$, ϕ_t, etc. are exponentially stable. (If $\hat{\theta}_t \notin D_s$ for some t, then it should be projected into D_s, e.g., by reducing the step size.)

2) The prediction error $\tilde{z}_t$ is asymptotically stationary and bounded above[2] for all $\bar{\theta} \in D_s$.

3) The reasonable *regularity conditions* (A.3–A.5 of [10]) on the functions which define the recursions (11) and on the predictor equations, including the equations for the intermediate variables, are satisfied for all $\hat{\theta}_t \in D_s$ when 2) holds. (These conditions can be checked before applying the RPE algorithm.)

4) The ODE (12a) and (12b) (parameterized by t) is asymptotically stable in the limit as $t \to \infty$ for all $\theta(\tau, t) \in D_s$.

B. Remarks

1) The stability of (12a) and (12b) is investigated via the Lyapunov function $V(\theta, t) - V(\theta^*, t)$ where with $\theta(\tau, t)$ again abbreviated as θ (not the true θ),

$$V(\theta, t) = \tfrac{1}{2}E\{\tilde{z}_{t|\theta}^T \Lambda_{t|\theta}^{-1} \tilde{z}_{t|\theta} + \log\det \Lambda_{t|\theta}\} \tag{15}$$

and $V(\theta^*, t)$ denotes the global minimum value (assumed to be unique) of $V(\theta, t)$ with respect to θ. Here, with an assumption that $\Lambda_{t|\theta}$ is uniformly bounded below, the function $V(\theta, t) - V(\theta^*, t)$ is positive definite for all t and $\theta \in D_s$. Note that in the limit as $t \to \infty$, the function $V(\theta, t)$ tends to infinity for θ outside the boundary of D_s. With appropriate differentiability assumptions

$$V'(\theta, t) = E\{(\tilde{z}'_{t|\theta})^T \Lambda_{t|\theta}^{-1} \tilde{z}_{t|\theta} + \tfrac{1}{2}\phi_{t|\theta} - \tfrac{1}{2}\mu_{t|\theta}^T\}^T = -f^T(\theta, t) \tag{16}$$

and

$$\frac{d}{d\tau}V[\theta(\tau, t), t] = V'[\theta(\tau, t), t]\frac{d}{d\tau}\theta(\tau, t) = -f^T[\theta(\tau, t), t] R^{-1}(\tau, t) f[\theta(\tau, t), t]. \tag{17}$$

Since $R^{-1}(\tau, t)$ is a positive definite matrix for all τ, t, as well as for $t \to \infty$, the function $V(\theta, t) - V(\theta^*, t)$ is decreasing outside the set $D_c(t) = \{\theta \mid f(\theta, t) = 0\}$ which coincides with the set of stationary points of (12a) and (12b). Therefore, in the limit as $t \to \infty$, $\theta(\tau, t)$ converges to the set $\bar{D}_c$ (which is a subset of D_s).

2) In order to establish the relation between the set $\bar{D}_c$ and the set of local minima of $J_{t|\theta}$, we require an asymptotic ergodicity condition[3]

$$\lim_{t\to\infty}\left[\frac{1}{t}J_{t|\theta} - E\left\{\frac{1}{t}J_{t|\theta}\right\}\right] = 0, \qquad \text{with probability 1.} \tag{18}$$

Also, we require an additional condition that

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{t} V(\theta, k) = \lim_{t\to\infty} V(\theta, t) \tag{19}$$

which is ensured if $V(\theta, t)$ is uniformly bounded from above for all t, and $\lim_{t\to\infty} V(\theta, t)$ exists. Under the conditions (18), (19) the following relation can be derived

$$\lim_{t\to\infty}\frac{1}{t}J_{t|\theta} = \lim_{t\to\infty} V(\theta, t), \qquad \text{with probability 1} \tag{20}$$

and hence the set $\bar{D}_c$ is coincident with the set of local minima of $J_{t|\theta}$.

3) For the case when $J_{t|\theta}$ has a unique minimum, the RPE algorithm is then, under the conditions of the convergence result and (18) and (19), asymptotically equivalent to any off-line prediction error scheme. For this case, the RPE estimate has the attractive features of the off-line estimate as outlined in [8] and [9].

4) The above convergence results are perhaps significant only in that they tell us that the particular recursive approximate prediction error algorithm of the previous section is one that should be studied more closely, since under various reasonable conditions, the recursive scheme leads to the same result as an off-line minimization.

5) Specializations of the above convergence theory for the case of scalar ARMA models is given in [7].

