Dedicated to G. J. Chaitin
for his 50th Birthday

Recursively Enumerable Reals and Chaitin Ω Numbers*

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Abstract. A real α is called recursively enumerable if it is the limit of a recursive, increasing, converging sequence of rationals. Following Solovay [23] and Chaitin [10] we say that an r.e. real α *dominates* an r.e. real β if from a good approximation of α from below one can compute a good approximation of β from below. We shall study this relation and characterize it in terms of relations between r.e. sets. Solovay's [23] Ω-like numbers are the maximal r.e. real numbers with respect to this order. They are random r.e. real numbers. The halting probability of a universal self-delimiting Turing machine (Chaitin's Ω number, [9]) is also a random r.e. real. Solovay showed that any Chaitin Ω number is Ω-like. In this paper we show that the converse implication is true as well: any Ω-like real in the unit interval is the halting probability of a universal self-delimiting Turing machine.

1 Introduction and Notation

Algorithmic information theory, as developed by Chaitin [8, 9, 11], Kolmogorov [16], Solomonoff [22], Martin-Löf [19], and others (see Calude [4]), gives a satisfactory description of the quantity of information of individual finite strings and infinite sequences. The same quantity of information may be organised in various ways; in order to quantify the degree of organisation of the information in a string or a sequence, Bennett [2], Juedes, Lathrop, and Lutz [13], and others, have considered the computational depth. Roughly speaking, the computational depth of an object is the amount of time required for an algorithm to derive the

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object from its shortest description. Bennett [2] showed that the characteristic sequence χ_K of the halting problem is strongly deep, while no random sequence is strongly deep. Investigating this matter further, Juedes, Lathrop, and Lutz [13] have considered the notion of "usefulness" of infinite sequences. A sequence $\mathbf{x}$ is useful if all recursive sequences can be computed with oracle access to $\mathbf{x}$ within a fixed recursive time bound. For example χ_K is useful, while no recursive or random sequence is useful.

It is well known that the halting probability of a universal self-delimiting Turing machine, called Chaitin Ω number (see Chaitin [9, 12], Calude [4]), is random, but χ_K is not; Ω and χ_K contain the same quantity of information but codified in vastly different ways. As we noted before, χ_K is useful but Ω is not useful in the sense of Juedes, Lathrop, and Lutz [13]. However, when one is interested in approximating sequences[1] Ω is more "useful" than χ_K; it is one of the aims of this paper to give a mathematical sense to this statement.

A real number is called *r.e.* if it is the limit of a recursive, increasing, converging sequence of rationals. R.e. reals are extensively used in computable analysis, see Weihrauch [25] and Ko [15]. We will characterize r.e. reals in various ways. In order to compare the "usefulness" of r.e. reals for approximation purposes, Solovay [23] (see also Chaitin [10]) has introduced a *domination* relation $\leq_{dom}$ between real numbers, which we shall define precisely in Section 3. Informally, a real α dominates a real β (written as $\beta \leq_{dom} \alpha$) if from a good approximation of α from below one can compute a good approximation of β from below. The relation $\leq_{dom}$ is transitive and reflexive, hence it naturally defines a partially ordered set $\langle \mathbf{R}_{r.e.}\,; \leq_{dom} \rangle$ whose elements are the $=_{dom}$-equivalence classes. We shall see that this partially ordered set is an upper semilattice. It has a minimum element which is the class containing exactly all recursive reals. We study this relation $\leq_{dom}$ further and characterize it in terms of certain reducibilities between r.e. sets. Solovay [23] (see also Chaitin [10]) called an r.e. real *Ω-like* if it dominates every r.e. real. He showed that every Chaitin Ω number is Ω-like. In this paper we prove the converse implication by showing that any Ω-like real in the unit interval is the halting probability of a universal self-delimiting Turing machine. Thus, the semilattice of $=_{dom}$–classes of r.e. reals under $\leq_{dom}$ has a maximum element, which is the equivalence class containing exactly all Chaitin Ω numbers. This shows the strength of all Ω's for approximation purposes: from a good approximation of Ω one can obtain a good approximation of any r.e. real, and no other reals have this property. Consequently, Ω contains more information than any non-Ω-like r.e. real.

