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RECURSIVE PROPERTIES OF ISOMORPHISM TYPES

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Abstract

For Γ a recursively enumerable set of formulae, a structure $\mathfrak{A}$ on a recursive universe is said to be "Γ-recursively enumerable" if the satisfaction predicate restricted to Γ is recursively enumerable (equivalently, if the formulae of Γ uniformly denote recursively enumerable relations on $\mathfrak{A}$).

For recursively enumerable sets $\Gamma_1 \subseteq \Gamma_2$ of formulae we shall, under certain conditions, characterize structures $\mathfrak{A}$ with the following properties.

1) Every isomorphism from $\mathfrak{A}$ to a Γ_1-recursively enumerable structure is a recursive isomorphism.

2) Every Γ_1-recursively enumerable structure isomorphic to $\mathfrak{A}$ is recursively isomorphic to $\mathfrak{A}$.

3) Every Γ_1-recursively enumerable structure isomorphic to $\mathfrak{A}$ is Γ_2-recursively enumerable.

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0. Introduction

A structure on a recursive universe is said to be *decidable* if the satisfaction predicate is recursive (equivalently, if all formulae uniformly denote recursive relations). It is said to be *recursive* (respectively *recursively enumerable*) if the satisfaction predicate restricted to atomic formulae is recursive (respectively, recursively enumerable).

Structures with certain recursive properties have been characterized algebraically. Some of these characterizations have been proved using very similar finite injury priority constructions. This paper presents the basic technique of these constructions. The concepts of recursive and decidable structures are generalized to "Γ-recursively enumerable" structures and the results presented in this context. Some of the results generalized are the characterizations of structures with the following recursive properties. (The results appear in this paper as the corollaries numbered.)

(Cor. 3.1) Every isomorphism from $\mathfrak{A}$ to a recursive structure is a recursive isomorphism (Ash and Nerode [1], Goncharov [3]).

(Cor 2.2) Every recursive structure isomorphic to $\mathfrak{A}$ is recursively isomorphic to $\mathfrak{A}$; that is, $\mathfrak{A}$ is "recursively categorical" (Goncharov [3]).

(Cor. 2.1) Every decidable structure isomorphic to $\mathfrak{A}$ is recursively isomorphic to $\mathfrak{A}$; that is, $\mathfrak{A}$ is "decidably categorical" (Nurtazin [6]).

(Cor. 1.2) Every recursive structure isomorphic to $\mathfrak{A}$ is a decidable structure (Nurtazin [6]).

(Cor. 1.1) Every isomorphism from $\mathfrak{A}$ to a recursive structure carries R (a relation on $\mathfrak{A}$) to a recursively enumerable relation; that is, R is "intrinsically recursively enumerable" on $\mathfrak{A}$ (Ash and Nerode [1]).

Our general results have as corollaries some results new to the literature (for example Corollaries 1.4 and 2.3).

We consider only structures in an effective language $\mathfrak{L}$ and with recursive universes. Without loss of generality we take the non-logical symbols of $\mathfrak{L}$ to be a list $\{P_i\colon i < \omega\}$ of predicates. We write $\mathfrak{A}$, $\mathfrak{B}$ for structures on universes A, B respectively. Some model-theoretic terminology used may need to be defined. *Form*($\mathfrak{L}$) is the set of all $\mathfrak{L}$-formulae. We denote a finite sequence $a_{i_0}, \ldots, a_{i_n}$ of elements of A by $\bar{a} \subseteq A$. The sequence $\bar{x}$ of variables corresponding to $\bar{a}$ is $\bar{x} = x_{i_0}, \ldots, x_{i_n}$. If f is a function from A to B we write $f(\bar{a})$ for the sequence $f(a_{i_0}), \ldots, f(a_{i_n})$. $f|_M$ is the restriction of f to M.

For $\Gamma \subseteq \text{Form}(\mathfrak{L})$ we define:

$\neg\Gamma = \{\neg\phi(\bar{x})\colon \phi(\bar{x}) \in \Gamma\}$,

$\bigwedge\Gamma = \{\phi_1(\bar{x}_1) \wedge \cdots \wedge \phi_n(\bar{x}_n)\colon \phi_i(\bar{x}_i) \in \Gamma\}$, that is, all finite conjunctions,

$\exists\Gamma = \{\exists\bar{x}\phi(\bar{x}, \bar{y})\colon \phi(\bar{x}, \bar{y}) \in \Gamma\}$, and

$\forall\Gamma = \{\forall\bar{x}\phi(\bar{x}, \bar{y})\colon \phi(\bar{x}, \bar{y}) \in \Gamma\}$.

We say $\psi(\bar{a}, \bar{y})$ is *an atom of the Lindenbaum algebra* $B(\text{Th}\langle\mathfrak{A}, \bar{a}\rangle)$ if for every formula $\phi(\bar{a}, \bar{y})$ if $\mathfrak{A} \models \exists\bar{y}(\psi(\bar{a}, \bar{y}) \wedge \phi(\bar{a}, \bar{y}))$ then $\mathfrak{A} \models \forall\bar{y}(\psi(\bar{a}, \bar{y}) \to \phi(\bar{a}, \bar{y}))$. An equivalent definition is: if $\bar{a}_1, \bar{a}_2 \subseteq A$ are such that $\mathfrak{A} \models \psi(\bar{a}, \bar{a}_1)$ and $\mathfrak{A} \models \psi(\bar{a}, \bar{a}_2)$, then there is an automorphism $f\colon \mathfrak{A} \cong \mathfrak{A}$ such that $f(\bar{a}) = \bar{a}$ and $f(\bar{a}_1) = \bar{a}_2$.

$\{\ \}$ is the empty set.

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Let Γ be a recursively enumerable (r.e.) subset of Form($\mathfrak{L}$). An $\mathfrak{L}$-structure $\mathfrak{A}$ is Γ-*recursively enumerable* (Γ-r.e.) if it has a recursive universe and the satisfaction predicate restricted to Γ is recursively enumerable. The following are equivalent formulations of the definition.

i) The set of sentences $\{\phi(\bar{a}): \phi(\bar{x}) \in \Gamma, \bar{a} \subseteq A$ and $\mathfrak{A} \models \phi(\bar{a})\}$ is r.e.

ii) There is a partial effective procedure which when applied to any $\bar{a} \subseteq A$ and $\phi(\bar{x}) \in \Gamma$ terminates if and only if $\mathfrak{A} \models \phi(\bar{a})$.

Listed below are some examples of Γ-r.e. structures.

1) A structure $\mathfrak{A}$ with an r.e. relation R. $\mathfrak{A}$ is Γ-r.e. for $\Gamma = \{R\}$.

2) An r.e. structure $\mathfrak{A}$. $\mathfrak{A}$ is Γ-r.e. for $\Gamma = \{P_i: i < \omega\}$.

3) A recursive (rec.) structure $\mathfrak{A}$. $\mathfrak{A}$ is Γ-r.e. for $\Gamma = \{P_i: i < \omega\} \cup \{\neg P_i: i < \omega\}$.

4) A decidable (dec.) structure $\mathfrak{A}$. $\mathfrak{A}$ is Γ-r.e. for $\Gamma = $ Form($\mathfrak{L}$).

Notice that if $\mathfrak{A}$ is Γ-r.e. then it is also $\exists \wedge \Gamma$-r.e.

Let $\Gamma_1 \subseteq \Gamma_2$ be r.e. sets of $\mathfrak{L}$-formulae. Our first theorem characterizes structures $\mathfrak{A}$ with the property that every Γ_1-r.e. structure isomorphic to $\mathfrak{A}$ is Γ_2-r.e. The characterization is, however, subject to a certain decidability assumption.

For $\Sigma_1, \Sigma_2 \subseteq$ Form($\mathfrak{L}$) we say $\mathfrak{A}$ is $\Sigma_1 \to \Sigma_2$-*decidable* if there is an effective procedure which when applied to any $\bar{a} \subseteq A$, $\psi(\bar{x}, \bar{y}) \in \Sigma_1$ and $\phi(\bar{x}, \bar{y}) \in \Sigma_2$ determines whether or not $\mathfrak{A} \models \forall\bar{y}(\psi(\bar{a}, \bar{y}) \to \phi(\bar{a}, \bar{y}))$. Taking $\bar{y}$ to be the empty sequence and ϕ any formula such that $\mathfrak{A} \models \neg\phi(\bar{a})$ we see that if $\mathfrak{A}$ is $\Sigma_1 \to \Sigma_2$-dec. it is Σ_1-rec. (that is, $\Sigma_1 \cup \neg\Sigma_1$-r.e.). Similarly, taking ψ to be any formula such that $\mathfrak{A} \models \psi(\bar{a})$, we see that $\mathfrak{A}$ is Σ_2-rec. We may concern ourselves only with sets Σ_1 and Σ_2 for which such formulae ψ and ϕ exist because the results in which we use this definition (Theorems I, II and III) are otherwise trivially true.

THEOREM I. *Let* Γ_1, Γ_2 *be r.e. sets with* $\{P_i: i < \omega\} \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq$ Form($\mathfrak{L}$) *and* $\exists \wedge \Gamma_1$ *rec. in* Γ_2. *If* $\mathfrak{A}$ *is* $(\exists \wedge \Gamma_1) \to (\Gamma_2 - (\exists \wedge \Gamma_1))$-*dec., the following are equivalent.*

1) *Every* Γ_1-*r.e. structure isomorphic to* $\mathfrak{A}$ *is* Γ_2-*r.e.*

2) *For some* $\bar{a} \subseteq A$ *there is an effective procedure which when applied to any* $\bar{b} \subseteq A - \bar{a}$ *and* Γ_2-*formula* $\phi(\bar{x}, \bar{y})$ *such that* $\mathfrak{A} \models \phi(\bar{a}, \bar{b})$, *produces a* $\exists \wedge \Gamma_1$-*formula* $\psi(\bar{x}, \bar{y})$ *such that* $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$ *and* $\mathfrak{A} \models \forall\bar{y}(\psi(\bar{a}, \bar{y}) \to \phi(\bar{a}, \bar{y}))$.

