CAEPR
Working Paper
#2018-006

Recursive Utility and Thompson Aggregators, I: Constructive Existence Theory for the Koopmans Equation

Robert Becker
Indiana University

Juan Pablo Rincon-Zapatero
Universidad Carlos III de Madrid

July 16, 2018

Recursive Utility and Thompson Aggregators I: Constructive Existence Theory for the Koopmans Equation*

Robert A. Becker
Department of Economics
Indiana University
Bloomington, IN 47405
USA

Juan Pablo Rincón-Zapatero
Departamento de Economia
Universidad Carlos III de Madrid
28903 Getafe (Madrid)
Spain

July 16, 2018

Abstract

We reconsider the theory of Thompson aggregators proposed by Marinacci and Montrucchio [34]. We prove a variant of their Recovery Theorem establishing the existence of extremal solutions to the Koopmans equation. We apply the constructive Tarski-Kantorovich Fixed Point Theorem rather than the nonconstructive Tarski Theorem employed in [34]. We also obtain additional properties of the extremal solutions. The Koopmans operator possesses two distinct order continuity properties. Each is sufficient for the application of the Tarski-Kantorovich Theorem. One version builds on the order properties of the underlying vector spaces for utility functions and commodities. The second form is topological. The Koopmans operator is continuous in Scott's [40] induced topology. The least fixed point is constructed with either continuity hypothesis by the partial sum method. This solution is a concave function whenever the Thompson aggregator is concave and also norm continuous on the interior of its effective domain.

JEL Codes: D10, D15, D50,E21

Keywords: Recursive Utility, Thompson Aggregators, Koopmans Equation, Koopmans operator, Order Continuity, Tarski-Kantorovich Fixed

*This paper includes results appearing the first half of our 2017 working paper [11], "Recursive Utility and Thompson Aggregators." The working paper's second half covers our concave operator uniqueness theory. We thank the participants at the European Workshop in General Equilibrium Theory (Univeristy of Glasgow, June 2016 and Paris School of Economics, Paris 1, June 2018) for their helpful comments. Particular thanks go to Łukasz Balbus, Gaetano Bloise, Kevin Reffett, Yiannis Vailakis, and Łukasz Woźny for helpful conversations and suggestions. Comments by Karl Shlag and other partipants at a January 2018 University of Vienna theory workshop presentation are also gratefully acknowledged. RAB thanks Kevin Reffett for our many educational conversations over the last decade on order theory and its applications. The usual caveat applies.

Point Theorem, Extremal Solutions, Least Fixed Point Theory, Scott Topology, and Scott Continuous Koopmans operator.

1 Introduction

Recursive utility functions defined for discrete time, deterministic, and infinite horizon intertemporal choice problems have been studied intensively since their introduction by Koopmans ([25], [26], and [27]). Koopmans, Diamond and Williamson [28] extended that work. Koopmans showed a recursive utility function satisfied a particular functional equation, known today as the *Koopmans equation*. This equation relates the utility function to an *aggregator function* in two real variables, current consumption and future utility.

Lucas and Stokey [30] proposed taking the aggregator as the primitive concept. Using that function, the *Koopmans equation* in the unknown utility function is defined and a unique solution (in an appropriate function space) is sought. This solution *recovers* a unique recursive utility function representation of the underlying preference relation defined on the commodity space. This existence and uniqueness problem is solved by setting up a fixed point problem for the *Koopmans operator*. It is a selfmap defined on the given space of potential utility functions representing the underlying preference relation. They appeal to Banach's Contraction Mapping Principle. This yields yields the existence and uniqueness of the Koopmans equation's solution. In order to do so, the aggregator function must be carefully restricted in order to prove the Koopmans operator satisfies Blackwell's sufficient conditions for a contraction mapping. Generalizations of their approach are the subject of Boyd [14] and the monograph by Becker and Boyd [9]. Their work lays out the *recovery theory* for the class of *Blackwell aggregators*. More recently a literature on local contractions has expanded the recovery theorems for unbounded Blackwell aggregators that goes beyond the treatment in Becker and Boyd [9]. Rincón-Zapatero and Rodriguez-Palmero ([38],[39]) initiated the local contraction theory with additional results subsequently obtained by Martins-da-Rocha and Vailakis ([32],[33]).

The contraction mapping approach links to successive approximations as the tool for finding the solution as the limit of iterations of the Koopmans operator. The initial seed for this iterative procedure does not matter since any such initial condition's limit function is the same. When the initial seed is the zero function and the aggregator is bounded from below, then it turns out that the sequence of iterates approximates the fixed point from below. The calculation of the iterates in this scenario is called the *partial sum technique* in the literature. This approach is particularly fruitful because the Koopmans operator turns out to be a monotone operator and this sequence of iterates is a monotone sequence.

The monotonicity of the Koopmans operator holds for aggregator classes other than the Blackwell family.[1] Le Van and Vailakis [29] explore partial sum techniques for aggregators which might not be bounded from below. Their Koopmans operator is monotone. They also require the aggregator satisfy a Lipschitz condition in its second argument. They admit some cases where that

[1] Both Boyd [14] and Becker and Boyd [9] examine Blackwell aggregators that are not bounded from below. Partial sum techniques are applicable, but the implementation is sensitive to the structure of the underlying commodity space and the exclusion of consumption sequences taking the utility value negative infinity.

Lipschitz condition corresponds to no discounting, or even upcounting, in contrast to the usual discounted case. In the more common discounted case the Lipschitz condition is the utility discount factor for future utility. This Lipschitz constant's magnitude lies between 0 and 1. They find existence and uniqueness of the Koopmans operator's fixed point with an additional assumption.[2] Existence by a partial summation argument yields a particular utility function. Assuming the Lipshchitz contsant reflects the discounting of future utilties, then uniqueness holds provided a type of transversality condition obtains.

Marinacci and Montrucchio [34] introduced the new class of *Thompson aggregators* to distinguish them from the Blackwell aggregators. The Koopmans operator remains a monotone operator in this case. Thompson aggregators are economically reasonable, but fail to satisfy some properties required by various forms of contraction theorems in proving the Koopmans equation has a unique solution. The major issue concerns the Lipschitz condition required for admission to the Blackwell class. The Lipschitz condition on future utility fails altogether for one Thompson class.[3] In another Thompson class, a Lipschitz condition holds, but corresponds to upcounting, or possibly no discounting of future utility.[4] In both situations the contraction property breaks down. New techniques must be introduced in order to associate utility functions with Thompson aggregators.

They present a partial sum argument to prove there is a solution to the Koopmans equation. They also analyze the solutions of an auxilliary difference equation in utility values for each given consumption stream. This equation is shown to have a unique solution (with a restricted domain for that function). That solution is continuous in the product topology. This observation is useful for optimal growth problems. However, their product continuity proof depends on their existence and uniqueness theorems as well as the constructive derivation of the corresponding extremal fixed points.[5] Our approach rests on a direct analysis of the Koopmans operator's properties and how those features contribute to solving the Koopmans functional equation in a constructive manner.

We reexamine Marinacci and Montrucchio's existence result in our paper. We provide a rigourous foundation for the partial sum technique's success in proving the Koopmans equation has at least one solution for a given Thompson aggregator. The missing ingredient in their work concerns confirmation the Koopmans operator is order continuous. Moreover, important qualitative properties then follow from the partial sum approach.

The underlying monotone methods are not, by themself, sufficiently powerful to conclude the Koopmans equation has a unique solution. Iteration of the

[2] See Le Van and Vailakis ([29], Theorem 1, property *(b), p. 197*).

[3] In this case the interesting contribution by Le Van and Vailakis [29] does not apply.

[4] This case may overlap with the aggregator conditions in Le Van and Vailakis [29]. However, they also impose a strong condition, their assumption ($W5$), that might fail for a Thompson aggregator.

[5] See Marinacci and Montrucchio's [34] Theorem 5 and its corresponding proof. Bloise and Vailakis [13] follow their reasoning based on the the existence of extremal fixed points. They verify the product lower (upper) semicontinuity obtain for the least (greatest) fixed points.

Koopmans operator starting from a non-zero function may produce a different limit function than the one obtained by partial summation. The solution obtained by partial summation is the Koopmans operator's *least (smallest) fixed point*. We claim it should be singled out as the equation's *principal solution*.

Blackwell's sufficient condition for a contraction mapping assumes for recursive utility applications the Koopmans operator is a monotone self map. This property alone is sufficient, in many examples, to prove the existence of a solution in the stated function space. Marinacci and Montrucchio [34] separate the question of existence of a solution to the Koopmans equation from the determination of whether or not that solution is unique in the given space of possible utility representations.[6] Their existence proof turns on an application of the well-known Tarski Fixed Point Theorem [41] that yields the Koopmans operator's *least* and *largest fixed points*. These are the operator's *extremal fixed points*. Marinacci and Montrucchio [34] define an underlying space of possible utility functions that is an order interval in a space of bounded functions forming a Dedekind complete Riesz space. Their order interval is a complete lattice in the partial order induced by this function space.

Tarski's Theorem is nonconstructive. Marinacci and Montrucchio's iterative scheme "computing" the extremal fixed points by successive approximation may fail to yield those solutions. The missing ingredient is the requirement that the Koopmans operator enjoy an order continuity property. Our paper verifies this property holds in their setting. Absent such a proof, the extremal fixed points may only be found through transfinite induction. Although this is an iterative procedure, it is hardly a constructive one. Hence, it is desirable to prove a constructive version of their result in order to provide a foundation for computing approximate solutions to the Koopmans equation derived from Thompson aggregators. The notion of a constructive procedure as used here means successive approximations indexed on the natural numbers.

We consider two forms of order continuity. The first concept is the Koopmans operator satisfy *monotone sup/inf-preservation for monotone sequences (nondecreasing/nonincreasing, respectively)*. The second is *monotone nets sup-preservation*. The first is a purely order theoretic property dependent on order properties of the commodity vector space and the vector space of possible utility functions. The second property is topological as well as order theoretic. The particular topology is known as the *Scott (induced) topology*. This structure is important in the literature on foundations of computational theory. We exposit its main features in Section 5.3. The topological theory only applies to constructing the least fixed point belonging to the Koopmans operator on the given order interval. By contrast, *sup/inf-preservation for monotone sequences* constructs both the least and the largest fixed points, albeit starting from distinct initial seeds.

There are qualitative advantages to our constructive approach built on order

[6] They also use a contraction mapping with respect to the Thompson metric for their uniqueness results. Martins-da-Rocha and Vailakis ([32],[33]) also study Thompson aggregators and obtain the existence of unique solutions using a combination of local contraction with Thompson metric space structure.

continuity for the Koopmans operator. Marinacci and Montrucchio ([34], p. 1785) suggest this, but do not clearly defend it in their paper. They claim ([34], p. 1790) the largest one is sup norm upper semicontinuous as a real-valued function defined on the underlying commodity space. Given their claimed iterative procedure the least fixed point is likewise norm lower semicontinuous. We verify these conditions by application of the Tarski-Kantorovich Fixed Point Theorem which secures a foundation for constructing fixed points by iterative schemes. The order continuity property of the Koopmans operator also implies its set of fixed points is a countably chain complete partially ordered set. This property is the constructive analog of Tarski's conclusion that the set of fixed points is a complete lattice in its own right in the induced partial order. The countably chain complete property of the fixed point set is demonstrated for the general form of the Tarski-Kantorovich Theorem in Balbus, Reffett, and Woźny [6].

Section 2 offers a brief review of concepts on partially ordered sets, lattices, and positive cones in real Banach spaces. Next we recall the Tarski-Kantorovich Theorem and related concepts. The aggregator axioms and basic theory derived from Marinacci and Montrucchio [34] follow in Section 3. The fourth section includes our version of the Marinacci and Montrucchio existence theorem, which we term a *Recovery Theorem*. We separate the uniqueness question from the existence problem as each problem draws on different ways of combining order theoretic and topological structures derived from the model's formulation. In particular, special properties of Banach spaces are important in our approach to the uniqueness problem which are subsidiary in the existence theory. Moreover, our existence arguments accommodate economies capable of sustained growth whereas the uniqueness theory developed in our working paper only admits capital accumulation models with bounded growth paths.[7] We show by means of an example that the Koopmans equation may have multiple solutions.[8] Hence, some additional restrictions beyond those sufficient for existence are required for an adequate uniqueness theory. The least fixed point for concave Thompson aggregators is shown to exhibit concavity and weak continuity properties which are not necessarily found in the largest fixed point. We identify in Section 5 the least fixed point as the Koopmans operator's *principal solution* or *principal fixed point* following Kantorovich's [22] usage. Topological order continuity is one of the criteria we consider in our support for distinguishing the least and largest fixed points. We conclude with some thoughts for future research and comments on the necessity to extend our results to searching for continuous solutions to the Koopmans equation with weaker topologies for the commodity space than the norm topologies featured here. [9]

[7] Our uniqueness theory [11] draws on the second part of our working paper on Thompson aggregators. Our methods differ substantially from those in Marinacci and Montrucchio's working paper [35] based on contraction mappings in function spaces endowed with the Thompson metric.

[8] Bloise and Vailakis [13] also provide examples where uniqueness fails. Their first example is instructive in so far as our example is, for a different aggregator, making the same point. The extremal solutions might not agree when evaluated at the zero consumption sequence.

[9] Signigicant progress has been made on this point by Martins-da-Rocha and Vailakis ([32],

2 Mathematical Preliminaries

2.1 Posets, Lattices, and the Tarski-Kantorovich Theorem

A set X is said to be **partially ordered**, or a **poset**, if it is nonempty and for certain pairs (x, y) in $X \times X$ there is a binary relation $x \leq y$ which is reflexive, transitive, and antiysmmetric.[10]

A poset X is a **lattice** provided each pair of elements has a **supremum (sup, meet)** and an **infimum (inf, join)**. Standard lattice notation for sups and infs is followed: $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$. A **complete lattice** is a lattice in which each nonempty subset Y has a supremum $\bigvee Y$ and an infimum $\bigwedge Y$. The element $x \in Y$ is called **greatest,** or **largest** (**smallest,** or **least)** in Y if and only if $y \leq x$ $(x \leq y)$ respectively, for all $y \in Y$. Note that a complete lattice has a greatest element (**top**) and and a bottom element (**bottom**). An **order interval** in X, denoted by $\langle \underline{x}, \bar{x} \rangle \subseteq X$, is defined by $\underline{x} \leq \bar{x}$, $\underline{x} \neq \bar{x}$, and $x \in \langle \underline{x}, \bar{x} \rangle$ if and only if $\underline{x} \leq x \leq \bar{x}$. Clearly $\underline{x}$ is the least element of the order interval while $\bar{x}$ is the corresponding largest element.