IV. Relation to the Modified EKF Algorithm

In this section we demonstrate that the RPE algorithm of Section II specialized for the case of linear innovations

[2] At the expense of some additional regularity conditions, this requirement (condition A.2 of [10]) can be relaxed so that Gaussian noise models are not excluded.

[3] Interpretations in terms of asymptotic stability of models is given in [8] and [9].

TABLE I

Eq. No.	Modified EKF Algorithm	RPE Algorithm†
(T.1)	$\hat{\theta}_{t+1} = \hat{\theta}_t + L_t\hat{\varepsilon}_t$	$\hat{\theta}_{t+1} = \hat{\theta}_t + L_t^*\hat{\varepsilon}_t$
(T.2)	$L_t = (C_tP_t^2 + D_tP_t^3)^TS_t^{-1}$	$L_t^* = P_t^*(C_tW_t + D_t)^TS_t^{*-1}$ or $L_t^* = P_t^*\Psi_tS_t^{*-1}$
(T.3)	$S_t = \Lambda + D_tP_t^3D_t^T + [D_t(P_t^2)^TC_t^T] + [D_t(P_t^2)^TC_t^T]^T$	$S_t^* = \Lambda + \Psi_t^TP_t^*\Psi_t$
(T.4)	$P_{t+1}^2 = (A_t - K_tC_t)P_t^2 + (M_t - K_tD_t)P_t^3$ or $P_{t+1}^2 = A_tP_t^2 + M_tP_t^3 - K_tS_tL_t^T$	$W_{t+1} = (A_t - K_tC_t)W_t + (M_t - K_tD_t)$
(T.5)	$P_{t+1}^3 = P_t^3 - L_tS_tL_t^T$	$P_{t+1}^* = P_t^* - L_t^*S_t^*L_t^{*T}$

† With modified time indexes.

models can be derived from the modified EKF of Ljung [3]. In fact, the modified EKF is an RPE algorithm with a particular way of computing the search direction. As shown below, neglecting terms which asymptotically approach zero in the EKF leads to a simpler RPE algorithm, viz., the appropriate specialization of the schemes of Section II.

The linear innovations model under consideration is described by

$$x_{t+1} = A(\theta)x_t + B(\theta)u_t + K(\theta)\varepsilon_t, \qquad x_0 = 0 \tag{21a}$$

$$z_t = C(\theta)x_t + \varepsilon_t \tag{21b}$$

where x_t, z_t, and u_t are vectors of dimensions n, m, and p, respectively, and the sequence $\{\varepsilon_t\}$ consists of independent random vectors with zero mean and covariance

$$E\{\varepsilon_t\varepsilon_s^T\} = \Lambda\delta_{ts}. \tag{22}$$

The prediction error $\hat{\varepsilon}_t = \tilde{z}_t$ is given from

$$\hat{x}_{t+1} = A_t\hat{x}_t + B_tu_t + K_t\hat{\varepsilon}_t \tag{23}$$

$$\hat{\varepsilon}_t = z_t - C_t\hat{x}_t \tag{24}$$

with the abbreviations $A(\hat{\theta}_t) = A_t$, etc., where $\hat{\theta}_t$ is the estimate of θ at time t and $\hat{x}_t$ is the estimate of x_t.

The modified EKF algorithm[4] [3] associated with the model (21a), (21b) and (22) is given by (23) and (24) and the recursions in Table I, as is the RPE algorithm for comparison. The following definitions apply:

$$M_t = M(\hat{\theta}_t, \hat{x}_t, u_t, \hat{\varepsilon}_t) = \frac{\partial}{\partial\theta}[A(\theta)\hat{x}_t + B(\theta)u_t + K(\theta)\hat{\varepsilon}_t]_{\theta=\hat{\theta}_t} \tag{25}$$

$$D_t = D(\hat{\theta}_t, \hat{x}_t) = \frac{\partial}{\partial\theta}[C(\theta)\hat{x}_t]_{\theta=\hat{\theta}_t} \tag{26}$$

$$W_t = \frac{\partial\hat{x}_t}{\partial\hat{\theta}_t}; \quad \Psi_t = -\left(\frac{\partial\hat{\varepsilon}_t}{\partial\hat{\theta}_t}\right)^T = (C_tW_t + D_t)^T. \tag{27}$$

In the algorithms, the estimate $\hat{\theta}_t$ is constrained such that $|\lambda_i[A_t - K_tC_t]| < 1$ for all i, t where $\lambda_i[X]$ denotes the eigenvalues of X. Note that the quantities L_t, S_t, and P_t^3 are parallel to L_t^*, S_t^* and P_t^*. The quantity $P_t^2(P_t^3)^{-1}$ is roughly parallel to W_t.

