Recursive Random-Sampling Strategy for a Digital Wattmeter

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***Abstract*—A recursive, random-sampling strategy is proposed for the implementation of a digital broadband wattmeter. In this strategy each sampling instant is obtained by adding to the preceding one a predetermined constant lag plus a random increment. In order to correlate the measurement uncertainty to the bandwidth, the asymptotic mean-square error arising from the sampling strategy and the filtering algorithm is evaluated and analyzed; it has been shown that the proposed sampling strategy does not limit the bandwidth of the instrument if an appropriate statistical distribution of the random increments is selected. The theoretical results are compared with those obtained by simulating the measurement process.**

I. Introduction

When the primary objective is the design of a broadband sampling wattmeter and the waveform of the input signals can be almost periodic, it is convenient to use a random asynchronous sampling strategy. In fact, the bandwidth of a strategy using asynchronous equally spaced samples is certainly smaller than the sampling frequency [1], [2]. In a previous paper [3] we proposed a random asynchronous sampling strategy which does not limit the bandwidth of the input signals; in this hypothesis the bandwidth is limited only by that of the sample-and-hold (S/H) devices and of the input signal-conditioning circuits. The experimental results obtained with a prototype wattmeter confirmed the theoretical results [4]. This strategy had however the drawback of requiring a pair of S/H devices and ADC's for each channel (voltage and current). In this paper a new random-sampling strategy, of a recursive type, is proposed. It has the same property of the previous one, i.e., it does not limit the bandwidth of the wattmeter (Section IV), and has the advantage that it can be implemented using only one S/H and one ADC for each channel (Section II).

To evaluate the performance of the proposed wattmeter we use a criterion which allows comparing, in terms of accuracy and bandwidth, the recursive sampling strategy with all nonrecursive ones [2], [5], [6]. By using this criterion, uncertainty is quantified by the asymptotic, mean-square error of the output referred to the true value of the measurand, i.e., mean power. This error takes into account not only all the possible measurement occurrences, but also the variability of the unknown nuisance parameters introduced by the measurement method (Section II). The asymptotic mean-square error, evaluated considering the frequency content of the instantaneous power, satisfies the superposition principle, in the sense that the square of the rms value of any spectral component contributes additively through a weighting coefficient to the final error (Section III). These weighting coefficients are picked up from a continuous function of frequency, which we call the weighting function; this function, therefore, characterizes the performance of the instrument also with respect to the bandwidth. The expression of the weighting function for the proposed recursive, random-sampling strategy is deduced and discussed, and the results of the theoretical study are compared with simulation findings (Section IV).

Manuscript received May 14, 1992; revised August 10, 1992. This work was supported by the Italian Ministry of University, Scientific and Technological Research.

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IEEE Log Number 9204513.

II. The Recursive Sampling Strategy

Let us suppose that the instantaneous power $p(t)$ has a discrete spectrum and can be expressed by a generalized Fourier series in the form:

$$p(t) = P_0 + \sum_{\substack{q=-\infty \\ q \neq 0}}^{+\infty} P_q \exp(j\omega_q t), \tag{1}$$

where $\omega_{-q} = -\omega_q$, $P_{-q} = P_q^*$ coincides with the complex conjugate of P_q, and the different ω_q do not necessarily have a common submultiple. In the hypothesis that the different angular frequencies do not have a common submultiple, the signal is called almost-periodic. The DC quantity P_0, which constitutes the true value of the measurand, is defined by:

$$P_0 = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} p(t)\, dt. \tag{2}$$

When the converted signal is periodic with period $T_1 = 2\pi/\omega_1$, then $\omega_q = q\omega_1$, and the DC quantity P_0 can be calculated without considering the limit in (2) if T_0 coincides with the period of the signal ($T_0 = T_1$).

