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REDUCED CONTRAGREDIENT LIE ALGEBRAS AND PC LIE ALGEBRAS

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Abstract

Using the theory of standard pentads, we can embed an arbitrary finite-dimensional reductive Lie algebra and its finite-dimensional completely reducible representation into some larger graded Lie algebra. However, it is not easy to find the structure of the "larger graded Lie algebra" from the definition in general cases. Under these, the first aim of this paper is to show that the "larger graded Lie algebra" is isomorphic to some PC Lie algebra, which are Lie algebras corresponding to special standard pentads called pentads of Cartan type. The second aim is to find the structure of a PC Lie algebra.

Introduction

Using the theory of standard pentads, an arbitrary finite-dimensional reductive Lie algebra and its representation can be embedded into some graded Lie algebra. The term "standard pentad" is defined as the following.

Definition 0.1 (standard pentad, [3, Definition 2.2]). Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \otimes V \to V$ a representation of $\mathfrak{g}$ on V, $\mathcal{V}$ a submodule of $\mathrm{Hom}(V, \mathbb{C})$ and B a non-degenerate invariant bilinear form on $\mathfrak{g}$ all defined over $\mathbb{C}$. When a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B)$ satisfies the following conditions, we call it a standard pentad:

- the restriction of the canonical bilinear form $\langle \cdot, \cdot \rangle : V \times \mathrm{Hom}(V, \mathbb{C}) \to \mathbb{C}$ to $V \times \mathcal{V}$ is non-degenerate,
- there exists a linear map $\Phi_\rho : V \otimes \mathcal{V} \to \mathfrak{g}$, called a Φ-map, satisfying an equation

$$B(a, \Phi_\rho(v \otimes \phi)) = \langle \rho(a \otimes v), \phi \rangle$$

  for any $a \in \mathfrak{g}$, $v \in V$, $\phi \in \mathcal{V}$.

From a standard pentad, we can construct a graded Lie algebra.

Theorem 0.2 ([3, Theorem 2.15]). *For any standard pentad* $(\mathfrak{g}, \rho, V, \mathcal{V}, B)$*, there exists a graded Lie algebra*

$$L(\mathfrak{g}, \rho, V, \mathcal{V}, B) = \bigoplus_{n \in \mathbb{Z}} V_n,$$

called the Lie algebra associated with $(\mathfrak{g}, \rho, V, \mathcal{V}, B)$*, satisfying the conditions that*

$$V_0 \simeq \mathfrak{g}$$

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as Lie algebras, that

$$V_{-1} \simeq \mathcal{V}, \quad V_1 \simeq V$$

as $\mathfrak{g}$-modules via the isomorphism of Lie algebras $V_0 \simeq \mathfrak{g}$ and that the restriction of bracket product $[\cdot,\cdot] : V_1 \times V_{-1} \to V_0$ is induced by the Φ-map $\Phi_\rho : V \otimes \mathcal{V} \to \mathfrak{g}$.

When a Lie algebra $\mathfrak{g}$ is finite-dimensional, it is known that any pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B)$ is standard. Thus, we can "embed" a finite-dimensional reductive Lie algebra[1] and its representation into some graded Lie algebra in the sense of Theorem 0.2. Here, we have a problem how to find the structure of a Lie algebra of the form $L(\mathfrak{g}, \rho, V, \mathcal{V}, B)$. In this paper, we shall consider this problem under some assumptions.

Now, as special cases of standard pentads, we give the notion of pentads of Cartan type. A pentad of Cartan type is a pentad which has a finite-dimensional commutative Lie algebra and its finite-dimensional diagonalizable representation. We can describe an arbitrary pentad of Cartan type by two positive integers r, n and three matrices A, D, Γ as $P(r, n; A, D, \Gamma)$. We denote the Lie algebra associated with $P(r, n; A, D, \Gamma)$ by $L(r, n; A, D, \Gamma)$, and moreover, we call a Lie algebra of the form $L(r, n; A, D, \Gamma)$ a PC Lie algebra. For detail on pentads of Cartan type, see [4].

We have two aims of this paper. The first aim is to show that an arbitrary Lie algebra of the form $L(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B)$ with a finite-dimensional reductive Lie algebra $\mathfrak{g}$ and its finite-dimensional completely reducible representation (ρ, V) is isomorphic to some PC Lie algebra (Theorem 2.1). And, moreover, the second aim is to find the structure of PC Lie algebras (Theorem 3.2). In [4, Theorem 3.9], we have obtained a way how to describe the structure of $L(r, n; A, D, \Gamma)$ under the assumption that $\Gamma \cdot {}^tD \cdot A \cdot D$ is invertible. In this paper, we shall find the structure of $L(r, n; A, D, \Gamma)$ without any assumptions on r, n and A, D, Γ.