In the following section we review some fundamental notions and facts from algorithmic information theory. In Section 3 we give various characterizations of r.e. reals and introduce the domination relation and prove basic facts about it. We compare it with Turing reducibility and characterize it in terms of another reducibility between sets of strings. In Section 4 we show that every Ω–like real is in fact the halting probability of a universal self–delimiting Turing machine.

[1] As in constructive mathematics, see Bridges and Richman [3], Weihrauch [25] and Ko [15], and many other areas.

We close this section by introducing some notation. By $\mathbf{N}, \mathbf{Q}$ and $\mathbf{R}$ we denote the set of nonnegative integers, the set of rational numbers, and the set of real numbers, respectively. If X and Y are sets, then $f : X \stackrel{o}{\to} Y$ denotes a possibly partial function defined on a subset of X. Let $\Sigma = \{0, 1\}$ denote the binary alphabet; Σ^* is the set of (finite) binary strings and Σ^ω is the set of infinite binary sequences. The length of a string x is denoted by $|x|$. Let $string_n$ ($n \geq 0$) be the nth string under the quasi-lexicographical ordering on Σ^*. For a sequence $\mathbf{x} = x_0 x_1 \cdots x_n \cdots \in \Sigma^\omega$ and an integer number $n \geq 0$, $\mathbf{x}(n)$ denotes the initial segment of length $n + 1$ of $\mathbf{x}$ and x_i denotes the ith digit of $\mathbf{x}$, i.e., $\mathbf{x}(n) = x_0 x_1 \cdots x_n$. Lower case letters k, l, m, n will denote nonnegative integers, and x, y, z strings. By $\mathbf{x}, \mathbf{y}, \cdots$ we denote infinite sequences from Σ^ω; finally, we reserve α, β, γ for reals. Capital letters are used to denote subsets of Σ^*. For a set $A \subseteq \Sigma^*$, we denote by χ_A the infinite characteristic sequence of A, that is, $(\chi_A)_n = 1$ if $string_n \in A$ and $(\chi_A)_n = 0$ otherwise.

We assume that the reader is familiar with Turing machine computations, including oracle computations. We use K to denote the halting problem, that is, $string_n \in K$ if and only if the nth Turing machine halts on the input $string_n$. We say that a language A is Turing reducible to a language B, and we write $A \leq_T B$, if there is an oracle Turing machine M such that $M^B(string_n) = (\chi_A)_n$, for all n. For further notation we refer to Calude [4].

2 Complexity and Randomness

In this section, we review some fundamentals of algorithmic information theory that we will use in this paper. We are especially concerned with self-delimiting (Chaitin/program-size) complexity and algorithmic randomness. The advantage of the self-delimiting version of the descriptive complexity is that it gives a precise characterization of algorithmic probability and random sequences.

A *self-delimiting* Turing machine is a deterministic Turing machine such that the program set $\mathrm{PROG}_M = \{x \in \Sigma^* \mid$ on input x, the machine M halts after finitely many steps$\}$ is prefix-free, i.e., a set of strings with the property that no string in it is a proper prefix of another. It follows by Kraft's inequality that, for every self-delimiting Turing machine M,

$$\Omega_M = \sum_{x \in \mathrm{PROG}_M} 2^{-|x|} \leq 1.$$

The number Ω_M is called the *halting probability* of M. In what follows we will omit the adjective "self-delimiting", since this is the only type of Turing machine considered in this paper.

Definition 1. Let M be a Turing machine. The *program-size complexity* of the string $x \in \Sigma^*$ (relative to M) is $H_M(x) = \min\{|y| \mid y \in \Sigma^*,\ M(y) = x\}$, where $\min \emptyset = \infty$.