The digital wattmeter described here is based on a recursive, random-sampling strategy in which every sam-

0018-9456/92$3.00

pling instant t_i is obtained by adding to the preceding one a predetermined constant lag T_c plus a random increment X_iT_c:

$$\mathbf{t}_i = \mathbf{t}_{i-1} + T_c(1 + \mathbf{X}_i). \quad (3)$$

The random variables are written in boldface letters, and the continuous random variable X_i is the ith of a set of stationary independent increments with a common characteristic function denoted by:

$$\Phi(\omega T_c) = E\{\exp(j\omega T_c \mathbf{X}_i)\}, \quad (4)$$

where $E\{\cdot\}$ is the expected value. In (4) we have considered the argument ωT_c, since X_i contributes to the sampling instant through the multiplicative factor T_c. The set of the sampling instants can be interpreted as a stochastic point process. The lag T_c is not correlated with any spectral component of the instantaneous power $p(t)$; therefore, this recursive, random-sampling strategy is of an asynchronous type. Obviously, if $\mathbf{X}_i = 0$ in (3), the sampling strategy becomes an equally spaced type losing the characteristic of recursivity.

The lag T_c in (3) has been introduced so that the time interval between two successive samples is never smaller than T_c; therefore, if the value selected for T_c is not less than the maximum data acquisition and processing time of the digital hardware adopted, a real-time instrument can be obtained using only one S/H device and one ADC for each of the two channels (voltage and current).

The discrete output of the instrument is an estimate of P_0 and can be expressed by:

$$\hat{\mathbf{P}}_k = \sum_{i=0}^{N-1} a_i\, p(\mathbf{t}_{k-i}) \quad (5)$$

which defines any linear time-invariant finite-impulse response filter (FIR filter); the coefficients a_i must be conveniently selected to achieve the prescribed filter characteristics [7]. The index k is an integer which marks a generic output randomly picked up from a sequence of $2h + 1$ successive outputs ($-h \le k \le +h$), each of which has an equal chance of being selected [2], [3]. So we can introduce $\mathbf{k}$ as a discrete random variable uniformly distributed in the interval $(-h, +h)$.

The initial one of the N sampling instants used to generate the first value of the output sequence ($k = -h$) is given by:

$$\mathbf{t}_{-h-N+1} = \tau + T_c(1 + \mathbf{X}_{-h-N+1}), \quad (6)$$

where τ is the unknown shift between the initial sampling instant and the time origin of $p(t)$. In the asynchronous case any realization of τ is independent of the instantaneous power $p(t)$, and it is strictly related to the turn-on instant of the instrument. Therefore, any actual value of τ may be assumed as a representation of a continuous random variable uniformly distributed in some generic time interval ($-T/2$, $+T/2$, T being unknown) [6].

Beginning at the first instant (6) and applying (3) recursively we obtain:

$$\mathbf{t}_{\mathbf{k}-i} = \left[(\mathbf{k}-i+h+N) + \sum_{r=-h-N+1}^{\mathbf{k}-i} \mathbf{X}_r\right] T_c + \boldsymbol{\tau}. \quad (7)$$

By substituting (1) and (7) into (5) we can write:

$$\hat{\mathbf{P}}_{\mathbf{k}} = \sum_{i=0}^{N-1} a_i P_0 + \sum_{\substack{q=-\infty \\ q\neq 0}}^{+\infty} P_q \cdot \exp\left(j\omega_q\left[(\mathbf{k}+h+N)T_c + \boldsymbol{\tau}\right]\right) \sum_{i=0}^{N-1} a_i \cdot \exp(-j\omega_q i T_c) \exp\left(j\omega_q T_c \sum_{r=-h-N+1}^{\mathbf{k}-i} \mathbf{X}_r\right). \quad (8)$$