Suppose that $Y \subseteq X$ and let X be a poset. The set Y is called a **chain** (of X) if and only if Y is nonempty and for all $x, y \in Y$, one of the two conditions $x \leq y$ or $y \leq x$ holds. If the chain is countable, then it is called a **countable chain**. Let $\{x^n\}_{n=0}^{\infty} \subset X$ be a monotone sequence (either $x^n \leq x^{n+1}$, or $x^n \geq x^{n+1}$ for each n). The monotone sequence $\{x^n\}_{n=0}^{\infty}$ is **increasing (decreasing)** when $x^n \leq x^{n+1}$ $(x^n \geq x^{n+1})$ for each n. A monotone sequence is a countable chain. The supremum and infimum of a monotone sequence are denoted in lattice notation as follows:

$$\bigvee_n x^n = \sup_n x^n; \text{ and } \bigwedge_n x^n = \inf_n x^n.$$

The subscript n in the meet and join notation is omitted when the index set is clearly understood from the context. If, for every chain $Y \subseteq X$, we have $\inf Y \equiv \bigwedge Y \in X$ and $\sup Y \equiv \bigvee Y \in X$, then X is said to be a **chain complete poset**. If this condition obtains only for every countable chain $Y \subseteq X$, then X is said to be a **countably chain complete poset**. If Y has greatest and smallest elements, then monotone sequences $\{x^n\} \subseteq Y$ are countably chain complete posets in Y.

A function $F : X \to X$ is said to be a **self-map on** X. By $F^N(x)$, we are denoting the $N^{th}-$iteration of F with initial seed x. That is, $F^N(x) = F\left(F^{N-1}(x)\right)$ for each natural number N and $F^0(x) \equiv x$. This self-map is said to be **monotone** whenever $x, y \in X$ and $x \leq y$, then $F(x) \leq F(y)$. Some writers refer to a monotone self-map as an **isotone self-map or an increasing self-map**. A point $x^* \in X$ with $F(x^*) = x^*$ is a fixed point of the self-map, F. The set of all fixed points of this self-map is denoted $\operatorname{fix}(F)$.

[33]) for Blackwell (and related) aggregators where local contraction arguments work well to recover a unique solution to the Koopmans equation.

[10] Birkhoff [12]and Davey and Priesley [16] cover the basic properties of posets and lattices.

The classical Tarski Fixed Point Theorem [41] asserts that a monotone self-map on a complete lattice has a nonempty set of fixed points. Moreover, there is a smallest and a largest fixed point. These are the **extremal fixed points**. The set of all fixed points forms a complete lattice in the induced order (the partial order inherited from X). Successive approximations iterating the monotone self-map by **transfinite induction** yields the largest fixed point with initial seed the top element, and the smallest fixed point when the bottom element is the initial seed.[11] Iteration using transfinite induction is not a constructive procedure in any sense of that term. The Tarski-Kantorovich Theorem is similar to Tarski's result, but combines a weaker property for the self-map's domain with a stronger order continuity condition imposed on the operator. That property implies the operator is a monotone self-map.

We consider two distinct forms of order continuity. The first is defined entirely in terms of the underlying order properties of our domain's (and range's) function space. This approach, introduced below, implies the set of fixed points is a countably chain complete subset of the operator's domain. The successive approximation procedure used in this result is constructive in so far as the iterations are indexed on the natural numbers in contrast to the transfinite iterative procedure underlying Tarski's Theorem. The second order continuity idea is topological and its recursive utility application is new.[12] This is the notion of continuity when the order interval of possible utility functions is endowed with Scott's induced topology. This topology's definition and the development of its properties as applied to the Koopmans operator are deferred to Section 5.3. Scott's topological structure yields a constructive foundation for the operator's least fixed point. We argue in Section 5 that this result reenforces the arguments supporting the least fixed point as the operator equation's **principal solution**.

Definition 1 *A self-map F defined on a countably chain complete poset X with the greatest element $\bar{x}$ and smallest element $\underline{x}$ is **monotonically sup-preserving** if for any increasing $\{x^n\}$ we have*

$$F\left(\bigvee x^n\right) = \bigvee F(x^n),$$

*and **monotonically inf-preserving** if for any decreasing $\{x^n\}$, we have*

$$F\left(\bigwedge x^n\right) = \bigwedge F(x^n).$$

*F is said to be **monotonically sup/inf-preserving** if and only if it is both mononically sup-preserving and monotonically inf-preserving.*

[11] Cousot and Cousot [15] provide a so-called constructive proof without monotonic sup-inf continuity. However, their argument employes transfinite induction. Echenique [17] simplifies their proof while maintaining a tranfinite induction argument. Gierz ([18], p.20) sketches an iterative least fixed point theorem that applies to a monotone self-map on complete lattice. However, that proof also employs transfinite induction indexed by the ordinals.

[12] See Vassilakis [42] for economic and game theoretic applications of Scott domains and Scott continuity (in terms of sequencs as opposed to nets).

Evidently, a monotonically sup (respectively, inf)-preserving self map on the ordered space X must be an increasing self-map. The sup-inf preservation property is a type of **order continuity** introduced in Kantrovich's [22] seminal article on monotone methods with successive approximations.[13] In the case of a monotonically increasing sequence the sup is regarded as the sequence's limit and continuity is taken to mean $F(\sup\{x^n\}) = \sup[\{F(x^n)\}]$ where the countable chain is denoted $\{x^n\}$. Likewise for the inf of a decreasing sequence. Some authors (e.g. Granas and Dugundji [19]) refer to order continuity as used here by the term $\sigma -$ **order continuity** to stress the restriction to countable chains and also drop the monotonicity requirement for the sequences. *The conclusions of the Tarksi-Kantorovich Theorem based on iteration indexed on the natural numbers can fail without order continuity. Davey and Priestley ([16], p.93) offer an elementary counterexample.*

The **Tarski-Kantorovich Fixed Point Theorem (TK FPT)** as refined by Balbus, Reffett and Woźny ([6], Theorem 7), states the following:[14]

Theorem 2 *Suppose that X is a countably chain complete partially ordered set with the greatest element, $\bar{x}$, and the smallest element, $\underline{x}$. Let F be a monotone self-map on X.*

1. *If F is monotonically inf-preserving; then $\bigwedge F^N(\bar{x})$ is the* ***greatest fixed point*** *of F, denoted x^{∞};*

2. *if F is monotonically sup-preserving; then $\bigvee F^N(\underline{x})$ is the* ***least fixed point*** *of F, denoted x_{∞}.*

3. *fix(F) is a nonempty countably chain complete poset in X.*

The result that fix(F) is a countably chain complete poset in X is due to Balbus, Reffett, and Woźny [6]. It is the analog of Tarski's result that fix(F) is a complete lattice in the induced order. The Tarski-Kantorovich theorem tells us that successive approximations (iteration of F indexed on the natural numbers) initiated at either the smallest or greatest element of the set X produces the smallest or largest fixed point in the limit, respectively. Moreover, it is clear that $x_{\infty} \leq x^{\infty}$. If x^* is any other fixed point for F, and $\underline{x} \leq x^*$, then $\underline{x} \leq F(\underline{x}) \leq F(x^*) = x^*$. Iteration produces the sequence $\left\{F^N(\underline{x})\right\}_{N=1}^{\infty}$ such that for each N, $F^N(\underline{x}) \leq x_{\infty} \leq x^*$ and $F^N(\underline{x}) \nearrow F(x_{\infty}) = x_{\infty} \leq x^*$. Hence, the fixed point x_{∞} is the **least fixed point (LFP)**. Likewise, x^{∞} is

[13] This notion of order continuity is an order theoretic concept for Riesz spaces. It is NOT a topological idea, although it is related to continuity of F in the Scott topology [40], as presented in Section 5.3. See Aliprantis and Border [1] for the Riesz space version of order continuity based on convergent nets. Vulikh [44] develops many themes from Kantorovich's [22] article.

[14] Granas and Dugundji([19], p. 26) name this result. The earliest published version is found in Kantorovich [22]. Baranga [8] presents it as the "Kleene Fixed Point Theorem." Jachymski et al ([20], p. 249) argues it is equivalent to the TK FPT. Also, see Stoltenberg-Hansen, et al ([43], p. 21) on Kleene's Fixed Point Theorem. Kamihigashi et al [21] apply the Kleene Fixed Point Theorem to dynamic programming.

the **greatest fixed point (GFP)**. The notation $F^N(\underline{x}) \nearrow F(x_\infty)$ indicates that $F^N(\underline{x})$ **approximates the LFP from below for each** N. Likewise, $F^N(\bar{x}) \searrow F(x^\infty) = x^\infty$ says $F^N(\bar{x})$ **approximates the GFP from above.**

2.2 Positive Cones and Nonlinear Operators in Riesz Spaces

Let E denote a real vector space. The zero element in E is denoted by θ. A nonempty subset P of E is said to be a **cone** if $x \in P$, then $\lambda x \in P$ for each scalar $\lambda \geq 0$.[15] In particular this definition of a cone implies $\theta \in P$. A cone induces a partial order on the vectors belonging to E. A vector x is said to be **positive**, written $x \geq \theta$, provided $x \in P$. The cone is then called the **positive cone** of E and is denoted by E^+ in the sequel. The standard partial relation expressing $x \geq y$ whenever $x, y \in E$ is defined by requiring $x - y \in E^+$. Write $x > \theta$ whenever $x \geq \theta$ and $x \neq \theta$. Likewise, $x > y$ provided $x \geq y$ and $x \neq y$.

Our application requires the vector spaces are Riesz spaces where E is equipped with the partial order derived from the cone E^+. A Riesz space is a partially ordered vector space that is also a lattice.[16] For each element $x \in E$, we define its **positive part**, x^+, its **negative part** x^-, and its **absolute value**, $|x|$, by the formulas:

$$x^+ = x \vee \theta, x^- = x \wedge \theta\,, \text{ and } |x| = x \vee (-x)\,.$$

An **order interval** in the Riesz space E is a set of the form $\langle x, y\rangle = \{z \in E : x \leq z \leq y\}$. A subset G of a Riesz space is **order bounded from above** if there is a $y \in E$ such that $z \leq y$ for each $z \in G$. The dual notion that this subset is order bounded from below is defined similarly. A subset of a Riesz space is **order bounded** if it is contained in an order interval. E is **order complete**, or **Dedekind complete**, if every nonempty subset that is order bounded from above has a supremum (and dually, every nonempty subset that is order bounded from below has an infimum).

Suppose further that E is a real Banach space. The notation $x >> \theta$ means $x \in int(E^+)$, where $int\,(E^+)$ denotes the norm interior of the cone E^+. Of course, this latter inequality is only meaningful when $int\,(E^+) \neq \varnothing$ — a strong topological restriction on the underlying Banach space. An arbitrary cone P contained in E with nonempty interior in its norm topology is said to be a **solid cone**. The positive cones turns out to be solid in our applications.

We consider an abstract nonlinear operator, denoted by A, that is positive on E^+. That is, it is a self-map: $A : E^+ \to E^+$. We also write this as $AE^+ \subseteq E^+$. The operator A is said to be **monotone (isotone, increasing) on** E^+ if $x \leq y, (x, y \in E^+)$ implies $Ax \leq Ay$. It is **antitone** whenever $Ax \geq Ay$ instead.

[15] This definition is not used by all authors (see Aliprantis and Tourky [4]). In our application P is defined in terms of a given partial order on a function space. Additional properties that might be imposed, such as P is norm closed, must be verified. Likewise, many cones are also convex and this property may be verified in examples.

[16] Riesz spaces are also known as vector lattices. Consult Aliprantis and Border ([1], Chapter 8) for a thorough review of Riesz spaces. We follow their terminology. All Riesz spaces appearing in our paper are **Archimedean**.

The Koopmans operator is shown in Section 4 to be **monotone** whenever the aggregator is also monotone in its arguments.

Given a nonlinear operator satisfying $AE^{+} \subseteq E^{+}$ we are concerned with the *existence of fixed points* as well as whether or not there is a *unique solution in the cone* E^{+}. The **operator equation** is $Ax = x$ with $x \in E^{+}$; a solution is a fixed point of the operator, A. In some applications there may be a trivial fixed point, θ. We are only interested in **nontrivial fixed points** $x \in E^{+}$ with $x \neq \theta$. The Koopmans operator does not admit a trivial fixed point under our assumptions.

The present paper addresses the existence of a solution in the cone E^{+}. We do this by showing the operator is an order contiuous self-map on a particular order interval in that cone. Application of the TK FPT yields extremal fixed points. Our uniqueness arguments are found in Becker and Rincón-Zapatero [11].

All spaces in this paper are complete normed Riesz spaces. They are also **Banach lattices**. That is, they are Riesz spaces which are Banach spaces whose norms are also lattice norms.[17] A norm $\|\bullet\|$ on a Riesz space is a **lattice norm** provided for each point x and y, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Indeed, the spaces on which the Koopmans operator acts turn out to be *abstract* $M-$ *spaces*, **or** $AM-$*spaces* **with an order unit**. $AM-$*spaces* are Banach lattices for which $\|x \vee y\| = \max\{\|x\|, \|y\| \text{ for each } x, y \in E^{+}\}$. An $AM-$ *space* E possesses an **order unit** whenever there exists an element $e \in E$, $e > \theta$, such that for each $x \in E$ there is a scalar $\lambda > 0$ satisfying $|x| \leq \lambda e$. If an $AM-$ *space* has a unit, then its lattice norm is defined for each $x \in E$ by $\|x\|_{\infty} = \inf\{\lambda > 0 : |x| \leq \lambda e\}$. This norm is equivalent to the given norm on E. One advantage to this setup is that the positive cone of an $AM-$ *space* with unit is norm-closed, convex and solid.[18]

3 Recursive Utility Theory for The Thompson Aggregator Class

Our development of Marinacci and Montrucchio's [34] Recovery Theorem begins with the defining properties of Thompson aggregators. We introduce a minor revision to their continuity axiom and emphasize the concavity of the aggregator in developing important additional properties of the principal solution to the Koopmans equation. Our continuity condition is critical to verifying the Koopmans operator is sup/inf preserving on its domain. This is the key step

[17] See Aliprantis and Border [1], Aliprantis and Burkinshaw [2], Meyer-Nieberg [36], Peressini [37], and Vulikh [44] for details on Riesz spaces and Banach lattices.

