University of Wisconsin
Computer Sciences Department
1210 West Dayton Street
Madison, Wisconsin 53706

RECURSIVE PROPERTIES OF ABSTRACT COMPLEXITY CLASSES

by

L. H. Landweber* and E. L. Robertson**

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ABSTRACT

It is proven that complexity classes of abstract measures of complexity need not be recursively enumerable. However, the complement of each class is shown to be r.e. The results are extended to complexity classes determined by partial functions, and the properties of these classes are investigated.

Properties of effective enumerations of complexity classes are studied. For each measure another measure with the same complexity classes is constructed such that almost every class admits an effective enumeration of efficient devices.

Finally complexity classes are shown not to be closed under intersection.

INTRODUCTION

The study of abstract complexity measures began (Blum [1]) with general answers, unfortunately largely negative, to questions which had arisen in the study of specific models of computation. Although occasionally too inclusive, the appealing generality of the axiomatic approach has created a field significant in its own right.

One of the most investigated aspects of abstract complexity theory involves classes of functions which may be computed within a given bound on complexity. These complexity classes were initially studied by Hartmanis and Stearns [4] for the number of steps taken by a specific Turing machine model. Almost all the investigations of complexity classes have centered on their order structure under set-theoretic inclusion [1,2,6]. A notable exception is a result of Borodin [2] generalizing one of Hartmanis and Stearns [4] concerning the recursive enumerability of functions in a class. We answer negatively an open question about the total generality of Borodin's result, but give other characterizations for classes which hold for all measures and bounding functions. In particular we show that while complexity classes need not be recursively enumerable, they are, with respect to several definitions, no more complicated than complements of r.e. sets.

Borodin's method for presenting (enumerating indices for) a class of functions, when this is possible, does not enumerate algorithms which are within the desired complexity bound. In Section 3, we show that this must be the case, and further investigate the "quality" of presentations of classes.

A surprising result is that all measures "almost" have "good" presentations.

That is, for any measure of complexity, there is another measure, having almost the same complexity classes such that if the class of functions computable within any complexity g is recursively enumerable, then there is an enumeration of the class in the new measure such that all algorithms enumerated operate within complexity g. The proof involves a detailed consideration of complexity classes of partial functions as discussed in Section 2.

Finally, we disprove a widely held assumption about the closure of the family of classes under intersection.

1. Basic Definitions and Notations

Notation:

$\mathcal{R}$ and $\mathcal{P}$ are the classes of recursive and partial recursive functions respectively. We use f, g, h ... to denote functions known to be total and lower case Greek (α, τ, γ) for arbitrary partial functions (with the exception of β defined below). $\gamma(n)\downarrow(\uparrow)$ means that γ is defined (undefined) at n. If $\alpha, \gamma \in \mathcal{P}$, "$\alpha \leq \gamma$ almost everywhere (a.e.)" means $(\exists x)(\forall y > x)[\alpha(y) \leq \gamma(y) \vee \gamma(y)\uparrow]$ and "$\alpha \leq \gamma$ infinitely often (i.o.)" means $(\forall x)(\exists y > x)[\alpha(y) \leq \gamma(y) \vee \gamma(y)\uparrow]$. "For sufficiently (arbitrarily) large $f \in \mathcal{R}$, A(f)" means $(\exists g \in \mathcal{R})(\forall f \in \mathcal{R})[f \geq g \text{ a.e.} \Longrightarrow A(f)]$ $((\forall g \in \mathcal{R})(\exists f \in \mathcal{R})[f \geq g \text{ a.e. } \& \ A(f)])$. S^c is the complement of S.

A measure of computational complexity is a pair $\langle\varphi, \Phi\rangle$; $\varphi = \{\varphi_i\}$ a Godel numbering of the partial precursive functions (Rogers [9]) and $\Phi = \{\Phi_i\}$ satisfying Blum's [1] axioms:

i) $\Phi_i(n)\downarrow \Longleftrightarrow \varphi_i(n)\downarrow$.

ii) the predicate $\Phi_i(n) = m$ is recursive (decidable).

Definition: For $t \in \mathcal{R}$, let

$$R_t^{\Phi} = \{f \mid f \in \mathcal{R} \wedge (\exists i)[\varphi_i = f \wedge \Phi_i \leq t \text{ a.e.}]\}$$

R_t^{Φ} is the $\langle\varphi, \Phi\rangle$ - computational complexity class of t.

Definition: For $\tau \in \mathcal{P}$, let

$$P_\tau^\Phi = \{\gamma \mid \gamma \in \mathcal{P} \wedge (\exists_i)[\varphi_i = \gamma \wedge \Phi_i \leq \tau \text{ a.e.}] \wedge (\forall x)[\tau(x)\downarrow \Rightarrow \gamma(x)\downarrow]\}$$

P_τ^Φ is the partial $\langle\varphi, \Phi\rangle$ -complexity class of τ.

Fact (Blum [1]). For any measure $\langle\varphi,\Phi\rangle$ there is a unique function $\beta \in \mathcal{R}$ such that, for all i, $\varphi_{\beta(i)} = \Phi_i$. Although this is an exception to the convention above, it will be used throughout (written β^Φ if ambiguities arise).

Definition: B is a presentation of $C \subseteq \mathcal{P}$ if

$$C = \{\varphi_i \mid i \in B\}.$$

$C \subseteq \mathcal{P}$ is recursively enumerable (r.e.) if there is a presentation of it which is r.e. (as a subset of N). C is recursively presentable (r.p.) if it has an r.e. presentation.

Observe that, by padding the indices of an r.e. presentation, a recursive presentation may be obtained which preserves most interesting properties of the r.e. presentation.

Definition: $\langle \Phi, \varphi \rangle$ has the parallel computation property if there is a recursive function $h(x, y)$ such that for any $\varphi_i, \varphi_j, \varphi_{h(i,j)}$ satisfies:

$$\varphi_{h(i,j)}(n) = \begin{cases} \varphi_i(n) & \text{if } \Phi_i(n) \leq \Phi_j(n) \\ \varphi_j(n) & \text{o.w.} \end{cases}$$

$$\Phi_{h(i,j)}(n) = \min(\Phi_i(n), \Phi_j(n))$$

Definition: $\langle \Phi, \varphi \rangle$ is proper if for all total $\varphi_i, \Phi_i \in R^{\Phi}_{\Phi_i}$. $\langle \Phi, \varphi \rangle$ is effectively proper if it is proper and there is a recursive h such that for φ_i total, $\varphi_{h(i)} = \Phi_i$ and $\Phi_{h(i)}(n) \leq \Phi_i(n)$ a.e.n.