In order to obtain a scale factor equal to one, it is necessary to impose that:

$$\sum_{i=0}^{N=1} a_i = 1. \quad (9)$$

III. The Performance Analysis

An appropriate characterization of the output uncertainty can be obtained by evaluating the statistical parameters of the output $\hat{\mathbf{P}}_{\mathbf{k}}$, i.e., the mean value $E\{\hat{\mathbf{P}}_{\mathbf{k}}\}$ and the mean-square error $E\{(\hat{\mathbf{P}}_{\mathbf{k}} - P_0)^2\}$. In order to incorporate all the *a priori* chances and also to avoid the influence of the conventional time origin on the instrument performance, the number $(2h + 1)$ of the output states and the excursion T of the initial shift τ must be sufficiently large and theoretically must tend to infinite. Therefore, we consider the asymptotic statistical parameters, i.e., the asymptotic mean:

$$\overline{P} = \lim_{\substack{h\to\infty \\ T\to\infty}} E\{\hat{\mathbf{P}}_{\mathbf{k}}\} \quad (10)$$

and the asymptotic mean-square error:

$$e^2 = \lim_{\substack{h\to\infty \\ T\to\infty}} E\{(\hat{\mathbf{P}}_{\mathbf{k}} - P_0)^2\}. \quad (11)$$

In the following only the errors arising from the sampling strategy and the filtering procedure are considered [8], because this paper aims to deduce the specific properties of the proposed sampling strategy and to compare the strategy with the other ones. It can be shown (see Appendix) that the output of the instrument is asymptotically unbiased ($\overline{P} = P_0$) and, consequently, the asymptotic mean-square error coincides with the asymptotic variance ($e^2 = \sigma^2$) whose final expression is:

$$\sigma^2 = 2\sum_{q=1}^{\infty} |p_q|^2\, W^2(f_q T_c). \quad (12)$$

This equation shows, according to the superposition principle, that the contribution of the squared rms value of each harmonic component f_q is weighted by the nonnegative coefficient $W^2(f_q T_c)$. The sequence of the weighting coefficients $W^2(f_1 T_c)$, $W^2(f_2 T_c)$, $\cdots$ can be derived from a continuous weighting function $W^2(fT_c)$ by determining its values at the successive normalized frequencies f_1T_c, f_2T_c, $\cdots$ The lag T_c being a constant not correlated with any spectral component of the instantaneous power, the products f_1T_c, f_2T_c, $\cdots$ can assume every positive real value.

Previously it was found that (12) is valid for other nonrecursive sampling strategies [3], [4]; we can now hypothesize that a corresponding weighting function can be deduced for each sampling strategy, recursive or not, random or equally spaced. Thus the behavior of the weighting function $W^2(fT_c)$ as a function of fT_c and the possible points at which the weighting coefficients are evaluated completely describe the performance of each sampling strategy with the associated filtering algorithm.

IV. The Weighting Function

The final expression of the weighting function for the recursive sampling strategy defined by (3) can be deduced from (A26) (see Appendix) by replacing the discrete variable f_q by a continuous one f:

$$W^2(fT_c) = \sum_{i=0}^{N-1} a_i^2 + 2\sum_{r=1}^{N-1}\sum_{i=r}^{N-1} a_i a_{i-r} \cdot \mathrm{Re}\,[\exp(j2\pi r f T_c)\,\Phi^r(2\pi f T_c)]. \quad (13)$$

The weighting function depends on the coefficients a_i of the FIR filter and on the time-shifted characteristic function of the random increments. Assuming that the random increments are uniform between 0 and b, the characteristic function becomes:

$$\Phi(2\pi fT_c) = E\{\exp(j2\pi fT_c\mathbf{X}_i)\} = \exp(j\pi fbT_c)\,\mathrm{sinc}\,(bfT_c). \quad (14)$$

In this hypothesis the mean sampling interval is $(1 + b/2)T_c$, and the mean response time of the wattmeter, in which the N samples of the input signals are taken in order to estimate the mean power, results in $T_m = (N - 1)(1 + b/2)T_c$. By using an N-point rectangular window to realize the FIR filter ($a_i = 1/N$), (13) after some manipulations can be rewritten as follows (see in Appendix (A27)):

$$W^2(fT_c) = \frac{1}{N} + \frac{2}{N^2}\sum_{r=1}^{N-1}(N - r) \cdot \cos\left[2\pi r\left(1 + \frac{b}{2}\right)fT_c\right]\mathrm{sinc}^r\,(bfT_c). \quad (15)$$