[4] More precisely, this is a modified one step ahead predictor algorithm.

A. Remarks

1) The computational burden associated with the RPE algorithm is less in equations (T.3) and (T.4) of Table I than for the EKF scheme. (The difference for each iteration is $mnr + m^2n + mr^2$ multiplications where n, m, and r are the state, measurement, and parameter vector dimensions, respectively. The number of multiplications in the RPE equations of Table I is $3(rm^2 + r^2m) + n^2(r + m) + rm + 2mnr + r^3$. With $m = 1$, $r = 2$, and $n = 4$, for example, there is a reduction of 17 percent in the number of multiplications in going from the EKF scheme to the RPE scheme. With $m = n = r = 4$, the reduction is of 26 percent.)

2) Kalman filters with an adaptive Kalman gain have been studied by a number of authors; see for example [5]. The RPE scheme of this section appears to be the most straightforward approach with guaranteed convergence. In this case the model is (21a), (21b) where $A(\theta)$, $B(\theta)$, and $C(\theta)$ are in fact independent of θ and are known, and the columns of $K(\theta)$ go to make up θ. The RPE algorithm of this section is relatively simple since $A_t = A$, $B_t = B$, $C_t = C$, $D_t = 0$, and $M_t = (\partial/\partial\theta)[K(\theta)\hat{\varepsilon}_t]_{\theta=\hat{\theta}_t}$. It is coincident with the RML2 scheme [6] and [7] for single input-single output signal models with unknown "moving average" parameters associated with the noise but with known autoregressive parameters and known moving average parameters associated with known inputs.

3) The convergence analysis for the EKF algorithm is given in [3] and the asymptotic equivalence of the two schemes for the special case when $C(\theta) = \theta^TH$, $A(\theta) = F + (G + K)\theta^TH$, $K(\theta) = K$, and $B(\theta) = B$ is pointed out. This theory is now generalized for the innovations model (21a) and (21b) along somewhat different lines so as to yield an alternative derivation of the RPE algorithm.

B. Derivation of RPE Algorithm via EKF Theory

The EKF algorithm can be reexpressed in terms of $\tilde{W}_t$ where $\tilde{W}_t = W_t - \hat{W}_t$, $\hat{W} = P^2(P^3)^{-1}$.[5] Setting

$$C_t\tilde{W}_tP_t^3 \equiv 0, \qquad C_tW_tP_t^3W_t^TC_t^T \equiv 0$$

[5] The matrix W_t is defined by (T.4) of the RPE algorithm, but with A_t, etc., evaluated using EKF $\hat{\theta}_t$.

then yields the RPE algorithm. The details are as follows.

From the definition of $\hat{W}_t$ and the EKF algorithm,

$$\hat{W}_{t+1} = (A_t - K_tC_t)\hat{W}_t + (M_t - K_tD_t) + P_{t+1}^2[(P_{t+1}^3)^{-1} - (P_t^3)^{-1}] \quad (28)$$

and

$$\tilde{W}_{t+1} = (A_t - K_tC_t)\tilde{W}_t - P_{t+1}^2[(P_{t+1}^3)^{-1} - (P_t^3)^{-1}]. \quad (29)$$

Rewriting S_t in terms of $\tilde{W}_t$ yields after a tedious derivation,

$$S_t = \Lambda + \Psi_t^T P_t^3 \Psi_t - (C_tW_tP_t^3W_t^TC_t^T + C_t\tilde{W}_tP_t^3D_t^T + D_tP_t^3\tilde{W}_t^TC_t^T). \quad (30)$$

Similar treatment for L_t and P_t^3 gives

$$L_t = P_t^3\Psi_tS_t^{-1} - (P_t^3\tilde{W}_t^TC_t^TS_t^{-1}) \quad (31)$$

$$P_{t+1}^3 = P_t^3 - P_t^3\Psi_tS_t^{-1}\Psi_t^TP_t^3 + (P_t^3\Psi_tS_t^{-1}C_t\tilde{W}_tP_t^3 + P_t^3\tilde{W}_t^TC_t^TS_t^{-1}\Psi_t^TP_t^3 - P_t^3\tilde{W}_t^TC_t^TS_t^{-1}C_t\tilde{W}_tP_t^3). \quad (32)$$

Setting $C_t\tilde{W}_tP_t^3 \equiv 0$ and $C_tW_tP_t^3W_t^TC_t^T \equiv 0$ in (30), (31), and (32) leads to the RPE algorithm with $P_t^3 \equiv P_t^*$, $L_t \equiv L_t^*$ and $S_t \equiv S_t^*$.