PROOF. We first show that 2) $\Rightarrow$ 1). Let $\mathfrak{B}$ be a Γ_1-r.e. structure and $f: \mathfrak{A} \cong \mathfrak{B}$. We show that $\mathfrak{B}$ is Γ_2-r.e.

Let $\bar{a}$ be the sequence described in 2) and $\{\phi_1(\bar{a}, \bar{a}_1), \Phi_2(\bar{a}, \bar{a}_2), \ldots\}$ an effective listing of all Γ_2-sentences true in $\mathfrak{A}$ ($\mathfrak{A}$ is Γ_2-r.e.). Apply the effective procedure described in 2) to each element of this list to obtain another effective list $\{\psi_1(\bar{a}, \bar{y}_1), \psi_2(\bar{a}, \bar{y}_2), \ldots\}$ of formulae such that $\forall i < \omega$, $\mathfrak{A} \models \psi_i(\bar{a}, \bar{a}_i)$ and $\mathfrak{A} \models \forall \bar{y}_i(\psi_i(\bar{a}, \bar{y}_i) \to \phi_i(\bar{a}, \bar{y}_i))$. $\mathfrak{B}$ is Γ_1-r.e. and therefore $\exists \bigwedge \Gamma_1$-r.e. Thus there is an effective list of all $\exists \bigwedge \Gamma_1$-sentences $\phi(\bar{b})$ true in $\mathfrak{B}$. If for some $i < \omega$ and $\bar{b} \subseteq B$ $\psi_i(f(\bar{a}), \bar{b})$ is a member of this list, then list $\phi_i(f(\bar{a}), \bar{b})$ as true in $\mathfrak{B}$. We show that this (effective) process lists precisely those Γ_2-sentences true in $\mathfrak{B}$.

If the process decides $\phi_i(f(\bar{a}), \bar{b})$ is true in $\mathfrak{B}$, then $\mathfrak{B} \models \psi_i(f(\bar{a}), \bar{b})$. But $\mathfrak{A} \models \forall \bar{y}_i(\psi_i(\bar{a}, \bar{y}_i) \to \phi_i(\bar{a}, \bar{y}_i))$ and f: $\mathfrak{A} \cong \mathfrak{B}$. Therefore $\mathfrak{B} \models \forall \bar{y}_i(\psi_i(f(\bar{a}), \bar{y}_i) \to \phi_i(f(\bar{a}), \bar{y}_i))$; and so $\mathfrak{B} \models \phi_i(f(\bar{a}), \bar{b})$.

If $\phi(f(\bar{a}), \bar{b})$ is a Γ_2-sentence true in $\mathfrak{B}$, then, since f: $\mathfrak{A} \cong \mathfrak{B}$, $\mathfrak{A} \models \phi(\bar{a}, f^{-1}(\bar{b}))$. Therefore, since $\mathfrak{A}$ is Γ_2-r.e., for some $i < \omega$, $\phi(\bar{a}, f^{-1}(\bar{b})) = \phi_i$. Then $\mathfrak{A} \models \psi_i(\bar{a}, f^{-1}(\bar{b}))$ and, as before, we have $\mathfrak{B} \models \psi_i(f(\bar{a}), \bar{b})$. Thus, the process will list $\phi(f(\bar{a}), \bar{b})$ as true in $\mathfrak{B}$. This shows that $\mathfrak{B}$ is Γ_2-r.e.

We now show that $\neg 2) \Rightarrow \neg 1)$ via a finite injury priority construction. Let $\mathfrak{A}$ be a $(\exists \bigwedge \Gamma_1) \to (\Gamma_2 - (\exists \bigwedge \Gamma_1))$-dec. structure on recursive universe $A = \{a_0, a_1, \ldots\}$ satisfying $\neg 2)$. We shall construct a Γ_1-r.e. $\mathfrak{B}$ in stages and a g: $\mathfrak{B} \cong \mathfrak{A}$ such that $\mathfrak{B}$ is not Γ_2-r.e., thus showing that 1) is false. $\mathfrak{B}$ will be a structure on recursive universe $B = \{b_0, b_1, \ldots\}$.

At each stage s of our construction we shall define a partial map g_s: $B \to A$ so that $g = \lim_s g_s$ exists, and is a surjection from B to A. At stage s we define a finite set Σ^s of Γ_1-sentences $\phi(\bar{b})$ as follows. Since $\mathfrak{A}$ is Γ_1-r.e. there is an effective list of all Γ_1-sentences $\phi(\bar{a})$ true in $\mathfrak{A}$. We define $\Sigma^{-1} = \{\ \}$ and $\Sigma^s = \Sigma^{s-1} \cup \{\phi(\bar{b})$: $\phi(\bar{x}) \in \Gamma$, $\bar{b} \subseteq \mathrm{dom}(g_s)$ and $\phi(g_s(\bar{b}))$ has been listed as true in $\mathfrak{A}$ by this stage$\}$. We write $\bigwedge \Sigma^s$ for the conjunction of all the sentences in Σ^s. $\mathfrak{B}$ is defined to be the structure on B satisfying all the quantifier-free sentences in $\Sigma = \bigcup_s \Sigma^s$. We will ensure that g: $\mathfrak{B} \cong \mathfrak{A}$.

Let $W_0, W_1, \ldots$ be a list of all r.e. subsets of Γ_2-sentences $\phi(\bar{b})$ with $\bar{b} \subseteq B$. The idea of the construction is to use the hypothesis to diagonalize over the W_e: $e < \omega$ to ensure that none of them lists precisely those Γ_2-sentences true in $\mathfrak{B}$. W_e^s denotes the subset of W_e calculated by stage s.

Our requirements are:

P_e^1: $b_e \in \mathrm{dom}(g)$,

P_e^2: $a_e \in \mathrm{ran}(g)$, and

Q_e: $W_e \neq \{\phi(\bar{b})$: $\phi(\bar{x}) \in \Gamma_2$, $\bar{b} \subseteq B$ and $\mathfrak{B} \models \phi(\bar{b})\}$.

The basic action to meet the requirements Q_e is this. If at stage s we have $\phi(\bar{b}) \in W_e^s$ with $\bar{b} \subseteq \mathrm{dom}(g_s)$, at stage $s+1$ we attempt to define g_{s+1} so that $\mathfrak{A} \models \neg\phi(g_{s+1}(\bar{b}))$; and therefore $\mathfrak{B} \models \neg\phi(\bar{b})$. A problem arising from this action is that since $g_{s+1}(\bar{b}) \neq g_s(\bar{b})$, we run the risk that $g = \lim_s g_s$ may not exist. Another problem is that as we want $\Sigma = \bigcup_s \Sigma^s$ to be the Γ_1-diagram of $\mathfrak{B}$, we must arrange that no sentence of Σ^s is made false by our definition of g_{s+1}.

DEFINITIONS. P_e^1 *requires attention at stage* $s+1$ if $b_e \notin \mathrm{dom}(g_s)$.

P_e^1 *is injured at stage* $s+1$ if $g_{s+1}(b_e) \neq g_s(b_e)$,

P_e^2 *requires attention at stage* $s+1$ if $a_e \notin \mathrm{ran}(g_s)$.

P_e^2 *is injured at stage* $s+1$ if $g_{s+1}^{-1}(a_e) \neq g_s^{-1}(a_e)$.

In order to meet the requirements Q_e we define at some stages s sequences $\bar{d}_e^s \subseteq B$ with the intention that if $\{\phi(\bar{b})\colon \phi(\bar{x}) \in \Gamma_2, \bar{b} \subseteq B \text{ and } \mathfrak{B} \models \phi(\bar{b})\} \subseteq W_e$, then $\bar{d}_e = \lim_s \bar{d}_e^s$ exists, and there is a formula $\theta(\bar{x}) \in \Gamma_2$ such that $\theta(\bar{d}_e) \in W_e$, but $\mathfrak{B} \not\models \theta(\bar{d}_e)$.

At any stage s when $P_0^1, \ldots, P_e^1, P_0^2, \ldots, P_e^2$ do not require attention we define for $\bar{b} \subseteq \mathrm{dom}(g_s)$ the *splitting of* dom(g_s) *with respect to* Q_e *and* $\bar{b}$ to be $\mathrm{dom}(g_s) = \bar{b}_0, \bar{b}_1, \bar{b}_2$ where:

$\bar{b}_0 = b_0, b_1, \ldots, b_e, g_s^{-1}(a_0), \ldots, g_s^{-1}(a_e), \ \bigcup \overline{d_{e'}^s}$ with the union taken over all $e' < e$ for which $\overline{d_{e'}^s}$ is defined,

$\bar{b}_1 = \bar{b} - \bar{b}_0$, and

$\bar{b}_2 = \mathrm{dom}(g_s) - (\bar{b}_0 \cup \bar{b}_1)$.

A sentence $\phi(\bar{b}) \in W_e^{s+1}$ *can be used to attack* Q_e *at stage* $s+1$ *if*

1) $\bar{b} \subseteq \mathrm{dom}(g_s)$,

2) $\phi(\bar{x}) \notin \exists \bigwedge \Gamma_1$, and

3) $\mathfrak{A} \models \exists \bar{x}_1 (\exists \bar{x}_2 \bigwedge \Sigma^s(g_s(\bar{b}_0), \bar{x}_1, \bar{x}_2) \wedge \neg\phi(g_s(\bar{b}_0), \bar{x}_1))$ where $\bar{b}_0, \bar{x}_1, \bar{x}_2$ correspond to $\bar{b}_0, \bar{b}_1, \bar{b}_2$ in the splitting of dom(g_s) with respect to Q_e and $\bar{b}$.