[18] The positive cone of an AM-space has a nonempty norm interior provided it is an AM-space with unit. See Aliprantis and Tourky ([4], p. 64) and Peressini ([37], p. 183). A Banach lattice has an order unit if and only if that order unit is an interior point of the space's positive cone. In this case, the original sup norm and lattice norm topologies are equivalent. See Meyer-Nieberg ([36], Corollary 1.2.14 for details).

differentiating our work from Marinacci and Montrucchio [34].[19]

3.1 Defining Properties of Thomson Aggregators

The class of Thompson aggregators is delineated by the following four basic assumptions.

Definition 3 *$W:\mathbb{R}^2_+ \to \mathbb{R}$ is said to be a* ***Thompson aggregator*** *if it satisfies properties (T1) – (T4):*

(T1) *$W \geq 0$, continuous, and monotone: $(x,y) \leq (x',y')$ implies $W(x,y) \leq W(x',y')$;*

(T2) *$W(x,y) = y$ has at least one nonnegative solution for each $x \geq 0$;*

(T3) *$W(x,\bullet)$ is concave at 0 for each $x \geq 0$, that is*

$$W(x,\mu y) \geq \mu W(x,y) + (1-\mu) W(x,0)$$

for each $\mu \in [0,1]$ and each $(x,y) \in \mathbb{R}^2_+$;

(T4) *$W(x,0) > 0$ for each $x > 0$.*

Our definition of a Thompson aggregator builds in the assumption that it is jointly continuous in (x,y) over $\mathbb{R}^2_+$. Marinacci and Montrucchio [34] prove that their definition of a Thompson aggregator is jointly continuous with y restricted to the open interval $(0,\infty)$. For technical reasons we require joint continuity as well as continuity at $y=0$ whatever value is assumed by x. They admit this in their Recovery Theorem's formal assumptions. We prefer to build this joint continuity assumption directly into the definition of a Thompson aggregator as the known examples satisfy it. *This assumption is critical to the verification that the Koopmans operator enjoys the order continuity property required for the Tarski-Kantorovich Theorem's application (and the corresponding proof for Scott continuity found in Section 5.3).* This condition also shows up in our demonstration that there is an upper semicontinuous (lower semicontinuous) extremal solution to the Koopmans equation using the Tarski-Kantorovich Theorem. For these reasons our existence argument differs from the parallel one given by Marinacci and Montrucchio built on the nonconstructive Tarski's Fixed Point Theorem [41]. In addition, they impose two additional properties formalized here as assumptions $(T5)$ and $(T6)$.[20] These conditions are essential ingredients to the proof of their recovery theorem. Both properties further restrict the class of Thompson aggregators from which an underlying recursive utility representation is possible.

The first additional condition imposed by Marinacci and Montrucchio is the aggregator be $\gamma-$ **subhomogeneous**.

[19] Marinacci and Montrucchio [35], p.2 appeal to Kantorovich's [22] original fixed point theorem for an existence result, but do not verify the required order continuity property obtains for the Koopmans operator.

[20] Their limit condition is assumed in their results, but it is not listed as a separate axiom.

(T5) *W is $\gamma-$ subhomogeneous — there is some $\gamma > 0$ such that:*

$$W\left(\mu^{\gamma}x, \mu y\right) \geq \mu W\left(x, y\right)$$

for each $\mu \in (0,1]$ and each $(x, y) \in \mathbb{R}_{+}^{2}$.

The *standard positive homogeneity* of degree γ aggregator functional form corresponds to the case where

$$W\left(\mu x, \mu y\right) = \mu^{\gamma} W\left(x, y\right)$$

for each $\mu > 0$ and each $(x, y) \in \mathbb{R}_{+}^{2}$. If the defining inequality in $(T5)$ is an equality, then we say W is $\gamma-$homogeneous. We turn to the second property required for the recovery theorem's proof.

(T6) *W satisfies the **MM-Limit Condition**: for a given $\alpha \geq 1$ and $\gamma > 0$* (from $(T5)$),

$$\lim_{t\to\infty} \frac{W\left(1, t\right)}{t} < \alpha^{-1/\gamma}, \tag{1}$$

with $t > 0$.

The parameter α in $(T6)$ is the economy's maximum possible consumption growth factor in applications. Condition $(T6)$ turns out to be an important **joint restriction** on the preferences embodied in the aggregator function as well as on the underlying commodity space, as might arise from properties of technologies in production economies and/or endowments in exchange economies.[21] Condition (1) may not obtain for an arbitrarily chosen member of the Thompson class given the α parameter's value. Joint restrictions of this type routinely appear in treatments of the Blackwell aggregator class. What is certainly true under assumptions $(T1) - (T5)$ is that

$$L \equiv \lim_{y\to\infty} \frac{W(1, y)}{y} \tag{2}$$

exists as the ratio $W\left(x, y\right)/y$ is decreasing in y and bounded below by zero as formally demonstrated by Marinacci and Montrucchio [34]. But, this limit, L, could be larger or smaller than $\alpha^{-1/\gamma}$. Certainly if $L = 0$, then (1) holds. We list satisfaction of the MM Limit Condition as an explicit axiom that might, or might not, obtain for a particular aggregator in order to emphasize that some restrictions may apply on the underlying model's deep preference and technology parameters.

[21] Becker and Boyd [9] cover many examples of models with commodity spaces arising in interemporal choice and consistent with our Thompson aggregator specification.

3.2 Examples of Thompson Aggregators

There are two important sources for examples. The KDW aggregator (defined below) has parameterizations placing it outside the Blackwell class and firmly in the Thompson family. There are also many new examples based on the Constant Elasticity of Substitution functional form for utility functions and production functions commonly studied in microeconomic theory. Both the CES and KDW examples illustrate the fine properties of Thompson aggregators that are also required to meet $(T5)$ and $(T6)$.

3.2.1 CES Aggregators

Standard utility theory for two, or more, goods suggests the CES class as a potential source for aggregators. Certainly, CES utility functions over two dated consumption goods, one good corresponding to today's consumption, and the other to tomorrow's consumption, are reasonable and widely applied in equilibrium theory. Indeed, these forms are often taken as the standard specifications! The Fisherian inspired reinterpretation of the aggregator's second argument as *future utility*, is the economic basis for our interest in aggregator models! This suggests introducing the corresponding class of **CES aggregators** defined by the formula:

$$W(x,y) = (1-\beta)\, x^{\rho} + \beta y^{\rho}, \text{ for } 0 < \rho \leq 1. \tag{3}$$

The parameter β is restricted — $0 < \beta < 1$. Note that this family of functions is positively homogeneous of degree ρ. The **elasticity of substitution** is $\sigma := 1/(1-\rho)\,; \rho \neq 1$. The restriction $0 < \rho < 1$ is required to insure W is both a positively homogeneous and concave function in the variables $(x, y) \in \mathbb{R}^2_+$ with $W(x,y) \geq 0$ and $W(0,0) = 0$. These aggregator functions are unbounded from above. This is an important point for developing an appropriate recovery theorem. Verification of property $(T3)$ also follows from the fact W is jointly concave in (x, y), a fact that may NOT be true for an arbitrary Thompson aggregator. This joint concavity condition plays a critical role in proving the smallest fixed point is a concave function on the commodity space. This property is critical for working with concave optimization techniques in optimal growth settings. The other Thompson aggregator criteria are met when $\sigma > 1$. A**ssume** this restriction applies without further notice. The CES Thompson aggregators are readily shown to satisfy $(T5)$ and $(T6)$. Note that $(T5)$ holds for $\gamma = 1$.

Routine calculations show that for the CES aggregator $W(x, \bullet)$ does **not** satisfy a Lipschitz condition in $y \geq 0$ whenever $0 < \rho < 1$. Just compute $W_2 \equiv \partial W/\partial y$ and note $\sup_{y \geq 0} W_2(x,y) = +\infty$. This aggregator specification fails to exhibit the discounting property qualifying it for Blackwell aggregator status.[22]

Marinacci and Montrucchio [34] introduce a four parameter family of aggre-

[22] Recall, this is the requirement $0 < \sup_{y \geq 0} W_2(x,y) < 1$ for differentiable aggregators such as the examples developed here.

gators which are variants of the CES class: set

$$W(x,y) = \left(x^{\eta} + \beta y^{\xi}\right)^{1/\rho}, \tag{4}$$

where $\eta, \xi, \rho, \beta > 0$. Conditions $(T1)$ and $(T4)$ always hold. If $\xi \leq 1$, then this aggregator IS a Thompson aggregator in two cases:

(i) $\xi < \rho$, or

(ii) $\xi = \rho$ and $\beta < 1$.

Property $(T5)$ holds with $\gamma = \xi/\eta$, provided $\xi \leq \rho$. In this case, the aggregator is γ-subhomogeneous. Property $(T3)$ follows provided $\xi \leq 1$ and $\xi \leq \rho$. Notice that this aggregator is jointly concave provided $\eta \leq 1, \rho \leq 1$ and $\xi < 1$ as well. For example for $\beta = 1, \eta = 1, \rho = 11$ and $\xi = 1/2$, then $W(x,y) = x + \sqrt{y}$ is Thompson.

3.2.2 KDW Aggregators

Koopmans, Diamond, and Williamson [28] introduced an interesting aggregator. We refer to it as the **KDW aggregator**. It is defined by the formula

$$W(x,y) = \frac{\delta}{d} \ln\left(1 + ax^{b} + dy\right)$$

where $a, b, d, \delta > 0$. This aggregator satisfies $(T5)$ with $\gamma = b^{-1}$ and also satisfies $(T6)$.[23]

The KDW aggregator fails to satisfy the required Blackwell contraction condition when $\delta \geq 1$. Recall this aggregator always satisfies a Lipschitz condition in its second argument. Assumption $(T5)$ holds for the KDW aggregator.[24] The KDW aggregator is an example of a $\gamma-$subhomogeneous (with $\gamma = b^{-1}$) aggregator that is NOT a homogeneous aggregator, like members in the CES family. This example also illustrates why *(T5)* only requires $\gamma > 0$. IF the parameter $0 < b < 1$ (so the KDW aggregator is concave in x for each y), then $\gamma > 1$ must hold for the aggregator to satisfy *(T5)*. It is interesting to note that $(T5)$ applies to both current consumption and future utility arguments, whereas the question of discounting or not is a property of the future utility argument alone as well as parameter δ's magnitude.

The KDW aggregator satisfies $(T6)$. That is, the limit $L = 0$ in (1). Here, just notice for $x = 1$,

$$\frac{W(1,y)}{y} = \frac{\ln(1 + a + dy)}{y} \to 0 \text{ as } y \to \infty$$

for **any** $a, d \geq 0$. In this case, $(T6)$ holds for *any* $a \geq 1$.

[23] Becker and Rincón-Zapatero [11] includes detailed calculations supporting our claims.
[24] See Becker and Rincón-Zapatero [11] for the detailed calculations.

4 Recovery Theory: Marinacci and Montrucchio's Theorem

4.1 The Setup

Marinacci and Montrucchio [34] prove a Recovery Theorem for Thompson aggregators whenever the underlying commodity space is the positive cone of a principal ideal of the vector space of all real-valued sequences, s, with the usual coordinatewise partial order and corresponding definitions of sup and inf. The space s is a Dedekind complete Riesz space. Let $C = \{c_t\}_{t=1}^{\infty}$ denote an element of s. Given a non-zero vector $\omega \in s^+$, the set

$$A_\omega = \{C \in s : |C| \leq \lambda\omega \text{ for some scalar } \lambda > 0\}$$

defines a **principal ideal** in s. Use the notation 0 for the zero vector of this commodity space and reserve θ for the real-valued zero function, $\theta(C) = 0$, defined on this space.The positive cone of A_ω is:

$$A_\omega^+ = \{C \in A_\omega : C \geq 0\}.$$

This is the commodity space in the anticipated economic applications.[25] It is also a Dedekind complete Riesz space in its induced order viewed as a subset of s.

We generally consider two cases of this commodity space on economic grounds: the first occurs when $\omega = (1, 1, \ldots)$, and $A_\omega = \ell_\infty$, the vector space of all bounded real-valued sequences. The second case arises in the general exponential model where $\omega = (\alpha, \alpha^2, \ldots)$ for $\alpha \geq 1$. In the latter situation we recall $l_\infty \subset A_\omega \subset s$ when $\alpha > 1$. Our version of Marinacci's and Montrucchio's recovery theory applies to exponential models where $\alpha \geq 1$. Thus, we always assume the vector ω is strictly positive in each component. This implies ω is an **order unit** in the space A_ω. Furthermore, for each $C \in A_\omega$,

$$\|C\|_\infty = \inf\{\lambda > 0 : |C| \leq \lambda\omega\}$$

defines a lattice norm. Here, λ is a scalar; note $|C| = \{|c_t|\}_{t=1}^{\infty}$.

Following ideas drawn from Boyd [14], and further developed in Becker and Boyd [9], weighted norms are introduced on this principal ideal. These norms are deduced using strictly positive real-valued weight functions defined on A_ω. These weight functions are expressed in particular functional forms in aggregator models. These functions are specifically chosen to be well- adapted to the application at hand. There are two distinct uses of weight functions. First, we use the lattice norm inherited from the given principal ideal to define a weighted norm on the set A_ω that turns it into a Banach space in its own right. Second, we introduce another weight function to form a space of bounded functions according to this weight function. These functions are real-valued and defined on

[25] Becker and Boyd [9] illustrate a range of applications and present arguments for postulating this positive cone as the commodity space.

A_ω^+, the positive cone of A_ω. Think of these functions as possible trial utility functions on the underlying commodity space. We seek a solution to the Koopmans operator equation in this function space. Marinacci and Montrucchio's weight function is chosen in this latter case in order to construct a particular order interval of trial functions on which the solution to Koopmans' equation is sought.