C. Justification

The terms $C_t\tilde{W}_tP_t^3$ and $C_tW_tP_t^3W_t^TC_t^T$ approach zero asymptotically as the following lemma shows. Now since the calculation of these terms adds considerably to the calculations, it makes sense to try a scheme in which they are set to zero. Of course, the resulting RPE algorithm will have a somewhat different search direction than the modified EKF scheme viewed as an RPE algorithm, but the heuristics that lead to the algorithm can only be justified by a convergence analysis and experimental verification. The convergence analysis of the previous section, of course, specializes to yield that the REP scheme of this section converges under reasonable conditions not restated here. Also since RML2 of [6] and [7] belongs to the class of algorithms here and is well studied experimentally, we have reasonable experimental confirmation of the usefulness of the heuristics above.

Lemma: Let the matrices $(A_t - K_tC_t)$, $(M_t - K_tD_t)$, C_t, and D_t be bounded above uniformly in t. Then with $|\lambda_i[A_t - K_tC_t]| < 1$,

1) $P_t^3 \to 0$, as $t \to \infty$
2) $C_t\tilde{W}_tP_t^3 \to 0$ and $C_tW_tP_t^3W_t^TC_t^T \to 0$, as $t \to \infty$.

Proof Outline: 1) From the EKF equations

$$S_t = \Lambda + D_tP_t^3D_t^T + D_t(P_t^2)^TC_t^T + C_tP_t^2D_t^T = \Lambda + \hat{\Psi}_t^TP_t^3\hat{\Psi}_t - (C_tP_t^2)(P_t^3)^{-1}(C_tP_t^2)^T$$

where

$$\hat{\Psi}_t \triangleq [C_tP_t^2(P_t^3)^{-1} + D_t]^T.$$

Hence

$$S_t \le \Lambda + \hat{\Psi}_t^TP_t^3\hat{\Psi}_t.$$

Define

$$\bar{P}_{t+1} = \bar{P}_t - \bar{P}_t\hat{\Psi}_t\bar{S}^{-1}\hat{\Psi}_t^T\bar{P}_t$$
$$\bar{S}_t = \Lambda + \hat{\Psi}_t^T\bar{P}_t\hat{\Psi}_t.$$

For each t where $\bar{P}_t = P_t^3$,

$$(\bar{P}_{t+1} - \bar{P}_t) - (P_{t+1}^3 - P_t^3) = P_t^3\hat{\Psi}_t(S_t^{-1} - \bar{S}_t^{-1})\hat{\Psi}_t^TP_t^3 \ge 0$$

and, therefore, $\bar{P}_t \ge P_t^3 \ge 0$.

Applying the matrix inversion lemma yields $\bar{P}_{t+1}^{-1} = \bar{P}_t^{-1} + \hat{\Psi}_t\Lambda^{-1}\hat{\Psi}_t^T$ which in turn implies that[6]

$$\bar{P}_t = \left[\bar{P}_0^{-1} + \sum_{k=0}^{t}\hat{\Psi}_k\Lambda^{-1}\hat{\Psi}_k^T\right]^{-1} \to 0, \quad \text{as } t \to \infty.$$

Since $\bar{P}_t \ge P_t^3 \ge 0$, then $P_t^3 \to 0$ as $t \to \infty$.

2) Under the conditions of the lemma, (T.4) of the EKF algorithm ensures that $P_t^2 \to 0$ as $P_t^3 \to 0$. Therefore, $C_tP_t^2 + D_tP_t^3 = C_t\hat{W}_tP_t^3 + D_tP_t^3 = \Psi_t^TP_t^3 - C_t\tilde{W}P_t^3 \to 0$ as $t \to \infty$. But, under the conditions of the lemma, W_t and, hence, Ψ_t are bounded above, giving that $\Psi_t^TP_t^3 \to 0$ as $t \to \infty$. The results 2) follow.

V. Further Simplifications

In this section we restrict attention to the innovations signal models of [4] and [5]:

$$x_{t+1} = (F + G\Theta)x_t + Kz_t + Bu_t \quad (33)$$

$$z_t = \Theta x_t + \varepsilon_t \quad (34)$$

where x_t, z_t, and u_t are vectors of dimensions n, m, and p, respectively, and the sequence $\{\varepsilon_t\}$ consists of independent random vectors with zero mean and known covariance. The unknown parameter elements are organized into an unknown $m \times n$ matrix $\Theta^T = [\theta_1\,\theta_2 \cdots \theta_m]$ or as a vector $\theta^T = [\theta_1^T\theta_2^T \cdots \theta_m^T]$.

The algorithm defined by the model (33) and (34) and by an appropriate scheme for estimating θ contains as a subclass of algorithms the output error identification methods and the model reference adaptive schemes. The adaptive observers are also closely related. The detailed reference list is found in [4].