It was shown by Chaitin [9] that there is a self-delimiting Turing machine U that is *universal*, in the sense that, for every self-delimiting Turing machine M, there is a constant c_M (depending upon U and M) with the following property: if $x \in \mathrm{PROG}_M$, then there is an $\tilde{x} \in \mathrm{PROG}_U$ such that $U(\tilde{x}) = M(x)$ and $|\tilde{x}| \leq |x| + c_M$. Clearly, every universal machine produces every string. For two universal machines U and V, we see $H_U(x) = H_V(x) + O(1)$. The halting probability Ω_U of a universal machine U is called *Chaitin Ω number*; for more about Ω_U see Bennett [1], Calude, Salomaa [7], Calude, Meyerstein [6]. In the rest of the paper, unless stated otherwise, we will use a fixed universal machine U and will omit the subscript U in $H_U(x)$ and Ω_U. We will also abuse our notation by identifying the real number Ω with the infinite binary sequence which corresponds to Ω (i.e., the infinite[2] binary expansion of Ω without "0.").

We conclude this section with a brief discussion of (algorithmically) random infinite binary sequences.[3] Random sequences were originally defined by Martin-Löf [19] using constructive measure theory. Complexity-theoretic characterizations of random sequences have been obtained by Chaitin [9] (see also Levin [18], Schnorr [20]). A Martin–Löf test is an r.e. set $A \subseteq \mathbf{N} \times \Sigma^*$ satisfying the following measure-theoretical condition: $\mu(A^{(i)}) \leq 2^{-i}$, for all $i \in \mathbf{N}$ where we define $A^{(i)} = \{\alpha \in \Sigma^\omega \mid \text{there is a finite prefix } x \text{ of } \alpha \text{ with } (i,x) \in A\}$.[4] Here μ is the usual product measure on Σ^ω given by $\mu(\{w\}\Sigma^\omega) = 2^{-|w|}$, for $w \in \Sigma^*$. An infinite sequence $\mathbf{x}$ is *random* if, for every Martin-Löf test A, $\mathbf{x} \notin \bigcap_{i \geq 0} A^{(i)}$. This can also be expressed via the program-size complexity: $\mathbf{x}$ is random if and only if there exists a constant $c > 0$ such that $H(\mathbf{x}(n)) > n - c$, for every integer $n > 0$, and if and only if $\lim_{n\to\infty}(H(\mathbf{x}(n)) - n) = \infty$.

Theorem 2. (Chaitin [9]) *For every universal machine U, the halting probability Ω_U is random.*

3 R.E. Reals and Domination

It is the aim of this section to compare the information contents of r.e. reals. A real α is called *r.e.* if there is a recursive, increasing sequence of rationals which converges to α.[5] We start with several characterizations of r.e. reals.

For a prefix-free set $A \subseteq \Sigma^*$ we define a real number by $2^{-A} = \sum_{x \in A} 2^{-|x|}$, which, due to Kraft's inequality, lies in the interval $[0,1]$. For a set $X \subseteq \mathbf{N}$ we define the number $2^{-X-1} = \sum_{n \in X} 2^{-n-1}$. This number also lies in the interval $[0,1]$. If we disregard all finite sets X, which lead to rational numbers 2^{-X-1}, we get a bijection $X \mapsto 2^{-X-1}$ between the class of infinite subsets of $\mathbf{N}$ and the real numbers in the interval $(0,1]$. If $0.\mathbf{y}$ is the binary expansion of a real α with infinitely many ones, then $\alpha = 2^{-X_\alpha - 1}$ where $X_\alpha = \{i \mid y_i = 1\}$. Clearly, if X_α is r.e., then the number $2^{-X_\alpha-1}$ is r.e., but the converse is not true as the

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[2] This expansion is unique since by Theorem 2, Ω is random and, hence, irrational.