The $\alpha - norm$, $\|\bullet\|_\alpha$, is defined for elements of A_ω by the formula:

$$\|C\|_\alpha = \sup_{t\geq 1}\left|\frac{c_t}{\alpha^t}\right|. \tag{5}$$

The normed vector space $\ell_\infty(\alpha)$ is defined by the pair $(A_\omega, \|\bullet\|_\alpha)$ where $\alpha \geq 1$. We note that the sequences in this space are $\alpha - norm - bounded$ since $(|c_t|/\alpha^t) \leq \lambda < +\infty$. This is so as $C \in A_\omega$ means there is some scalar $\lambda > 0$ such that $|c_t| \leq \lambda\alpha^t$ for each t. Hence, $\|C\|_\alpha \leq \lambda < +\infty$ whenever $C \in A_\omega$. This normed space is a vector lattice with the usual pointwise operations for join and meet of two vectors. The positive cone of this space is denoted by $\ell_\infty^+(\alpha)$, which is just A_ω^+ with the relative $\alpha - norm$ topology. The space $\ell_\infty(\alpha)$ is also a Banach lattice, so its positive cone is also $\alpha - norm$ closed. The lattice norm is equivalent to the $\alpha - norm$. This positive cone is also convex and has a nonempty $\alpha - norm$ interior. The latter fact follows from the observation that $\ell_\infty(\alpha)$ is an AM-space with unit vector ω.

We turn to the second weight function. We need to define a set of possible, or trial, real-valued utility functions with common domain $\ell_\infty^+(\alpha)$. These trial utility functions must also be bounded in an appropriately defined norm. The next weight function enters at this stage in order to define a suitable space of "bounded" real-valued functions on the commodity space.

First, define a **weight function**, φ_γ following Marinacci and Montrucchio's [34] specification. For each $C \in \ell_\infty^+(\alpha)$ define φ_γ by the formula:

$$\varphi_\gamma(C) = (1 + \|C\|_\alpha)^{1/\gamma}. \tag{6}$$

This weight function is uniformly continuous and convex on $\ell_\infty^+(\alpha)$ with respect to the $\alpha - norm$ topology.[26] Here, the parameter $\gamma > 0$ appearing in the weight function is taken from $(T5)$. This weight function as well the $\alpha - norm$ entangle preference and technology parameters — the growth rate α is derived from a model's technology side while the parameter γ comes from the model's preference side.

Definition 4 *A function $U : \ell_\infty^+(\alpha) \to \mathbb{R}$ is $\varphi_\gamma-$ bounded provided*

$$\|U\|_\gamma := \sup_{C\in\ell_\infty^+(\alpha)} \frac{|U(C)|}{(1 + \|C\|_\alpha)^{1/\gamma}} < +\infty.$$

[26] The norm $\|\bullet\|_\alpha$ is a uniformly continuous real-valued function defined on the set A_ω. See Aliprantis and Burkinshaw ([3], p. 218). Hence, the function $\varphi_\gamma(C)$ is continuous as the composition of the continuous functions $1 + \|C\|_\alpha$ and $\phi(x) = x^{1/\gamma}$ for $x > 0$.

The set of all $\varphi_\gamma -$ bounded real-valued functions with domain $\ell_\infty^+(\alpha)$ is denoted by F_γ^α.

The **zero function,** θ, is defined by $\theta(C) = 0$ for each C. The zero function is the origin in the vector space F_γ^α. The space F_γ^α is a Dedekind complete Riesz space. Clearly the weight function φ_γ satisfies $\varphi_\gamma(\theta) = 1$ and $\varphi_\gamma(C) \geq 1$ for each C. Moreover, $\left\|\varphi_\gamma\right\| = 1$ as well and φ_γ is an order unit in F_γ^α.

The space

$$C_\gamma^\alpha := \left\{U \in F_\gamma^\alpha : U \text{ is } \|\bullet\|_\alpha - \text{ continuous on } \ell_\infty^+(\alpha)\right\}$$

is a closed subspace of F_γ^α. However, this space is **not** a complete lattice. The corresponding positive cone, denoted $\left(C_\gamma^\alpha\right)^+$, is a solid cone. Its weighted sup norm interior is nonempty since the weight function $\varphi_\gamma \in \left(C_\gamma^\alpha\right)^+$ is an order unit. This property is important for the Recovery Theorem and the conclusion that the greatest fixed point of the Koopmans operator is a sup norm upper semicontinuous $\varphi_\gamma -$ *bounded* real-valued function on $\ell_\infty^+(\alpha)$.

The aggregator approach to recovering recursive utility representations of an underlying preference relation defined on the given commodity space is expressed in terms of a functional equation. This equation takes the aggregator function as the primitive concept. The **Koopmans equation for recursive utility** is

$$U(C) = W(c_1, U(SC)). \tag{7}$$

Define the **shift operator** $S : \ell_\infty^+(\alpha) \to \ell_\infty^+(\alpha)$ according to the rule $C = \{c_1 c_2, c_3, \ldots\} \mapsto SC = \{c_2, c_3, \ldots\}$. A solution of this equation is a recursive utility function representation of the preference relation. Of course, it all depends on what is meant by a solution. Proving this functional equation has a solution turns on recasting the problem as demonstrating a corresponding nonlinear operator, known as the **Koopmans operator** (denoted by T_W) has a fixed point in the desired function space of possible solutions. The Koopmans operator (defined below) is formally defined given a function $U \in \left(F_\gamma^\alpha\right)^+$ by the following equation for each $C \in \ell_\infty^+(\alpha)$:

$$(T_W U)(C) = W(c_1, U(SC)).$$

If $T_W U = U$, then U is a solution to the Koopmans equation and defines a recursive representation of the underlying preference relation.

The Koopmans operator enjoys a monotonicity property whenever the aggregator is specified by a member of the Thompson aggregator class.

Lemma 5 *If W is a Thompson aggregator, then $T_W : \left(F_\gamma^\alpha\right)^+ \to \left(F_\gamma^\alpha\right)^+$ is a monotone operator.*

Proof. Suppose $U, U' \in \left(F_\gamma^\alpha\right)^+$ and $U \geq U'$. Then $T_W U \geq T_W U'$ since for each $C \in \ell_\infty^+(\alpha)$ we have $T_W U(C) = W(c_1, U(SC)) \geq W(c_1, U'(SC)) = T_W U'(C)$ as W is monotone increasing according to $(T1)$. ■

A fixed point of the Koopmans operator belongs to $\left(F_{\gamma}^{\alpha}\right)^{+}$. *It may fail to possess any useful analytically or economically important properties***.** For example, this fixed point may not be continuous, or even upper semicontinuous. Hence, *we seek at least one solution with mathematical properties appropriate for analyzing an intertemporal choice model.* Showing the model has an optimal solution is the first step in this analysis. Some form of continuity for the objective function is usually required to demonstrate an optimum exists.

Our objective is to show the Koopmans equation (7) has at least one economically interesting solution in the space $\left(F_{\gamma}^{\alpha}\right)^{+}$ using the monotonicity property of the Koopmans operator when the aggregator belongs to the Thompson class, is $\gamma-$ subhomogeneous and satisfies the MM Limit Condition. By an economically interesting solution we mean one that enjoys some form of continuity property. In fact, we show there are extremal solutions. The smallest fixed point is a lower semicontinuous function while the largest is an upper semicontinuous solution in the space $\left(F_{\gamma}^{\alpha}\right)^{+}$. The smallest and largest solutions define an order interval of fixed points. This set, fix(T_W), is also a countably chain complete subset of $\left(F_{\gamma}^{\alpha}\right)^{+}$. The formal statement of these facts is the **Marinacci and Montrucchio** [34] **Recovery Theorem**.

Theorem 6 *(Marinacci and Montrucchio [34]). Suppose W is a Thompson aggregator satisfying (T5) and (T6).*

1. *There is a $\|\bullet\|_{\alpha}-$ upper semicontinuous function $U^{\infty} \in \left(F_{\gamma}^{\alpha}\right)^{+}$ such that $T_W U^{\infty} = U^{\infty}$.*
2. *There is a $\|\bullet\|_{\alpha}-$ lower semicontinuous function $U_{\infty} \in \left(F_{\gamma}^{\alpha}\right)^{+}$ such that $T_W U_{\infty} = U_{\infty}$.*
3. *U_{∞} is the least fixed point, U^{∞} is the greatest fixed point, and fix(T_W) is a countably chain complete subset of $\left(F_{\gamma}^{\alpha}\right)^{+}$.*

Our proof of this Recovery Theorem is based on verifying the hypotheses of the Tarski-Kantorovich Theorem are met on an appropriately chosen order interval in the positive cone $\left(F_{\gamma}^{\alpha}\right)^{+}$. This order interval is denoted $\left\langle \theta, U^{T} \right\rangle$, where θ is the zero function in $\left(F_{\gamma}^{\alpha}\right)^{+}$ and U^{T}, called *"U-top,"* is defined below.

Of course we require $U^{T} \in \left(F_{\gamma}^{\alpha}\right)^{+}$ as well. The desired order interval has the property $T_W \theta \geq \theta$ and $T_W U^{T} \leq U^{T}$ with $T_W : \left\langle \theta, U^{T} \right\rangle \rightarrow \left\langle \theta, U^{T} \right\rangle$. Evidently $T_W \theta \geq \theta$ since for each $C \in \ell_{\infty}^{+}(\alpha)$ we have $T_W \theta (C) = W(c_1, 0) \geq 0$ as $\theta (SC) = 0$.

The defining characteristics of U^{T} are summarized based on the corresponding analysis in Marinacci and Montrucchio ([34]). The difference between our proofs is entirely concerned with the role played by Thompson criterion $(T1)$ in our argument. Modifying $(T1)$ allows us to consider two distinct interpretations of order continuity. Both turn out to support the rigorous foundation for successive approximations to construct the extremal solutions. The order theoretic approach is initiated in section 4.3, whereas the topological form is presented in section 5.3.

4.2 The Order Interval $\langle \theta, U^T \rangle$

The definition of U^T is the first order of business in this subsection. We consider a Thompson aggregator, W. We have already specified the order interval's bottom element is the zero-function, θ. Note that it is trivially a $\|\bullet\|_\alpha -$ continuous function and also belongs to C_γ^α (by $(T1)$). Marinacci and Montrucchio [34] define the function U^T as follows:

$$U^T(C) = W(1, y_\alpha)\, \varphi_\gamma(C).$$

Here, the element $y_\alpha > 0$ is the solution to $W(1, y_\alpha) = \alpha^{-1/\gamma} y_\alpha$ (shown to exist in [34] using the additional properties $(T5) - (T6)$). It is straightforward to verify $U^T \in (F_\gamma^\alpha)^+$. Clearly $U^T \geq \theta$ and $U^T(C) > 0$ whenever $C \neq 0$ and $\|U^T\|_\gamma = W(1, y_\alpha) < +\infty$. Furthermore, $U^T \in (C_\gamma^\alpha)^+$ follows from its definition.[27]

The next result (again, see Marinacci and Montrucchio [34] for the proof) is critical to showing the Koopmans operator is a self-map on the order interval $\langle \theta, U^T \rangle \subset (F_\gamma^\alpha)^+$.

Proposition 7 *If W is a Thompson aggregator satisfying (T5) and (T6), then $T_W U^T \leq U^T$.*

4.3 Proof of the Marinacci-Montrucchio Recovery Theorem

The formal proof of the Marinacci-Montrucchio Recovery Theorem depends on verifying the Koopmans operator satisfies the Tarski-Kantorovich Fixed Point Theorem's hypotheses. Three key requirements must hold. First, the Koopmans operator must be a self-map on $\langle \theta, U^T \rangle$. Second, this order interval must be a countably chain complete poset as a subset of $(F_\alpha^\gamma)^+$. Third, the Koopmans operator must be monotonically sup/inf-preserving. *This property rests on the joint continuity assumption* $(T1)$.[28] The monotonically sup/inf-preservation property is the order continuity condition satisfied by the Koopmans operator when only order properties of the underlying utility function and commodity spaces are assumed. A successive approximation calculation (indexed on the natural numbers) recovers at least one underlying utility function from the given Thompson aggregator.[29]

Marinacci and Montrucchio's [34] proof applies the Tarski's Fixed Point Theorem. This relies on the fact that the Koopmans operator is a monotone self-map defined on a complete lattice given by the order interval $\langle \theta, U^T \rangle \subset (F_\gamma^\alpha)^+$.

[27] Of course, this is true because the underlying $\|\bullet\|_\alpha -$ *norm* is a (uniformly) continuous function from $\ell_\infty^+(\alpha)$ into $\mathbb{R}_+$.

[28] This is **the** subtle difference between our version of the recovery theorem and Marinacci's and Montrucchio's theory.

[29] We comment below on the uniqueness question, but refer the reader to the literature and our working paper [11] for details on this important point.

It does NOT require any order continuity property for their demonstration. However, their proof is, strictly speaking, nonconstructive. They claim to successively approximate the extremal fixed points whose existence is guaranteed by the Tarski Fixed Point Theorem. For example, they obtain a *sequence* of approximate utility functions via iteration on the natural numbers from the initial seed, U^T. Their claim that this sequence's limit is the greatest solution does not follow from the nonconstructive Tarski Theorem alone.

The TK FPT, by contrast, is constructive in the sense that successive approximation indexed on the natural numbers IS the underlying method. The monotonically sup/inf-preserving (order continuity) property is the additional ingredient that allows us to know our procedure finds each extremal fixed point by interation on the natural numbers. This, in turn, has ramifications for deducing semicontinuity and other qualitative properties of these special fixed points.

There is another important formal difference between the hypotheses of the Tarski and Tarski-Kantrovich Theorems. The conditions on the domain X and on the self-map F for the TK FPT result are weaker than those in Tarski's Theorem. The underling poset X is no longer assumed to be a complete lattice. Our underlying vector space F_α^γ is not a complete lattice. However, it is both a Dedekind complete Riesz space and a Banach lattice. Hence, the order interval $\langle \theta, U^T \rangle$ is a complete lattice in the induced order inherited from F_α^γ. A monotonic sequence in this order interval is automatically order bounded and the sup or inf of such a sequence belongs to the order interval as well. Hence, our order interval is a countably chain complete poset. The TK FPT is available for a recovery proof provided the Koopmans operator is order continuous.

4.4 The Recovery Theorem's Formal Proof

The application of the TK FPT to the Koopmans operator turns on verifying it is monotonically sup/inf-preserving on the order interval $\langle \theta, U^T \rangle \subset (F_\alpha^\gamma)^+$ and that order interval is also a countably chain complete set.

Proposition 8 *Suppose W is a Thompson aggregator satisfying $(T5)$ and $(T6)$. Then the associated Koopmans operator is a monotonically sup/inf-preserving self-map on $\langle \theta, U^T \rangle$.*

Proof. Lemma 5 implies T_W is a monotone operator. It is obvious that $T_W\theta \geq \theta$ and $T_W U^T \leq U^T$ follows from Proposition 7.