Notation 0.3. Throughout this paper, all objects are defined over the complex number field $\mathbb{C}$. We use the following notations:

- $\mathrm{Span}(v_1, \ldots, v_n)$: a vector space spanned by $v_1, \ldots, v_n$,
- $\mathrm{M}(k, l; \mathbb{C})$: a set of matrices of size $k \times l$ whose entries belong to $\mathbb{C}$,
- $\mathrm{diag}(c_1, \ldots, c_m)$: a diagonal matrix of size m whose (i, i)-entry is c_i,
- δ_{ij}: the Kronecker delta.

Notation 0.4. We regard a representation ρ of a Lie algebra $\mathfrak{l}$ on U as a linear map

$$\rho : \mathfrak{l} \otimes U \to U$$

satisfying

$$\rho([a, b] \otimes u) = \rho(a \otimes \rho(b \otimes u)) - \rho(b \otimes \rho(a \otimes u))$$

for any $a, b \in \mathfrak{l}$ and $u \in U$. Moreover, we denote an ideal $\{a \in \mathfrak{l} \mid \rho(a \otimes u) = 0 \text{ for any } u \in U\}$ of $\mathfrak{l}$ by $\mathrm{Ann}\, U$.

[1] It is known that any finite-dimensional reductive Lie algebra has a non-degenerate invariant bilinear form (see [1, Chapter 1. §6.4 Proposition 5]).

1. PC Lie algebras and contragredient Lie algebras

The purpose of this section is to prepare some notion and notations we need to understand the statements of the main theorems, Theorems 2.1 and 3.2. For detail, refer [2] and [4].

Definition 1.1 (pentads of Cartan type, [4, Definition 2.4]). Let r, n be positive integers. Let $A \in \mathrm{M}(r, r; \mathbb{C})$ be an invertible square matrix, $D = (d_{ij}) \in \mathrm{M}(r, n; \mathbb{C})$ a matrix and $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_n) \in \mathrm{M}(n, n; \mathbb{C})$ an invertible diagonal matrix. Let $\mathfrak{h}^r$, $\mathbb{C}^\Gamma_D$, $\mathbb{C}^\Gamma_{-D}$ be vector spaces with dimensional r, n and n respectively, and take their bases $\{\epsilon_1, \dots \epsilon_r\}$, $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ respectively:

$$\mathfrak{h}^r = \mathrm{Span}(\epsilon_1, \dots, \epsilon_r), \quad \mathbb{C}^\Gamma_{-D} = \mathrm{Span}(e_1, \dots, e_n), \quad \mathbb{C}^\Gamma_D = \mathrm{Span}(f_1, \dots, f_n).$$

We regard $\mathfrak{h}^r$ as a commutative Lie algebra:

$$\mathfrak{h}^r \simeq \mathfrak{gl}^r_1$$

and define representations $\square^r_D$ and $\square^r_{-D}$ of $\mathfrak{h}^r$ on $\mathbb{C}^\Gamma_D$ and $\mathbb{C}^\Gamma_{-D}$ as:

$$\square^r_D(\epsilon_i \otimes e_j) = d_{ij} e_j, \quad \square^r_{-D}(\epsilon_i \otimes f_j) = -d_{ij} f_j$$

for any $i = 1, \dots, r$ and $j = 1, \dots, n$. Moreover, we define non-degenerate bilinear maps $B_A : \mathfrak{h}^r \times \mathfrak{h}^r \to \mathbb{C}$ and $\langle \cdot, \cdot \rangle^\Gamma_D : \mathbb{C}^\Gamma_D \times \mathbb{C}^\Gamma_{-D} \to \mathbb{C}$ as:

$$B_A(c_1\epsilon_1 + \dots + c_r\epsilon_r, c'_1\epsilon_1 + \dots + c'_r\epsilon_r) = \begin{pmatrix} c_1 & \cdots & c_r \end{pmatrix} \cdot {}^{\mathrm{t}}A^{-1} \cdot \begin{pmatrix} c'_1 \\ \vdots \\ c'_r \end{pmatrix}, \quad \langle e_i, f_j \rangle^\Gamma_D = \delta_{ij}\gamma_i$$

for $i, j = 1, \dots, n$. Under these, we define a standard pentad $(\mathfrak{h}^r, \square^r_D, \mathbb{C}^\Gamma_D, \mathbb{C}^\Gamma_{-D}, B_A)$ and denote it by $P(r, n; A, D, \Gamma)$. We call a standard pentad of the form $P(r, n; A, D, \Gamma)$ a pentad of Cartan type.