[3] The interested reader is referred to Calude [4] and Wang [24] for more details.

[4] See Calude [4] for a detailed motivation.

[5] Note that the property of being r.e. depends only on the fractional part of the real number.

Chaitin Ω numbers show. We characterize r.e. reals α in terms of prefix-free r.e. sets of strings[6] and in terms of the sets X_α.

Theorem 3. *For a real $\alpha \in (0,1]$ the following conditions are equivalent:*

1. *The number α is r.e.*
2. *There is a recursive, non–decreasing sequence of rationals $(a_n)_{n\geq 0}$ which converges to α.*
3. *The set $\{p \in \mathbf{Q} \mid p < \alpha\}$ of rationals less than α is r.e.*
4. *There is an infinite prefix-free r.e. set $A \subseteq \Sigma^*$ with $\alpha = 2^{-A}$.*
5. *There is a total recursive function $f : \mathbf{N}^2 \to \{0,1\}$ such that*
   (a) *If for some k, n we have $f(k,n) = 1$ and $f(k,n+1) = 0$ then there is an $l < k$ with $f(l,n) = 0$ and $f(l,n+1) = 1$.*
   (b) *We have: $k \in X_\alpha \iff \lim_{n\to\infty} f(k,n) = 1$.*

In order to compare the information contents of r.e. reals, Solovay [23] has introduced the following definition.

Definition 4. (Solovay [23] and Chaitin [10]) The real α is said to *dominate* the real β if there are a partial recursive function $f : \mathbf{Q} \stackrel{o}{\to} \mathbf{Q}$ and a constant $c > 0$ with the property that if p is a rational number less than α, then $f(p)$ is (defined and) less than β, and it satisfies the inequality $c(\alpha - p) \geq \beta - f(p)$. In this case we write $\alpha \geq_{dom} \beta$ (or $\beta \leq_{dom} \alpha$).

Roughly speaking, a real α dominates a real β if there is an effective way to get a good approximation to β from below from any good approximation to α from below. For r.e. reals this can also be expressed as follows.

Lemma 5. *An r.e. real α dominates an r.e. real β if and only if there are recursive, non-decreasing sequences (a_i) and (b_i) of rationals and a constant c with $\lim_n a_n = \alpha$, $\lim_n b_n = \beta$, and $c(\alpha - a_n) \geq \beta - b_n$, for all n.*

Theorem 6. (Solovay [23]) *Let $\mathbf{x}, \mathbf{y} \in \Sigma^\omega$ be two infinite binary sequences such that both $0.\mathbf{x}$ and $0.\mathbf{y}$ are r.e. reals and $0.\mathbf{x} \geq_{dom} 0.\mathbf{y}$. Then, $H(\mathbf{y}(n)) \leq H(\mathbf{x}(n)) + O(1)$.*

Next, we formulate a few results which will be useful in discussing the lattice structure of r.e. reals.

Lemma 7. *Let α, β and γ be r.e. reals. Then the following conditions hold:*

1. *The relation $\geq_{dom}$ is reflexive and transitive.*
2. *For every α, β one has $\alpha + \beta \geq_{dom} \alpha$.*
3. *If $\gamma \geq_{dom} \alpha$ and $\gamma \geq_{dom} \beta$, then $\gamma \geq_{dom} \alpha + \beta$.*

The second and third statement are true also if addition is replaced by multiplication and only positive r.e. reals are considered. By Theorem 6 we obtain

[6] Note that the prefix-free r.e. sets $A \subseteq \Sigma^*$ are exactly the domains of self-delimiting Turing machines.

Corollary 8. *The sum of a random r.e. real and an r.e. real is a random r.e. real.*

Also, the product of a positive random r.e. real with a positive r.e. real is a random r.e. real. Hence, addition (and multiplication on positive numbers) preserves the property of being random and r.e. in contrast with the fact that addition and multiplication do not preserve randomness. For example, if α is a random number, then $1-\alpha$ is random as well,[7] but $\alpha + (1-\alpha) = 1$ is not random.