For $f = 0$ and $f \to \infty$ we obtain $W^2(0) = 1$ and $W^2(fT_c) \to 1/N$, respectively; further, for $bfT_c = p$, with p a positive integer, we get $W^2(p/b) = 1/N$. The plot of the weighting function for $N = 10$ and for different values of b ($b = 0.5, 1, 2$) is shown in Fig. 1. This very low value of N has been chosen only to show the shape of the weighting function more clearly. We note that the function oscillates around the value $1/N$. For increasing values of b, i.e., the range of the random increments $\mathbf{X}_i$ in (3), the maximum of the weighting function, excluding the value at the null frequency, reduces in magnitude and shifts towards the origin. Obviously, the overshoot above $1/N$ must not exceed an acceptable threshold in order to contain the contribution of each spectral component to the asymptotic variance through the corresponding weighting coefficients. On the other hand, higher values of b increase the mean response time of the wattmeter, and consequently b must be selected as small as possible.

By excluding the values near the origin, (15) can be approximated for N sufficiently large by the following relation (see in Appendix (A28)):

$$W^2(fT_c) \approx \frac{1}{N}\,\frac{1 - \mathrm{sinc}^2(bfT_c)}{1 + \mathrm{sinc}^2(bfT_c) - 2\,\mathrm{sinc}^2(bfT_c)\cos\left[2\pi\left(1 + \frac{b}{2}\right)fT_c\right]}. \quad (16)$$

The frequency corresponding to the maximum W^2_{Max} of the weighting function given by (16) does not depend on the number N, but is influenced only by the selected range b. The amplitude of this maximum is inversely proportional to N; thus the product of W^2_{Max} and the mean response time of the wattmeter normalized to T_c ($T_m/T_c \approx N[1 + b/2]$) is independent of N and can be plotted as a function of b (Fig. 2). This product attains a minimum for $b \approx 1.5$, where $W^2_{\mathrm{Max}} \approx 1.5/N$; therefore, this value of the range b allows us optimal control of both the asymptotic variance and the mean response time of the wattmeter. However, the choice of b is not critical, as it can be confirmed by the flatness of the minimum in Fig. 2. Finally, we have simulated the measurement process in order to estimate the weighting coefficients at different frequencies of sinusoidal inputs. Fig. 3 plots, for $b = 1.5$ and $N = 10$, the weighting function (solid line) corresponding to (15), the approximated one (dashed line) from (16) and the weighting coefficients (solid dots) estimated by simulation considering $4*10^3$ successive outputs of the wattmeter. This figure emphasizes both the optimal approximation gained by using (16) and the goodness of the comparison between the estimated weighting coefficients and the corresponding points of the theoretical weighting function.

By referring to our previous papers which describe a nonrecursive random-sampling strategy with a uniform distribution of the sampling instants within each interval T_c [3], [5], [6], we can observe that the shape of the weighting function given by (15) is different from that of the nonrecursive one only for the presence of a small rip-

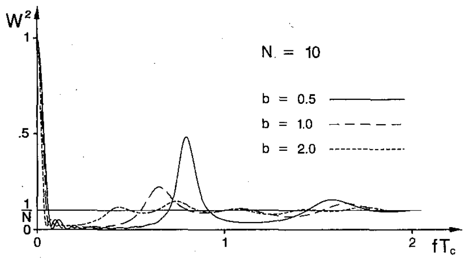

Fig. 1. Plot of the weighting function (15) for $N = 10$ and three different values of b.

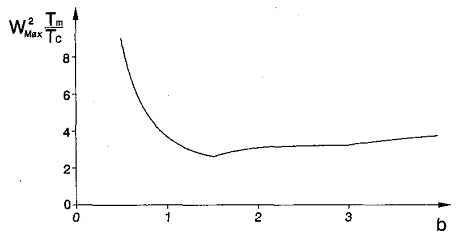

Fig. 2. Product between the absolute maximum (W^2_{Max}) of the weighting function approximated by the (16) and the mean response time T_m of the wattmeter, normalized to T_c, versus the range b.