Definition 1.2 (Cartan matrix of a pentad of Cartan type, [4, Definition 2.15]). For a pentad of Cartan type $P(r, n; A, D, \Gamma)$, put

$$C(A, D, \Gamma) = \Gamma \cdot {}^{\mathrm{t}}D \cdot A \cdot D.$$

We call $C(A, D, \Gamma)$ the Cartan matrix of $P(r, n; A, D, \Gamma)$.

Definition 1.3 (PC Lie algebras, [4, Definition 3.6]). For a pentad of Cartan type $P(r, n; A, D, \Gamma)$, we denote its corresponding graded Lie algebra (see Theorem 0.2) by $L(r, n; A, D, \Gamma)$. We call a Lie algebra of the form $L(r, n; A, D, \Gamma)$ a PC Lie algebra.

Remark 1.4. The structure of a PC Lie algebra $L(r, n; A, D, \Gamma)$ is independent to the diagonal matrix Γ and to the order of column vectors in D (see [4, Propositions 1.7 and 2.6]).

Moreover, we need to recall some notion of graded Lie algebras due to Kac in [2].

Definition 1.5 (transirivity, [2, p.1275, Definition 2]). A graded Lie algebra

$$G = \bigoplus_{i=-\infty}^{+\infty} G_i$$

is said to be transitive if:

- for $x \in G_i$, $i \geq 0$, $[x, G_{-1}] = \{0\}$ implies $x = 0$,
- for $x \in G_i$, $i \leq 0$, $[x, G_1] = \{0\}$ implies $x = 0$.

DEFINITION 1.6 (CONTRAGREDIENT LIE ALGEBRAS, [2, p.1279]). Let $A = (A_{ij})$ $i, j = 1, \dots, n$ be a matrix with elements from $\mathbb{C}$. Let G_{-1}, G_1, G_0 be vector spaces with bases $\{F_i\}$, $\{E_i\}$, $\{H_i\}$ respectively ($i = 1, \dots, n$). We define a structure of local Lie algebra on $\hat{G}(A) := G_{-1} \oplus G_0 \oplus G_1$ by

$$[E_i, F_j] = \delta_{ij} H_i, \quad [H_i, H_j] = 0, \quad [H_i, E_j] = A_{ij} E_j, \quad [H_i, F_j] = -A_{ij} F_j. \tag{1.1}$$

Then, we call the minimal graded Lie algebra $G(A) = \bigoplus_{i \in \mathbb{Z}} G_i$ with local part $\hat{G}(A)$ a contragredient Lie algebra, and the matrix A its Cartan matrix.

DEFINITION 1.7 (REDUCED CONTRAGREDIENT LIE ALGEBRAS, [2, p.1280]). Let $G(A)$ be a contragredient Lie algebra with Cartan matrix A and Z the center of $G(A)$. We call a factor Lie algebra $G(A)/Z$ a reduced contragredient Lie algebra with Cartan matrix A.

2. Representations of finite-dimensional reductive Lie algebras and PC Lie algebras

In this section, we shall give the first main theorem of this paper. The following theorem tells us the importance of PC Lie algebras.

Theorem 2.1. *Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra, (ρ, V) a finite-dimensional completely reducible representation of $\mathfrak{g}$, B a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ all defined over $\mathbb{C}$. Then a pentad $(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B)$ is standard, and moreover, the corresponding Lie algebra $L(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B)$ is isomorphic to some PC Lie algebra up to grading. That is, an arbitrary finite-dimensional reductive Lie algebra and its arbitrary finite-dimensional completely reducible representation can be embedded into some PC Lie algebra.*

Proof. Since $\mathfrak{g}$ is finite-dimensional, we have that the pentad $(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B)$ is standard (see [3, Lemma 2.3]). Denote the center part of $\mathfrak{g}$ by $\mathfrak{z}$ and the semisimple part of $\mathfrak{g}$ by $\mathfrak{s}$. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ and a fundamental system ψ of the root system R with respect to $(\mathfrak{s}, \mathfrak{h})$. We regard V and $\mathrm{Hom}(V, \mathbb{C})$ as subspaces of $L(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B)$ in the sense of Theorem 0.2. If we take a non-zero root vector X_γ of a root $\gamma \in R$ and define $\mathfrak{g}$-submodules $\underline{V} \subset V$ and $\overline{\mathcal{V}} \subset \mathrm{Hom}(V, \mathbb{C})$ by

$$\underline{V} = \{y \in V \mid [X_{-\alpha}, y] = 0 \quad \text{for any } \alpha \in \psi\},$$
$$\overline{\mathcal{V}} = \{\eta \in \mathrm{Hom}(V, \mathbb{C}) \mid [X_\alpha, \eta] = 0 \quad \text{for any } \alpha \in \psi\},$$

then we have an isomorphism of Lie algebras up to grading:

$$L(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B) \simeq L\left(\mathfrak{z} \oplus \mathfrak{h}, \rho|_{\mathfrak{z} \oplus \mathfrak{h}}, \left(\sum_{\alpha \in \psi} \mathbb{C} X_\alpha \oplus \underline{V}\right), \left(\sum_{\alpha \in \psi} \mathbb{C} X_{-\alpha} \oplus \overline{\mathcal{V}}\right), B|_{(\mathfrak{z} \oplus \mathfrak{h}) \times (\mathfrak{z} \oplus \mathfrak{h})}\right)$$

using [4, Theorem 3.27 and (3.17) in its proof], chain rule ([3, Theorem 3.26]) and the assumption that the bilinear form B is symmetric. Thus, to prove our claim, it suffices to show that the representation $\rho\mid_{\mathfrak{z}\oplus\mathfrak{h}}$ of $\mathfrak{z}\oplus\mathfrak{h}$ on $\sum_{\alpha\in\psi}\mathbb{C}X_\alpha\oplus\underline{V}$ is simultaneously diagonalizable (see [4, Proposition 2.5]). Here, note that the representation restricted to $\mathfrak{h}$ on $\sum_{\alpha\in\psi}\mathbb{C}X_\alpha\oplus\underline{V}$ is simultaneously diagonalizable from the well-known properties of Cartan subalgebras. Thus, if we assume that the representation $(\mathfrak{z}\oplus\mathfrak{h}, \sum_{\alpha\in\psi}\mathbb{C}X_\alpha\oplus\underline{V})$ is not simultaneously diagonalizable, then there exist an element $\epsilon\in\mathfrak{z}$, an element $a\in\mathbb{C}$ and an element $Y\in\underline{V}$ such that

$$\rho(\epsilon\otimes Y)-aY\neq 0,\quad \rho(\epsilon\otimes(\rho(\epsilon\otimes Y)-aY))-a(\rho(\epsilon\otimes Y)-aY)=0. \tag{2.1}$$

Using these, define a non-zero proper vector subspace U of V by

$$U=\{y\in V\mid \rho(\epsilon\otimes y)-ay=0\}.$$

Since ϵ belongs to the center part of $\mathfrak{g}$, U is a $\mathfrak{g}$-submodule of V. Then from the assumption that the representation ρ of $\mathfrak{g}$ on V is completely reducible, we have a non-zero proper $\mathfrak{g}$-submodule W of V such that

$$V=U\oplus W.$$

If we take elements $u\in U$ and $w\in W$ such that

$$Y=u+w,$$

then we have that

$$0\neq\rho(\epsilon\otimes Y)-aY=(\rho(\epsilon\otimes u)-au)+(\rho(\epsilon\otimes w)-aw)=\rho(\epsilon\otimes w)-aw\in U\cap W$$

from (2.1). It is a contradiction. □

Example 2.2. For $m=0,1,2,\ldots$, we denote by $m\Lambda_1$ the irreducible representation of $\mathfrak{sl}_2$ on $(m+1)$-dimensional vector space $V(m+1)$. For example, the adjoint representation of $\mathfrak{sl}_2$ on itself is $(\mathrm{ad},\mathfrak{sl}_2)=(2\Lambda_1,V(3))$. Denote the Killing form of $\mathfrak{sl}_2$ by $K_{\mathfrak{sl}_2}$. Then we have an isomorphism of Lie algebras

$$L(\mathfrak{sl}_2,m\Lambda_1,V(m+1),\mathrm{Hom}(V(m+1),\mathbb{C}),K_{\mathfrak{sl}_2})\simeq L\left(1,2;\begin{pmatrix}1/8\end{pmatrix},\begin{pmatrix}2&-m\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix}\right)$$

up to grading (see [4, (3.21) in Theorem 3.28]). That is, the representation $(m\Lambda_1,V(m+1))$ of $\mathfrak{sl}_2$ can be embedded into the PC Lie algebra associated with

$$P\left(1,2;\begin{pmatrix}1/8\end{pmatrix},\begin{pmatrix}2&-m\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix}\right). \tag{2.2}$$

From Theorem 2.1, it is natural for us to ask the structure of PC Lie algebras.

3. Structure of PC Lie algebras

Using the notion and notations recalled in section 1, we can describe the structure of a given PC Lie algebra. For this, we shall start with the following lemma.

Lemma 3.1. *We identify an m-tuple* $(x_1,\ldots,x_m)\in\mathfrak{gl}_1^m$ *with a row vector* $\begin{pmatrix}x_1&\cdots&x_m\end{pmatrix}\in\mathrm{M}(1,m;\mathbb{C})$. *For an arbitrary pentad of Cartan type* $P(r,n;A,D,\Gamma)$ *and its corresponding*

Lie algebra $L(r,n;A,D,\Gamma)$, we have the following claims.