Let $\hat{\Theta}$ and $\hat{\theta}$ denote estimates[7] of Θ, θ; then the prediction equations are

$$\hat{x}_{t+1} = F\hat{x}_t + (G + K)\hat{z}_t + Bu_t + K\hat{\varepsilon}_t \quad (35a)$$

$$\hat{z}_t = \hat{\Theta}\hat{x}_t \qquad \hat{\varepsilon}_t = z_t - \hat{z}_t. \quad (35b)$$

[6] If

$$\sum_{k=0}^{\infty}\hat{\Psi}_k\Lambda^{-1}\hat{\Psi}_k^T \ne \infty$$

then a modification like the one introduced in (11b) is required.

[7] The estimate is time varying in the RPE algorithm, but for simplicity of notation the subscript t is omitted.

Using the RPE scheme for estimating θ we need to calculate Ψ_t. In this case, $\Psi_t \triangleq -(\partial\hat{\varepsilon}_t/\partial\theta)^T$ is readily calculated by taking derivatives in (35a) and (35b) and introducing the definitions

$$n\updownarrow\overset{n}{\overleftrightarrow{[W_t^i]}} = \frac{\partial\hat{x}_t}{\partial\theta_i} \quad n\updownarrow\overset{m}{\overleftrightarrow{[\Psi_t^i]}} = -\left(\frac{\partial\hat{\varepsilon}_t}{\partial\theta_i}\right)^T. \tag{36}$$

A. Algorithm for Ψ_t

$$W_{t+1}^i = FW_t^i + K(\Psi_t^i)^T \quad (\Psi_t^i)^T = e_i\hat{x}_t^T + \hat{\Theta}W_t^i \quad W_0^i = 0 \tag{37}$$

for $i = 1, 2, \cdots, m$ where e_i is a zero vector of appropriate size m save that the ith element is unity, and

$$(\Psi_t)^T = m\updownarrow\overbrace{[(\Psi_t^1)^T(\Psi_t^2)^T\cdots(\Psi_t^m)^T]}^{n\times m}. \tag{38}$$

Motivated as in the scalar measurement case where RML2 algorithm [6] and [7] is employed for calculating Ψ_t, we demonstrate a parallel simplification for the multivariable model of (35a) and (35b). The simplification replaces Ψ_t by an estimate $\hat{\Psi}_t$ with the property that, if $\hat{\Theta}_t \to \Theta$ as $t \to \infty$, then $\hat{\Psi}_t \to \Psi_t$ as $t \to \infty$, and also that the convergence theory of Section III is unchanged by the substitution of $\hat{\Psi}_t$ for Ψ_t, as it demands that $\hat{\Psi}_{t|\theta} \to \Psi_{t|\theta}$ as $t \to \infty$ for all $\theta \in D_s$.

The calculation of $\hat{\Psi}_t$ is simplified in the sense that it requires less computational burden than for the direct scheme of (37). The computational saving is illustrated after the presentation of the algorithm and its convergence analysis.

B. Algorithm for $\hat{\Psi}_t$

$$\hat{\Psi}_{t+1}^i = (F + G\hat{\Theta})\hat{\Psi}_t^i + K\bar{z}_t^i + B\bar{u}_t^i \tag{39a}$$

$$\hat{\Psi}_t^T = [(\hat{\Psi}_t^1)^T(\hat{\Psi}_t^2)^T\cdots(\hat{\Psi}_t^m)^T] \tag{39b}$$

where

$$\eta_{t+1}^i = (F + G\hat{\Theta})\eta_t^i - Ge_i[z_t^T u_t^T] \tag{40a}$$

$$\begin{bmatrix}\bar{z}_t^i\\ \bar{u}_t^i\end{bmatrix} = \begin{bmatrix}z_t\\ u_t\end{bmatrix}e_i^T - (\eta_t^i)^T\hat{\Theta}^T \tag{40b}$$

and $\eta_0^i = 0, \hat{\Psi}_0^i = 0$ for $i = 1, 2, \cdots, m$. The matrices $[\hat{\Psi}_t^i]$, $[\eta_t^i]$, and $[(\bar{z}_t^i)^T(\bar{u}_t^i)^T]$ are of dimensions $n \times m$, $n \times (m + p)$, and $m \times (m + p)$, respectively.