Suppose $\{U^N\} \equiv \{U^N\}_{N=1}^\infty$ is a sequence of φ_γ — bounded functions in the order interval $\langle \theta, U^T \rangle \subset (F_\alpha^\gamma)^+$. Clearly both the sup and inf of this sequence exist as elements of $\langle \theta, U^T \rangle$. This implies $\{U^N\}$ is a a countably chain complete set in $\langle \theta, U^T \rangle$ provided it is a chain. Therefore, $\langle \theta, U^T \rangle$ is a countably chain complete poset follows immediately as $\{U^N\}$ may be an arbitrarily chosen countable chain in $\langle \theta, U^T \rangle$.

The order interval $\langle \theta, U^T \rangle$ evidently contains a smallest and largest element. Now suppose $\{U^N\}$ is any monotone increasing sequence of functions in $\langle \theta, U^T \rangle$.

By countable chain completeness, we find $\bigvee U^N$ exists since each $U^N \leq U^T$. Hence, there is a function $U = \bigvee U^N \in \langle \theta, U^T \rangle$. In fact, $U^N \nearrow U$ pointwise on $\ell_\infty^+(\alpha)$. That is $\lim_{N\to\infty} U^N(C) = U(C)$ for each $C \in \ell_\infty^+(\alpha)$. Since W is increasing in its second argument and *continuous in its second argument*, $(T1)$ implies for each $C \in \ell_\infty^+(\alpha)$ the following equalities:

$$
\begin{aligned}
\bigvee [T_W U^N](C) &= \bigvee W\left(c_1, U^N(SC)\right) \text{ (by definition of } T_W) \\
&= \lim_N W\left(c_1, U^N(SC)\right) \text{ (by the monotone property of } W \text{ in } (T1)) \\
&= W\left(c_1, \lim_N U^N(SC)\right) \text{ (by continuity of } W \text{ in } (T1)) \\
&= W(c_1, U(SC)) \\
&= T_W\left(\bigvee U^N\right)(C).
\end{aligned}
$$

Hence, the Koopmans operator is monotonically sup-preserving. Apply the analogous argument for monotone decreasing sequences $\{U^N\}$, bounded below by the zero function. This shows that T_W is also monotonically inf-preserving. Hence, the Koopmans operator is monotonically sup/inf-preserving. ■

This Proposition's proof seemingly depends only on assumption $(T1)$. However, the other properties come into play when verifying T_W is a monotone self map on the order interval $\langle \theta, U^T \rangle \subset (F_\alpha^\gamma)^+$.

Our take on the Marinacci-Montrucchio Recovery Theorem proof appears below.

Proof. (1): Iterate T_W using U^T as the initial seed. That is, for each natural number, N, let

$$U^N = T_W U^{N-1} \text{ and } U^0 \equiv U^T.$$

Clearly for each $N \geq 1$,

$$\theta \leq U^N \leq U^{N-1} \leq \cdots \leq U^1 \leq U^T.$$

Hence, there is a function U^∞ such that

$$U^\infty = \bigwedge_N U^N \in \langle \theta, U^T \rangle$$

since $\langle \theta, U^T \rangle$ is a countably chain complete subset of $(F_\alpha^\gamma)^+$.

The function U^T is $\|\bullet\|_\alpha -$ continuous on $\ell_\infty^+(\alpha)$. Hence, since, by $(T1)$, W is a continuous function on $\mathbb{R}_+^2$, the function $U^1 = T_W U^T$ is also a $\|\bullet\|_\alpha -$ continuous function on $\ell_\infty^+(\alpha)$, and so on for each U^N. Hence, U^∞ is a $\|\bullet\|_\alpha -$ upper-semicontinuous real-valued function on $\ell_\infty^+(\alpha)$ as it is the pointwise infimum of continuous functions. Proposition 8 shows that T_W is monotonically sup-inf-preserving. Therefore, T_W satisfies the hypotheses of the TK FPT. Hence, we may conclude by that Theorem that U^∞ is a fixed point of the Koopmans operator. That is,

$$T_W U^\infty = U^\infty.$$

The fixed point $U^{\infty} = \bigwedge_N T_W^N U^T$ is the largest fixed point of the Koopmans operator in $\langle \theta, U^T \rangle$.

(2): A parallel argument establishes that there is also a smallest fixed point in $\langle \theta, U^T \rangle$, denoted $U_{\infty} = \bigvee_N T_W^N \theta$ found by iterating T_W with the initial seed, θ, the zero function. Moreover, U_{∞} is a $\|\bullet\|_{\alpha}-$ lower-semicontinuous real-valued function on $\ell_{\infty}^{+}(\alpha)$.

(3): Let $\text{fix}(T_W)$ denote the nonempty set of fixed points belonging to our Koopmans operator in the order interval $\langle \theta, U^T \rangle$. Balbus, et al (see [6], Theorem 7, p. 109) implies $\text{fix}(T_W)$ is a countably chain complete poset in $\langle \theta, U^T \rangle$. ∎

Recall $T_W^N \theta \nearrow U_{\infty}$ says that each $T_W^N \theta$ approximates U_{∞} from below. The partial sum technique encapsulates the observation that $T_W^N \theta$ approximates U_{∞} from below. This construction of U_{∞} receives special attention in Section 5.

The Recovery Theorem's proof implies that **IF** $U^{\infty} = U_{\infty} \equiv U^*$, then $U^* \in (F_{\alpha}^{\gamma})^+$ is the unique $\|\bullet\|_{\alpha}-$ continuous φ_{γ}-bounded real-valued function in the order interval $\langle \theta, U^T \rangle$ satisfying the Koopmans equation when W is a Thompson aggregator. That is, in this situation $U^* \in (C_{\alpha}^{\gamma})^+$ as well! *Uniqueness of the solution in the larger space $(F_{\alpha}^{\gamma})^+$ implies that the solution is also a $\|\bullet\|_{\alpha}-$ continuous and φ_{γ}-bounded real-valued function*! The interesting problem at this point is to provide conditions under which there is a unique $\|\bullet\|_{\alpha}-$ continuous and φ_{γ}-bounded solution to this aggregator's Koopmans equation. The uniqueness question is addressed in the literature.[30] We point out some subtle issues that must be addressed in setting up the uniqueness problem and additionally motivate our interpretation (in Section 5) of the least fixed point as the operator equation's *principal solution*.

There are two issues. First, the extremal fixed points may never be identical on the domain, $\ell_{\infty}^{+}(\alpha)$. This may obtain whenever $W(0,0) = 0$ holds, a common property enjoyed by the CES and KDW aggregators. The following example illustrates this point with a CES aggregator (3) assuming $0 < \rho < 1$. Let $\alpha = 1$ and identify ℓ_{∞}^{+} and $\ell_{\infty}^{+}(\alpha)$. Note there is a unique $y^* > 0$ such that $W(1, y^*) = y^*$. Choose a natural number N. Compute $T_W^N \theta(C)$ and evaluate this expression at $C = 0$ to obtain:

$$T_W^N \theta(0) = W(0,0) = 0.$$

Hence, passing to the limit we find $U_{\infty}(0) = 0$. On the other hand, calculation of $U^{\infty}(0)$ proceeds as follows by computing the iterates directly for this CES

[30] See our working paper [11] for our uniqueness approach via concave operator theory.

aggregator:

$$
\begin{aligned}
&T_W^0 U^T(0) \equiv U^T(0) = y^* \varphi_\gamma(0) = y^* \text{ as } \varphi_\gamma(0) = 1;\\
&T_W U^T(0) = W(0, y^*) = \beta (y^*)^\rho;\\
&T_W^2 U^T(0) = W(0, W(0, y^*)) = \beta [\beta (y^*)^\rho]^\rho = \beta^{1+\rho} (y^*)^{\rho^2};\\
&\qquad\qquad \vdots\\
&T_W^N U^T(0) = W(0, W(0, W(0, \ldots, W(0, y^*) = \beta^{\left(1+\rho+\rho^2+\cdots+\rho^{N-1}\right)} (y^*)^{\rho^N}.
\end{aligned}
$$

Clearly $\rho^N \to 0$ as $N \to \infty$ implies $\lim_N (y^*)^{\rho^N} = 1$ and

$$
U^\infty(0) = \lim_{N\to\infty} T_W^N U^T(0) = \beta^{\left(\frac{1}{1-\rho}\right)} > 0,
$$

and hence, $U^\infty(0) > U_\infty(0) = 0$. The extremal fixed points of the Koopmans operator cannot agree on the entire domain, ℓ_∞^+. The Koopmans operator, defined for all consumption sequences in ℓ_∞^+, is **NOT** uniquely determined by the aggregator function! However, this does not mean we cannot say something useful about the subset of consumption sequences where the extremal fixed points deliver the same utility value. The papers by Marinacci and Montrucchio ([34],[35]), Bloise and Vailakis [13], and Becker and Rincón-Zapatero [11] prove uniqueness theorems by further restricting the commodity space's domain for the utility functions. Each of the cited papers find uniqueness on subsets of the commodity space that exlude consumption sequences with zero components. For example, uniqueness typically obtains on the norm interior of the commodity space's positive cone.

The second issue concerns the interpretation of multiple solutions to the operator equation when the extremal fixed points are unequal. Either one of the extremal fixed points is a strictly increasing transformation of the other, or neither is a strictly increasing transformation of the other. In the first case, there is no fundamental economic difference between the two utility representations. If one is a utility representation of the underlying (and hidden) intertemporal preference relation, then so is the other. Multiple solutions to the Koopmans equation may not be an issue from an economic persective. On the other hand, if the two distinct extremal solutions are NOT ordinally equivalent, then we know spurious solutions exist. These solutions do not represent the underlying preference relation. In this case, we argue next the least fixed point is the operator equation's principal solution. It possesses economically important characteristics (e.g. concavity) not necessarily provable for the greatest solution absent a uniqueness theorem for the Koopmans equation.

5 The Principal Fixed Point of $T_W U = U$.

The Recovery Theorem yields two interesting fixed points, U_∞ and U^∞. The former is lower semicontinuous and the latter upper semicontinuous in the commodity space's norm topology. These continuity properties were found from the pointwise convergence of $T_W^N \theta(C) \nearrow U_\infty(C)$ and $T_W^N U^T(C) \searrow U^\infty(C)$. The latter fixed point's upper semicontinuity property suggests a standard optimal growth problem with this upper semicontinuous objective, U^∞, has an optimal program provided the feasible consumption sequences form a norm compact subset of the commodity space. Unfortunately, this norm compactness property does not generally hold for infinite horizon problems. Hence, the norm upper semicontinuity property enjoyed by U^∞ is not, in itself, particularly useful for the purposes of optimal growth theory. The LFP, U_∞, is norm lower semicontinuous. However, it is norm continuous on the interior of its effective domain in the commodity space when it is also a concave function. Our standard CES and KDW aggregator examples imply U_∞ is concave. This norm continuity also has some implications for weak continuity and brings us a step closer to an application in optimal growth theory since feasible paths are typically compact in the commodity space's product topology.

5.1 Approximation of U_∞ From Below: A Computational Perspective

The successive approximation of the function U_∞ for a given consumption profile $C \in \ell_\infty^+(\alpha)$ yields the following *partial sum* relations:

$$
\begin{aligned}
&(T_W \theta)(C) = U^1(C) = W(c_1, 0);\\
&\left(T_W^2 \theta\right)(C) = U^2(C) = \left(T_W U^1\right)(C) = W(c_1, W(c_2, 0));\\
&\qquad\qquad \cdots\\
&\left(T_W^N \theta\right)(C) = \left(T_W U^{N-1}\right)(C) = W(c_1, W(c_2, W(c_3, \ldots, W(c_N, 0)\cdots))),
\end{aligned}
$$

and so on. Since $\left(T_W^N \theta\right)(C) \nearrow U_\infty(C)$, we find

$$0 = \theta(C) \leq U^1(C) \leq U^2(C) \leq \cdots \leq U_\infty(C).$$

Rewriting this in terms of the aggregator, we have the nondecreasing sequences of "finite horizon" approximations of the infinite horizon value $U_\infty(C)$ in terms of the underlying aggregator:

$$0 \leq W(c_1, 0) \leq W(c_1 W(c_2, 0)) \leq \cdots \leq U_\infty(C).$$

That is, successive approximations starting from the zero function θ provides an approximation, from below, for the value $U_\infty(C)$. Each approximation incorporates the consumption of a finite number of consecutive periods. These initial segments of the consumption sequence may be interpreted as consumption over a finite horizon of length N. That is, more consumption periods are incorporated in the N^{th} approximation than its predecessors. In this sense, there is more information in $W(c_1 W(c_2, 0))$ about $U_\infty(C)$ than provided by $W(c_1, 0)$, and so on. This theoretical computation of $U_\infty(C)$ starts with *no information* about $U_\infty(C)$ as $\theta(C) = 0$ for each C. This interpretation is consistent with the computer science literature on theoretical computation and successive approximation.[31]

Our first reason for proposing U_∞ as the principal fixed point is that it is the pointwise limit of an increasing sequence of functions which are strictly smaller than it. Information about the value $U_\infty(C)$ improves with each iteration. And, each step in the iteration requires knowledge of a finite number of coordinates of the given consumption sequence and the form of the aggregator function only. Iteration from the top element fails on this issue as the function itself needs to input the exact value $\|C\|_\alpha$, which depends on knowing the entire infinite horizon consumption stream. Iteration from U^T also requires calculation of the particular value of the aggregator, $W(1, y_\alpha)$, in addition to inputing the aggregator's functional form! It would seem more information must be secured to carry through the iteration initiated at U^T than at θ. Calculating the value $U_\infty(C)$, or a "good approximation of that value," requires a finite number of consecutive consumption dates. Calculating $U^\infty(C)$, or a "good approximation of that value," requires inputing the complete sequence, C. From a theoretical computational perspective the principal fixed point approximations of $U_\infty(C)$ offer some informational advantages over the succession of approximations to $U^\infty(C)$.

5.2 The Principal Fixed Point is a Monotone Concave Function

Lemma 9 *U_∞ is a monotone function: $C \geq C'$ implies $U_\infty(C) \geq U_\infty(C')$.*

Proof. Thompson property $(T1)$ implies that each term of the partial sum, $U^{N+1} = T_W^N \theta$ is a monotone function of the consumption sequence. Fix consumption sequences $C \geq C'$. That is, $c_t \geq c_t^{'}$ for each t. Then $W(c_1, 0) \geq W\left(c_1^{'}, 0\right)$. Likewise,

$$W(c_1 W(c_2, 0)) \geq W\left(c_1^{'} W\left(c_2^{'}, 0\right)\right),$$

and so on for the successive indices N. It readily follows that the limiting function, U_∞ is monotone. ■

A similar argument shows that U^∞ is also a monotone function of C.