In the following we discuss the lattice structure of r.e. reals under the domination relation. For two reals α and β, $\alpha =_{dom} \beta$ denotes the conjunction $\alpha \geq_{dom} \beta$ and $\beta \geq_{dom} \alpha$. For a real α, let $[\alpha] = \{\beta \in \mathbf{R} \mid \alpha =_{dom} \beta\}$; $\mathbf{R}_{r.e.} = \{[\alpha] \mid \alpha \text{ is an r.e. real}\}$. By Lemma 7 the least upper bound of any two classes containing r.e. reals α and β, respectively, is the class containing the r.e. real $\alpha + \beta$. We conclude

Theorem 9. *The structure $\langle \mathbf{R}_{r.e.}\,; \leq_{dom}\rangle$ is an upper semi-lattice.*

Now we compare the domination relation $\leq_{dom}$ on r.e. reals with the Turing reducibility. For every infinite sequence $\mathbf{x} = x_0x_1x_2\ldots \in \Sigma^\omega$ such that $0.\mathbf{x}$ is an r.e. real, let $A_\mathbf{x} = \{v \in \Sigma^* \mid 0.v \leq 0.\mathbf{x}\}$ and $A^\#_\mathbf{x} = \{string_n \mid x_n = 1\}$. Then, obviously, $A_\mathbf{x}$ is an r.e set which is Turing equivalent to $A^\#_\mathbf{x}$.[8] In the following, we establish the relationship between domination and Turing reducibility.

Lemma 10. *Let $\mathbf{x}, \mathbf{y} \in \Sigma^\omega$ be two infinite binary sequences such that both $0.\mathbf{x}$ and $0.\mathbf{y}$ are r.e. reals and $0.\mathbf{x} \geq_{dom} 0.\mathbf{y}$. Then $A_\mathbf{y} \leq_T A_\mathbf{x}$.*

Does the converse of Lemma 10 hold true? The negative answer will be given in Corollary 27. Let $\langle RE\,; \leq_T\rangle$ denote the upper semi-lattice structure of the class of r.e. sets under the Turing reducibility. A *strong homomorphism* from a partially ordered set $(X, \leq)$ to another partially ordered set $(Y, \leq)$ is a mapping $h : X \to Y$ such that

1. For all $x, \tilde{x} \in X$, if $x \leq \tilde{x}$, then $h(x) \leq h(\tilde{x})$.
2. For all $y, \tilde{y} \in Y$, if $y \leq \tilde{y}$, then there exist $x, \tilde{x}$ in X such that $x \leq \tilde{x}$ and $h(x) = y$, $h(\tilde{x}) = \tilde{y}$.

Lemma 10 shows that the mapping $0.\mathbf{x} \mapsto A_\mathbf{x}$ is a homomorphism. One can show that it is even a strong homomorphism.

Theorem 11. *There exists a strong homomorphism from $\langle \mathbf{R}_{r.e.}\,; \leq_{dom}\rangle$ onto $\langle RE\,; \leq_T\rangle$.*

We continue with the characterization of the domination relation between r.e. real numbers in terms of prefix-free r.e. sets of strings. We consider only infinite prefix-free r.e. sets. By $R.E.$ we denote the class of all infinite prefix-free r.e. subsets of Σ^*. First, we consider a relation between r.e. sets which is very close to the domination relation, but will turn out to be not equivalent.

[7] The number $1-\alpha$ does not need to be r.e. if α is r.e.

[8] Note that $A^\#_\mathbf{x}$ is not necessarily an r.e. set.