C. Rationale

Introducing q as the forward shift operator, we have that (37) in operator notation is

$$(\Psi_t^i)^T = \hat{D}(q^{-1})e_i\hat{x}_t^T = \hat{d}_i(q^{-1})\hat{x}_t^T \tag{41}$$

where $\hat{d}_i(q^{-1})$ is the ith column of $\hat{D}(q^{-1}) = [I_m - \hat{\Theta}(qI_n - F)^{-1}G]^{-1}$, and also that (40a) and (40b) in operator notation is

$$\begin{bmatrix}\bar{z}_t^i\\ \bar{u}_t^i\end{bmatrix} = \begin{bmatrix}z_t\\ u_t\end{bmatrix}e_i^T\hat{D}^T(q^{-1}) = \begin{bmatrix}z_t\hat{d}_i^T(q^{-1})\\ u_t\hat{d}_i^T(q^{-1})\end{bmatrix}. \tag{42}$$

If now $\hat{d}_i(q^{-1})$ is independent of t, as when $\hat{\theta}$ has converged to θ, then postmultiplying (35a) and (35b) by $\hat{d}_i^T(q^{-1})$ and reorganizing in terms of (41) and (42) gives recursions for Ψ_t^i as

$$\Psi_{t+1}^i = (F + G\hat{\Theta})\Psi_t^i + K\bar{z}_t^i + B\bar{u}_t^i. \tag{43}$$

To indicate that (43) is an approximation, Ψ_t^i is replaced by $\hat{\Psi}_t^i$. Fig. 1 depicts the algorithm where $W(q^{-1})$ is a known subsystem with parameters F, G, K, and B.

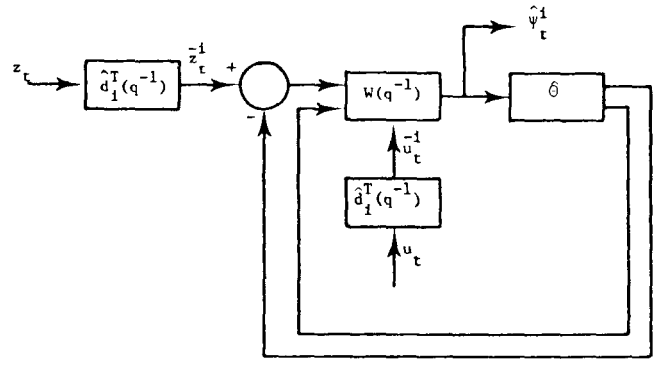

Fig. 1. Simplified RPE scheme.

D. Convergence Analysis

As already noted, with $\hat{\theta} \in D_s$ independent of time, the recursions (39) and (40a) and (40b) are identical to (37), which means that the stability analysis associated with the ordinary differential equation of the RPE algorithm as $t \to \infty$ is unaffected by the simplification. We conclude that the convergence analysis of an earlier section remains valid for the simplification.

E. Computational Burden

The computational burden associated with equations (39) and (40a) and (40b) is less than for equations (37). As an indication, the calculation of $\hat{\Psi}_t$ requires $n^2(4m^2 + mp) + n[2m^3 + 2m^2(p + 1) + mp] + (m^3 + m^2p)$ multiplications for each iteration, while the calculation of Ψ_t requires $n^3m + 2n^2m^2 + nm^2$ multiplications, where n, m, and p are the state, measurement, and input vector dimensions, respectively. Observe that in (39) and (40a) and (40b) the number of multiplications is proportional to n^2, and in (37) it is proportional to n^3. With $m = 2$, $n = 10$, and $p = 1$, for example, there is a reduction of 24 percent in the number of multiplications in going from (37) to (39) and (40a) and (40b). With $m = 2$, $n = 20$, and $p = 1$, the reduction is 30 percent.

F. Remarks

1) For scalar measurement models a further simplification to this algoritm is possible as follows. Let $d_{ij}(q^{-1})$ be the ijth element of

$$\hat{D}(q^{-1}) = [I_m - \hat{\Theta}(qI_n - F)^{-1}G]^{-1},$$

then (38) and (41) imply

$$(\Psi_t)^T = m\updownarrow\begin{bmatrix}\overleftrightarrow{d_{11}(q^{-1})\hat{x}_t^T} & \cdots & d_{1m}(q^{-1})\hat{x}_t^T\\ \vdots & & \\ d_{m1}(q^{-1})\hat{x}_t^T & \cdots & d_{mm}(q^{-1})\hat{x}_t^T\end{bmatrix}. \tag{44}$$