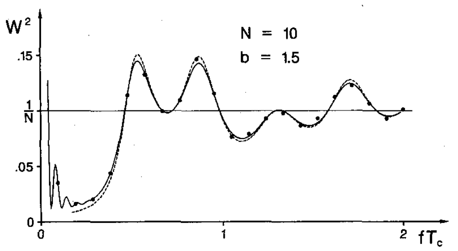

Fig. 3. Comparison between the theoretical weighting function (solid line), the approximated one (dashed line) and the weighting coefficients (black dots) estimated by simulation considering $4 \cdot 10^3$ successive outputs of the wattmeter.

ple around the value $1/N$ (Fig. 3). Therefore, both these sampling strategies do not introduce any limitation in frequency. The recursive, random-sampling strategy has however two advantages over the nonrecursive one: the value of the range b is not critical, and only one S/H and one ADC are required for each channel.

V. Conclusions

The digital wattmeter described here is based on a recursive, random-sampling strategy in which two successive sampling instants cannot differ by an interval smaller than a predetermined value T_c; this interval makes possible an instrument which can operate in real time with only one S/H and one ADC for each of the two channels. To evaluate the performance of the instrument, the asymptotic variance of the output was computed, and it was shown that it can be deduced with the same simple formula previously obtained for other sampling strategies and filtering algorithms. In fact, the asymptotic variance can be expressed as the sum of the contributions of each spectral component of the instantaneous power, weighted by coefficients which can easily be deduced from a continuous function which we call the weighting function. Therefore, the comparison between the different sampling strategies and filtering algorithms can be made by comparing the weighting functions. The recursive random strategy has been implemented by supposing the random-sampling increments uniformly distributed within a finite time interval correlated with a predetermined lag time T_c. It was also shown that the optimum value of this time interval, for an excellent compromise between the measurement uncertainty, quantified through the asymptotic variance, and the measurement time, is not critical. From the shape of the weighting function corresponding to this optimum time interval it can be deduced that the bandwidth of the instrument is not limited by the sampling strategy itself, i.e., by the conversion time of the ADC devices, but only by the bandwidth of the S/H devices and of the input conditioning circuits. The contribution to the uncertainty of the spectral components of the converted signal depends only on the size N of the window used to design the FIR filters, i.e., on the time available for each measurement.

Appendix

By assuming:

$$\mathbf{Y}_r = (1 + \mathbf{X}_r)T_c, \tag{A1}$$

(8) with the condition expressed by (9) can be written in the following simple manner:

$$\hat{\mathbf{P}}_{\mathbf{k}} = P_0 + \sum_{i=0}^{N-1} a_i \sum_{\substack{q=-\infty \\ q \neq 0}}^{+\infty} P_q \exp(j\omega_q \tau) \sum_{i=0}^{N-1} a_i \mathbf{Y}_{\mathbf{k}i}, \tag{A2}$$

where

$$\mathbf{Y}_{\mathbf{k}i}(\omega_q) = \exp\left(j\omega_q \sum_{r=-h-N+1}^{\mathbf{k}-i} \mathbf{Y}_r\right). \tag{A3}$$

Let us introduce the characteristic function of $\mathbf{Y}_r$:

$$\Phi_Y(\omega_q) = E\{\exp(j\omega_q \mathbf{Y}_r)\} = \exp(j\omega_q T_c)\,\Phi(\omega_q T_c) \tag{A4}$$

where $E\{\cdot\}$ is the expected value, and $\Phi(\omega_q T_c)$ is defined by (4). The expected value of $\mathbf{Y}_{\mathbf{k}i}$ (A3), taking into account that random sums of random variables [9] are involved, can be derived by using the identity:

$$E\{\mathbf{Y}_{\mathbf{k}i}(\omega_q)\} = E\{E\{\mathbf{Y}_{\mathbf{k}i}(\omega_q) \| \mathbf{k}\}\} \tag{A5}$$

where $E\{\cdot \| \cdot\}$ is the conditional expected value; consequently, for (A4) and recalling that the sequence of identical distributed random variables $\mathbf{Y}_r$ is independent of $\mathbf{k}$, we have:

$$E\{\mathbf{Y}_{\mathbf{k}i}(\omega_q)\} = \Phi_Y^{N-1}(\omega_q) \frac{1}{2h+1} \frac{\Phi_Y^{2h+1}(\omega_q) - 1}{\Phi_Y(\omega_q) - 1} \tag{A6}$$

due to the properties of the geometric progression. By recalling that $|\Phi_Y(\omega_q)| \leq 1$, the limiting value of (A6) for

h tending to infinity is:

$$\lim_{h\to\infty} E\{\mathbf{Y_{ki}}(\omega_q)\} = \begin{cases} 1 & \text{when } \Phi_Y(\omega_q) = 1 \\ 0 & \text{elsewhere.} \end{cases} \quad \text{(A7)}$$

Now we can deduce the expected value of $\hat{\mathbf{P}}_\mathbf{k}$. From (A2) one gets:

$$E\{\hat{\mathbf{P}}_\mathbf{k}\} = P_0 + \sum_{\substack{q=-\infty \\ q\neq 0}}^{+\infty} P_q \operatorname{sinc}(f_q T) \sum_{i=0}^{N-1} a_i \cdot E\{\mathbf{Y_{ki}}(\omega_q)\} \quad \text{(A8)}$$

because $\omega_q = 2\pi f_q$ and:

$$E\{\exp(j\omega_q\tau)\} = \frac{1}{T}\int_{-T/2}^{+T/2} \exp(j\omega_q t)\, dt = \operatorname{sinc}(f_q T) \quad \text{(A9)}$$

since τ is independent of $\mathbf{Y_{ki}}(\omega_q)$. By recalling (A7), the limiting value of $E\{\hat{\mathbf{P}}_\mathbf{k}\}$ for h and T tending to infinite, i.e., the asymptotic mean, is

$$\overline{P} = \lim_{\substack{h\to\infty \\ T\to\infty}} E\{\hat{\mathbf{P}}_\mathbf{k}\} = P_0 \quad \text{(A10)}$$

due to the fact that $\lim_{T\to\infty} \operatorname{sinc}(f_q T) = 0$, since f_q is always different from zero.

Now we can deduce the square error $(\hat{\mathbf{P}}_\mathbf{k} - P_0)^2$:

$$\begin{aligned}(\hat{\mathbf{P}}_\mathbf{k} - P_0)^2 &= \sum_{\substack{q,q'=-\infty \\ q,q'\neq 0}}^{+\infty} P_q P_{q'} \exp[j(\omega_q + \omega_{q'})\tau] \cdot \sum_{i,i'=0}^{N-1} a_i a_{i'} \mathbf{Y_{ki}}(\omega_q)\, \mathbf{Y_{ki'}}(\omega_{q'}) \\ &= \sum_{\substack{q,q'=-\infty \\ q,q'\neq 0}}^{+\infty} P_q P_{q'} \exp[j(\omega_q + \omega_{q'})\tau] \cdot \left[\sum_{i=0}^{N-1} a_i^2 \mathbf{Y_{ki}}(\omega_q + \omega_{q'}) + 2\sum_{\substack{i,i'=0 \\ i>i'}}^{N-1} a_i a_{i'} \mathbf{Y_{ki}}(\omega_q + \omega_{q'})\, \mathbf{Z_{kii'}}\right]\end{aligned} \quad \text{(A11)}$$

where we have distinguished the contributions to the sums for $i = i'$ and $i \neq i'$. Besides we have set:

$$\mathbf{Z_{kii'}} = \exp\left[j\omega_{q'} \sum_{r=\mathbf{k}-i+1}^{\mathbf{k}-i'} \mathbf{Y}_r\right] \quad \text{(A12)}$$

and we have taken into account that changing the indexes q with q' and/or i with i' does not alter the expressions.