(i) *We have equations*

$$[V_{-1},V_1] = \mathrm{Span}(\textit{the row vectors of } (\Gamma\cdot{}^tD\cdot A)) = \left\{\begin{pmatrix}c_1 & \cdots & c_n\end{pmatrix}\cdot\Gamma\cdot{}^{\mathrm{t}}D\cdot A \,\middle|\, c_1,\dots,c_n\in\mathbb{C}\right\}$$

and

$$\dim[V_{-1},V_1] = \mathrm{rank}\,D.$$

(ii) *We have equations*

$$\begin{aligned}\mathrm{Ann}\,\mathbb{C}_D^\Gamma &= \left\{c_1\epsilon_1+\cdots+c_r\epsilon_r \,\middle|\, \begin{pmatrix}c_1 & \cdots & c_r\end{pmatrix}\cdot D = \begin{pmatrix}0 & \cdots & 0\end{pmatrix}, c_1,\dots,c_r\in\mathbb{C}\right\}\\ &= \left\{\begin{pmatrix}c_1 & \cdots & c_r\end{pmatrix} \,\middle|\, \begin{pmatrix}c_1 & \cdots & c_r\end{pmatrix}\cdot D = \begin{pmatrix}0 & \cdots & 0\end{pmatrix}, c_1,\dots,c_r\in\mathbb{C}\right\}\end{aligned}$$

and

$$\dim\mathrm{Ann}\,\mathbb{C}_D^\Gamma = r-\mathrm{rank}\,D.$$

(iii) *We have equations*

$$\begin{aligned}&[V_{-1},V_1]\cap\mathrm{Ann}\,\mathbb{C}_D^\Gamma\\ &= \left\{\begin{pmatrix}c_1 & \cdots & c_n\end{pmatrix}\cdot\Gamma\cdot{}^{\mathrm{t}}D\cdot A \,\middle|\, \begin{pmatrix}c_1 & \cdots & c_n\end{pmatrix}\cdot C = \begin{pmatrix}0 & \cdots & 0\end{pmatrix}, c_1,\dots,c_n\in\mathbb{C}\right\}\end{aligned}$$

and

$$\dim([V_{-1},V_1]\cap\mathrm{Ann}\,\mathbb{C}_D^\Gamma) = \mathrm{rank}\,D-\mathrm{rank}\,C,$$

where $C = C(A,D,\Gamma)$ is the Cartan matrix of $P(r,n;A,D,\Gamma)$.

Proof. (i) The vector space $[V_{-1},V_1]$ is spanned by $h_i\in\mathfrak{h}^r$ $(i=1,\dots,r)$, which are identified with the i-th row vectors of the matrix $\Gamma\cdot{}^tD\cdot A$ $(i=1,\dots,r)$ (see [4, Proposition 2.11, Definition 2.12]). Thus, we have that

$$\begin{aligned}[V_{-1},V_1] = \mathrm{Span}(h_1,\dots,h_n) &= \mathrm{Span}(\text{the row vectors of } \Gamma\cdot{}^tD\cdot A)\\ &= \left\{\begin{pmatrix}c_1 & \cdots & c_n\end{pmatrix}\cdot\Gamma\cdot{}^tD\cdot A \,\middle|\, c_1,\dots,c_n\in\mathbb{C}\right\}.\end{aligned}$$

Moreover, since both $\Gamma\in\mathrm{M}(n,n;\mathbb{C})$ and $A\in\mathrm{M}(r,r;\mathbb{C})$ are invertible, we have an equation

$$\dim[V_{-1},V_1] = \mathrm{rank}(\Gamma\cdot{}^tD\cdot A) = \mathrm{rank}\,D.$$

(ii) This claim has been proved in [4, Proposition 2.25].

(iii) This claim follows from (i) and (ii) immediately.

This completes the proof. □

Using Lemma 3.1, we can describe the structure of an arbitrary PC Lie algebra using reduced contragredient Lie algebras. This is the second main theorem.