Lemma 10 *Suppose W is a jointly concave and increasing Thompson aggregator in (x, y). If U is a concave function in $(\mathcal{F}_\alpha^\gamma)^+$, then $T_W U$ is a concave function in $(\mathcal{F}_\alpha^\gamma)^+$.*

Proof. Suppose $U \in (\mathcal{F}_\alpha^\gamma)^+$ is a concave function. Clearly $T_W U \in (\mathcal{F}_\alpha^\gamma)^+$. Let $C^0 \neq C^1$ be two consumption sequences in in $\ell_\infty^+(\alpha)$ and let $\lambda \in [0, 1]$. Then

$$T_W U(\lambda C^1 + (1-\lambda) C^0) = W(\lambda c_1^0 + (1-\lambda) c_1^1, U(S(\lambda C^1 + (1-\lambda) C^0)).$$

[31] See Stoletenberg-Hansen, et al ([43], p.23).

Note that $S(\lambda C^1 + (1-\lambda)C^0) = \lambda SC^1 + (1-\lambda)SC^0$ and U concave yield

$$U(\lambda SC^1 + (1-\lambda)SC^0) \geq \lambda U(SC^1) + (1-\lambda)U(SC^0).$$

Since W is increasing in both arguments

$$W(\lambda c_1^1+(1-\lambda)c_1^0, U(S(\lambda C^1+(1-\lambda)C^0)) \geq W(\lambda c_1^1+(1-\lambda)c_1^0, \lambda U(SC^1)+(1-\lambda)U(SC^0)).$$

Finally, since W is also jointly concave and

$$(\lambda c_1^1+(1-\lambda)c_1^0, \lambda U(SC^1)+(1-\lambda)U(SC^0)) = \lambda(c_1^1, U(SC^1))+(1-\lambda)(c_1^0, U(SC^0)),$$

is a convex combination of $\left(c_1^1, U\left(SC^1\right)\right)$ and $\left(c_1^0, U\left(SC^0\right)\right)$, we have

$$W(\lambda c_1^1+(1-\lambda)c_1^0, U(S(\lambda C^1+(1-\lambda)C^0)) \geq \lambda W(c_1^1, U(SC^1))+(1-\lambda)W(c_1^0, U(SC^0)).$$

Thus

$$T_W U(\lambda C^1 + (1-\lambda)C^0) \geq \lambda T_W U(C^1) + (1-\lambda)T_W U(C^0),$$

and $T_W U \in (\mathcal{F}_\alpha^\gamma)^+$ is a concave function. ■

Evidently θ is a concave function. The Lemma implies $T_W\theta = U^1$ is also concave. Iterate the Koopmans operator with initial seed θ to obtain the sequence $\left\{U^N\right\}_{N=1}^\infty$, where $U^N = T_W U^{N-1}$ and $U^0 \equiv \theta$. The Lemma implies for a given natural number N that $U^N = T_W U^{N-1}$ is concave whenever U^{N-1} is concave. By mathematical induction U^{N+1} is concave provided U^N is concave. Therefore, U^N is a concave function for each N. The next result proves that U_∞ inherits this concavity property. The argument turns entirely on the pointwise convergence of the iterates $U^N(C)$ to $U_\infty(C)$ that is a by-product of the Recovery Theorem's proof.

Proposition 11 *Suppose W is a jointly concave and an increasing Thompson aggregator in (x, y). The principal fixed point U_∞ is a real-valued monotone concave function.*

Proof. For each natural number N concavity of U^N implies

$$U^N(\lambda C^1 + (1-\lambda)C^0) \geq \lambda U^N(C^1) + (1-\lambda)U^N(C^0),$$

where $C^0 \neq C^1$ are two consumption sequences in $\ell_\infty^+(\alpha)$ and $\lambda \in [0,1]$. Let $C^\lambda = \lambda C^1 + (1-\lambda)C^0$ denote this convex combination. As $U^N(C) \nearrow U_\infty(C)$ holds for each C, it holds, in particular, when $C = C_1, C_2$ and C^λ, respectively. Taking the limits in the previous inequality implies for U_∞ that:

$$U_\infty(\lambda C^1 + (1-\lambda)C^0) \geq \lambda U_\infty(C^1) + (1-\lambda)U_\infty(C^0).$$

This proves U_∞ is concave. ■

The concavity of U_∞ has implications for proving that function is continuous in the norm topology of $\ell_\infty^+(\alpha)$, at least on the norm-interior of its effective domain. The **effective domain** of U_∞ is the set:

$$\left\{C \in \ell_\infty^+(\alpha) : U_\infty(C) > -\infty\right\}.$$

The effective domain of the nonnegative concave function U_∞ is $\ell_\infty^+(\alpha)$. U_∞ is also a **proper function** on $\ell_\infty^+(\alpha)$. First, its effective domain is nonempty. Second, U_∞ is not identically $+\infty$ since it is a φ_γ — bounded function on $\ell_\infty^+(\alpha)$. In particular, it is proper on the set $\ell_\infty^+(\alpha)$ since it takes a finite value at each $C \in \ell_\infty^+(\alpha)$ with $\|C\|_\alpha < +\infty$.

The LFP, U_∞, is norm continuous on the interior of its effective domain, $\ell_\infty^{++}(\alpha)$. By U_∞ concave, it suffices to show that U_∞ is bounded below by some constant, μ, on a neighborhood of some point $C^* \in \ell_\infty^{++}(\alpha)$, the norm interior of the positive cone $\ell_\infty^+(\alpha)$.[32] But this trivially follows from the fact $U_\infty \geq \theta$! The point $\omega = (\alpha, \alpha^2, \ldots)$ for some $\alpha \geq 1$ is an order unit in this positive cone and it is also a norm interior point, that is $\omega \in \ell_\infty^{++}(\alpha)$. Define the α-norm open set

$$B = \left\{C \in \ell_\infty^+(\alpha) : \|C - \omega\|_\alpha < \varepsilon\right\},$$

where $\varepsilon > 0$ is chosen so that B is contained in the positive cone's norm interior.

Lemma 12 *1. U_∞ is a concave function that is bounded below on B.*

2. U_∞ is a norm continuous function on the interior of its effective domain, $\ell_\infty^{++}(\alpha)$.

Proof. The proof follows by verifying the conditions of Theorem 5.43 for convex functions in Aliprantis and Border ([1], p.188). Just observe the convex function $-U_\infty$ is bounded from above by 0 for each $C \in B$. The second property follows at once. ■

The continuity property deduced from concavity is not, by itself, sufficient to prove an optimum exists in an optimal growth model when the consumption possibility set is contained in the positive cone $\ell_\infty^+(\alpha)$. We still need to weaken the topology and show that U_∞ is weakly upper semicontinuous in the same topology for which feasible consumption sequences form a compact set.

However, this result does get us partway to verifying weak upper semicontinuity obtains for this utility function. For example, consider the case where $\alpha = 1$. That is, let $\ell_\infty^+ = \ell_\infty^+(1)$. The norm-dual of ℓ_∞ is the set ba of bounded-additive set functions on the positive integers. Since U_∞ is concave it is norm continuous on the interior of its effective domain by the previous lemma.

Recall the **hypograph** of U_∞ is the set of all nonnegative real-valued bounded sequences C and real numbers r such that $r \leq U_\infty(C)$. Since U_∞ is a concave function, then the hypograph is a convex subset of $\ell_\infty^+ \times \mathbb{R}$. See Aliprantis and Border ([1]. p. 254). Norm continuity on each closed subset of ℓ_∞^{++} implies that there are corresponding closed convex subsets contained in the hypograph.

Now consider the weak topology of the dual pair (ℓ_∞, ba). Each nonempty norm-closed convex set in ℓ_∞ is also a nonempty weakly closed convex set (c.f. Aliprantis and Border ([1], Theorem 5.98, p. 214). Now consider the hypograph of U_∞ where the domain of U_∞ is further restricted to a nonempty closed convex subset of ℓ_∞^{++} where it is also norm continuous. It follows that U_∞ is also weakly upper semicontinuous on that restricted domain! Indeed, it is

[32] See Aliprantis and Border ([1], Theorems 5.42 and 5.43, p. 188).

upper semicontinuous in the relative product topology (or, topology of coordinatewise convergence) on that domain since each coordinate linear functional $p^t = (0, 0, \ldots, 1, 0, \ldots) \in \ell_1 \subset ba$.

This is not, by itself, a final answer to proving U_∞ is upper semicontinuous in a topology on ℓ_∞^+ sufficiently weak to insure compactness of the feasible consumption sequences as well. Indeed, this result only resolves weak upper semicontinuity on a proper subset of this utility function's effective domain.[33] However, this argument indicates this problem may have a positive resolution using the product topology by further exploiting the full implications that U_∞ is concave. Technology based conditions might also prove useful in combination with the principal solution's concavity property in obtaining a weak upper semicontinuity property on the feasible consumption alternatives.

The weak upper semicontinuity property of U_∞ just established does not extend to U^∞. The induction argument proving U_∞ is concave fails. The initial iterate, $T_W U^T$, is not a concave function. The weight function φ_γ is a convex function in C since the $\alpha - norm$ is a convex function. Hence, the top function, U^T, is a convex function of C. This does NOT exclude the possibility that U^∞ is concave! Indeed, when the Koopmans equation has a unique solution in the order interval $\langle \theta, U^T \rangle$ it will turn out that U^∞ is concave, at least on the interior of its domain, whenever U_∞ is also concave.

Iteration of the Koopmans operator starting from θ yields a product lower semicontinuous limit function U.[34] This same approach fails when applied to iteration of the Koopmans operator from U^T. The initial seed function, U^T, is not a product continuous function. The weight function's $\|\bullet\|_\alpha - norm$ continuity appearing in U^T depends directly on the continuity of the $\|\bullet\|_\alpha - norm$ on $\ell_\infty^+(\alpha)$. To see the problem, just let $\alpha = 1$ and once again set $\ell_\infty^+ = \ell_\infty^+(1)$. Identify $\|\bullet\|$ and $\|\bullet\|_1$. Suppose the corresponding $\|\bullet\| - norm$ is a continuous function in the product topology. Let $C^1 = \{1, 0, 0, 0, \ldots\}$, $C^2 = \{0, 1, 0, 0, \ldots\}$, and so on. Each sequence belongs to ℓ_∞^+ and $\left\|C^N\right\| = 1$ for each N. But $\left\{C^N\right\} \to 0$ in the product topology (equivalent to the topology of coordinatewise convergence). Hence, if $\|\bullet\|$ is a continuous function in this topology, $\lim_N \left\|C^N\right\| = 0$ as well. This contradicts the property $\left\|C^N\right\| = 1$ for each N. Hence, $\|\bullet\|$ is not continuous in the product topology.[35] We note the sup norm is weakly lower semicontinuous on $\ell_\infty^+(\alpha)$.[36] Hence, $T_W U^T$ is weakly lower semincontinuous as well. given W is continuous (by $(T1)$). However, this observation does not help resolve the issue of proving U^∞ is weakly upper semicontinuous.

Marinacci and Montrocchio ([34], p. 1801) propose a different, qualitative, solution to this weak continuity problem. They recall Boyd's Lemma ([14],

[33] Bloise and Vailakis [13] show the GFP is product upper semicontinous on weakly compact order intervals in the positive cone of ℓ_∞. This result is based on a study of solutons to the auxillary difference equation, which in turn depends on the constructive existence of the GFP.

[34] W is jointly continuous by *(T1)*. and $T_W \theta = U^1$ is is product continuous. By induction, so is each iterate U^N. The pointwise limit function U_∞ is also product lower semicontinuous.

[35] See Majumdar [31] for a closely related discussion about why compactness of feasible sets fails in the one-sector growth model when the commodity space ℓ_∞ is given its sup norm topology.

[36] See Aliprantis and Border ([1], Lemma 6.22, p. 235).

Lemma 2), that the relative product topology and the $\beta - norm$ topology on the commodity space $\ell_\infty^+(\beta)$ coincides with the relative product topology on each $\alpha - bounded$ subset of $\ell_\infty^+(\alpha)$ whenever $\beta > \alpha \geq 1$. They go on to argue that the smallest and largest fixed points coincide on a particular *proper* subset of $\ell_\infty^+(\alpha)$ (relevant to their uniqueness theory) and hence this unique solution (on the particular subset) must be continuous in the relative product topology provided β is sufficiently close to α. This resolution of the weak continuity problem is useful (and was exploited by Boyd [14] and Becker and Boyd [9] for the Blackwell aggregator family). Their proposed resolution also turns on the successive approximations and pointwise convergence properties underlying the construction of the extremal fixed points. Indeed, the uniqueness of the solution in $\langle \theta, U^T \rangle$ is critical for their argument to be valid. Hence, this approach in the Thompson theory may need further development for applications in optimal growth theory.

5.3 Scott Continuity of T_W and Construction of Its Principal Fixed Point

The existence of the Koopmans operator's existence of the LFP, U_∞, only required monotonic sup-preservation. The successive approximation argument concludes the nondecreasing sequence $\{T_W^n \theta\}$ converges pointwise to U_∞. That is, for each $C = \{c_t\}_{t=1}^\infty$, the *"partial sums"* $T_W^N \theta(C) \nearrow U_\infty(c_1, c_2, c_3, \ldots)$. The underlying order continuity property is a Riesz space concept. It is purely order theoretic; no topological meaning is associated to a convergent sequence (or, more generally, net).[37]

The TK FPT least fixed point construction may be recast in terms of the lower limit of the monotone sequence $\{T_W^N \theta\}$ and a property of the Koopmans operator that is analogous to lower semicontinuity. These twin notions are implicit in the monotonic sup-preservation property. First, we rewrite $\{T_W^N \theta\} \nearrow U_\infty$ as

$$\liminf_N \left(T_W^N \theta\right) \equiv \sup_N \left(\inf_{K \geq N} T_W^K \theta \right) = \bigvee\nolimits_N T_W^N \theta \equiv U_\infty.$$

Then monotonic sup-preservation is the same as stating:

$$T_W \left(\liminf_N \left(T_W^N \theta\right) \right) = \liminf_N \left(T_W^N \theta\right),$$

or $T_W(\vee_N T_W^N \theta) = T_W(U_\infty) = U_\infty$. Abstract these conditions to apply to nets and to describe a topological continuity idea. The possibility for unifying order and topological properties for the Koopmans operator falls into place.