Q_e *is injured* at stage $s+1$ if $g_{s+1}(\overline{d_e^s}) \neq g_s(\overline{d_e^s})$. In this case we say that $\overline{d_e^{s+1}}$ is undefined. (Otherwise $\overline{d_e^{s+1}} = \overline{d_e^s}$.)

CONSTRUCTION. *Stage* 0. Define $g_0\colon b_0 \to a_0$.

Stage $s+1$. Arrange the requirements in order of decreasing priority as $P_0^1, P_0^2, Q_0, P_1^1, P_1^2, Q_1, \ldots$ and look for the one of highest priority requiring attention at this stage. Attack the requirement as follows.

If the requirement is a P_e^1, attack it by defining $g_{s+1}(b_e)$ to be the least $a_n \notin \mathrm{ran}(g_s)$; and $g_{s+1}|_{\mathrm{dom}(g_s)} = g_s$.

If the requirement is a P_e^2, attack it by defining $g_{s+1}^{-1}(a_e)$ to be the least $b_n \notin \mathrm{dom}(g_s)$; and $g_{s+1}|_{\mathrm{dom}(g_s)} = g_s$.

If the requirement is a Q_e, we choose the first $\phi(\bar{b}) \in W_e^{s+1}$ which can be used to attack Q_e at this stage and define g_{s+1} as follows.

$g_{s+1}(\bar{b}_0) = g_s(\bar{b}_0)$.

$g_{s+1}(\bar{b}_1) =$ the least sequence $\bar{a}_1$ disjoint from $g_s(\bar{b}_0)$ such that $\mathfrak{A} \models \exists \bar{x}_2 \wedge \Sigma^s(g_s(\bar{b}_0), \bar{a}_1, \bar{x}_2) \wedge \neg\phi(g_s(\bar{b}_0), \bar{a}_1)$. There is such an $\bar{a}_1$ by property 3 of $\phi(\bar{b})$.

$g_{s+1}(\bar{b}_2) =$ the least sequence $\bar{a}_2$ disjoint from $g_s(\bar{b}_0) \cup \bar{a}_1$ such that $\mathfrak{A} \models \bigwedge \Sigma^s(g_s(\bar{b}_0), \bar{a}_1, \bar{a}_2)$. There is such an $\bar{a}_2$ by definition of $\bar{a}_1$. Define $\overline{d_e^{s+1}} = \bar{b}$.

We conclude the proof with the following remarks.

1) *The construction is effective.* The only problem is to decide when and how to attack a requirement Q_e. At stage $s+1$ we only need consider whether or not one of the (finite number of) requirements Q_e with $e \leq s+1$ requires attention. To do this we run through the (finite number of) sentences in W_e^{s+1} asking if there are any that can be used to attack Q_e at this stage. This can be done effectively since $\mathfrak{A}$ is $(\exists \wedge \Gamma_1) \to (\Gamma_2 - (\exists \wedge \Gamma_1))$-decidable.

Once we have decided to attack a requirement Q_e, we need to find the sequences $\bar{a}_1$, $\bar{a}_2$. To do this we run through all sequences of the right length from A until we find sequences that fit the bill. Once again the decidability assumption on $\mathfrak{A}$ allows us to do this effectively.

2) *Each requirement Q_e is attacked at most a finite number of times.* Assume this to be true for the requirements $Q_0, \ldots, Q_{e-1}$. We show it to be true for Q_e. Let t be a stage after which none of $Q_0, \ldots, Q_{e-1}$ are ever attacked. Then by the fact that $\bigcup \overline{d_e^s} \subseteq b_0$ in the definition of the splitting of dom(g^s), Q_e is never injured after this stage. Thus Q_e is attacked at most once after stage t.

LEMMA 1. *$g = \lim_s g_s$ exists, and g: $\mathfrak{B} \cong \mathfrak{A}$.*

PROOF. We show that for any $e < \omega$, g is re-defined on b_e and a_e at most a finite number of times; and therefore $\lim_s g_s(b_e)$ and $\lim_s g_s^{-1}(a_e)$ exist. Let t be a stage after which none of $Q_0, \ldots, Q_{e-1}$ are ever attacked. Then, as in the last proof, since b_e, $g_s^{-1}(a_e) \in \bar{b}_0$, P_e^1 and P_e^2 are never injured after this stage. Thus $\lim_s g_s(b_e) = g_t(b_e)$ and $\lim_s g_s^{-1}(a_e) = g_t^{-1}(a_e)$. Therefore $g = \lim_s g_s$ exists.

Since for any $e < \omega$, P_e^1 and P_e^2 are attacked at some stage; g: $B \to A$ is a surjection.

We now show that for every predicate P in $\mathfrak{L}$, $\mathfrak{B} \models P(\bar{b})$ if and only if $\mathfrak{A} \models P(g(\bar{b}))$.

Let $\mathfrak{A} \models P(\bar{b})$. Then, by definition of $\mathfrak{B}$, $P(\bar{b}) \in \Sigma$; that is, $P(\bar{b}) \in \Sigma^s - \Sigma^{s-1}$ for some s. By definition of Σ^s, $\mathfrak{A} \models P(g_s(\bar{b}))$. If at some further stage $t+1$ we re-define g on $\bar{b}$, we make sure that $\mathfrak{A} \models \bigwedge \Sigma^t(g_{t+1}(\text{dom}(g_t)))$. Therefore $\mathfrak{A} \models P(g_{t+1}(\bar{b}))$. Thus $\mathfrak{A} \models P(g(\bar{b}))$.

Let $\mathfrak{A} \models P(g(\bar{b}))$. Let s be a stage when g has taken on its final value on $\bar{b}$. Since $\mathfrak{A}$ is Γ_1-r.e., there is a stage t $(\geq s)$ by which $P(g(\bar{b}))$ has been listed as true in $\mathfrak{A}$. Then, by definition of Σ^t, $P(\bar{b}) \in \Sigma^t \subseteq \Sigma$. Thus $\mathfrak{B} \models P(\bar{b})$.

LEMMA 2. *$\mathfrak{B}$ is a Γ_1-r.e. structure.*

PROOF. By applying to any Γ_1-formula ϕ the argument applied to the predicate P in Lemma 1, we see that $\phi(\bar{b}) \in \Sigma$ if and only if $\mathfrak{A} \models \phi(g(\bar{b}))$. Since g: $\mathfrak{B} \cong \mathfrak{A}$, we therefore have $\Sigma = \{\phi(\bar{b})\colon \phi(\bar{x}) \in \Gamma_1,\ \bar{b} \subseteq B \text{ and } \mathfrak{B} \models \phi(\bar{b})\}$. Σ is r.e. (by Remark 1), and therefore $\mathfrak{B}$ is Γ_1-r.e.

LEMMA 3. *$\mathfrak{B}$ is not Γ_2-r.e.*

PROOF. Suppose $\mathfrak{B}$ is Γ_2-r.e. We show that statement 2) of the theorem holds. Let e be least with $W_e = \{\phi(\bar{b})\colon \phi(\bar{x}) \in \Gamma_2,\ \bar{b} \subseteq B \text{ and } \mathfrak{B} \models \phi(\bar{b})\}$. Consider a stage s_0 after which none of $Q_0, \ldots, Q_{e-1}$ are ever attacked and $P_0^1, \ldots, P_e^1$, $P_0^2, \ldots, P_e^2$ do not require attention.

Q_e is not injured after stage s_0. Suppose Q_e were attacked at some stage $t \geqslant s_0$ by a sentence $\phi(\bar{b})$. Then $\bar{b} = \bar{d}_e^t$ and as Q_e is never injured, $g_t(\bar{b}) = g(\bar{b})$. Thus $\phi(\bar{b}) \in W_e$ and $\mathfrak{A} \models \neg\phi(g(\bar{b}))$. Therefore, since $g\colon \mathfrak{B} \cong \mathfrak{A}$, $\mathfrak{B} \models \neg\phi(\bar{b})$. This contradicts the choice of W_e. Thus Q_e is not attacked after stage s_0.

Let $\bar{b}_0$ be as in the splitting of $\mathrm{dom}(g_{s_0})$ with respect to Q_e. By the choice of s_0, $\bar{b}_0$ is the same in the splitting of $\mathrm{dom}(g_s)$ with respect to Q_e for any $s \geqslant s_0$. Therefore $g_{s_0}(\bar{b}_0) = g(\bar{b}_0) = \bar{a}_0$, say.

Let $\phi(\bar{b}_0, \bar{b})$ be a Γ_2-sentence true in $\mathfrak{B}$. We show that there is a $\exists \wedge \Gamma_1$-formula $\psi(\bar{x}_0, \bar{x})$ such that $\mathfrak{B} \models \psi(\bar{b}_0, \bar{b})$ and $\mathfrak{A} \models \forall\bar{x}(\psi(\bar{a}_0, \bar{x}) \to \phi(\bar{a}_0, \bar{x}))$. If $\phi(\bar{b}_0, \bar{b})$ is a $\exists \wedge \Gamma_1$-sentence, take $\psi(\bar{x}_0, \bar{x}) = \phi(\bar{x}_0, \bar{x})$ where $\bar{x}_0, \bar{x}$ correspond to $\bar{b}_0, \bar{b}$. This clearly fits the bill.

If $\phi(\bar{b}_0, \bar{b})$ is a $\Gamma_2 - (\exists \wedge \Gamma_1)$-sentence, consider a stage $s \geqslant s_0$ of our construction when

i) $\phi(\bar{b}_0, \bar{b}) \in W_e^s$ (such a stage exists by choice of W_e), and

ii) $\bar{b} \subseteq \mathrm{dom}(g_s)$.