Theorem 3.2. *Let $P(r,n;A,D,\Gamma)$ be a pentad of Cartan type and $C = C(A,D,\Gamma) = (C_{ij})_{i,j=1,\dots,n}$ its Cartan matrix. Let $G'(C)$ be the reduced contragredient Lie algebra with Cartan matrix C. Then there exist vector spaces U_0', Z, $\Delta\subset L(r,n;A,D,\Gamma)$ and a $\mathbb{Z}$-grading* *of* $L(r,n;A,D,\Gamma)$

$$L(r,n;A,D,\Gamma) = \bigoplus_{m\in\mathbb{Z}} U_m$$

such that

$$U_0 = U_0' \oplus Z \oplus \Delta, \quad \dim Z = \operatorname{rank} D - \operatorname{rank} C, \quad \dim \Delta = r - \operatorname{rank} D$$

and

$$\begin{aligned} &U_0' \oplus \bigoplus_{m\neq 0} U_m \simeq G'(C), && [Z, L(r,n;A,D,\Gamma)] = \{0\}, \\ &[U_m, U_{-m}] \subset U_0' \oplus Z, && \textit{the action of } \Delta \textit{ on } U_m \textit{ is diagonalizable} \end{aligned} \tag{3.1}$$

for all $m \in \mathbb{Z}$.

That is, any PC Lie algebra is of the form

$$L(r,n;A,D,\Gamma) \simeq (G'(C) \oplus Z) \oplus \Delta$$

for some vector space $Z \subset$ (the center of $L(r,n;A,D,\Gamma)$) and Δ satisfying

$$\begin{gathered} [L(r,n;A,D,\Gamma), L(r,n;A,D,\Gamma)] = G'(C) \oplus Z, \\ \dim Z = \operatorname{rank} D - \operatorname{rank} C, \quad \dim \Delta = r - \operatorname{rank} D. \end{gathered}$$

proof of Theorem 3.2. Let

$$L(r,n;A,D,\Gamma) = \bigoplus_{m\in\mathbb{Z}} V_m$$

be the canonical $\mathbb{Z}$-grading of $L(r,n;A,D,\Gamma)$,

$V_{-1} \simeq \mathbb{C}^{\Gamma}_{-D} = \mathrm{Span}(f_1,\dots,f_n)$, $\quad V_0 \simeq \mathfrak{h}^r = \mathrm{Span}(\epsilon_1,\dots,\epsilon_r)$, $\quad V_1 \simeq \mathbb{C}^{\Gamma}_{D} = \mathrm{Span}(e_1,\dots,e_n)$,

and denote its bracket product by $[\cdot,\cdot]$. Take a complementary subspace Δ to $[V_{-1}, V_1]$ in V_0:

$$V_0 = \mathfrak{h}^r = \mathfrak{gl}_1^r = [V_{-1}, V_1] \oplus \Delta.$$

Moreover, put

$$Z = [V_{-1}, V_1] \cap \operatorname{Ann} \mathbb{C}^{\Gamma}_{D}$$

and take a complementary subspace V_0' to Z in $[V_{-1}, V_1]$:

$$[V_{-1}, V_1] = V_0' \oplus Z = V_0' \oplus ([V_{-1}, V_1] \cap \operatorname{Ann} \mathbb{C}^{\Gamma}_{D}).$$

Summarizing,

$$V_0 = [V_{-1}, V_1] \oplus \Delta = V_0' \oplus Z \oplus \Delta. \tag{3.2}$$

Then, from Lemma 3.1, we have equations:

$$\dim V_0' = \operatorname{rank} C, \quad \dim Z = \operatorname{rank} D - \operatorname{rank} C, \quad \dim \Delta = r - \operatorname{rank} D. \tag{3.3}$$

Let us denote the canonical surjection from V_0 to V_0' with respect to the decomposition (3.2) by p. Under these, to prove our claim, it is sufficient to show that a Lie algebra

$$L''(r,n;A,D,\Gamma) = V_0' \oplus \bigoplus_{m\in\mathbb{Z}\setminus\{0\}} V_m$$

with bracket product $[\cdot,\cdot]''$ defined by

$$[x_k, y_l]'' = \begin{cases} [x_k, y_l] & (k+l \neq 0) \\ p([x_k, y_l]) & (k+l=0) \end{cases}, \text{ where } x_k \in V_k,\ y_l \in V_l\ (k,l \neq 0),\ x_0, y_0 \in V_0',$$

is isomorphic to a reduced contragredient Lie algebra $G'(C)$ with Cartan matrix C. We can easily check that the bilinear map $[\cdot,\cdot]''$ satisfies the axioms of Lie algebras.