Definition: $\{g_i\}$ is class determining for $\langle \varphi, \Phi \rangle$ if:

1. $g_i \in \mathcal{P}$ for all i.
2. $(\forall f \in \mathcal{R})(\exists g_i)\ R^{\Phi}_{g_i} = R^{\Phi}_{f}$.
3. $\{g_i\}$ is r.e.
4. "$g_i(x) = y$" is decidable for all i, x, y,

In [6] it is shown that:

Fact: For any measure $\langle \varphi, \Phi \rangle$, there is a class determining $\{g_i\}$.

2. Recursive Properties of Complexity Classes

Properties of (r.e.) classes (of sets) have been extensively studied (Rice [7,8], Dekker and Myhill [3]). Various authors have investigated the recursive properties of complexity classes (Young [10], Hartmanis and Stearns [4], Borodin [2]). In particular, for any $<\varphi,\Phi>$ and t, R_t^Φ has been shown r.e. if it contains all almost everywhere zero functions or all finite invariants of one of its members. We display a measure containing a non-r.e. complexity class, indicating that some restriction is necessary to insure the recursive enumerability of complexity classes. This result was independently obtained by Forbes Lewis [5]. This measure is proper and has the parallel computation property, both of which have been suggested as candidates for axioms. Consequently, the recursive enumerability of complexity classes is independent of both these properties. (The standard tape measure, where inputs are always scanned, has r.e. complexity classes, the parallel computation property and is proper.) Since it seems reasonable to expect classes to be r.e., the determination of suitable axioms which imply this is an important open problem. The remainder of the section is devoted to proving that the 'recursive complexity' of classes is at most Π_1 (complement of r.e.) and to a discussion of partial classes.

Theorem 2.1. For any r.e. $S \subseteq N$, there is a measure $\langle \varphi, \hat{\Phi} \rangle$ and a recursive t such that

1. $\langle \varphi, \hat{\Phi} \rangle$ is effectively proper.
2. $\langle \varphi, \hat{\Phi} \rangle$ has the parallel computation property.
3. $R_t^{\hat{\Phi}}$ is r.e. iff S^C is r.e.
4. $R_t^{\hat{\Phi}}$ has no infinite r.p. subset iff S^C is immune.*

Proof:

Let $\langle \varphi, \Phi \rangle$ be the standard tape measure for Turing machines. This measure is effectively proper and has the parallel computation property. Furthermore given any φ_i we may effectively find a φ_j such that $\varphi_i = \varphi_j$ and $\Phi_j(n) = \max(\Phi_i(n), 5)$. Let S be any r.e. set, say the domain of φ_z.

We define a new measure $\langle \varphi, \hat{\Phi} \rangle$ which satisfies 1. - 4.

A recursive function g enumerating certain classes of functions is required. $\langle -, - \rangle$ denotes a standard 1-1 pairing function: $N \times N \rightarrow N$.

$$\varphi_{g(i)}(n) = \begin{cases} 2 & \text{if } n = \langle i, 0 \rangle \\ 1 & \text{if } n = \langle i, j \rangle,\ j \geq 1 \qquad \Phi_z(i) \leq n \\ 0 & \text{otherwise} \end{cases}$$

Also let i_0 be an index of the identically zero function. Now define

$$\hat{\Phi}_{i_0}(n) \equiv 0$$

$$\hat{\Phi}_{g(i)}(n) = \varphi_{g(i)}(n)$$

$$\hat{\Phi}_i(n) = \max(\Phi_i(n), 5) \text{ otherwise}$$

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* S is immune if it is infinite but has no infinite r.e. subset.

We now show that $<\varphi, \hat{\Phi}>$ satisfies 1. - 4.

1. Let $\varphi_{h(i)} = \max(5, \Phi_i)$. Define

$$f(i) = \begin{cases} i & \text{if } i \in \{i_0 \cup \text{Rng}(g)\} \\ h(i) & \text{otherwise} \end{cases}$$

(Rng(f) is the range of function f)

Clearly this f is as required for $\hat{\Phi}$ to be effectively proper.

2. Φ has the parallel computation capability say via h, and define, by cases, $\hat{h}(i,j)$ such that $<\varphi, \hat{\Phi}>$ has this capability.

i \ j	$j \notin \text{Rng}(g) \cup \{i_0\}$	$j \in \text{Rng}(g)$	$j = j_0$
$i \notin \text{Rng}(g) \cup \{i_0\}$	$h(i, j)$	j	i_0
$i \in \text{Rng}(g)$	i	$\begin{cases} i_0 & \text{if } i \neq j \\ i & \text{if } i = j \end{cases}$	i_0
$i = i_0$	i_0	i_0	i_0

We leave to the reader the verification that this definition works in all cases.

3.-4. These follow directly from the fact that

$$S^C = \{i \mid (\exists j)[\varphi_j \in R^{\hat{\Phi}}_{\lambda x[0]} \wedge \varphi_j(<i,0>) = 2]\}$$

$$= \{i \mid \varphi_{g(i)} \in R^{\Phi}_{\lambda x[0]}\}$$

Q.E.D.

Notice that it is not necessary in the above construction to begin with the tape measure. Furthermore, any properties which hold for sufficiently large functions (such as density) are true in $<\varphi,\hat{\Phi}>$ iff they are true in $<\varphi,\Phi>$. The construction can be trivially modified to obtain, for any 'honest' t('t(x) = y' recursive), a measure in which R_t^Φ is not r.e.

Corollary 2.2. There is no effective procedure for deciding whether an arbitrary complexity class is r.e.

Corollary 2.3. There exists a measure $<\varphi,\Phi>$ such that every class R_g^Φ is r.p., but there is no effective way of obtaining a presentation given an index for g.

Proof: Let $<\varphi,\Phi>$ be any measure all of whose classes are r.p., and such that $\Phi_i(n) \geq 1$ and for some j, $\Phi_j \equiv 1$. Define f recursive by

$$\varphi_{f(i)}(n) = \begin{cases} 1 & \text{if } \Phi_i(i) > n \\ 0 & \text{if } \Phi_i(i) \leq n \end{cases},$$

so that $R^\Phi_{\varphi_{f(i)}} = \phi$ iff $\varphi_i(i)\downarrow$, and the result follows easily.

This construction can clearly be extended to classes not quite so trivial.

Note that in case R_t^Φ is r.e., it has a recursive presentation. This follows directly from Rogers' proof that all Gödel numberings are 'isomorphic' to the standard numbering and hence permit 'padding' (for any i, a $j > i$, $\varphi_j = \varphi_i$ can be effectively found).