Take $\psi(\bar{x}_0, \bar{x}) = \exists\bar{x}_2 \wedge \Sigma^s(\bar{x}_0, \bar{x}, \bar{x}_2)$, where $\bar{x}_0, \bar{x}, \bar{x}_2$ correspond to $\bar{b}_0, \bar{b}_1, \bar{b}_2$ in the splitting of $\mathrm{dom}(g_s)$ with respect to Q_e and $\bar{b}$. Clearly $\mathfrak{B} \models \psi(\bar{b}_0, \bar{b})$. We show that $\mathfrak{A} \models \forall\bar{x}(\psi(\bar{a}_0, \bar{x}) \to \phi(\bar{a}_0, \bar{x}))$. Since $s \geqslant s_0$, Q_e is not attacked at this stage. In particular $\phi(\bar{b}_0, \bar{b})$ cannot be used to attack Q_e at this stage. By i) and ii) above, this means that $\phi(\bar{b}_0, \bar{b})$ does not satisfy property 3; that is, $\mathfrak{A} \models \forall\bar{x}(\psi(\bar{a}_0, \bar{x}) \to \phi(\bar{a}_0, \bar{x}))$.

Thus for every Γ_2-sentence $\phi(\bar{b}_0, \bar{b})$ true in $\mathfrak{B}$, there is a $\exists \wedge \Gamma_1$-formula $\psi(\bar{x}_0, \bar{x})$ such that $\mathfrak{B} \models \psi(\bar{b}_0, \bar{b})$ and $\mathfrak{A} \models \forall\bar{x}(\psi(\bar{a}_0, \bar{x}) \to \phi(\bar{a}_0, \bar{x}))$. Therefore, for every Γ_2-sentence $\phi(\bar{a}_0, \bar{a})$ true in $\mathfrak{A}$, there is a $\exists \wedge \Gamma_1$-formula $\psi(\bar{x}_0, \bar{x})$ such that $\mathfrak{A} \models \psi(\bar{a}_0, \bar{a})$ and $\mathfrak{A} \models \forall\bar{x}(\psi(\bar{a}_0, \bar{x}) \to \phi(\bar{a}_0, \bar{x}))$; namely the formula corresponding to $\phi(\bar{b}_0, g^{-1}(\bar{a}))$. Given any $\phi(\bar{a}_0, \bar{a})$ true in $\mathfrak{A}$, we can find this $\exists \wedge \Gamma_1$-formula effectively using the fact that $\mathfrak{A}$ is $(\exists \wedge \Gamma_1) \to (\Gamma_2 - (\exists \wedge \Gamma_1))$-decidable.

This contradicts our hypothesis, thus proving Theorem I.

In Theorem I, Γ_2 was taken to be a set of formulae to simplify notation. The only property used is that every $\phi \in \Gamma_2$ is preserved under isomorphisms; that is, if $f\colon \mathfrak{B} \cong \mathfrak{A}$ then $\mathfrak{B} \models \phi(\bar{b})$ if and only if $\mathfrak{A} \models \phi(f(\bar{b}))$. Any relation R on $\mathfrak{A}$ may be treated as such a formula by interpreting it in any $\mathfrak{B}$ with $f\colon \mathfrak{B} \cong \mathfrak{A}$ as the relation $f^{-1}R = \{f^{-1}(\bar{a})\colon \bar{a} \in R\}$. We therefore have the following result.

THEOREM I′. *Let Γ_1 be an r.e. set with $\{P_i\colon i < \omega\} \subseteq \Gamma_1 \subseteq \mathrm{Form}(\mathfrak{L})$, and Γ_2 a set $\{R_i\colon i < \omega\}$ of relations on $\mathfrak{A}$. If $\mathfrak{A}$ is $(\exists \bigwedge \Gamma_1) \to \Gamma_2$-dec., the following are equivalent.*

1) *For every Γ_1-r.e. structure $\mathfrak{B}$ and isomorphism $f\colon \mathfrak{A} \cong \mathfrak{B}$, the relations $\{fR_i\}$ are uniformly r.e. on $\mathfrak{B}$.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which when applied to any $\bar{b} \subseteq A - \bar{a}$ and $R_i \in \Gamma_2$ such that $\mathfrak{A} \models R_i(\bar{a}, \bar{b})$ produces a $\exists \bigwedge \Gamma_1$-formula $\psi(\bar{x}, \bar{y})$ such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$ and $\mathfrak{A} \models \forall \bar{y}(\psi(\bar{a}, \bar{y}) \to R_i(\bar{a}, \bar{y}))$.*

Consider a language $\mathfrak{L}$ with a predicate $\equiv$ interpreted in $\mathfrak{A}$ as the identity relation. Then the sentences $\forall \bar{y}(\psi(\bar{a}, \bar{y}) \to \phi(\bar{a}, \bar{y}))$ and $\forall \bar{y}, \bar{z}(\psi(\bar{a}, \bar{y}) \wedge (\bar{a} \equiv \bar{z}) \to \phi(\bar{z}, \bar{y}))$ are equivalent in $\mathfrak{A}$. We may therefore replace $\phi(\bar{a}, \bar{y})$ by $\phi(\bar{y})$ throughout the definition of $\Sigma_1 \to \Sigma_2$-decidability and the statement of the theorem.

In the case $\Gamma_1 = \{P_i\colon i < \omega\} \cup \{\neg P_i\colon i < \omega\}$ and $\Gamma_2 = \Gamma_1 \cup \{R\}$ Theorem I′ reads as follows.

COROLLARY 1.1 (Ash and Nerode [1]). *Let $\mathfrak{A}$ be a structure with predicate $\equiv$ interpreted as the identity; and R a relation on $\mathfrak{A}$. If there is an effective procedure for determining, given any existential formula $\psi(\bar{x}, \bar{y})$ and $\bar{a} \subseteq A$, whether the implication $\psi(\bar{a}, \bar{x}) \to R(\bar{x})$ is true in $\mathfrak{A}$, the following are equivalent.*

1) *R is intrinsically r.e. on $\mathfrak{A}$.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which, when applied to any $\bar{b} \subseteq A - \bar{a}$ such that $\mathfrak{A} \models R(\bar{b})$, produces an existential formula $\psi(\bar{x}, \bar{y})$ such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$ and $\mathfrak{A} \models \forall \bar{y}(\psi(\bar{a}, \bar{y}) \to R(\bar{y}))$.*

Applying Theorem I to the sets $\Gamma_1 = \{P_i\colon i < \omega\} \cup \{\neg P_i\colon i < \omega\}$ and $\Gamma_2 = \mathrm{Form}(\mathfrak{L})$ we get

COROLLARY 1.2 (Nurtazin [6]). *For a dec. structure $\mathfrak{A}$ the following are equivalent.*

1) *Every recursive structure isomorphic to $\mathfrak{A}$ is decidable.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which, when applied to any $\bar{b} \subseteq A - \bar{a}$ and formula $\phi(\bar{x}, \bar{y})$ such that $\mathfrak{A} \models \phi(\bar{a}, \bar{b})$, produces an existential formula $\psi(\bar{x}, \bar{y})$ such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$ and $\mathfrak{A} \models \forall \bar{y}(\psi(\bar{a}, \bar{y}) \to \phi(\bar{a}, \bar{y}))$.*

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A structure is *n-recursive* if it is Γ-r.e. for Γ the set of all formulae with n or less alternations of quantifiers; that is, of the form $Q_1\bar{x}_1Q_2\bar{x}_2\cdots Q_m\bar{x}_m\phi(\bar{x}_1,\ldots,\bar{x}_m)$ with $m \leqslant n$, ϕ quantifier-free and each Q_i one of $\forall$, $\exists$.

Taking Γ_1 to be this Γ and $\Gamma_2 = \forall\Gamma_1$ Theorem I gives

COROLLARY 1.3 (Goncharov [3]). *For an $n+1$-rec. structure $\mathfrak{A}$, the following are equivalent.*

1) *Every n-rec. structure isomorphic to $\mathfrak{A}$ is $n+1$-rec.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which, when applied to any $\bar{b} \subseteq A - \bar{a}$ and $\forall\Gamma_1$-formula $\phi(\bar{x}, \bar{y})$ such that $\mathfrak{A} \models \phi(\bar{a}, \bar{b})$, produces a $\exists\Gamma_1$-formula $\psi(\bar{x}, \bar{y})$ such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$ and $\mathfrak{A} \models \forall\bar{y}(\psi(\bar{a}, \bar{y}) \to \phi(\bar{a}, \bar{y}))$.*

The next corollary has not appeared in the literature. For $\Gamma_1 = \{P_i: i < \omega\}$ and $\Gamma_2 = \Gamma_1 \cup \neg\Gamma_1$ the theorem gives

COROLLARY 1.4. *For a $\exists\bigwedge\Gamma_1$-rec. structure $\mathfrak{A}$ the following are equivalent.*

1) *Every r.e. structure isomorphic to $\mathfrak{A}$ is rec.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which, when applied to any $\bar{b} \subseteq A - \bar{a}$ and quantifier-free formula $\phi(\bar{x}, \bar{y})$ such that $\mathfrak{A} \models \phi(\bar{a}, \bar{b})$, produces a $\exists\bigwedge\Gamma_1$-formula $\psi(\bar{x},\bar{y})$ such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$ and $\mathfrak{A} \models \forall\bar{y}(\psi(\bar{a}, \bar{y}) \to \phi(\bar{a}, \bar{y}))$.*

In this case the $\exists\bigwedge\Gamma_1$-formulae are existential-positive formulae. Further results may be obtained by applying the theorem to other sets Γ_1 and Γ_2.