Now let us consider the following expected value:

$$\begin{aligned}E\{\mathbf{Y_{ki}}(\omega_q + \omega_{q'})\,\mathbf{Z_{kii'}}\} &= E\{E\{\mathbf{Y_{ki}}(\omega_q + \omega_{q'})\,\mathbf{Z_{kii'}} \| \mathbf{k}\}\} \\ &= E\{\mathbf{Y_{ki}}(\omega_q + \omega_{q'})\} \cdot \Phi_Y(\omega_q)^{i-i'}.\end{aligned} \quad \text{(A13)}$$

This last expression is a consequence of the fact that the set of random variables $\mathbf{Y}_r$ which appear in the definition of $\mathbf{Z_{kii'}}$ (A12) is independent of the set of variables $\mathbf{Y}_r$ which appear in $\mathbf{Y_{ki}}(\omega_q + \omega_{q'})$; besides:

$$E\{\mathbf{Z_{kii'}} \| \mathbf{k} = k\} = \Phi_Y(\omega_q)^{i-i'}. \quad \text{(A14)}$$

On the other hand, by remembering (A7), it follows:

$$\lim_{h\to\infty} E\{\mathbf{Y_{ki}}(\omega_q + \omega_{q'})\} = \begin{cases} 1 & \text{for } \Phi_Y(\omega_q + \omega_{q'}) = 1 \\ 0 & \text{elsewhere.} \end{cases} \quad \text{(A15)}$$

By recalling (A13) and by applying (A9) with $(\omega_q + \omega_{q'})$ instead of ω_q, we can now evaluate the mean-square error, i.e., the expected value of (A11):

$$\begin{aligned}E\{(\hat{\mathbf{P}}_\mathbf{k} - P_0)^2\} = \sum_{\substack{q,q'=-\infty \\ q,q'\neq 0}}^{+\infty} P_q P_{q'} \operatorname{sinc}[(f_q + f_{q'})T] &\cdot \left[\sum_{i=0}^{N-1} a_i^2 E\{\mathbf{Y_{ki}}(\omega_q + \omega_{q'})\}\right. \\ &+ 2\sum_{\substack{i,i'=0 \\ i>i'}}^{N-1} a_i a_{i'} \cdot E\{\mathbf{Y_{ki}}(\omega_q + \omega_{q'})\} \\ &\left.\cdot\, \Phi_Y(\omega_q)^{i-i'}\right].\end{aligned} \quad \text{(A16)}$$

Taking into account (A14) and that

$$\lim_{T\to\infty} \operatorname{sinc}[(f_q + f_{q'})T] = \begin{cases} 0 & \text{for } f_q \neq -f_{q'} \\ 1 & \text{for } f_q = -f_{q'} \end{cases} \quad \text{(A17)}$$

and consequently the contribution to the asymptotic, mean-square error is non-null only for $f_q = -f_{q'}$ (and so $q' = -q$ being $f_{-q} = -f_q$), we conclude from (A16) that:

$$\lim_{\substack{h\to\infty \\ T\to\infty}} E\{(\hat{\mathbf{P}}_\mathbf{k} - P_0)^2\} = 2\sum_{q=1}^{\infty} |P_q|^2 W^2(f_q T_c). \quad \text{(A18)}$$

We have considered that $\Phi_Y(0) = 1$, $P_{-q} = P_q^*$ and:

$$W^2(f_q T_c) = \sum_{i=0}^{N-1} a_i^2 + 2\sum_{\substack{i,i'=0 \\ i>i'}}^{N-1} a_i a_{i'} R_e[\Phi_Y(\omega_q)^{i-i'}] \quad \text{(A19)}$$

represents a generic weighting coefficient. By noting that:

$$\sum_{\substack{i,i'=0 \\ i>i'}}^{N-1} = \sum_{i=1}^{N-1}\sum_{i'=0}^{i-1} = \sum_{\substack{r=1 \\ r=i-i'}}^{N-1}\sum_{i=r}^{N-1} \quad \text{(A20)}$$

and by recalling (A4), (A19) becomes:

$$W^2(f_q T_c) = \sum_{i=0}^{N-1} a_i^2 + 2\sum_{r=1}^{N-1}\sum_{i=r}^{N-1} a_i a_{i-r} \cdot \operatorname{Re}[\exp(j2\pi r f_q T_c)\Phi^r(2\pi f_q T_c)]. \quad \text{(A21)}$$