The next two theorems indicate that in some sense Theorem 2.1 gives the best possible characterization of complexity classes.

Theorem 2.4. For any measure $\langle\varphi,\Phi\rangle$ and complexity class R_g^Φ, there is a presentation of R_g^Φ whose complement is r.e.

Proof: Let $B_i (i = 0,1,2,\ldots)$ be a recursive, denumerable set of indices for φ_i such that $i \neq j$ implies $B_i \cap B_j = \phi$ and $\bigcup_i B_i$ is recursive.

Enumerate a set S in stages as follows:

Stage n. Set $a_n \leftarrow 0$. For $0 \leq i \leq n$:

1. if $\Phi_i(a_i) \leq g(a_i)$, set $a_i \leftarrow a_i + 1$
2. if $g(a_i) < \Phi_i(a_i) \leq n$, set $a_i \leftarrow a_i + 1$ and enumerate the next smallest unenumerated member of B_i.
3. if $\Phi_i(a_i) > \max(n, g(a_i))$, enumerate the next smallest unenumerated number of B_i.

Let $E = (\cup B_i)^C \cup S$. E is r.e. Claim E^C is a presentation of R_g^Φ. To see this note that if φ_i is total, then B_i is completely enumerated iff $\Phi_i > g$ i.o.. If φ_i is not total, B_i is completely enumerated.

Notation: W_i is the i^{th} r.e. set, i.e., W_i is the domain of φ_i.

Corollary 2.5. There is an $f \in \mathcal{R}$ such that, if φ_j is total, $(W_{f(j)})^C$ is a presentation of $R_{\varphi_j}^\Phi$.

Theorem 2.6. For any measure $<\varphi,\Phi>$ and complexity class R_g^Φ, $\mathscr{P} - R_g^\Phi$ is r.e.

Proof: Define an effective enumeration $e_1, e_2, \ldots$ of indices such that φ_i total implies there is an e_j, $\varphi_{e_j} = \varphi_i$ iff $\varphi_i \notin R_g^\Phi$. $\varphi_{e_i}(x)$ is defined as follows:

1. If $\varphi_i(x)$ is defined go to 2. else $\varphi_{e_i}(x)$ is undefined.

2. $\varphi_{e_i}(x)$ equals $\varphi_i(x)$ if for each $0 \leq k \leq x$ either a. or b. holds. Otherwise $\varphi_{e_i}(x)$ is undefined.

   a. $(\exists m < x)[(\Phi_k(m) \leq \max(x, g(m)) \wedge \varphi_k(m) \neq \varphi_i(m)) \vee (\Phi_k(m) > \max(x, g(m))]$

   b. $(\exists m \geq x)[(\Phi_k(m) \leq g(m) \wedge \varphi_k(m) \neq \varphi_i(m)) \vee (\Phi_k(m) > g(m))]$

Now if $\varphi_i \in R_g^\Phi$ claim that φ_{e_i} is not total. Let $\varphi_j = \varphi_i$ such that there is an ℓ satisfying: $x > \ell$ implies $\Phi_j(x) \leq g(x)$. Then

$$\varphi_{e_i}(\max(\Phi_j(0), \ldots, \Phi_j(\ell), j, \ell))$$

is undefined so φ_i does not appear among $\{\varphi_{e_j}\}$.

Conversely if φ_i total & $\varphi_i \notin R_g^\Phi$, then one of a. or b. holds for every φ_k, x so $\varphi_i = \varphi_{e_i}$. To prove this, assume there are $\bar{k}$, $\bar{x}$ such that neither a. nor b. holds. Then

$$(\forall m < \bar{x})[(\Phi_{\bar{k}}(m) \leq \max(\bar{x}, g(m)) \Rightarrow \varphi_{\bar{k}}(m) = \varphi_i(m)) \wedge \Phi_{\bar{k}}(m) \leq \max(\bar{x}, g(m))]$$

$$(\forall m \geq \bar{x})[(\Phi_{\bar{k}}(m) \leq g(m) \Rightarrow \varphi_{\bar{k}}(m) = \varphi_i(m)) \wedge \Phi_{\bar{k}}(m) \leq g(m)]$$

Hence $(\forall m)\ \varphi_{\bar{k}}(m) = \varphi_i(m)$ and $(\forall m > \bar{x})\ \Phi_{\bar{k}}(m) \leq g(m)$ so $\varphi_i \in R_g^{\Phi}$.

The proof is completed by enumerating $\mathcal{P} - \mathcal{R}$ together with $\{e_i\}$.

Q.E.D.

<u>Corollary 2.7.</u> There is an $f \in \mathcal{R}$ such that φ_j total $\Rightarrow W_{f(j)}$ is a presentation of $\mathcal{P} - R_{\varphi_j}^{\Phi}$.

Of course $\mathcal{R} - R_t^{\Phi}$ is never r.p., since this would imply $\mathcal{R}$ is r.p. (enumerate $(\mathcal{R} - R_t^{\Phi}) \cup R_{t'}^{\Phi}$, where $R_t^{\Phi} \subset R_{t'}^{\Phi}$ is r.p.). If R is any r.p. class of total functions, we may present $\mathcal{P} - R$ by a construction similar to that above.

It is possible to extend most of the above to partial classes P_τ, and we feel it is valuable to exhibit some of these extensions at this point. Some of the results, for example Theorem 2.10, have been claimed but never proven in the literature. Furthermore, the techniques developed have proven very fruitful and are used later in the paper.

First we illustrate a technique making use of explicit knowledge about when functions are undefined, in a simple extension of a result of Borodin.

Proposition 2.8. For any measure $<\varphi,\Phi>$, there is a function b^{Φ} such that, for all $\psi \in \mathcal{P}$.

$$[\psi = 0 \text{ a.e.}] \Rightarrow (\exists j)[\varphi_j = \psi \wedge \Phi_j \leq b^{\Phi} \text{ a.e.}]$$

Proof: Let $\{d_i\}$ be an effective enumeration of all finite functions. Define h such that

$$\varphi_{h(i)}(x) = \begin{cases} 0 & \text{if } x \notin \text{domain } (d_i) \\ d_i(x) & \text{if } x \in \text{domain } (d_i) \wedge d_i(x) \neq 0 \\ \uparrow & \text{if } x \in \text{domain } (d_i) \wedge d_i(x) = 0 \end{cases}$$

Now $b^{\Phi}(x) = \max(0, \max\{\Phi_{h(j)}(x): \; j \leq x \wedge (d_j(x) \neq 0 \vee x \notin \text{domain } (d_j))\})$.