2

THEOREM II. *Let Γ be an r.e. set with $\{P_i: i < \omega\} \subseteq \Gamma \subseteq$ Form($\mathfrak{L}$). If $\mathfrak{A}$ is $(\exists\bigwedge\Gamma) \to (\exists\bigwedge\Gamma)$-dec the following are equivalent.*

1) *Every Γ-r.e. structure isomorphic to $\mathfrak{A}$ is recursively isomorphic to $\mathfrak{A}$.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which when applied to any $\bar{b} \subseteq A - \bar{a}$ produces a $\exists\bigwedge\Gamma$-formula $\psi(\bar{a}, \bar{y})$ which is an atom of the Lindenbaum algebra $B(\mathrm{Th}\langle\mathfrak{A}, \bar{a}\rangle)$ such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$.*

PROOF. We first show that 2) $\Rightarrow$ 1). Let $\mathfrak{B}$ be a Γ-r.e. structure and $f\colon \mathfrak{A} \cong \mathfrak{B}$. We describe by a back and forth argument a rec. map $g\colon \mathfrak{A} \cong \mathfrak{B}$. At each stage we define sequences $c_1,\ldots,c_n \in A$, $d_1,\ldots,d_n \in B$ and define $g\colon c_i \to d_i$. We ensure that $c_1,\ldots,c_n$ and $d_1,\ldots,d_n$ satisfy the same formulae in $\mathfrak{A}$ and $\mathfrak{B}$.

Let $\bar{a}$ be the sequence described in 2). Notice that $\psi(\bar{a}, \bar{y})$ is an atom of $B(\mathrm{Th}\langle\mathfrak{A}, \bar{a}\rangle)$ if and only if $\psi(f(\bar{a}), \bar{y})$ is an atom of $B(\mathrm{Th}\langle\mathfrak{B}, f(\bar{a})\rangle)$.

Stage 0. Define $g(\bar{a}) = f(\bar{a})$. Clearly $\bar{a}$, and $g(\bar{a})$ satisfy the same formulae.

Stage 2_{s+1}. Suppose we have g defined on $\bar{a}, c_1, \ldots, c_n$ at this stage by

$$g(\bar{a}, c_1, \ldots, c_n) = f(\bar{a}), d_1, \ldots, d_n.$$

Take c_{n+1} to be the least $a \in A - \{\bar{a}, c_1, \ldots, c_n\}$. We wish to find d_{n+1}. Apply the effective procedure described in 2) to $c_1, \ldots, c_{n+1}$ to obtain an atom $\exists\bar{x}\phi(\bar{a}, x_1, \ldots, x_{n+1}, \bar{x})$ of $B(\mathrm{Th}\langle\mathfrak{A}, \bar{a}\rangle)$ such that $\mathfrak{A} \models \exists\bar{x}\phi(\bar{a}, c_1, \ldots, c_{n+1}, \bar{x})$. Therefore $\mathfrak{A} \models \exists\bar{x}x_{n+1}\phi(\bar{a}, c_1, \ldots, c_n, x_{n+1}, \bar{x})$. Since $\bar{a}, c_1, \ldots, c_n$ and $f(\bar{a})$, $d_1, \ldots, d_n$ satisfy the same formulae, we therefore have $\mathfrak{B} \models \exists\bar{x}x_{n+1}\phi(f(\bar{a}), d_1, \ldots, d_n, x_{n+1}, \bar{x})$. Thus there is a $b \in B$ such that $\mathfrak{B} \models \exists\bar{x}\phi(f(\bar{a}), d_1, \ldots, d_n, b, \bar{x})$. Since $\mathfrak{B}$ is Γ-r.e. and therefore $\exists \wedge \Gamma$-r.e., we can find this b effectively. Define $g(c_{n+1}) = b$, that is, $b = d_{n+1}$. $\bar{a}, c_1, \ldots, c_{n+1}$ and $f(\bar{a}), d_1, \ldots, d_{n+1}$ satisfy the same formulae since $\exists\bar{x}\phi(\bar{a}, x_1, \ldots, x_{n+1}, \bar{x})$ is an atom.

Stage 2_{s+2}. Suppose we have $g(\bar{a}, c_1, \ldots, c_n) = f(\bar{a}), d_1, \ldots, d_n$. Take d_{n+1} to be the least $b \in B - \{f(\bar{a}), d_1, \ldots, d_n\}$. We wish to find c_{n+1}. Notice that each sequence $b \subseteq B - f(\bar{a})$ satisfies an atom of $B(\mathrm{Th}\langle\mathfrak{B}, f(\bar{a})\rangle)$; namely the one produced by applying the procedure of 2) to $f^{-1}(\bar{b})$. From an r.e. list of sequences of length $n + 1$ from $A - \bar{a}$, produce an r.e. list of atoms of $B(\mathrm{Th}\langle\mathfrak{A}, \bar{a}\rangle)$ by applying the effective procedure of 2) to each sequence. As was noticed previously, $d_1, \ldots, d_{n+1}$ satisfies one of these. Since $\mathfrak{B}$ is $\exists \wedge \Gamma$-r.e. we can find effectively the atom $\exists\bar{x}\phi(\bar{a}, x_1, \ldots, x_{n+1}, \bar{x})$ such that $\mathfrak{B} \models \exists\bar{x}\phi(f(\bar{a}), d_1, \ldots, d_{n+1}, \bar{x})$. Since $\bar{a}, c_1, \ldots, c_n$ and $f(\bar{a}), d_1, \ldots, d_n$ satisfy the same formulae, arguing as before, there is an $a \in A$ such that $\mathfrak{A} \models \exists\bar{x}\phi(\bar{a}, c_1, \ldots, c_n, a, \bar{x})$. $\mathfrak{A}$ is $\exists \wedge \Gamma$-r.e., and we can therefore find this a effectively. Define $c_{n+1} = a$. Once again, $\bar{a}, c_1, \ldots, c_{n+1}$, and $f(\bar{a}), d_1, \ldots, d_{n+1}$ satisfy the same formulae.

This process clearly describes a recursive map g: $\mathfrak{A} \cong \mathfrak{B}$.

Notice that in the above proof, at stages 2_{s+1} and 2_{s+2}, we used only the fact that $\bar{a}, c_1, \ldots, c_n$ and $f(\bar{a}), d_1, \ldots, d_n$ satisfy the same $\exists \wedge \Gamma$-formulae. We say that a formula $\psi(\bar{a}, \bar{y})$ is *an atom of the* $\exists \wedge \Gamma$-*part of* $B(\mathrm{Th}\langle, \bar{a}\rangle)$ if given any $\exists \wedge \Gamma$-formula $\psi(\bar{x}, \bar{y})$ if $\mathfrak{A} \models \exists\bar{y}(\psi(\bar{a}, \bar{y}) \wedge \phi(\bar{a}, \bar{y}))$ then $\mathfrak{A} \models \forall\bar{y}(\psi(\bar{a}, \bar{y}) \to \phi(\bar{a}, \bar{y}))$.

Consider the following statement.

2′) For some $\bar{a} \subseteq A$ there is an effective procedure which when applied to any $\bar{b} \subseteq A - \bar{a}$ produces a $\exists \wedge \Gamma$-formula $\psi(\bar{a}, \bar{y})$ which is an atom of the $\exists \wedge \Gamma$-part of $B(\mathrm{Th}\langle\mathfrak{A}, \bar{a}\rangle)$ such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$.

Clearly 2) $\Rightarrow$ 2′). We show that 2′) $\Rightarrow$ 2) by showing that the atoms $\psi(\bar{a}, \bar{y})$ of 2′) are in fact atoms of the whole of $B(\mathrm{Th}\langle\mathfrak{A}, \bar{a}\rangle)$.

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Let $\psi(\bar{a}, \bar{y})$ be an atom of 2′). Let $\bar{a}_1, \bar{a}_2 \subseteq A - \bar{a}$ be such that $\mathfrak{A} \models \psi(\bar{a}, \bar{a}_1)$ and $\mathfrak{A} \models \psi(\bar{a}, a_2)$. We show that $\psi(\bar{a}, \bar{y})$ is an atom of $B(\mathrm{Th}\langle \mathfrak{A}, \bar{a}\rangle)$ by showing that there is an automorphism $g\colon \mathfrak{A} \cong \mathfrak{A}$ such that $g(\bar{a}) = \bar{a}$ and $g(\bar{a}_1) = \bar{a}_2$. At stage 0 define $g(\bar{a}, \bar{a}_1) = \bar{a}, \bar{a}_2$, and define the map at stages 2_{s+1} and 2_{s+2} as was done previously.

This shows that 2) $\Leftrightarrow$ 2′).

We now show that $\neg$ 2′) $\Rightarrow \neg$1) via a finite injury priority construction. Let $\mathfrak{A}$ be a $(\exists \wedge \Gamma) \to (\exists \wedge \Gamma)$-dec. structure on rec. universe $A = \{a_0, a_1, \ldots\}$ satisfying $\neg$ 2′). We shall construct in stages a Γ-r.e. $\mathfrak{B}$ on rec. universe $B = \{b_0, b_1, \ldots\}$ isomorphic but not recursively isomorphic to $\mathfrak{A}$.

As in Theorem I we shall define at each stage s a partial map $g_s\colon B \to A$ so that $g = \lim_s g_s$ is a surjection from B to A. We shall also define finite sets Σ^s and Δ^s of $\exists \wedge \Gamma$-sentences $\psi(\bar{b})$ with $\bar{b} \subseteq B$. $\mathfrak{B}$ is defined to be the structure on B satisfying all the quantifier-free sentences in $\Sigma = \bigcup_s \Sigma^s$. We will ensure that $g\colon \mathfrak{B} \cong \mathfrak{A}$.

Let $\phi_0, \phi_1, \ldots$ be a list of all $\exists \wedge \Gamma$-sentences $\phi(\bar{b})$ with $\bar{b} \subseteq B$. Let $f_0, f_1, \ldots$ be a list of all partial rec. functions from B to A. As in Theorem I, the idea is to diagonalize over the $f_e\colon e < \omega$ to ensure that none of them is an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. f_e^s is the subset of f_e computed by stage s.