[37] Vulikh [44] covers varies forms of order convergence in Riesz spaces based on nets and sequences. Kantorovich's [22] fundamental paper defines the sequential version of order convergence. He defines a sequence's upper and lower limit first. Both are well-defined in a Dedeind complete Riesz space based on its order structure alone. A sequence has an order limit if its upper and lower limits agree. Order continuous functions are defined in terms of these upper and lower limits.

Scott [40] proposes a topology for a complete lattice by abstracting the notions of a lower limit for sequences and lower semicontinuity for functions.[38] His **induced topology** permits consideration of continuous self-maps on the given complete lattice. The literature following Scott's fundamental paper refers to the induced topology as the **Scott topology**. Assigning the Scott topology to $\langle \theta, U^T \rangle$ turns that set into a $T_0 - space$: given the points U and V in $\langle \theta, U^T \rangle$, there is a Scott open set containing one and not the other point. The space $\langle \theta, U^T \rangle$ endowed with its Scott topology is neither a T_1 space nor a T_2 space. Convergent nets may have more than one limit!

There are two ways to define the Scott topology. One specifies the open sets directly. The other defines the class of convergent nets and their limits. Sequences hardly suffice in this setup. Both approaches are found in the literature. We specify the net convergence class.[39] It is an analytical approach that directly links to our proof that the Koopmans operator is Scott continuous. Both descriptions of Scott's topology are presented in Gierz et al [18].[40] Scott's [40] original paper also develops both approaches. We closely follow Gierz et al's net convergence class presentation.

This topolgical structure, adapted to our setting, is presented below. Next, we prove the Koopmans operator is a Scott continuous self-map on the order interval $\langle \theta, U^T \rangle$. This order interval's complete lattice structure plays an integral role in this demonstration. The monotonic sequence $\{T_w^n \theta\}$ once again constructs the principal fixed point, U_∞, by successive approximations. This is a surprising conclusion given that we must use nets to describe the topology since sequences do not suffice. However, monotonic sequences are particular monotonic nets where the natural numbers form the directed index set. Scott introduced his topology to further the development of computational theory. A similar construction of the largest fixed point, U^∞, is **not** available using the Scott topological structure! Scott's topological setup abstracts properties enjoyed by real-valued lower semicontinuus functions defined on a metric space and may differ from related properties characteristic of upper semicontinuous functions. For this reason, we argue that the Scott continuity property of the Koopmans operator, and the subsequent fixed point theory (closely associated with the TK FPT), form another rationale for calling the smallest fixed point, U_∞, the principal solution to the operator equation, $T_W U = U$, for $U \in \langle \theta, U^T \rangle$.

A **net** $u : \Lambda \to \langle \theta, U^T \rangle$ is a mapping from a directed set, Λ, to the complete lattice $\langle \theta, U^T \rangle$. Denote the net by setting $u(\lambda) = U^\lambda \in \langle \theta, U^T \rangle$. The set Λ (with generic elements λ, μ, and ν) is the net's **index set**. This set is **directed** by a binary relation $\geq$ which is reflexive and transitive. Moreover, if λ and μ are elements of Λ, then there is a $\nu \in \Lambda$ such that $\nu \geq \lambda$ and $\nu \geq \mu$. Write this net as $\left(U^\lambda\right)_{\lambda \in \Lambda}$ or, when the meaning is clear, as $\left(U^\lambda\right)$. We say that $\left(U^\lambda\right)$

[38] Scott's formal constructions apply to a broader class of partially ordered sets. We stick with the stronger complete lattice setup that fits our theoretical model.

[39] Kelly ([23], [24]) shows how to specify a topology by describing convergent nets on the given space. This amounts to defining a Kuratowski closure operator.

[40] See Gierz, et al ([18], pp. 131-138) for detailed motivation, formal definitions of Scott open sets, and the formal development of his topology via net convergence.

is a net in $\left\langle \theta, U^T \right\rangle$. This net is **monotonic (isotonic)** when $\mu \geq \lambda$ implies $U^\mu \geq U^\lambda$. Monotone nets play an important role in Scott's topological theory.

For any net $\left(U^\lambda\right)$ in $\left\langle \theta, U^T \right\rangle$ define the net's **lower limit**, or lim inf, by

$$\liminf_\lambda \left(U^\lambda\right) = \sup_\lambda \left[\inf_{\mu \geq \lambda} U^\mu\right]. \tag{8}$$

Scott [40] refers to the lower limit of the net as its **principal limit**. We adopt this terminology as well and justify it below. Note that if $\left(U^\lambda\right)$ is a monotonic net in $\left\langle \theta, U^T \right\rangle$, then $\liminf_\lambda \left(U^\lambda\right) = \sup_\lambda \left(U^\lambda\right)$. This follows as the sup exists in $\left\langle \theta, U^T \right\rangle$ since this order interval is a complete lattice in its induced order inherited from the underlying space of possible utility functions. Clearly monotonicity of $\left(U^\lambda\right)$ imples $\inf_{\mu \geq \lambda} U^\mu = U^\lambda$ exists as well for each $\lambda \in \Lambda$.

The Scott topology is defined in terms of the definition of the class of Scott convergent nets. Let $\mathcal{S}$ denote the class of those pairs $\left(\left(U^\lambda\right), U\right)$ such that

$$U \leq \liminf_\lambda \left(U^\lambda\right). \tag{9}$$

For such a pair we say that U is an $\mathcal{S}-$ *limit* of the net $\left(U^\lambda\right)$ and we denote this limit

$$\left(U^\lambda\right) \xrightarrow{\mathcal{S}} U. \tag{10}$$

The convergence conditions and inequality (9) hold pointwise for each $C \in \ell_\infty^+(\alpha)$. That is, (9) is equivalent to the pointwise condition:

$$U(C) \leq \liminf_\lambda \left(U^\lambda(C)\right) = \sup_\lambda \left[\inf_{\mu \geq \lambda} U^\mu(C)\right]. \tag{11}$$

The monotonic net $\left(U^\lambda\right)$ has the property $\left(U^\lambda\right) \xrightarrow{\mathcal{S}} U^\mu$ for each $\mu \in \Lambda$. That is, each U^μ is an $\mathcal{S}-$ *limit* of the net $\left(U^\lambda\right)$! The reason is simple: each $U^\mu \leq \sup_\lambda \left(U^\lambda\right)$; hence $U^\mu \leq \liminf_\lambda \left(U^\lambda\right)$. This shows a net's $\mathcal{S}-$ *limit* may not be unique. For an arbitrary net in $\left\langle \theta, U^T \right\rangle$ we refer to the particular limit function, $\liminf_\lambda \left(U^\lambda\right)$, as the net's **principal limit** to distinguish it from other points in $\left\langle \theta, U^T \right\rangle$ which are also limits for this net.

This description of net convergence defines the Scott topology on the complete lattice $\left\langle \theta, U^T \right\rangle$. The Koopmans operator is **Scott continuous** if and only if for each $\left(U^\lambda\right) \xrightarrow{\mathcal{S}} U$, the corresponding values $\left(T_W U^\lambda\right) \xrightarrow{\mathcal{S}} T_W U$. That is, the **abstract lower semicontinuity property** holds (pointwise):

$$\left(T_W U\right) \leq \liminf_\lambda \left(T_W U^\lambda\right) \tag{12}$$

whenever $\left(U^\lambda\right) \xrightarrow{\mathcal{S}} U$. Writing out the pointwise version of the above inequality in terms of the underlying Thompson aggregator yields the condition

$$W\left(c_1, U(SC)\right) \leq \liminf_\lambda \left[W\left(c_1, U^\lambda(SC)\right)\right]. \tag{13}$$

Proposition 13 *(Gierz et al ([18], Proposition II-2.1, p. 157). The Koopmans operator is a Scott continuous self-map on* $\left\langle \theta, U^T \right\rangle$ *if and only if it is an order-preserving (monotone) operator and for any net* $\left(U^\lambda\right)$ *in* $\left\langle \theta, U^T \right\rangle$ *such that* $\liminf_\lambda \left(U^\lambda\right)$ *and* $\liminf_\lambda \left(T_W U^\lambda\right)$ *both exist,*

$$T_W \left(\liminf_\lambda U^\lambda \right) \leq \liminf_\lambda \left(T_W \left(U^\lambda \right) \right) \tag{14}$$

Inequality (14) expresses the abstract lower semicontinuity inequality (12) for the case where U *is the net's principal limit.* Note that $\liminf_\lambda \left(U^\lambda\right)$ and $\liminf_\lambda \left(T_W U^\lambda\right)$ both exist since $\left\langle \theta, U^T \right\rangle$ is a complete lattice in its induced order. The nets appearing in this proposition may, or may not, be monotonic. The pointwise analog of (14) expressed in terms of the Thompson aggregator is

$$W \left(c_1, \liminf_\lambda U^\lambda (SC) \right) \leq \liminf_\lambda W \left(c_1, U^\lambda (SC) \right), \tag{15}$$

where

$$\begin{aligned} T_W \left(\liminf_\lambda U^\lambda (C) \right) &= W \left(c_1, \liminf_\lambda U^\lambda (SC) \right), \text{ and} \\ \liminf_\lambda \left(T_W \left(U^\lambda (C) \right) \right) &= \liminf_\lambda W \left(c_1, U^\lambda (SC) \right). \end{aligned}$$

Since the Koopmans operator is known to be a monotone operator it suffices to verify (14) obtains for an arbitrary convergent net of functions in $\left\langle \theta, U^T \right\rangle$ in order to conclude the Koopmans operator is Scott continuous.

Observe that if $\left(U^\lambda\right) \xrightarrow{\mathcal{S}} U$, then $U \leq \liminf_\lambda U^\lambda$, so T_W monotone implies

$$T_W U \leq T_W \left(\liminf_\lambda U^\lambda \right).$$

Hence, if (14) also holds, then the previous inequality yields

$$T_W U \leq T_W \left(\liminf_\lambda U^\lambda \right) \leq \liminf_\lambda \left(T_W \left(U^\lambda \right) \right),$$

which is the abstract lower semicontinuity inequality (12) and T_W is Scott continuous.

Proposition 14 *(Scott Continuity Proposition)* T_W *is a Scott continuous self-map on* $\left\langle \theta, U^T \right\rangle$.

Proof. We prove the pointwise inequality (15) obtains. Fix a consumption sequence $C \in \ell_\infty^+ (\alpha)$. Note that Thompson aggregator property *(T1)* requires the aggregator function, $W(x, y)$, to be jointly continuous on $\mathbb{R}_+^2$. In particular, given $c_1 \geq 0$, the function $W(c_1, \bullet)$ is a lower semicontinuous function on $\mathbb{R}_+^* = [0, +\infty]$, the nonnegative extended real numbers endowed with its usual topology. Now consider $\left(U^{\lambda}(C)\right)$ and $\left(U^{\lambda}(SC)\right)$ as defining nets in $\mathbb{R}_{+}^{*}$. In fact, the values taken by the nets for each index are nonnegative real numbers as each U^{λ} is $\varphi_{\gamma}-$*bounded.* Indeed, each $U^{\lambda} \leq U^{T}$ implies $\left\|U^{\lambda}\right\|_{\gamma} \leq \left\|U^{T}\right\|_{\gamma}$. This lower semicontinuity property for the aggregator implies:

$$W\left(c_1, \liminf_{\lambda} U^{\lambda}(SC)\right) \leq \liminf_{\lambda} W\left(c_1, U^{\lambda}(SC)\right),$$

which is (15). Therefore (14) holds and T_W is Scott continuous by the previous Proposition. ■

Gierz et al ([18], p. 157) show that Scott continuity is also equivalent to sup preservation for directed sets (which can be taken to be nets). We highlight this property for *monotonic nets* as this sets up the fixed point argument proving $\mathrm{fix}(T_W)$ is nonempty.

Definition 15 *The Koopmans operator T_W is said to* ***preserve the supremum of the monotonic net*** $\left(U^{\lambda}\right)$ *in* $\left\langle\theta, U^{T}\right\rangle$ *whenever*

$$T_W\left(\liminf_{\lambda}\left(U^{\lambda}\right)\right) = \liminf_{\lambda}\left(T_W U^{\lambda}\right). \tag{16}$$

Put differently, T_W preserves the supremum of monotonic nets provided that

$$T_W\left(\sup\left(U^{\lambda}\right)\right) = \sup\left(T_W U^{\lambda}\right).$$

These suprema correspond to the principal limits of the monotonic nets $\left(U^{\lambda}\right)$ and $\left(T_W U^{\lambda}\right)$, where the latter net is also monotonic as T_W is a monotone operator. Notice that if this property holds for arbitrary monotone nets, then it holds in particular for monotonic (nondecreasing) sequences, such as $\left\{T_W^N \theta\right\}$. This observation is the key to reducing the existence of a fixed point for the Koopmans operator to the application of the TK FPT (for *monotonically* sup $-$*preserving sequences*). The smallest fixed point, U_{∞}, is constructed as before by iteration of T_W indexed on the natural numbers with initial seed θ. The existence of the smallest fixed point by successiven approximations is available even though sequences do not suffice to describe the Scott topology. Hence, the key step in showing this construction applies is the following Corollary to the Scott Continuity Proposition.