Corollary 2.9. b^{Φ} is recursive.

Theorem 2.10. For any measure $<\varphi,\Phi>$ and all sufficiently large $\varphi_j \in \mathcal{P}$ ($\varphi_j \geq b^{\Phi}$ a.e.), $P^{\Phi}_{\varphi_j}$ has an r.e. presentation.

Proof: Define f such that f is 1-1 and

$$\varphi_{f(i,j,u,v)}(x) = \begin{cases} \varphi_i(x) & \text{if } A(i,j,u,v,x) \wedge B(i,j,v,x) \\ 0 & \text{otherwise} \end{cases}$$

Where

$$A(i,j,u,v,x) \equiv (\forall x' < x)\left[\Phi_j(x') \le x \Rightarrow \begin{pmatrix} (x' \le u \Rightarrow \Phi_i(x') \le v) \\ \wedge \\ (x' > u \Rightarrow \Phi_i(x') \le \varphi_j(x')) \end{pmatrix}\right]$$

$$B(i,j,v,x) \equiv \Phi_i(x) \le \max(v, \Phi_j(x), \varphi_j(x))^*$$

Claim: 1) $\Phi_i \le \varphi_j$ a.e. $\wedge$ $(\varphi_j \downarrow \Rightarrow \varphi_i \downarrow) \Rightarrow (\exists u,v)[\varphi_{f(i,j,u,v)} = \varphi_j]$

2) $\Phi_i > \varphi_j$ i.o. $\Rightarrow (\forall u,v)(\exists z)(\forall x)[x \ge z \Rightarrow \neg A(i,j,u,v,x)]$

$\Rightarrow (\forall u,v)\,[\varphi_{f(i,j,u,v)} = 0$ a.e.$]$.

3) $(\forall uvx)\ [\varphi_j(x)\downarrow \Rightarrow \varphi_{f(i,j,u,v)}(x)\downarrow]$

Observe that $\varphi_{f(i,j,u,v)}(x)$ is computed first by checking A, then checking B: $\Phi_j(x') \le x$ guarantees that A can be checked. If $\varphi_j(x)$ is defined, the truth or falsity of B will eventually be ascertained. If $\varphi_j(x)$ is undefined, then $\varphi_i(x)\downarrow$ eventually results in an answer to B. The only problem arises if $\varphi_j(x)\uparrow$ and $\varphi_i(x)\uparrow$; but then unless A is false, $\varphi_{f(i,j,u,v)}(x)$ is undefined. But $\varphi_{f(i,j,u,v)}(x)\uparrow$ does not disqualify $\varphi_{f(i,j,u,v)}$ from P_{φ_i}, since we have assumed $\varphi_j(x)\uparrow$.

Assume $\Phi_i \le \varphi_j$ a.e. $(\varphi_j \downarrow \Rightarrow \varphi_i \downarrow)$. Let u be such that $x > u \Rightarrow \Phi_i(x) \le \varphi_j(x)$. Then a v satisfying claim 1. is

$$\max\{\Phi_i(x) \mid x \le u \wedge \varphi_j(x)\downarrow\}.$$

Assume $\Phi_i > \varphi_j$ i.o. Then for any u, there is $z > u$ such that there is an x', $u < x' < z$, for which $\Phi_j(x') < z$ and $\Phi_i(x') > \varphi_j(x')$. This z satisfies claim 2. Claim 3 follows directly from the method described above for computing $\varphi_{f(\,)}$.

*B is defined in this manner for use later.

Hence, $\{f(i,j,u,v) | i,u,v \in N\}$ is an r.e. presentation of $P^{\Phi}_{\varphi_j}$.

Q.E.D.

An analogous result to Theorem 2.4 may be proved for partial classes P^{Φ}_{γ} if the stage construction is done in a "zig-zag" manner, so as to avoid getting stuck if $\gamma(n)$ is undefined. The proof requires more bookkeeping details but involves the same basic approach. By a much more intricate argument than Theorem 2.6, it is possible to present $\mathcal{P} - P^{\Phi}_{\varphi_i}$.

Theorem 2.11. For any measure $<\varphi,\Phi>$ and any $\tau \in \mathcal{P}$; $\mathcal{P} - P^{\Phi}_{\tau}$ has an r.e. presentation.

Proof: The following is a sketch of a stage in the operation of a device which enumerates a presentation of $\mathcal{P} - P^{\Phi}_{\tau}$. Say $\tau = \varphi_j$.

Stage n.

1) If $(\forall x)[x \leq n \Rightarrow \Phi_j(x) > n]$, go to stage $n+1$

2) Enumerate functions diverging at some value where τ converges. This requires listing the domain of τ, which is done in stages corresponding to the stages of the larger device.

3) Enumerate the index of an algorithm which is equal to φ_n if indeed $\varphi_n \in \mathcal{P} - P^{\Phi}_{\tau}$, and which is almost everywhere undefined otherwise. A device for listing the domain of τ appears again, now built into this algorithm.

Observe that the above is effective as in Corollary 2.6, except that the enumerating function is not necessarily total. For example, if φ_i is everywhere undefined, $\mathcal{P} - P_\tau^\Phi = \phi$ and thus the enumerating function cannot be defined at any value.

The last clause of the definition of P_τ^Φ (i.e. $\tau(x)\downarrow \Rightarrow \varphi_i(x)\downarrow$) requires that the domain of each function if P_τ^Φ be at least the domain of τ. Since functions that are undefined at certain places are not necessarily excluded from P_τ^Φ, it is natural to wonder why $\varphi_i \varepsilon P_\tau^\Phi$ may not diverge at a finite number of values where τ converges. First of all, under the present definition $R_t^\Phi = P_t^\Phi$ if t is total, which would not be the case otherwise. But more significantly, the analogue to Corollary 2.7 does not hold for the weakened definition of a partial class, although Theorem 2.6 may be shown to hold non-effectively. An effective procedure yielding devices to enumerate the less restrictive partial classes would yield an effective enumeration of $\{i \mid \text{domain } \varphi_i \text{ finite}\}$ which is known not to be r.e.

3. Quality of Presentations

Given that a complexity class is r.e., one may then ask questions regarding the complexity of the devices enumerated. For example, if R_g^{Φ} is r.e., is there an enumeration $e_0, e_1, \ldots$ of it such that for all j, $\varphi_{e_j} \leq g$ a.e.? We believe that such questions pertaining to the quality of presentations are for more important than that of the existence of r.e. presentations, which in any case is only relevant to classes at the lower end of the hierarchy for somewhat artificial measures. The following results provide a rather complete characterization of measures with respect to the complexity of enumerations of their classes.