Take elements $h_i = [e_i, f_i] \in V_0$ (see [4, Definition 2.12]) and put $h_i' = p(h_i) \in V_0'$ for $i = 1,\ldots,n$. Then the Lie algebra $L''(r,n;A,D,\Gamma)$ is generated by $\{f_i, h_i', e_i \mid i = 1,\ldots,n\}$ with relations

$$[h_i', e_j]'' = C_{ij}e_j, \qquad [h_i', f_j]'' = -C_{ij}f_j, \qquad [e_i, f_j]'' = \delta_{ij}h_i'$$

for all $i,j = 1,\ldots,n$ (see [4, Proposition 2.13]). On the other hand, we take $\{F_i, H_i, E_i \mid i = 1,\ldots,n\}$ a basis of $\hat{G}(C) = G_{-1} \oplus G_0 \oplus G_1$, which is the local part of a contragredient Lie algebra $G(C) = G(C(A,D,\Gamma))$, satisfying the equations (1.1). Define a linear map ϕ : $\hat{G}(C) \to V_{-1} \oplus V_0' \oplus V_1$ by

$$\phi(H_i) = h_i', \qquad \phi(E_i) = e_i, \qquad \phi(F_i) = f_i$$

for $i = 1,\ldots,n$. This linear map ϕ is a surjective homomorphism between the local parts of $G(C)$ and of $L''(r,n;A,D,\Gamma)$. We can compute the kernel of ϕ using Lemma 3.1 (iii) as follows:

$$\begin{aligned}
\operatorname{Ker}\phi &= \left\{c_1H_1 + \cdots + c_nH_n \in \hat{G}(C) \,\middle|\, \phi(c_1H_1 + \cdots + c_nH_n) = 0,\ c_1,\ldots,c_n \in \mathbb{C}\right\} \\
&= \left\{c_1H_1 + \cdots + c_nH_n \in \hat{G}(C) \,\middle|\, c_1h_1' + \cdots + c_nh_n' = 0 \in L''(r,n;A,D,\Gamma),\ c_1,\ldots,c_n \in \mathbb{C}\right\} \\
&= \left\{c_1H_1 + \cdots + c_nH_n \in \hat{G}(C) \,\middle|\, c_1h_1' + \cdots + c_nh_n' \in Z = [V_{-1}, V_1] \cap \operatorname{Ann}\mathbb{C}_D^\Gamma,\ c_1,\ldots,c_n \in \mathbb{C}\right\} \\
&= \left\{c_1H_1 + \cdots + c_nH_n \in \hat{G}(C) \,\middle|\, c_1h_1 + \cdots + c_nh_n \in Z = [V_{-1}, V_1] \cap \operatorname{Ann}\mathbb{C}_D^\Gamma,\ c_1,\ldots,c_n \in \mathbb{C}\right\} \\
&= \left\{c_1H_1 + \cdots + c_nH_n \in \hat{G}(C) \,\middle|\, \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \cdot \Gamma \cdot {}^tD \cdot A \in Z,\ c_1,\ldots,c_n \in \mathbb{C}\right\} \\
&= \left\{c_1H_1 + \cdots + c_nH_n \in \hat{G}(C) \,\middle|\, \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \cdot C = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix},\ c_1,\ldots,c_n \in \mathbb{C}\right\} \\
&= (\text{the center of } G(C)).
\end{aligned}$$

Thus, we have an isomorphism of local Lie algebras:

$$\begin{aligned}
&(\text{the local part of } L''(r,n;A,D,\Gamma)) \simeq V_{-1} \oplus V_0' \oplus V_1 \\
&\simeq \hat{G}(C)/(\text{the center of } G(C)) \simeq (\text{the local part of } G'(C)).
\end{aligned}$$

Here, both graded Lie algebras $L''(r,n;A,D,\Gamma) = V_0' \oplus \bigoplus_{m\in\mathbb{Z}\setminus\{0\}} V_m$ and $G'(C) = \bigoplus_{m\in\mathbb{Z}} G_m'$ are transitive except their local parts, i.e. they satisfy the condition in Definition 1.5 for $i \neq 0, \pm1$. Indeed, the transitivity for $|m| \geq 2$ of $L''(r,n;A,D,\Gamma)$ comes from the construction of the Lie algebra associated with a standard pentad (see [3, Definitions 2.9, 2.12]), and, one of $G'(C)$ comes from the construction of minimal Lie algebras (see [2, pp.1276–1278, Proposition 4]). We can extend the isomorphism between the local parts of $L''(r,n;A,D,\Gamma)$ and of $G'(C)$ to the isomorphism between the whole graded Lie algebras:

$$L''(r, n; A, D, \Gamma) \simeq G'(C)$$

by a similar way to the proof of [4, Theorem 1.5]. Thus, we have our claim. □

We retain to use the notations in Theorem 3.2. When a pentad $P(r, n; A, D, \Gamma)$ has the invertible Cartan matrix, it has already been shown that we have an isomorphism of Lie algebras:

$$L(r, n; A, D, \Gamma) \simeq \mathfrak{gl}_1^{r-n} \oplus G(C(A, D, \Gamma)) \quad \text{(see [4, Theorem 3.9])}.$$