Our requirements are:

P_e^1: $b_e \in \mathrm{dom}(g)$,

P_e^2: $a_e \in \mathrm{ran}(g)$,

P_e^3: If $\phi_e = \phi_e(\bar{b})$, say, and $\mathfrak{B} \models \phi_e(\bar{b})$, then $\phi_e(\bar{b}) \in \Sigma$, and

Q_e: $f_e\colon \mathfrak{B} \to \mathfrak{A}$ is not an isomorphism.

Definitions. The four definitions for P_e^1 and P_e^2 are as in Theorem I.

P_e^3 requires attention at stage $s + 1$ if $\phi_e = \phi_e(\bar{b})$ say, has $\bar{b} \subseteq \mathrm{dom}(g_s)$ and $\phi_e \notin \Sigma^s \cup \Delta^s$.

In order to meet the requirements Q_e we define at some stages s sequences $\overline{d_e^s} \subseteq B$ with the intention that if f_e is a total function, $\bar{d}_e = \lim_s \overline{d_e^s}$ exists, and there is a formula $\theta(\bar{x}) \in \exists \wedge \Gamma$ such that precisely one of $\mathfrak{B} \models \theta(\bar{d}_e)$, $\mathfrak{A} \models \theta(f_e(\bar{d}_e))$ is true.

Q_e requires attention at stage $s + 1$ if $\overline{d_e^s}$ is undefined.

Q_e is injured at stage $s + 1$ if $g_{s+1}(\overline{d_e^s}) \neq g_s(\overline{d_e^s})$. In this case we say that $\overline{d_e^{s+1}}$ is undefined. (Otherwise $\overline{d_e^{s+1}} = \overline{d_e^s}$.)

At any stage s when $P_0^1, \ldots, P_e^1, P_0^2, \ldots, P_e^2$ do not require attention we define the *splitting of* $\mathrm{dom}(g_s)$ *with respect to* Q_e as $\mathrm{dom}(g_s) = \bar{b}_0, \bar{b}_1, \bar{b}_2$ where:

$\bar{b}_0 = b_0, b_1, \ldots, b_e, g_s^{-1}(a_0), \ldots, g_s^{-1}(a_e), \bigcup \overline{d_{e'}^s}$ (as in Theorem I);

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$\bar{b}_1 = (\mathrm{dom}(f_e^{s+1}) \cap \mathrm{dom}(g_s)) - \bar{b}_0$; and
$\bar{b}_2 = \mathrm{dom}(g_s) - (\bar{b}_0 \cup \bar{b}_1)$.

A $\exists \bigwedge \Gamma$-sentence $\phi(\bar{b})$ *can be used to attack* Q_e *at stage* $s+1$ if

1) $\bar{b} \subseteq \mathrm{dom}(g_s)$ and $\bar{b}_0 \subseteq \mathrm{dom}(f_e^{s+1})$;

2) $\mathfrak{A} \models \exists \bar{x}_1, \bar{x}_2(\bigwedge \Sigma^s \wedge \phi(g_s(\bar{b}_0), \bar{x}_1, \bar{x}_2))$, and

3) $\mathfrak{A} \models \neg(\forall \bar{x}_1(\exists \bar{x}_2 \wedge \Sigma^s(g_s(\bar{b}_0), \bar{x}_1, \bar{x}_2) \to \exists \bar{x}_2(\bigwedge \Sigma^s \wedge \phi(g_s(\bar{b}_0), \bar{x}_1, \bar{x}_2))))$

where $\bar{b}_0, \bar{x}_1, \bar{x}_2$ correspond to $\bar{b}_0, \bar{b}_1, \bar{b}_2$ in the splitting of $\mathrm{dom}(g_s)$ with respect to Q_e.

CONSTRUCTION. *Stage* 0. Define $g_0: b_0 \to a_0$; $\Sigma^0 = \Delta^0 = \{\ \}$.

Stage $s+1$. Arrange the requirements in order of decreasing priority as $P_0^1, P_0^2, P_0^3, P_1^1, P_1^2, P_1^3, \ldots$ and look for the one of highest priority requiring attention at this stage.

If the requirement is a P_e^1 or P_e^2, attack it as was done in Theorem I; and define $\Sigma^{s+1} = \Sigma^s$ and $\Delta^{s+1} = \{\ \}$. If the requirement is a P_e^3, check the requirements $Q_{e'}$ with $e' < e$ to see if there is one that requires attention at this stage, such that ϕ_e can be used to attack $Q_{e'}$ at this stage. If there is none, attack P_e^3 as follows. (Let $\phi_e = \phi_e(\bar{b})$.) If $\mathfrak{A} \models \phi_e(g_s(\bar{b}))$, define $\Sigma^{s+1} = \Sigma^s \cup \{\phi_e\}$ and $\Delta^{s+1} = \Delta^s$. If $\mathfrak{A} \models \neg\phi_e(g_s(\bar{b}))$, define $\Delta^{s+1} = \Delta^s \cup \{\phi_e\}$ and $\Sigma^{s+1} = \Sigma^s$. In both cases define $g_{s+1} = g_s$.

If there is a $Q_{e'}$ with $e' < e$, requiring attention such that $\phi_e(\bar{b})$ can be used to attack $Q_{e'}$ at this stage, then attack $Q_{e'}$ as follows.

Case 1. If $\mathfrak{A} \not\models \exists \bar{x}_2(\bigwedge \Sigma^s \wedge \phi_e(f_{e'}^{s+1}(\bar{b}_0), f_{e'}^{s+1}(\bar{b}_1), \bar{x}_2))$, where $\bar{b}_0, \bar{b}_1, \bar{x}_2$ correspond to $\bar{b}_0, \bar{b}_1, \bar{b}_2$ in the splitting of $\mathrm{dom}(g_s)$ with respect to $Q_{e'}$, then define g_{s+1} as follows.

$g_{s+1}(\bar{b}_0) = g_s(\bar{b}_0)$.

$g_{s+1}(\bar{b}_1) =$ the least sequence $\bar{a}_1$ disjoint from $g_s(\bar{b}_0)$ such that $\mathfrak{A} \models \exists \bar{x}_2(\bigwedge \Sigma^s \wedge \phi_e(g_s(\bar{b}_0), \bar{a}_1, \bar{x}_2))$. There is such an $\bar{a}_1$ by property 2 of $\phi_e(\bar{b})$.

$g_{s+1}(\bar{b}_2) =$ the least sequence $\bar{a}_2$ disjoint from $g_s(\bar{b}_0) \cup \bar{a}_1$ such that $\mathfrak{A} \models \bigwedge \Sigma^s \wedge \phi_e(g_s(\bar{b}_0), \bar{a}_1, \bar{a}_2)$. There is such an $\bar{a}_2$ by definition of $\bar{a}_1$.

Case 2. If $\mathfrak{A} \models \exists \bar{x}_2(\bigwedge \Sigma^s \wedge \phi_e(f_{e'}^{s+1}(\bar{b}_0), f_{e'}^{s+1}(\bar{b}_1), \bar{x}_2))$, define g_{s+1} as follows.

$g_{s+1}(\bar{b}_0) = g_s(\bar{b}_0)$.

$g_{s+1}(\bar{b}_1) =$ the least sequence $\bar{a}_1$ disjoint from $g_s(\bar{b}_0)$ such that $\mathfrak{A} \models \exists \bar{x}_2 \wedge \Sigma^s(g_s(\bar{b}_0), \bar{a}_1, \bar{x}_2) \wedge \neg(\exists \bar{x}_2(\bigwedge \Sigma^s \wedge \phi_e(g_s(\bar{b}_0), \bar{a}_1, \bar{x}_2)))$. There is such an $\bar{a}_1$ by property 3 of $\phi_e(\bar{b})$.

$g_{s+1}(\bar{b}_2) =$ the least sequence $\bar{a}_2$ disjoint from $g_s(\bar{b}_0) \cup \bar{a}_1$ such that $\mathfrak{A} \models \bigwedge \Sigma^s(g_s(\bar{b}_0), \bar{a}_1, \bar{a}_2)$.

In both cases define $\Sigma^{s+1} = \Sigma^s$, $\Delta^{s+1} = \{\ \}$ and $\overline{d_e^{s+1}} = \bar{b}_0, \bar{b}_1$.

We conclude the proof with the following remarks.

1) The construction is recursive. The proof is similar to the corresponding proof in Theorem I.

2) Each requirement Q_e is attacked at most a finite number of times. The proof is as in Theorem I.

LEMMA 1. $g = \lim_s g_s$ *exists and* $g: \mathfrak{B} \cong \mathfrak{A}$.

PROOF. Arguing as in Lemma 1 of Theorem I, $g = \lim_s g_s$ exists and is a surjection from B to A.

We show that for every predicate P in $\mathfrak{L}$, $\mathfrak{B} \models P(\bar{b})$ if and only if $\mathfrak{A} \models P(g(\bar{b}))$. If $\mathfrak{B} \models P(\bar{b})$, just as in Lemma 1 of Theorem I, $\mathfrak{A} \models P(g(\bar{b}))$.

Let $\mathfrak{A} \models P(g(\bar{b}))$ and $P(\bar{b}) = \phi_e$. Consider the stage s at which g takes on its final value on $\bar{b}$. By the construction, since $g_s \neq g_{s-1}$, $\Delta^s = \{\ \}$. Since $\mathfrak{A} \models P(g(\bar{b}))$, $P(\bar{b}) \notin \Delta^t$ for any $t \geqslant s$. If $P(\bar{b}) \notin \Sigma^s$, then P_e^3 requires attention at stage $s + 1$, and at some further stage is attacked. In either case $\phi_e(\bar{b}) \in \Sigma$, and therefore $\mathfrak{B} \models \phi_e(\bar{b})$.