Corollary 16 *T_W preserves the supremum of each monotonic net $\left(U^{\lambda}\right)$ in $\left\langle\theta, U^{T}\right\rangle$.*

Proof. Let $\left(U^{\lambda}\right)$ be a monotonic net in $\left\langle\theta, U^{T}\right\rangle$ with its principal Scott limit $\vee_{\lambda} U^{\lambda}$. The net $\left(T_W U^{\lambda}\right)$ is also a monotonic net in $\left\langle\theta, U^{T}\right\rangle$ since T_W is monotone. Its principal Scott limit is $\vee_{\lambda} T_W U^{\lambda}$. Since T_W is a Scott continuous self-map on $\left\langle\theta, U^{T}\right\rangle$, inequality (14) holds in the following form:

$$T_W\left(\bigvee_{\lambda} U^{\lambda}\right) \leq \bigvee_{\lambda}\left(T_W U^{\lambda}\right).$$

The converse inequality follows since T_W is a monotone operator. To see this, note that for each index $\mu \in \Lambda$,

$$\bigvee_{\lambda} U^{\lambda} \geq U^{\mu},$$

and by T_W monotone,

$$T_W \left(\bigvee_{\lambda} U^{\lambda} \right) \geq T_W U^{\mu}.$$

The supremum of the righthand side, after changing back to the λ index notation, is just $\vee_{\lambda} T_W U^{\lambda}$. Hence,

$$T_W \left(\bigvee_{\lambda} U^{\lambda} \right) \geq \bigvee_{\lambda} \left(T_W U^{\lambda} \right).$$

Therefore,

$$T_W \left(\bigvee_{\lambda} U^{\lambda} \right) = \bigvee_{\lambda} \left(T_W U^{\lambda} \right),$$

and the Koopmans operator preserves the supremum of monotonic nets. ■

The main result in this section is the existence of a smallest or least fixed point for the Koopmans operator and its construction by successive approximations.[41]

Theorem 17 *(Least Fixed Point Existence and Construction Theorem) The Scott continuous Koopmans operator has a least fixed point, U_{∞}. Moreover, $U_{\infty} = \vee_N T_W^N \theta$ and it is constructed by successive approximations indexed on the natural numbers.*

Proof. The existence and construction of U_{∞} follows from the TK FPT since T_W preserves the supremum of each monotonic net $\left(U^{\lambda}\right)$ in $\left\langle \theta, U^T \right\rangle$. In particular, this holds for the monotonic sequence $\left\{T_W^N \theta\right\}$. Hence, $U_{\infty} = \vee_N T_W^N \theta = T_W U_{\infty}$ and $U_{\infty} \in \text{fix}(T_W)$.

Suppose that $U \in \text{fix}(T_W)$. Then $\theta \leq U$ and T_W monotone implies $T_W \theta \leq T_W U = U$. Iterate this to yield the inequality $T_W^N \theta \leq U$. Hence, passing to the limit we find $U_{\infty} \leq U$ and U_{∞} is the least fixed point of the Koopmans operator acting on $\left\langle \theta, U^T \right\rangle$. ■

The sequence $\left\{T_W^N \theta\right\}$ has many Scott limits besides its principal limit, U_{∞}. But NONE of the other Scott limits, such as $T_W^N \theta$, are also fixed points. That

[41] This result appears in Gierz et al ([18], p. 160). Their constructive proof is basically the same as the proof of the Tarski-Kantorovich Theorem. We have linked to Tarski-Kantorovich by explicitly proving the monotonic net sup preservation property for the Koopmans operator. Thus, we formally show this condition implies the hypothesis for the Tarksi-Kantorovich Theorem's construction of the Koopmans operator's smallest fixed point. A related approach oriented to computation theory for abstract operators on complete partially ordered spaces is in Stoltenberg, et al ([43], p. 42).

is, the LFP is the only Scott limit of $\left\{T_W^N\theta\right\}$ that is also a fixed point of the Koopmans operator.

Marinacci and Montrucchio's Recovery Theorem says that $U^\infty = \wedge_N T_W^N U^T$ is the GFP. This is demonstrated by showing the antitone sequence $\left\{T_W^N U^T\right\}$ is inf-preserving: $T_W\left(\wedge_N T_W^N U^T\right) = \wedge_N T_W^N U^T$. This inf-preservation property required by the TK FPT does NOT have an analog in the Scott topology approach.[42] The antitone sequence $\left\{T_W^N U^T\right\} \searrow U^\infty$ fails to satisfy the monotonic net sup-preservation property simply because it is not monotone (as used in the Scott topology setup). Scott continuity acting alone yields the inequality $T_W U^\infty \leq \liminf_N \left(T_W^N U^T\right) = U^\infty$. Absent a form of inf-preservation, Scott's topological structure does not imply U^∞ is a fixed point for the Koopmans operator. Even though we know from the TK FPT that U^∞ is the GFP of T_W, this is not provable from Scott continuity alone.

Now suppose that we establish U^∞ as the GFP (say, by invoking the TK-FPT based proof of the Recovery Theorem). Then we observe that every function $U \in \langle\theta, U^\infty\rangle$ is also a Scott limit of the sequence $\left\{T_W^N U^T\right\}$. In particular, each $U \in$fix(T_W), including the LFP, is a Scott limit of $\left\{T_W^N U^T\right\}$. Therefore, the sequence $\left\{T_W^N U^T\right\}$ does not have a *unique* Scott limit which is also a fixed point, unlike the LFP theory's case. We cannot reasonably say that the GFP is *constructed* as the unique Scott limit of $\left\{T_W^N U^T\right\}$ which is also a fixed point.

The fact that the fixed point U_∞ is shown to exist as a consequence of verifying the Koopmans operator is Scott continuous provides us with a topological, as well as order-theoretic, defense for considering this fixed point as the operator equation's principal solution.

The Least Fixed Point Existence and Construction Theorem does not yield either the existence of the GFP nor any statement about fix(T_W) other than it is nonempty and U_∞ is its smallest element. By contrast, the TK FPT constructions yield the extremal fixed points and fix(T_W) is a countably chain complete poset. The Least Fixed Point Existence and Construction Theorem's hypotheses are stronger than monotonicity of T_W assumed in Tarski's Theorem [41]. The formal argument is also more elementary (by reduction to the monotonic sup-preservation of sequences) in comparison to Tarski's Theorem.[43] In particular, the constructive TK FPT proof based on successive approximations by iteration over the natural numbers is certainly more elementary than the recent "constructive" versions for Tarski's Theorem due to Cousot and Cousot [15] and Echenique [17] obtained by interating over the ordinals.

6 Concluding Comments

Our existence theory is not yet ready for applications to the Ramsey optimal growth model. Yet, we report progress on that front using the iterative construction of the principal solution to the Koopmans equation. There are two

[42] The inf-preservation property is the analog of saying the Koopmans operator is upper semicontinuous. However, this concept is not meaningful in the Scott topology.

[43] See the comments in Gierz et al ([18], p. 160).

questions: First, how far can the topology on $\ell_\infty^+(1)$, for example, be weakened to provide product compact feasible sets in standard growth models AND a product upper-semicontinuous U^∞. If U_∞ is concave and finite on $\ell_\infty^+(1)$ and there is a unique solution to the Koopmans operator, then we can conclude it is also continuous on the positive cone's norm interior in the corresponding weak topology. This points to the second problem concerning the development of sufficient conditions for a unique solution to the Koopmans equation provided one exists in the first place.

Martins-Da-Rocha and Vailakis [33] prove uniqueness theorems on a similar domain to the one found in Marinacci and Montrucchio's uniqueness theory (and ours too), but employing weaker topologies that would be consistent with proving optimal programs exist. Their results turn on checking the local contraction mapping theorems originating in Rincón-Zapatero and Rodriguez-Palmero ([38], [39]). Marinacci and Montrucchio [34] do this for the Thompson case using the Thompson metric topology. Martins-da-Rocha and Vailakis ([32], [33]) also follow that programmatic use of the Thompson metric in their uniqueness theories.

The importance of uniqueness theory is now clear. It seeks a unique way to match a given aggregator and a unique utility function solution to the Koopmans equation. It also yields properties of the solution suitable for proving existence and uniqueness of optimal solutions in optimal capital accumulation models. We leave development of our uniqueness theory to another paper.[44]

[44] See the working paper by Becker and Rincón-Zapatero [11] for a concave operator treatment of the uniqueness question.

References

[1] Charalmbos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide, 3^{rd}-Edition*, Springer, New York, 2006.

[2] Charalmbos D. Aliprantis and Own Burkinshaw, *Positive Operators*, Academic Press, San Diego, 1985.

[3] Charalmbos D. Aliprantis and Own Burkinshaw, *Principles of Real Analysis, 3^{rd}-Edition*, Academic Press, San Diego, 1998.

[4] Charalmbos D. Aliprantis and Rabee Tourky, *Cones and Duality*, American Mathematical Society, Providence, 2007.

[5] Łukasz Balbus, "On Non-Negative Recursive Utilities in Dynamic Programming with Nonlinear Aggregators and CES," Working Paper University of Zielona Góra, Poland, December 2015.

[6] Łukasz Balbus, Kevin Reffett, and Łukasz Woźny, "Time Consistent Markov Policies in Dynamic Economies with Quasi-Hyperbolic Consumers," *International Journal of Game Theory*, 44, 2015, pp. 83-112.

[7] Łukasz Balbus, "On Non-Negative Recursive Utilities in Dynamic Programming with Nonlinear Aggregators and CES," University of Zielora Góra Working Paper, June 21, 2016.

[8] Andrei Baranga, "The Contraction Principle as a Particular Case of Kleene's Fixed Point Theorem," *Discrete Mathematics*, 98, 1991, pp. 75-79.

[9] Robert A. Becker and John H. Boyd III, *Capital Theory, Equilibrium Analysis, and Recursive Utility*, Blackwell Publishing, 1997.

[10] Robert A. Becker, John H. Boyd, III, and Ciprian Foias, "The Existence of Ramsey Equilibrium," *Econometrica*, Vol. 59 (2), March 1991, pp. 441-460.

[11] Robert A. Becker and Juan Pablo Rincón-Zapatero, "Recursive Utility and Thompson Aggregators," Indiana University CAEPR Working Paper #XX, July 22, 2017 (revised August 29, 2017).

[12] Garrett Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, RI, 1940.

[13] Gaetano Bloise and Yiannis Vailakis, "Convex Dynamic Programming with (Bounded) Recursive Utility," *Journal of Economic Theory*, 173, 2018, pp. 118-141.

[14] John H. Boyd III, "Recursive Utility and the Ramsey Problem," *Journal of Economic Theory*, 50, 1990, pp. 326-345.

[15] Patrick Cousot and Radhia Cousot, "Constructive Versions of Tarski's Fixed Point Theorems," *Pacific Journal of Mathematics*, Vo. 82 (1), 1979, pp. 43-57.

[16] B.A. Davey and J.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, UK, 1990.

[17] Federico Echenique, "A Short and Constructive Proof of Tarski's Fixed Point Theorem," *International Journal of Game Theory*, Vol. 32(2), June 2005, pp. 215-218.

[18] Gierz, G., H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, and D.S. Scott (Gierz et al) *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, UK, 2003.

[19] Andrzej Granas and James Dugundji, *Fixed Point Theory*, Springer, New York, 2003.

[20] Jacek Jachymski, Leslaw Gajek, and Piotr Pokarowski, "The Tarski-Kantorovich Principle and the Theory of Iterated Function Systems," *Bulletin of the Australian Mathematical Society*, 61, 2000, pp. 247-261.

[21] Takashi Kamihigashi, Kevin Reffett, and Masayuki Yao, "An Application of Kleene's Fixed Point Theorem to Dynamic Programming: A Note," Working Paper, Reserach Institute for Economics and Business Administration, Kobe University, July 2014.

[22] Leonid Kantorovich, "The Method of Successive Approximations for Functional Equations," *Acta Mathematica*, Vo. 71, 1939, pp. 63-97.

[23] J.L. Kelly, "Convergence in Topology," *Duke Mathematics Journal*, 17, 1950, 277-283.

[24] J.L. Kelly, *General Topology*, Van Nostrand Reinhold, 1955.

[25] Tjalling C. Koopmans, "Stationary Ordinal Utility and Impatience," *Econometrica*, 28, 1960, 287-309.

[26] Tjalling C. Koopmans, "Representation of Preference Orderings Over Time,"in *Decision and Organization* (CB. McGuire and Roy Radner, eds.), North-Holland, Amsterdam, 1972.

[27] Tjalling C. Koopmans, "Representation of Preference Orderings with Independent Components of Consumption,"in *Decision and Organization* (CB. McGuire and Roy Radner, eds.), North-Holland, Amsterdam, 1972.

[28] Tjalling C. Koopmans, Peter A. Diamond, and Richard E. Williamson, "Stationary Utility and Time Perspective," *Econometrica*, 82, 1964, pp. 82-100.

[29] Coung Le Van and Yiannis Vailakis, "Recursive Utility and Optimal Growth with Bounded or Unbounded Returns," *Journal of Economic Theory*, 123, 2005, 187-209.

[30] Robert E. Lucas and Nancy L. Stokey, "Optimal Growth with Many Consumers," *Journal of Economic Theory*, 32, 1984, pp. 139-171.

[31] Mukul Majumdar, "Some Remarks on Optimal Growth with Intertemporally Dependent Preferences in the Neoclassical Model, *Review of Economic Studies*, 42, 1975, pp. 147-153.

[32] V. Felipe Martins-da-Rocha and Yiannis Vailakis, "Existence and Uniqueness of a Fixed Point for Local Contractions," *Econometrica*, Pol. 78 (3), May 2010, pp. 1127-1141.

[33] V. Felipe Martins-da-Rocha and Yiannis Vailakis, "Fixed Point for Local Contractions: Applications to Recursive Utility," *International Journal of Economic Theory*, 9, 2013, pp. 23-33.

[34] Massimo Marinacci and Luigi Montrucchio, "Unique Solutions for Stochastic Recursive Utilities," *Journal of Economic Theory*, 145, 2010, pp. 1776-1804.

[35] Massimo Marinacci and Luigi Montrucchio, "Unique Tarski Fixed Points," Working Paper No. 604, University of Bocconi, July 2017.

[36] Peter Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, New York, 1991.

[37] Anthony L. Peressini, *Ordered Topological Vector Spaces*, Harper & Row, New York, 1967.

[38] Juan Pablo Rincón-Zapatero and Carlos Rodriguez-Palmero, "Existence and Uniqueness to the Bellman Equation in the Unbounded Case," *Econometrica*, 71 (5), September 2003, pp. 1519-1555.

[39] Juan Pablo Rincón-Zapatero and Carlos Rodriguez-Palmero, "Recursive Utility with Unbounded Aggregators," *Economic Theory*, 33, 2007, pp. 381-391.

[40] Dana Scott, "Continuous Lattices," in: *Toposes, Algebraic Geometry and Logic* (F.W. Lawvere, ed.), Springer Lecture Notes in Mathematics 274, Springer-Verlag, Berlin, 1972, pp. 97-136.

[41] A. Tarski, "A Lattice-Theoretical Fixpoint Theorem and Applications," *Pacific Journal of Mathamatics*, 3, 1955, pp. 285-309.

[42] Spyros Vassilakis, "Some Economic Applications of Scott Domains," *Mathematical Social Sciences*, 24, 1992, pp. 173-208.

[43] Viggo Stoltenberg-Hansen, Ingrid Lindström, and Edward R. Griffor, *Mathematical Domains*, Cambridge University Press, Cambridge, UK, 1994.

[44] B.Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, Wolters-Noordhoff Scientific Publications, Gronigen, Netherlands, 1967.