Definition 12. Let $A, B \in R.E.$ The set A *strongly simulates* B (shortly, $B \leq_{ss} A$) if there exist a partial recursive function $f : \Sigma^* \overset{o}{\to} \Sigma^*$ and a constant $c > 0$ such that $A = \mathrm{dom}\,(f)$, $B = f(A)$, and $|x| \leq |f(x)| + c$, for all $x \in A$. If $A \leq_{ss} B$ and $B \leq_{ss} A$, then we say that A and B are $\sim_{ss}$*-equivalent.*

One observes immediately that the relation $\leq_{ss}$ is reflexive and transitive. Hence, it defines a partially ordered set $\langle R.E._{ss}, \leq_{ss}\rangle$ where $R.E._{ss}$ is the set of $\sim_{ss}$-equivalence classes of $R.E.$ Our next goal is to see how the strong simulation relation $\leq_{ss}$ and $\leq_{dom}$ are related. The following lemma is straightforward.

Lemma 13. *If A, B are infinite prefix-free r.e. sets and $B \leq_{ss} A$, then 2^{-A} dominates 2^{-B}.*

The next result shows that the converse implication in Lemma 13 is in some sense true as well. It will also be important in the following section. Therefore we give its proof.

Theorem 14. *Let α be an r.e. real in the interval $(0, 1]$, and B be an infinite prefix-free r.e. set. If α dominates 2^{-B}, then there is an infinite prefix-free r.e. set A with $\alpha = 2^{-A}$ and $B \leq_{ss} A$.*

Proof. Let (y_i) be a one-to-one recursive enumeration of B and (a_n) be an increasing recursive sequence of positive rationals converging to α. In view of the domination property of α, there are an increasing, total recursive function $f : \mathbf{N} \to \mathbf{N}$ and a constant $c \in \mathbf{N}$ such that, for each $n \in \mathbf{N}$,

$$2^c \cdot (\alpha - a_n) \geq 2^{-B} - \sum_{i=0}^{f(n)} 2^{-|y_i|}\,. \tag{1}$$

Without loss of generality, we may assume that $a_0 \geq \sum_{i=0}^{f(0)} 2^{-|y_i|-c}$ (otherwise we may take a large enough c).

We construct a recursive sequence $(n_i)_{i\geq 0}$ of numbers and a recursive double sequence $(m_{i,j})_{i,j\geq 0}$ of elements in $\mathbf{N} \cup \{\infty\}$. These numbers will be the lengths of the strings in A. The numbers n_i serve in order to guarantee that $B \leq_{ss} A$. The numbers $m_{i,j}$ are used "to fill" the set A up in order to get exactly $\alpha = 2^{-A}$.

Construction of (n_i): We define $n_i = |y_i| + c$, for all i.

Begin of construction of $(m_{i,j})$.

Stage 0. Let $m_{i,j} = \infty$, for all $i \leq f(0)$ and $j \in \mathbf{N}$.

Stage s $(s \geq 1)$. If

$$a_s \leq \sum_{i=0}^{f(s)} 2^{-n_i} + \sum_{i=0}^{f(s-1)} \sum_{j=0}^{\infty} 2^{-m_{i,j}},$$

then let $m_{i,j} = \infty$, for all i with $f(s-1) < i \leq f(s)$ and $j \in \mathbf{N}$. Otherwise, let $m_{i,j} = \infty$, for all i with $f(s-1) < i < f(s)$ and $j \in \mathbf{N}$, and let $(m_{f(s),j})_{j\in\mathbf{N}}$ be recursively defined in such a way that

$$\sum_{j=0}^{\infty} 2^{-m_{f(s),j}} = a_s - \left(\sum_{i=0}^{f(s)} 2^{-n_i} + \sum_{i=0}^{f(s-1)} \sum_{j=0}^{\infty} 2^{-m_{i,j}} \right).$$

End of construction of $(m_{i,j})$.

One proves the following equation

$$\alpha = \sum_{i=0}^{\infty} \left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}} \right). \tag{2}$$

by distinguishing the following two cases:

Case 1. There are infinitely many s with $a_s = \sum_{i=0}^{f(s)} \left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}} \right)$. For this case, it is straightforward that the equation (2) holds.