Definition. R_g^{Φ} is h-presentable if it has an r.e. presentation S satisfying

$$i \in S \Rightarrow \Phi_i \leq h \text{ a.e.}$$

Such a presentation is said to be h-bounded.

Perhaps the ideal result would be a theorem stating that every R_g^{Φ} has a g-bounded presentation. Unfortunately this is not the case, although a weaker version does hold. This theorem was stated in [6]. Their proof, which is valid only for the tape measure, is given below for Proposition 3.6.

Theorem 3.1. For any $\langle\varphi,\Phi\rangle$, there is a recursive h such that for sufficiently large $g(x)$, R_g^{Φ} is hg-presentable.

Proof. Let $f(i,j,u,v)$ be as in Theorem 2.10. Then

$$\{f(i,j,u,v) \mid i,v,u \geq 0\}$$

is an r.e. presentation of $R^{\Phi}_{\varphi_j}$ if φ_j is total and $R^{\Phi}_{\varphi_j}$ contains all almost everywhere zero functions. Now let

$$h(y,x) = \max\{\Phi_{f(i,j,u,v)}(x) \mid i,j,u,v \leq x \wedge (\Phi_i(x) \leq y \vee$$
$$\neg A(i,j,u,v,x))\}$$

where

$$A(i,j,u,v,x) \equiv (\forall w < x)[\Phi_j(w) \leq x \Rightarrow (w \leq u \wedge \Phi_i(w) \leq v \vee$$
$$w > u \wedge \Phi_i(w) \leq \varphi_j(w))] .$$

First observe that h is recursive. This is true because: $\Phi_{f(i,j,u,v)}(x)$ figures in the computation of $h(y,x)$ only if either $\Phi_i(x) \leq y$ (so $\Phi_i(x) \downarrow$ and hence $\Phi_{f(i,j,u,v)}(x) \downarrow$) or $\neg A$ in which case $\varphi_{f(i,j,u,v)}(x)$ is 0 (so $\Phi_{f(i,j,u,v)}(x) \downarrow$).

Now if φ_j is total, $\Phi_{f(i,j,u,v)}(x) \leq h(\varphi_j(x),x)$ almost everywhere because $\Phi_{f(i,j,u,v)}(x)$ is included in the definition of $h(\varphi_j(x),x)$ for $x > \max(i,j,u,v)$. The proof is completed by requiring $\varphi_j(x) \geq x$ a.e.

Q.E.D.

The proof can be generalized to yield

Corollary 3.2. For any $<\varphi,\Phi>$, there is a recursive h such that for sufficiently large p.r. $\tau(x)$, P^{Φ}_{τ} is $h\tau$-presentable (with obvious generalization of hg-presentable to allow p.r. g).

The next theorem shows that Theorem 3.1 can not be strengthened to provide a g-bounded presentation for sufficiently large R_g^Φ. The proof provides an interesting application of the method used to prove Theorem 2.1.

Theorem 3.3. There is a measure $\langle\varphi^*,\Phi^*\rangle$ such that for arbitrarily large g, R_g^Φ is not g-presentable.

Proof: Let $\langle\varphi,\Phi\rangle$ be a measure. Let $\{g_i\}$ be a class determining set for $\langle\varphi,\Phi\rangle$. For each g_i obtain a recursive set of indices $\{e_0^i, e_1^i, e_2^i, \dots\}$ satisfying

1. $(\forall k)(\text{domain } g_i = \text{domain } \varphi_{e_k^i})$
2. $(\forall j,k)(\varphi_j = \varphi_{e_k^i} \Rightarrow \Phi_j > g_i \text{ a.e.})$
3. $(\forall k,\ell)(k \neq \ell \text{ and } g_i(0)\downarrow \Rightarrow \varphi_{e_k^i}(0) \neq \varphi_{e_\ell^i}(0))$.

Moreover the sets are chosen so that

4. $\bigcup_i \{e_0^i, e_1^i, \dots\}$ is recursive

and 5. $\varphi_{e_k^i}(0) = \varphi_{e_\ell^j}(0) \iff i = j \text{ and } k = \ell \text{ or } \varphi_{e_k^i}(0)\uparrow \text{ and } \varphi_{e_\ell^j}(0)\uparrow$.

The sets of indices can be effectively chosen using a trivial variation of the method of Blum [proof of Theorem 7 [1]]. Note that 5. can be satisfied by requiring $\varphi_{e_k^i}(0)$ to be a different power of the i^{th} prime for all k, in case $g_i(0)\downarrow$.

Define $<\varphi^*, \Phi^*>$ by

$$\varphi_i^* = \varphi_i \quad \text{for all} \quad i$$

$$\Phi^*_{e_k^i}(x) = \begin{cases} g_i(x) & \text{if } \Phi_k(k) > x \\ g_i(x) + 1 & \text{if } \Phi_k(k) \le x \end{cases}$$

$$\Phi_j^* = \Phi_j \quad \text{for } j \notin \{e_k^i\} .$$

$<\varphi^*, \Phi^*>$ is a measure because $\{g_i\}$ is a measured set, g_i and $\varphi_{e_k^i}$ have the same domain for all i and k and, $\{e_k^i\}$ is recursive.

For g_i total $\varphi_{e_k^i} \in R_{g_i}^{\Phi^*}$ iff $\varphi_k(k)\uparrow$. If g_i is large enough so that $R_{g_i}^{\Phi^*}$ contains all almost everywhere zero functions, then $R_{g_i}^{\Phi^*}$ has an r.e. presentation S. But S cannot be g_i-bounded because it would then have to contain $\{e_k^i \mid \varphi_k(k)\uparrow\}$ since no other definition of such a $\varphi_{e_k^i}$ can have a measure less than or equal to g_i a.e. . Hence S r.e. would imply $\{k \mid \varphi_k(k)\uparrow\}$ r.e.

The proof is completed by observing that $\{g_i\}$ contains arbitrarily large total functions and that $\varphi_{e_k^i}(0) = \varphi_{e_\ell^j}(0)$ if and only if $i = j$, $k = \ell$ and $g_i(0)\downarrow$ so that at most one measure of any total function is changed.

Q.E.D.

The proof immediately yields

Corollary 3.4. Let $<\varphi^*, \Phi^*>$ be as above. Then for arbitrarily large g_i, if $R_f^{\Phi^*} = R_{g_i}^{\Phi^*}$, $R_f^{\Phi^*}$ is not f-presentable.