With $m = 1$ the matrix $\hat{D}(q^{-1})$ is reduced to $d_{11}(q^{-1}) = (1/[1 - \theta^T(qI_n - F)^{-1}G])$, and $(\Psi_t)^T$ is given by $d_{11}(q^{-1})\hat{x}_t^T$. If $\hat{x}_t^T$ is chosen as

$$\hat{x}_t^T = [-z_{t-1} \cdots -z_{t-n_a} \mid u_{t-1}^T \cdots u_{t-n_b}^T \mid \hat{\varepsilon}_{t-1} \cdots \hat{\varepsilon}_{t-n_c}],$$

which is always possible, then $(\Psi_t)^T$ is determined by

$$\begin{aligned}(\Psi_t)^T &= [-d_{11}(q^{-1})z_{t-1} \cdots -d_{11}(q^{-1})z_{t-n_a} \mid d_{11}(q^{-1})u_{t-1} \\ &\quad \cdots d_{11}(q^{-1})u_{t-n_b} \mid d_{11}(q^{-1})\hat{\varepsilon}_t \cdots d_{11}(q^{-1})\hat{\varepsilon}_{t-n_c}] \\ &= [\alpha_t \alpha_{t-1} \cdots \alpha_{t-n_a+1} \mid \beta_t^T \beta_{t-1}^T \\ &\quad \cdots \beta_{t-n_b+1}^T \mid \gamma_t \gamma_{t-1} \cdots \gamma_{t-n_c+1}]\end{aligned}$$

where $\alpha_t = -d_{11}(q^{-1})z_{t-1}$, $\beta_t^T = d_{11}(q^{-1})u_{t-1}^T$, and $\gamma_t = d_{11}(q^{-1})\hat{\varepsilon}_t$. Thus the calculation of Ψ_t is reduced to the calculation of α_t, β_t, and γ_t, and their time shifts. It turns out that this class of algorithms contains the RML2 algorithm of [6] and [7].

There appears to be no corresponding simplification for the vector measurement case unless there is available an explicit expression for $\hat{D}(q^{-1})$, since the computation of α_t, β_t, and γ_t is done in terms of the coefficients of $d_{ij}(q^{-1})$.

However, for multivariable signal models described by $A(q^{-1})z_t = B(q^{-1})u_t + C(q^{-1})\varepsilon_t$ where

$$A(q^{-1}) = I + A_1 q^{-1} + \cdots + A_{n_a} q^{-n_a},$$

$B(q^{-1}) = B_1 q^{-1} + \cdots + B_{n_b} q^{-n_b}$, $C(q^{-1}) = I + C_1 q^{-1} + \cdots + C_{n_c} q^{-n_c}$, the transfer function matrix $D(q^{-1})$ coincides with $[C(q^{-1})]^{-1}$. For this case it is possible to use canonical forms where $C(q^{-1})$ is a diagonal matrix (canonical forms II and III of [17, p. 94]) and thus to have an explicit expression for $D(q^{-1})$. Notice that these canonical forms reduce the computation but at the same time lead to a less accurate parameter estimate [17, p. 108].

2) For scalar measurement models ($\Theta = \theta^T$), no external input ($B = 0$), and no internal feedback ($G = -K$), the system model is reduced to MA model, and the predictor is simply the familiar set of equations

$$\hat{x}_{t+1} = F\hat{x}_t + K\hat{\varepsilon}_t \tag{45a}$$

$$z_t = \theta^T \hat{x}_t + \hat{\varepsilon}_t. \tag{45b}$$

The calculation of Ψ_t, or rather $\hat{\Psi}_t$, employed in the RPE algorithm is achieved as

$$\hat{\Psi}_{t+1} = (F - K\hat{\theta}^T)\hat{\Psi}_t + K\hat{\varepsilon}_t. \tag{46}$$

In this case $\hat{D}(q^{-1}) = [1 + \hat{\theta}^T(qI_n - F)^{-1}K]^{-1}$,

$$\bar{z}_t = \hat{D}(q^{-1})z_t,$$

and $z_t = \hat{D}^{-1}(q^{-1})\hat{\varepsilon}_t$, and thus $\bar{z}_t$ is simply the pseudoinnovation $\hat{\varepsilon}_t$. It is this fact that allows us to achieve the *very simple* algorithm for Ψ_t, or rather its estimate $\hat{\Psi}_t$. This algorithm is coincident with the corresponding RML2 algorithm.

3) Without giving the details here, it is of interest that the RPE algorithm for this special case with Ψ_t replaced by $\hat{\Psi}_t$ is in fact identical to the algorithm one would achieve using ELS ideas to a somewhat unconventional nonminimal model. The conventional model for $\hat{\varepsilon}_t$ consists of the innovations model for z_k followed by the whitening filter is not suitable for application of extended filtering ideas since $\hat{\varepsilon}_t$ is not expressed directly as a linear function of the unknown parameters. However, in the case of scalar z_k and $\hat{\varepsilon}_k$, there is no loss of generality reversing the concatenation, and so achieving an unconventional model for $\hat{\varepsilon}_t$ in which $\hat{\varepsilon}_t$ is expressed directly as a linear function of the unknown parameters. For this model, the ELS approach leads to the RPE algorithm above. Fig. 2 depicts the situation.