LEMMA 2. $\mathfrak{B}$ *is a* Γ-*r.e. structure.*

PROOF. By applying to any $\exists \wedge \Gamma$-formula ϕ the argument applied to the predicate P above, we see that $\phi(\bar{b}) \in \Sigma$ if and only if $\mathfrak{A} \models \phi(g(\bar{b}))$. $g: \mathfrak{B} \cong \mathfrak{A}$; and by Remark 1, Σ is an r.e. set of $\exists \wedge \Gamma$-sentences $\phi(\bar{b})$. Thus $\mathfrak{B}$ is a $\exists \wedge \Gamma$-r.e. structure.

LEMMA 3. $\mathfrak{B}$ *is not recursively isomorphic to* $\mathfrak{A}$.

PROOF. Suppose $\mathfrak{B}$ is recursively isomorphic to $\mathfrak{A}$. We show that statement 2′) is true. Let e by least with $f_e: \mathfrak{A} \cong \mathfrak{B}$.

Consider a stage s_0 after which none of $Q_0, \ldots, Q_{e-1}$ are ever attacked and $P_0^1, \ldots, P_e^1$, $P_0^2, \ldots, P_e^2$ do not require attention. Let $\bar{b}_0$ be as in the splitting of $\operatorname{dom}(g_{s_0})$ with respect to Q_e. By the choice of s_0, $\bar{b}_0$ is the same in the splitting of $\operatorname{dom}(g_s)$ with respect to Q_e for any $s \geqslant s_0$. Therefore $g_{s_0}(\bar{b}_0) = g(\bar{b}_0) = \bar{a}_0$, say.

Q_e is not injured after stage s_0. Suppose Q_e were attacked at some stage $t \geqslant s_0$ by a sentence $\phi(\bar{b})$. Then $\bar{d}_e^t$ would be defined; and as Q_e is never injured, $\bar{d}_e = \bar{d}_e^t$ and $g(\bar{d}_e) = g_t(\bar{d}_e)$. Consider the formula $\theta(\bar{x}) = \exists \bar{y}(\bigwedge \Sigma^t \wedge \phi(\bar{x}, \bar{y}))$ where $\bar{x}$ corresponds to $\bar{b}_0$, $\bar{b}_1$ and $\bar{y}$ to $\bar{b}_2$ in the splitting of $\operatorname{dom}(g_t)$ with respect to Q_e. Since $g_t(\bar{b}_0, \bar{b}_1) = g(\bar{b}_0, \bar{b}_1)$ and $g: \mathfrak{B} \cong \mathfrak{A}$ we have that precisely one of $\mathfrak{B} \models \theta(\bar{d}_e)$, $\mathfrak{A} \models \theta(f_e(\bar{d}_e))$ is true. This contradicts the choice of f_e. Thus Q_e is not attacked after stage s_0.

Given any $\bar{c} \subseteq B - \bar{b}_0$, perform the effective construction up to a stage $s \geqslant s_0$ when $\bar{b}_0 \cup \bar{c} \subseteq \mathrm{dom}(g_s) \cap \mathrm{dom}(f_e^s)$. Consider the $\exists \wedge \Gamma$-formula $\psi(\bar{x}, \bar{y}) = \exists \bar{z} \wedge \Sigma^s(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x}, \bar{y}, \bar{z}$ correspond to $\bar{b}_0$, $\bar{c}$ and $\mathrm{dom}(g_s) - (\bar{b}_0 \cup \bar{c})$ respectively. Clearly $\mathfrak{B} \vDash \psi(\bar{b}_0, \bar{c})$. We show that $\psi(\bar{a}_0, \bar{y})$ is an atom of the $\exists \wedge \Gamma$-part of $B(\mathrm{Th}\langle \mathfrak{A}, \bar{a}_0 \rangle)$.

Consider the first stage $m + 1 > s$ when $\Sigma^{m+1} \neq \Sigma^s$; let $\Sigma^{m+1} - \Sigma^s = \{\sigma(\bar{b})\}$. By definition of attack on P^3, this means that $\mathfrak{A} \vDash \sigma(g_m(\bar{b}))$. This implies that the first two conditions in the definition of "$\sigma(\bar{b})$ can be used to attack Q_e at stage $m + 1$" are satisfied. However, Q_e is not attacked at stage $m + 1$. Therefore, condition 3 fails; that is,

$$\mathfrak{A} \vDash \forall \bar{x}_1 \big(\exists \bar{x}_2 \wedge \Sigma^s(\bar{a}_0, \bar{x}_1, \bar{x}_2) \to \exists \bar{x}_2 \big(\wedge \Sigma^s \wedge \sigma(\bar{a}_0, \bar{x}_1, \bar{x}_2)\big)\big)$$

where $\bar{x}_1$, $\bar{x}_2$ correspond to $\bar{b}_1$, $\bar{b}_2$ in the splitting of $\mathrm{dom}(g_m)$ with respect to Q_e. Since $\bar{c} \subseteq \bar{b}_1$ we may deduce from this that $\mathfrak{A} \vDash \forall \bar{y}(\exists \bar{x}' \wedge \Sigma^s(\bar{a}_0, \bar{y}, \bar{x}') \to \exists \bar{x}' \wedge \Sigma^{m+1}(\bar{a}_0, y, \bar{x}'))$ where $\bar{y}$, $\bar{x}'$ correspond to $\bar{c}$, $\mathrm{dom}(g_m) - (\bar{b}_0 \cup \bar{c})$. Applying this argument repeatedly, we see that for any $t \geqslant s$ $\mathfrak{A} \vDash \forall \bar{y}(\exists \bar{x}' \wedge \Sigma^s(\bar{a}_0, \bar{y}, \bar{x}') \to \exists \bar{x}' \wedge \Sigma^t(\bar{a}_0, \bar{y}, \bar{x}'))$ where $\bar{y}$, $\bar{x}'$ correspond to $\bar{c}$, $\mathrm{dom}(g_t) - (\bar{b}_0 \cup \bar{c})$. Call this property $\sharp$.

Consider any $\exists \wedge \Gamma$-formula $\phi(\bar{a}_0, \bar{y})$ consistent with $\psi(\bar{a}_0, \bar{y})$. If $\phi(\bar{a}_0, \bar{c}) \in \Sigma^t$ for some $t \geqslant s$, by property $\sharp$, $\mathfrak{A} \vDash \forall \bar{y}(\psi(\bar{a}_0, \bar{y}) \to \phi(\bar{a}_0, \bar{y}))$. If $\phi(\bar{a}_0, \bar{c}) \notin \Sigma$, then (by virtue of the fact that Δ is reduced to $\{\ \}$ at every stage a requirement P^1 or P^2 is attacked) there is a stage $t \geqslant s$ at which the P^3 corresponding to $\phi(\bar{a}_0, \bar{c})$ is the requirement of highest priority requiring attention. Since $\phi(\bar{a}_0, \bar{y})$ is consistent with $\psi(\bar{a}_0, \bar{y})$; by property $\sharp$, the first two conditions in the definition of "$\phi(\bar{a}_0, \bar{c})$ can be used to attack Q_e at stage t" are satisfied. However, Q_e is not attacked at stage t. Therefore, condition 3 fails. By property $\sharp$ this implies that $\mathfrak{A} \vDash \forall \bar{y}(\psi(\bar{a}_0, \bar{y}) \to \phi(\bar{a}_0, \bar{y}))$. Thus $\psi(\bar{a}_0, \bar{y})$ is an atom of the $\exists \wedge \Gamma$-part of $B(\mathrm{Th}\langle \mathfrak{A}, \bar{a}_0 \rangle)$.

We have described an effective procedure which when applied to any $\bar{c} \subseteq B - \bar{b}_0$ produces an atom $\psi(\bar{a}_0, \bar{y})$ of the $\exists \wedge \Gamma$-part of $B(\mathrm{Th}\langle \mathfrak{A}, \bar{a}_0 \rangle)$, such that $\mathfrak{B} \vDash \psi(\bar{b}_0, \bar{c})$. Notice that any $\bar{d} \subseteq A - \bar{a}_0$ satisfies one of these atoms; namely, the one corresponding to $g^{-1}(\bar{d})$. Apply this effective procedure to an r.e. list of all sequences in $B - \bar{b}_0$ to obtain an r.e. list of such atoms. Given any $\bar{d} \subseteq A - \bar{a}_0$ find an element from this list that $\bar{d}$ satisfies. Since $\mathfrak{A}$ is $\exists \wedge \Gamma$-rec., we can conduct the search effectively.

Thus statement 2′) is true, and therefore so is Theorem II.

In the proof of Lemma 3, all that was used to arrive at a contradiction is that for some $e < \omega$, $B \subseteq \mathrm{dom}(f_e)$ and $f_e \colon \mathfrak{B} \to f_e(\mathfrak{B})$ is an isomorphism. Thus we

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actually constructed a Γ-r.e. $\mathfrak{B}$ isomorphic to $\mathfrak{A}$ but not recursively *embeddable* in $\mathfrak{A}$.

Under the assumption that 2) is false we can in fact construct a countable number of Γ-r.e. structures $\mathfrak{B}_0, \mathfrak{B}_1, \ldots$ such that for each i, $\mathfrak{B}_i \cong \mathfrak{A}$, and for $i \neq j$, $\mathfrak{B}_i$ is not recursively embeddable in $\mathfrak{B}_j$.

The $\mathfrak{B}_i$ are constructed on universe $B = \{b_0, b_1, \ldots\}$ as follows. At each stage s we define, for each $i < \omega$, partial maps $g_i^s\colon B \to A$ and finite sets Σ_i^s, Δ_i^s of $\exists \wedge \Gamma$-sentences $\phi(\bar{b})$ with $\bar{b} \subseteq B$. $\mathfrak{B}_i$ is the structure on B satisfying the quantifier-free sentences in Σ_i. g_i will be an isomorphism from $\mathfrak{B}_i$ to $\mathfrak{A}$.