Proof. If $R_f^{\Phi^*} = R_{g_i}^{\Phi^*}$ is to be f-presentable, then $f > g_i$ a.e. since all other definitions of functions in $\{\varphi_{e_k^i} \mid \varphi_k(k) \uparrow\}$ have measures greater than g_i a.e.

But then $\{\varphi_{e_k^i}\}$ class is a subset of this, so that functions will be included which are not in $R_{g_i}^{\Phi^*}$.

Q.E.D.

On the other hand there is no hope that the previous construction can be extended to obtain a measure containing no g such that R_g^{Φ} is g-presentable. The proof of this uses the Union Theorem of [6]. A non-constructive proof using properties of ordinals can also be given.

Theorem 3.5. For any measure $<\varphi, \Phi>$, there are arbitrarily large functions g such that R_g^{Φ} is g-presentable.

Proof. Let $<\varphi, \Phi>$ be a measure and g_0 a recursive function such that R_{g_0} contains all almost everywhere zero functions. Let S_0 be an r.e. presentation of R_{g_0}. Assume g_i and S_i have been defined and satisfy

1) $(\forall x)\ g_i(x) > g_{i-1}(x)$;

2) S_i is an r.e. presentation of R_{g_i} ;

3) $S_i \supseteq S_{i-1}$; and

4) $(\forall j \in S_{i-1})(\Phi_j \leq g_i \text{ a.e.})$.

Define g_{i+1}, S_{i+1} from g_i, S_i so that 1) - 4) are satisfied.

The procedure can be effectively implemented. Given an index of a sufficiently large recursive function g_i, the method of Borodin yields an index of a recursive function h whose range S_i is a presentation of g_i. A slight modification yields a presentation containing S_{i-1}. g_{i+1} is given by

$$g_{i+1}(x) = \max(\{\Phi_{h(j)}(x) \mid j \le x\}, g_i(x)) + 1 .$$

$\{g_i\}$ is an r.e. self bounded set of functions so by the Union Theorem [6], there is a g such that

$$R_g = \bigcup_i R_{g_i} .$$

Claim that $S = \bigcup_i S_i$ is a g-bounded r.e. presentation of R_g.

If $e \in S$, then $e \in S_j$ for some j so $\Phi_e < g_{j+1}$ a.e. From the proof of the Union Theorem it follows that $\Phi_e < g$ a.e. so S is g-bounded. S is a presentation of R_g because each S_i is a presentation of R_{g_i} and $R_g = \bigcup_i R_{g_i}$.

Q.E.D.

After the above it is natural to ask if the construction may be revised to find for any $\langle\varphi,\Phi\rangle$, arbitrarily large g such that R_g^Φ is not g-presentable. A straightforward proof shows that a strong result to the contrary holds for the standard tape measure, presuming that the input is always read. The proof is essentially that given in [6] of Theorem 3.8.

Proposition 3.6. If $<\varphi, \Phi>$ is the standard tape measure, $(\forall i, x)(\Phi_i(x) \geq x)$, then for all $g \in \mathcal{R}$, R_g^{Φ} is g – presentable.

Proof. Let $g \in \mathcal{R}$ and assume $g(x) \geq x$. We simply describe a Turing machine algorithm for computing $\varphi_{h(i,u)}$ such that, as is usual in these arguments, the double enumeration of $\{h(i,u)\}$ is a g-bounded presentation of R_g^{Φ}.

Evaluation of $\varphi_{h(i,u)}(x)$:

1. In a finite control, determine if $\Phi_i(x') \leq \max(v, g(x'))$ for $x' \leq u$. If this fails output 0 and halt.

2. If $x \leq u$, output $\varphi_i(x)$ and halt, otherwise let the "available space" be the length of the input x.

3. For each x', $u < x' < x$: If it is possible to compute $g(x')$ within the available space, try to compute $\varphi_i(x')$ within $g(x')$ squares and if this fails halt with output 0.

4. Compute $\varphi_i(x)$ until either;

   4.1. The computation halts, then output $\varphi_i(x)$ and halt.

or 4.2. More than the available space is required, then go to 5.

or 4.3. An infinite loop using only the available space is detected, then halt with output 0.

5. Compute $g(x)$ until either;

   5.1. The computation halts, then halt with 0 output.

or 5.2. More than the available space is required, then increase the available space by one square and return to 4.

The reader can easily verify that this procedure takes less than $g(x)$ squares almost everywhere, and that if $\varphi_i(x)$ takes less than $g(x)$ squares for all $x > u$, then $\varphi_{h(i,u)} = \varphi_i$. (If $\varphi_i(x)$ takes more than $g(x)$ squares for some $x > u$, then eventually 3. results in $\varphi_{h(i,u)}$ being almost everywhere zero.)

But a far stronger result is possible, holding that every measure may be slightly modified, so that the above result holds.

The proof of Theorem 3.6 involves a rather intricate consideration of a method for enumerating partial classes (complexity classes of partial recursive functions) since it is not possible to effectively identify the total functions.

Theorem 3.7. For any measure $\langle\varphi,\Phi\rangle$, there is a measure $\langle\hat{\varphi},\hat{\Phi}\rangle$, which can be obtained effectively from $\langle\varphi,\Phi\rangle$, with the same complexity classes such that for all sufficiently large $g \in \mathcal{R}$, $R_g^{\hat{\Phi}}$ is g-presentable.

Proof: Recall the technique used in the proof of Theorem 2.8 where $f(i,j,u,v)$ satisfies

$$\{f(i,j,u,v) \mid i,u,v \geq 0\}$$

is an r.e. presentation of $P_{\varphi_j}^{\Phi}$ for sufficiently large φ_j. In particular

$$\varphi_{f(i,j,u,v)}(x) = \begin{cases} \varphi_i(x) \text{ if } A(i,j,u,v,x) \wedge B(i,j,v,x) \\ 0 \qquad \text{otherwise} \end{cases}$$

where $A \equiv (\forall x' < x)\ [(\Phi_j(x') \leq x) \Longrightarrow ((x' \leq u \Longrightarrow \Phi_i(x') \leq v)$

$\wedge (x' > u \Longrightarrow \Phi_i(x') \leq \varphi_j(x'))]$

$B \equiv \Phi_i(x) \leq \max(v, \Phi_j(x), \varphi_j(x))$

As before note that in computing $\varphi_{f(i,j,u,v)}(x)$, if no answer is received for $A \wedge B$, then it must be the case that $\varphi_i(x)$ and $\varphi_j(x)$ are both undefined so $\varphi_{f(i,j,u,v)}(x) \uparrow$.