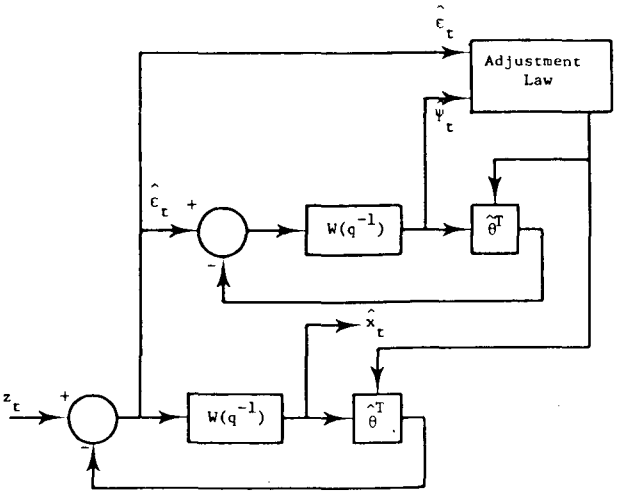

Fig. 2. RPE scheme achieved via extended least squares.

VI. Conclusions

The procedure of Söderström [6] to construct a recursive approximate maximum likelihood parameter identification algorithm for scalar observations from a special class of parameterized linear signal models and the convergence analysis due also to Söderström, Ljung, and Gustavsson [7] do in fact have very useful generalizations for more general prediction error indexes and innovations signal models. In the derivation of the off- and on-line prediction error schemes, there is a key role for convenient derivation of the Fisher information matrix by Goodwin and Payne [13].

The convergence theory is useful to justify the steps leading to the recursive prediction error schemes for it indicates that under reasonable conditions, the recursive scheme leads asymptotically to the same result as an off-line minimization. Couching the convergence results in these terms, there is immediately available a theory from Anderson, Moore, and Hawkes [9] of what happens when the signal generating system is not in the assumed model set.

A crucial restriction on the prediction error schemes proposed is that the pseudoinnovations $\tilde{z}_t$ and its parameter sensitivity matrix (vector) $\tilde{z}_t'$ be available. These quantities certainly are readily calculated on-line for the class of signal models of [4], [5] and the more general ones discussed in the paper, but for certain minimum variance control, or certain filtering problems, the sensitivity matrix may not be readily calculated. Further study for such cases is at present under way using approximations to the sensitivity matrix.

The simplifications possible for the recursive prediction

error scheme and the computational effort comparisons of the paper, suggest that these algorithms have some inherent advantage over the extended Kalman filters derived by augmenting the state variable of innovations models with the unknown parameter vector, as in Ljung [3]. Further case studies could confirm or dispel such a conclusion.

In a later paper, the RPE schemes of this paper (including RML2 algorithm) are modified so as to avoid the stability test at each iteration as a necessary step to ensure convergence. The simulation studies of this later paper further support the approach developed here.

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Artificial Intelligence Programming Languages for Computer Aided Manufacturing

CHUCK RIEGER, JONATHAN ROSENBERG,
AND HANAN SAMET, MEMBER, IEEE

Abstract—**Eight Artificial Intelligence programming languages (SAIL, LISP, MICROPLANNER, CONNIVER, MLISP, POP-2, AL, and QLISP) are presented and surveyed, with examples of their use in an automated shop environment. Control structures are compared, and distinctive features of each language are highlighted. A simple programming task is used to illustrate programs in SAIL, LISP, MICROPLANNER, and CONNIVER. The report assumes reader knowledge of programming concepts, but not necessarily of the languages surveyed.**

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C. Rieger and H. Samet are with the Department of Computer Science, University of Maryland, College Park, MD 20742.

J. Rosenberg is with the Department of Computer Science, Carnegie-Mellon University, Pittsburgh, PA 15213.

I. Introduction

EARLY INTEREST in computers and computing tended to revolve around the high speeds at which numerical calculations could be performed for such tasks as discrete analysis, simulation, payroll handling, and the like. These first applications were supported by such languages as FORTRAN, ALGOL, and COBOL. Today, numerical tasks are still prevalent, and there is a host of new languages. However, there has been a growing interest in the application of computers to computations which are less numeric and more symbolic in nature, in particular in applications in which the key problems are not the speed of multiply and divide hardware, but rather in the forms of data storage and control that are needed to carry out complex decision making and planning tasks.

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