Let $f_0, f_1, \ldots$ be a list of all partial recursive functions from B to B.

Our requirements are:

$P_{e,i}^1$: $b_e \in \mathrm{dom}(g_i)$,

$P_{e,i}^2$: $a_e \in \mathrm{ran}(g_i)$,

$P_{e,i}^3$: If $\phi_e = \phi_e(\bar{b})$ and $\mathfrak{B}_i \models \phi_e(\bar{b})$, then $\phi_e(\bar{b}) \in \Sigma_i$, and

$Q_{e,i,j}$: $f_e\colon \mathfrak{B}_i \to \mathfrak{B}_j$ is not an embedding.

The definitions are obtained by appropriate modifications to the previous definitions, as follows.

The *splitting of* $\mathrm{dom}(g_i^s)$ *with respect to* $Q_{e,i,j}$ *is*

$\bar{b}_0 =$ as before,

$\bar{b}_1 = (\mathrm{dom}(g_i^s) \cap f_e^{s+1^{-1}}(\mathrm{dom}(g_j^s))) - \bar{b}_0$, and

$\bar{b}_2 = \mathrm{dom}(g_i^s) - (\bar{b}_0 \cup \bar{b}_1)$.

Sentence $\phi(\bar{b})$ *can be used to attack* $Q_{e,i,j}$ at stage $s + 1$ if

1) $\bar{b} \subseteq \mathrm{dom}(g_i^s)$ and $\bar{b}_0 \subseteq f_e^{s+1^{-1}}(\mathrm{dom}(g_j^s))$;

2) and 3) are obtained from the previous ones by substituting g_i^s for g_s.

Impose some order on the requirements and perform the same construction.

Decide if $\mathfrak{A} \models \phi_e(g_i^s(\bar{b}))$ in order to attack $P_{e,i}^3$, and distinguish between the two cases in the attack of $Q_{e',i,j}$ by asking if

$$\mathfrak{A} \models \exists \bar{x}_2\big(\wedge \Sigma_i^s \wedge \phi_e\big(g_j^s\big(f_{e'}^{s+1}(\bar{b}_0, \bar{b}_1)\big), \bar{x}_2\big)\big).$$

The definition of g_i^{s+1} is obtained by replacing g_s by g_i^s and Σ^s by Σ_i^s in the previous definition.

The remarks are proved as in the previous case.

We therefore have

THEOREM II′. *If* $\mathfrak{B}$ *is* $(\exists \wedge \Gamma) \to (\exists \wedge \Gamma)$*-dec., the following are equivalent.*

1) *Statement* 2) *of Theorem* II *is false.*

2) *There is a* Γ*-r.e. structure* $\mathfrak{B}$ *isomorphic to* $\mathfrak{A}$ *but not recursively embeddable in* $\mathfrak{A}$.

3) *There are countably many* Γ*-r.e. structures* $\mathfrak{B}_0, \mathfrak{B}_1, \ldots$ *isomorphic to* $\mathfrak{A}$, *such that if* $i \neq j$, $\mathfrak{B}_i$ *is not recursively embeddable in* $\mathfrak{B}_j$.

By applying Theorem II to $\Gamma = \mathrm{Form}(\mathfrak{L})$, we get

COROLLARY 2.1 (Nurtazin [6]). *For a dec. structure $\mathfrak{A}$ the following are equivalent.*

1) *$\mathfrak{A}$ is decidably categorical.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which when applied to any $\bar{b} \subseteq A - \bar{a}$ produces a formula $\psi(\bar{a}, \bar{y})$ which is an atom of $B(\mathrm{Th}\langle \mathfrak{A}, \bar{a}\rangle)$, such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$.*

For $\Gamma = \{P_i\colon i < \omega\} \cup \{\neg P_i\colon i < \omega\}$ the theorem gives

COROLLARY 2.2 (Goncharov [3]). *For a 2-rec. structure $\mathfrak{A}$ the following are equivalent.*

1) *$\mathfrak{A}$ is recursively categorical.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which when applied to any $\bar{b} \subseteq A - \bar{a}$ produces an existential formula $\psi(\bar{a}, \bar{y})$ which is an atom of $B(\mathrm{Th}\langle \mathfrak{A}, \bar{a}\rangle)$, such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$.*

The next corollary is new to the literature. For $\Gamma = \{P_i\colon i < \omega\}$ we get

COROLLARY 2.3. *For a 2-rec. structure $\mathfrak{A}$ the following are equivalent.*

1) *Every r.e. structure isomorphic to $\mathfrak{A}$ is recursively isomorphic to $\mathfrak{A}$.*

2) *For some $\bar{a} \subseteq A$ there is an effective procedure which when applied to any $\bar{b} \subseteq A - \bar{a}$ produces an existential-positive $(\exists \wedge \Gamma)$ formula $\psi(\bar{a}, \bar{y})$ which is an atom of $B(\mathrm{Th}\langle \mathfrak{A}, \bar{a}\rangle)$, such that $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$.*

In this last corollary, the requirement that $\mathfrak{A}$ is 2-rec. is stronger than is necessary, that is, $(\exists \wedge \Gamma) \to (\exists \wedge \Gamma)$-decidability.

Further results may be obtained by applying the theorem to other sets Γ; for example the set of formulae with n alternations of quantifiers.

3

Consider a structure $\mathfrak{A}$ on $A = \{a_0, a_1, \ldots\}$. For each $i < \omega$ define a unary relation E_i on $\mathfrak{A}$ by: $\mathfrak{A} \models E_i(a_j)$ if and only if $a_i = a_j$.

Let Γ be an r.e. set with $\{P_i\colon i < \omega\} \subseteq \Gamma \subseteq \mathrm{Form}(\mathfrak{L})$, and $\bar{\Gamma} = \{E_i\colon i < \omega\}$. We show that if $\mathfrak{A}$ is $\bar{\Gamma}$-r.e., the following are equivalent.

i) Every isomorphism from $\mathfrak{A}$ to a Γ-r.e. structure is a rec. isomorphism.

ii) For every Γ-r.e. structure $\mathfrak{B}$ and isomorphism $f\colon \mathfrak{A} \cong \mathfrak{B}$, the relations $\{fE_i\}$ are uniformly r.e. on $\mathfrak{B}$.

PROOF. Clearly i) $\Rightarrow$ ii).

Let $\mathfrak{B}$ be a Γ-r.e. structure, and f: $\mathfrak{A} \cong \mathfrak{B}$. For $a_i \in A$ we wish to find $f(a_i)$. Use the fact that fE_i is r.e. to find $b \in B$ such that $\mathfrak{B} \models fE_i(b)$. This means that $\mathfrak{A} \models E_i(f^{-1}(b))$; that is, $f^{-1}(b) = a_i$. Thus $b = f(a_i)$.

Notice that ii) is statement 1) of Theorem I′ for $\Gamma_1 = \Gamma$ and $\Gamma_2 = \bar{\Gamma}$. Applying this theorem to Γ and $\bar{\Gamma}$ we get

THEOREM III. *Let* Γ *be an r.e. set with* $\{P_i: i < \omega\} \subseteq \Gamma \subseteq \text{Form}(\mathfrak{L})$ *and* $\bar{\Gamma}$ *as above. If* $\mathfrak{A}$ *is* $\exists \bigwedge (\bar{\Gamma} \cup \Gamma)$*-rec.; the following are equivalent.*

1) *Every isomorphism from* $\mathfrak{A}$ *to a* Γ*-r.e. structure is a recursive isomorphism.*

2) *For some* $\bar{a} \subseteq A$ *there is an effective procedure which when applied to any* $a_i \in A - \bar{a}$ *produces a* $\exists \bigwedge \Gamma$*-formula* $\psi(\bar{x}, y)$ *such that* $\mathfrak{A} \models \psi(\bar{a}, a_j)$ *if and only if* $a_i = a_j$.

In the case when $\Gamma = \{P_i: i < \omega\} \cup \{\neg P_i: i < \omega\}$, Theorem III gives

COROLLARY 3.1 (Ash and Nerode [1], Goncharov [3]). *For a* 1*-rec. structure* $\mathfrak{A}$*, the following are equivalent.*

1) *Every isomorphism from* $\mathfrak{A}$ *to a rec. structure is a rec. isomorphism.*

2) *For some* $\bar{a} \subseteq A$ *there is an effective procedure which when applied to any* $b \in A - \bar{a}$ *produces an existential formula* $\psi(\bar{x}, y)$ *such that*

$$\mathfrak{A} \models \forall(\psi(\bar{a}, y) \leftrightarrow y \equiv b).$$

Theorem III may be applied to other sets Γ to produce similar results. Some interesting cases are $\Gamma = \{P_i: i < \omega\}$ (to get a result for r.e. structures), and Γ the set of formulae with n alternations of quantifiers.

4. Addendum

The characterizations presented in the previous sections are all subject to certain decidability assumptions. Ash and Nerode [1] discuss some cases in which the decidability assumption in Theorem I′ may be reduced. Goncharov [4] has shown that in the general case these decidability assumptions cannot be completely removed.

Theorem II has been discussed in particular cases. LaRoche proved that a recursive Boolean algebra is recursively categorical if and only if it has a finite number of atoms. (A proof of this result may be found in Remmel [7].) Remmel [8] showed that a recursive linear order is recursively categorical if and only if it has a finite number of successivities. These conditions are equivalent, in these cases, to statement 2) of Theorem II. However, the results do not follow from Theorem II as there are linear orders and Boolean algebras which do not have any structures in their isomorphism type with the decidability required in Theorem II. Goncharov and Dzgoev [2] have generalized these results to produce a condition (branching) sufficient for a rec. structure not to be recursively categorical. Goncharov [5] uses this condition to characterize recursively categorical Abelian p-groups, and a similar condition to characterize recursively categorical structures in a language consisting only of unary predicates.

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