Case 2. The inequality $a_s < \sum_{i=0}^{f(s)} \left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}} \right)$ holds true for almost all $s \in \mathbf{N}$. Then the estimate "$\leq$" in (2) is obvious. For "$\geq$" observe that in this case there is a largest $s_0 \geq 0$ with $a_{s_0} = \sum_{i=0}^{f(s_0)} \left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}} \right)$. By (1) we have $\alpha - a_{s_0} \geq \sum_{i=f(s_0)+1}^{\infty} 2^{-n_i}$. Hence, by the construction, the estimation "$\geq$" in (2) follows.

Let $h : \mathbf{N} \to \{(i,j) \in \mathbf{N}^2 \mid m_{i,j} \neq \infty\}$ be a recursive bijection. We define a recursive sequence $(\tilde{n}_i)$ by $\tilde{n}_{2i} = n_i$ and $\tilde{n}_{2i+1} = m_{h(i)}$. By the Kraft-Chaitin theorem (see Chaitin [9], Calude, Grozea [5]) and (2), combined with $0 < \alpha < 1$, we can construct a one-to-one recursive sequence (x_i) of strings with $|x_i| = \tilde{n}_i$ such that the set $\{x_i \mid i \in \mathbf{N}\}$ is prefix-free. We set $A = \{x_i \mid i \in \mathbf{N}\}$ and, using (2), obtain $2^{-A} = \alpha$. Finally we define a recursive function $g : A \to B$ by $g(x_{2i}) = y_i$ and such that $|g(x_{2i+1})| \geq |x_{2i+1}|$, for all i (this is possible because B is infinite and r.e.). Obviously, $g(A) = B$, and $|x| \leq |g(x)| + c$, for all $x \in A$. This shows $B \leq_{ss} A$. □

Lemma 13 and Theorem 14 imply

Theorem 15. *The mapping h defined by $h(A) = 2^{-A}$ is a strong homomorphism from $\langle R.E._{ss}, \leq_{ss}\rangle$ onto $\langle \mathbf{R}_{r.e.}, \leq_{dom}\rangle$.*

But the next result shows that h cannot be one-to-one.

Theorem 16. *There exist infinite prefix-free r.e. sets A and B with $2^{-A} = 2^{-B} = 1$ but $A \not\leq_{ss} B$ and $B \not\leq_{ss} A$.*

However, by relaxing the strong simulation relation one can characterize the domination relation by a simulation relation between prefix-free r.e. sets. A sequence $D_0, D_1, D_2, \ldots$ of finite subsets of Σ^* is called a *strong array* (Soare [21]) if there is a total recursive function g such that with respect to a standard bijection ν from $\mathbf{N}$ onto the set of all finite subsets of Σ^* we have $D_i = \nu(g(i))$ for all i. An *effective, finite partition* of an infinite r.e. set A is a strong array $D_0, D_1, D_2, \ldots$ of finite, pairwise disjoint subsets of A with $\bigcup_i D_i = A$.

Definition 17. Let A and B be infinite, prefix-free, r.e. sets. We say that A *simulates* B if there are two effective, finite partitions (D_i) of A and (E_i) of B, respectively, and a constant $c > 0$ such that $c \cdot (2^{-D_i}) \geq 2^{-E_i}$, for all i.

Theorem 18. *Let A, B be infinite prefix-free r.e. sets. Then A simulates B if and only if 2^{-A} dominates 2^{-B}.*

4 Random R.E. Reals and Ω-like Reals

In this section, we study random r.e. reals and Ω-like reals, which were introduced by Solovay [23]. Chaitin [10] has given a slightly different definition. We start with Chaitin's definition.

Definition 19. (Chaitin [10]) An r.e. real α is called *Ω-like* if it dominates all r.e. reals.

Solovay's original manuscript [23] contains the following definition.