If the FIR filter is realized with a rectangular window

($a_i = 1/N$), the weighting coefficients can be written in the form:

$$W^2(f_q T_c) = \frac{1}{N} + \frac{2}{N^2} \sum_{r=1}^{N-1} (N - r) \cdot \text{Re} \left[\exp(j2\pi r f_q T_c) \Phi^r(2\pi f_q T_c)\right] = \frac{1}{N} + \frac{2}{N^2} \text{Re} \left\{ \sum_{r=1}^{N-1} (N - r) \cdot \exp(j2\pi r f_q T_c) \Phi^r(2\pi f_q T_c) \right\} \quad \text{(A22)}$$

where we have taken into account a simple property of the complex numbers. From this equation it can be deduced that $W^2(0) = 1$. To derive a simpler expression of the weighting coefficients we can proceed as follows. By imposing:

$$g = \exp(j2\pi r f_q T_c) \Phi^r(2\pi f_q T_c) \quad \text{(A23)}$$

and by recalling the expression of the arithmetic-geometric progression [10], (A22) becomes

$$W^2(f_q T_c) = \frac{1}{N} + \frac{2}{N^2} \text{Re} \left\{ \frac{g^{N+1} - Ng^2 + (N - 1)g}{(g - 1)^2} \right\}. \quad \text{(A24)}$$

By excluding the value $f_q T_c = 0$, where g is identically equal to one, one always gets $|g| < 1$. Consequently, if the number N of the samples of the input signals taken in order to estimate the mean power is sufficiently large, we can disregard the term g^{N+1}; besides, we can confound $N - 1$ with N in (A24). After simple manipulations we can deduce:

$$W^2(f_q T_c) \approx \frac{1}{N} \text{Re} \left\{ 1 + 2 \frac{-g^2 + g}{(g - 1)^2} \right\} = \frac{1}{N} \frac{1 - |g|^2}{1 + |g|^2 - 2 \,\text{Re}\,[g]}. \quad \text{(A25)}$$

By introducing (A23) we obtain:

$$W^2(f_q T_c) \approx \frac{1}{N} \frac{1 - |\Phi^r(2\pi f_q T_c)|^2}{1 + |\Phi^r(2\pi f_q T_c)|^2 - 2\text{Re}\left[\exp(j2\pi r f_q T_c) \Phi^r(2\pi f_q T_c)\right]}. \quad \text{(A26)}$$

If the common distribution of each random variable $\mathbf{X}_i$ defined in (3) is uniform within the interval $(0, b)$ so that the characteristic function of the random increments $\mathbf{X}_i T_c$ is given by (14) with $f = f_q$, from (A22) it can be deduced that the weighting coefficients can be written in the form:

$$W^2(f_q T_c) = \frac{1}{N} + \frac{2}{N^2} \sum_{r=1}^{N-1} (N - r) \cdot \text{Re} \left\{ \exp\left[j2\pi r \left(1 + \frac{b}{2}\right) f_q T_c \right] \cdot \text{sinc}^r (bf_q T_c) \right\} = \frac{1}{N} + \frac{2}{N^2} \sum_{r=1}^{N-1} (N - r) \cdot \cos\left[2\pi r \left(1 + \frac{b}{2}\right) f_q T_c \right] \text{sinc}^r (bf_q T_c). \quad \text{(A27)}$$

From (A26) it follows that this equation can be optimally approximated, for $f_q > 0$ and N sufficiently large, by the relation:

$$W^2(f_q T_c) \approx \frac{1}{N} \frac{1 - \text{sinc}^2 (bf_q T_c)}{1 + \text{sinc}^2 (bf_q T_c) - 2\text{sinc}^2 (bf_q T_c) \cos\left[2\pi \left(1 + \frac{b}{2}\right) f_q T_c \right]}. \quad \text{(A28)}$$

Acknowledgment

The authors are in debt to the Prof. C. Franchetti and Prof. U. Tiberio, University of Florence, Italy, for the important mathematical suggestions.

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