Define f' similarly

$$\varphi_{f'(i,j,u,v)}(x) = \begin{cases} \text{i)} \ \Phi_i(x) \qquad \text{if } A \wedge B \\ \text{ii)} \ \max(v, \Phi_j(x), \varphi_j(x)) \ \text{ if } A \wedge (\neg B) \\ \text{iii)} \ b(x) \qquad \text{if } \neg A \end{cases}$$

where b is as in Proposition 2.8. $\varphi_{f'(i,j,u,v)}(x)$ is undefined only in case both $\varphi_i(x)$ and $\varphi_j(x)$ are undefined since then no answer is received.

At last we are able to define a new measure $\langle \hat{\varphi}, \hat{\Phi} \rangle$

$$\hat{\varphi}_k = \varphi_{f(\pi_1^4(k), \pi_2^4(k)\pi_3^4(k)\pi_4^4(k))}$$

$$\hat{\Phi}_k = \varphi_{f'(\pi_1^4(k), \pi_2^4(k), \pi_3^4(k), \pi_4^4(k))}$$

$(\pi_\ell^4(\langle x_1, \ldots, x_4 \rangle) = x_\ell$, π_ℓ^4 recursive 1-1, onto).

Claim $<\hat{\varphi},\hat{\Phi}>$ is a measure. First the enumeration $\{\varphi_k\}$ is clearly effective [Rogers 9] since

$$\varphi_i = \hat{\varphi}_{<i,\beta(i),0,0>}$$

$(\Phi_i = \varphi_{\beta(i)})$ and f provide the required mappings. The close similarity between the definitions of f and f' make it bvious that $\hat{\varphi}_k(x) \downarrow$ if and only if $\Phi_k(x) \downarrow$ (ii) in the definition of f' only can occur if $7B$ so that $\max(v,\Phi_j(x))$ exists.).

We show how to decide whether

$$"\hat{\Phi}_{<i,j,u,v>}(x) = y".$$

```
if 7A(i,j,v,x) then if y = b(x) then T else F
                else begin
                     if y < max(v,Φj(x), φj(x)) then
                           if y = Φi(x) then T else F ;
                     if y = max(v,Φj(x),φj(x)) then
                           if y ≤ Φi(x) then T else F ;
                     if y > max(v,Φj(x),φj(x)) then F
                     end:
```

This algorithm is effective since testing A is a finite process, b is recursive, $y \leq \Phi_i(x)$ and $y \leq \max(v,\Phi_j(x),\varphi_j(x))$ are effectively decidable.

The algorithm is correct because of the following facts which the reader may easily verify.

a) $\neg A(i,j,u,v,x) \Rightarrow \hat{\Phi}_{<i,j,u,v>}(x) = b(x)$

b) $A \wedge \hat{\Phi}_{<>}(x) < \max(\ ) \Rightarrow (B \Rightarrow \hat{\Phi}_{<>}(x) = \Phi_i(x))$

c) $A \wedge \hat{\Phi}_{<>}(x) = \max(\ ) \Rightarrow [B \wedge \hat{\Phi}_{<>}(x) = \Phi_i(x) \vee$

$\neg B \wedge (\hat{\Phi}_{<>}(x) = \max(\ ) > \Phi_i(x))]$

d) $\neg(A \wedge (\hat{\Phi}_{<>}(x) > \max(\ )))$

a) - d) verify lines 1, 3-4, 5-6 and 7 respectively.

The proof is completed by showing that for all j $P^{\Phi}_{\varphi_j} = P^{\hat{\Phi}}_{\varphi_j}$ and for $\varphi_j \geq b$ a.e. $\{<i,j,u,v>\}$ is a φ_j-bounded presentation of $P^{\Phi}_{\varphi_j}$.

1. Assume $\varphi_i \in P^{\Phi}_{\varphi_j}$. Then there is a k, $\varphi_k = \varphi_i$, $\Phi_k \leq \varphi_j$ a.e. and $\varphi_j(x)\downarrow \Rightarrow \varphi_k(x)\downarrow$. Let u,v be such for $x > u$, $\Phi_k(x) \leq \varphi_j(x)$ and for $x \leq u$, $\varphi_k(x)\downarrow \Rightarrow \Phi_k(x) \leq v$. Then $\hat{\varphi}_{<k,j,u,v>} = \varphi_k$ and $\Phi_k = \hat{\Phi}_{<k,j,u,v>} \leq \varphi_j$ a.e. Hence $\varphi_i \in P^{\hat{\Phi}}_{\varphi_j}$.

2. Assume $\hat{\varphi}_{<i,j,u,v>} \in P^{\hat{\Phi}}_{\varphi_j}$. Without loss of generality assume that $\hat{\Phi}_{<i,j,u,v>} \leq \varphi_j$ a.e. There are two cases. First let $\varphi_j \geq b$ a.e. Then either a) $\hat{\varphi}_{<>} = \varphi_i$ and $\hat{\Phi}_{<>} = \Phi_i$, so $\Phi_i \leq \varphi_j$ a.e. and $\hat{\varphi}_{<>} \in P^{\Phi}_{\varphi_j}$ or b) $\hat{\varphi}_{<>} = 0$ a.e. so $\hat{\varphi}_{<>} \in P^{\Phi}_{\varphi_j}$ because it contains all almost everywhere zero functions whose domain includes that of φ_j (see definition of b). The second case occurs if $\varphi_j < b$ i.o. Then $\hat{\Phi}_{<>} < b$ i.o. so $\varphi_{<>} = \varphi_i$ and $\Phi_{<>} = \Phi_i$. But then $\Phi_i \leq \varphi_j$ a.e. and $\hat{\varphi}_{<>} \in P^{\hat{\Phi}}_{\varphi_j}$.

1. and 2. imply $P^{\Phi}_{\varphi_j} = P^{\hat{\Phi}}_{\varphi_j}$.

3. By Theorem 2.10, $\{f(i,j,u,v)\}$ is an r.e. presentation of $P^{\Phi}_{\varphi_j}$ if $\varphi_j \geq b$ a.e. But for all i,j,u,v, $\hat{\varphi}_{<i,j,u,v>} = \varphi_{f(i,j,u,v)}$ so 1. and 2. imply $\{<i,j,u,v>\}$ is an r.e. presentation of $P^{\hat{\Phi}}_{\varphi_j}$. It is easily to see that it is also φ_j-bounded.

Q.E.D.

The proof of Theorem 3.7 also yields

Corollary 3.8. For any measure $<\varphi,\Phi>$, there is a measure $<\hat{\varphi},\hat{\Phi}>$, with the same complexity classes, except that each class contains all finite invariants of $\lambda x[0]$ such that, for all $g \in \mathcal{R}$, $R^{\hat{\Phi}}_g$ is g-presentable.