Definition 20. (Solovay [23]) A recursive, increasing, and converging sequence (a_i) of rationals is called *universal* if for every recursive, increasing and converging sequence (b_i) of rationals there exists a number $c > 0$ such that $c \cdot (\alpha - a_n) \geq \beta - b_n$ for all n, where $\alpha = \lim_n a_n$ and $\beta = \lim_n b_n$.

Solovay called a real α Ω-like if it is the limit of a universal recursive, increasing sequence of rationals. We shall see that both definitions are equivalent. One implication follows immediately from Lemma 5.

Lemma 21. *If a real α is the limit of a universal recursive, increasing sequence of rationals, then it is Ω-like.*

By modifying slightly a proof of Solovay [23] we obtain the following result.

Theorem 22. *Let U be a universal machine. Every recursive, increasing sequence of rationals converging to Ω_U is universal.*

Thus, every Chaitin Ω number is Ω-like in Solovay's sense. The converse of Theorem 22 holds true even for Ω-like numbers in Chaitin's sense.

Theorem 23. *Let $0 < \alpha < 1$ be an Ω-like real. Then there exists a universal machine U such that $\Omega_U = \alpha$.*

Proof. Let V be a universal machine. Since α is Ω-like it dominates $2^{-\mathrm{PROG}_V}$. By Theorem 14 there exist a prefix-free r.e. set A with $2^{-A} = \alpha$, a recursive function $f : A \to \mathrm{PROG}_V$ with $A = \mathrm{dom}\,(f)$ and $f(A) = \mathrm{PROG}_V$, and a constant $c > 0$ with $|x| \leq |f(x)| + c$, for all $x \in A$. We define a machine U by $U(x) = V(f(x))$. The universality of V implies that also U is universal. We have $\alpha = 2^{-A} = 2^{-\mathrm{PROG}_U} = \Omega_U$. □

Theorem 24. *Let $0 < \alpha < 1$ be an r.e. real. The following conditions are equivalent:*

1. *For some universal Turing machine U, $\alpha = \Omega_U$.*
2. *The real α is Ω-like.*
3. *There exists a universal recursive, increasing sequence of rationals converging to α.*
4. *Every recursive, increasing sequence of rationals with limit α is universal.*

Proof. This follows from Lemma 21, Theorem 22, and Theorem 23. □

The following result was proved by Solovay [23] for Ω-like numbers. If follows immediately from Theorem 6 and Theorem 24.

Corollary 25. *Let U and V be two universal machines. Then $H(\Omega_U(n)) = H(\Omega_V(n)) + O(1)$.*

In analogy with Corollary 8 we obtain from Lemma 7 and Theorem 24 the following corollary.

Corollary 26. *The fractional part of the sum of an Ω number and an r.e. real is an Ω number.*

Also, the fractional part of the product of an Ω number with a positive r.e. real is an Ω number. Since Ω is random (Theorem 2) every Ω-like number is random (Solovay [23]). Now we can answer the question raised after Lemma 10. The sets A_Ω and A_{χ_K} are defined as before Lemma 10.

Corollary 27. *The following statements hold:*

1. $0.\chi_K \not\geq_{dom} \Omega$,
2. $A_\Omega =_T A_{\chi_K} =_T K$.

Proof. It is well-known that χ_K is not random, and hence, by Theorem 2, not an Ω number. Theorem 24 and the transitivity of $\geq_{dom}$ show the first claim. $A_\Omega \leq_T K =_T A_{\chi_K}$ is clear and $A_{\chi_K} \leq_T A_\Omega$ follows from Lemma 10. □

Does there exist a random r.e. real which is not Ω-like? We conjecture that this is true. Kucera [17] (see also Kautz [14]) has observed that $0'$ is the only r.e. degree which contains a random real.[9] But Corollary 27 shows that it splits into different $=_{dom}$-classes.

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[9] Here we identify a real $0.\mathbf{x}$ with the set $A^{\#}_{\mathbf{x}}$.

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