Proof: Change all occurrences of b(x) in the above to 0 .

Corollary 3.9. For any measure $<\varphi,\Phi>$, there is a measure $<\hat{\varphi},\hat{\Phi}>$ such that for all j , $P^{\Phi}_{\varphi_j} = P^{\hat{\Phi}}_{\varphi_j}$ and for $\varphi_j \geq b$ a.e., $P^{\hat{\Phi}}_{\varphi_j}$ is φ_j-presentable.

4. Closure Properties of Complexity Classes

Complexity classes are not closed under complementation. Indeed $\mathcal{R} - R_t^{\Phi}$ can never be a class. It is also easy to show that classes are not necessarily closed under finite unions. However, to date, the situation with respect to intersection has been generally misstated. While restrictions such as parallel computation imply closure under intersection the next theorem proves that this is not a measure theoretic property.*

Theorem 4.1. There is a measure $\hat{\Phi}$ such that for arbitrarily large g_1, g_2 $R_{g_1}^{\hat{\Phi}} \cap R_{g_2}^{\hat{\Phi}}$ is not a complexity class.

Proof: Let $<\varphi, \Phi>$ be any measure. Let $\{t\}$ be a class determining set for $<\varphi, \Phi>$ and choose an increasing recursive h satisfying

1) $R_t^{\Phi} \subsetneq R_{ht}^{\Phi}$ for all $t \in \{t\}$; 2) for each $t \in \{t\}$, there is a

*We are indebted to A. Borodin for the suggestion that our original counterexample was valid for arbitrarily large t.

recursive $\varphi_i \in R^{\Phi}_{ht} - R^{\Phi}_{t}$ for which $\Phi_i \leq ht$ a.e. and for all $\varphi_k = \varphi_i$ $\Phi_k > t$ a.e. The existence of h is given by the compression theorem [1].

For notational convenience let

$$F_e(f)(n) = \begin{cases} h\,f(n) & n \text{ even} \\ f(n) & m \text{ odd} \end{cases}$$

$$F_o(f)(n) = \begin{cases} f(n) & n \text{ even} \\ h\,f(n) & n \text{ odd} \end{cases}$$

Define a new acceptable enumeration of the p.r. functions by

$$\varphi_j(0) = \begin{cases} 3\varphi_{[j/4]}(0) & j \equiv 0 (\text{mod } 4) \text{ or } j \equiv 1 (\text{mod } 4) \\ 3\varphi_{[j/4]}(0) + 1 & j \equiv 2 (\text{mod } 4) \\ 3\varphi_{[j/4]}(0) + 2 & j \equiv 3 (\text{mod } 4) \end{cases}$$

$\hat{\varphi}_j(n) = \varphi_{[j/4]}(n)$ for $n > 0$, $j = 0, 1, 2, \ldots$, ([a/b] indicates integer division.).

$\hat{\Phi}$ is given by

$$\hat{\Phi}_{4j} = \hat{\Phi}_{4j+2} = F_e(\Phi_j)$$

$$\hat{\Phi}_{4j+1} = \hat{\Phi}_{4j+3} = F_o(\Phi_j).$$

Now for any recursive g there is a $t > g$, $t \in \{t\}$. We prove that $R^{\hat{\Phi}}_{F_e(ht)} \cap R^{\hat{\Phi}}_{F_o(ht)}$ is not a complexity class of $\langle \hat{\varphi}, \hat{\Phi} \rangle$.

Let $\varphi_i \in R^{\Phi}_{ht} - R^{\Phi}_{t}$, satisfy $\Phi_i \le ht$ a.e., for all k, $\varphi_k = \varphi_i$ implies $\Phi_k > t$ a.e.

I. $\hat{\varphi}_{4i} = \hat{\varphi}_{4i+1} \in R^{\hat{\Phi}}_{F_e(ht)} \cap R^{\hat{\Phi}}_{F_o(ht)}$.

II. a. $\hat{\varphi}_{4i+2} \notin R^{\hat{\Phi}}_{F_o(ht)}$ b. $\hat{\varphi}_{4i+3} \notin R^{\hat{\Phi}}_{F_e(ht)}$

III. $S = R^{\hat{\Phi}}_{F_o(ht)} \cap R^{\hat{\Phi}}_{F_e(ht)}$ is not a complexity class of $\langle\hat{\varphi}, \hat{\Phi}\rangle$.

<u>Proof of III</u>.

Assume $S = R^{\hat{\Phi}}_{g}$. Because by II, $\hat{\varphi}_{4i+2}, \hat{\varphi}_{4i+3} \notin S$ we have for any $j \ni \hat{\varphi}_{4j+2} = \hat{\varphi}_{4i+2}$ or $\hat{\varphi}_{4j+3} = \hat{\varphi}_{4i+3}$

$$\hat{\Phi}_{4j+2}, \hat{\Phi}_{4j+3} > g \text{ i.o.}$$

But $\hat{\varphi}_{4j+3} = \hat{\varphi}_{4i+3}$ iff $\hat{\varphi}_{4j+2} = \hat{\varphi}_{4i+2}$ iff $\hat{\varphi}_{4j} = \hat{\varphi}_{4i} = \hat{\varphi}_{4i+1} = \hat{\varphi}_{4j+1}$. Also for all j $\hat{\Phi}_{4j} = \hat{\Phi}_{4j+2}$, $\hat{\Phi}_{4j+1} = \hat{\Phi}_{4j+3}$. Hence for any $\hat{\varphi}_{4j} = \hat{\varphi}_{4j+1} = \hat{\varphi}_{4i+1} = \hat{\varphi}_{4i}$ we have

$$\hat{\Phi}_{4j}, \hat{\Phi}_{4j+1} > g \text{ i.o.}$$

so $\varphi_{4i} \notin R^{\hat{\Phi}}_{g}$ contradicting I.

Q.E.D.

5. CONCLUSION

One of the major problems facing workers in axiomatic complexity theory is to determine axioms which, together with Blum's axioms, restrict the class of measures to the standard examples (time, tape, reversals, etc.). Of the properties considered in this paper, all except g-presentability of classes hold for suitable moficiations of the standard measures (i.e., strong properness, closure under intersection, parallel computation, recursive enumerability of complexity classes.). The situation with respect to g-presentability is not yet clear. We do not know whether the proof for the tape measure (Proposition 3.6) can be modified to work for all standard measures.

None of the above properties is measure theoretic (true of all measures). Those axioms which are eventually accepted should probably imply these properties (except possibly g-presentability) as well as some important deep characteristics of the standard measures.

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