This result is a special case of Theorem 3.2. In fact, from the definition of Cartan matrices of a pentad of Cartan type, we can easily show that the data satisfy conditions that

$$r \geq n \quad \text{and} \quad \operatorname{rank} D = \operatorname{rank} C = n$$

when $C = C(A, D, \Gamma)$ is invertible. Under this situation, we have that

$$G'(C) \simeq G(C), \quad \dim Z = 0, \quad \dim \operatorname{Ann} \mathbb{C}_D^\Gamma = r - n$$

from Lemma 3.1 and the equations (3.3). Since we have $\dim[V_{-1}, V_1] + \dim \operatorname{Ann} \mathbb{C}_D^\Gamma = r = \dim V_0$ and $[V_{-1}, V_1] \cap \operatorname{Ann} \mathbb{C}_D^\Gamma = \{0\}$, we can take $\Delta = \operatorname{Ann} \mathbb{C}_D^\Gamma$. Thus, we have an isomorphism of Lie algebras:

$$L(r, n; A, D, \Gamma) \simeq G'(C) \oplus Z \oplus \Delta \simeq G(C) \oplus \{0\} \oplus \operatorname{Ann} \mathbb{C}_D^\Gamma \simeq \mathfrak{gl}_1^{r-n} \oplus G(C).$$

Corollary 3.3. *We retain to use the notations in* Theorem 3.2. *When a pentad of Cartan type $P(r, n; A, D, \Gamma)$ satisfies*

$$r = \operatorname{rank} D = \operatorname{rank} C(A, D, \Gamma),$$

the corresponding Lie algebra is isomorphic to a reduced contragredient Lie algebra:

$$L(r, n; A, D, \Gamma) \simeq G'(C(A, D, \Gamma)).$$

As we have seen in [4, Theorem 3.11], for any invertible matrix C, we can find a pentad of Cartan type $P(r, n; A, D, \Gamma)$ with Cartan matrix $C(A, D, \Gamma) = C$ such that its data satisfy the assumption of Corollary 3.3: $r = \operatorname{rank} D = \operatorname{rank} C = n$ (thus, $G'(C) \simeq G(C) \simeq L(r, n; A, D, \Gamma)$). However, it does not hold when C is non-invertible. There exists a non-invertible matrix C which do not have a pentad of Cartan type with Cartan matrix C satisfying the assumption of Corollary 3.3. As an example, we can take

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Indeed, if it has such a pentad $P(r, n; A, D, \Gamma)$, then we have that $r = 1$, $n = 2$ and that 1×1 matrix A, 1×2 matrix D, 2×2 matrix Γ satisfy

$$\Gamma \cdot {}^t D \cdot A \cdot D = C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to show that such A, D, Γ do not exist.

REMARK 3.4. Under the notations in Theorem 3.2, the vector space Z is contained in the center of $L(r, n; A, D, \Gamma)$. However, in general, Z and the center of $L(r, n; A, D, \Gamma)$ do not coincide.

Lemma 3.5. *Let $P(r,n;A,D,\Gamma)$ be a pentad of Cartan type. Assume that the i-th column vector of D is a zero-vector. Then corresponding elements $e_i \in \mathbb{C}^{\Gamma}_{D}$ and $f_i \in \mathbb{C}^{\Gamma}_{-D}$ belong to the center of $L(r,n;A,D,\Gamma)$.*

Proof. Denote $L(r,n;A,D,\Gamma) = \bigoplus_{n\in\mathbb{Z}} V_n$ and identify V_{-1}, V_0, V_1 with $\mathbb{C}^{\Gamma}_{-D}, \mathfrak{h}^r, \mathbb{C}^{\Gamma}_{D}$ respectively. Let us show that e_i belongs to the center of $L(r,n;A,D,\Gamma)$. From the assumption of our claim, it is clear that $[V_0, e_i] = \{0\}$. Take arbitrary elements $a \in V_0$ and $f \in V_{-1}$. Then we have an equation

$$B_A(a,[e_i,f]) = B_A(a,\Phi_{\square^r_D}(e_i\otimes f)) = \langle \square^r_D(a\otimes e_i), f\rangle^{\Gamma}_{D} = \langle 0, f\rangle^{\Gamma}_{D} = 0.$$

Since B_A is non-degenerate on $V_0 \simeq \mathfrak{h}^r$, we have that $[V_{-1}, e_i] = \{0\}$. Moreover, it holds that $[V_1, e_i] = \{0\}$. In fact, for any $f \in V_{-1}$, we have

$$[[V_1,e_i],f] \subset [[V_1,f],e_i] + [V_1,[e_i,f]] \subset [V_0,e_i] + [V_1,0] = \{0\}.$$

Since $[V_1,V_1] \subset V_2 \subset \mathrm{Hom}(V_{-1},V_1)$ (see [3, Definitions 2.9, 2.12]), we have that $[V_1,e_i] = \{0\}$. By a similar argument, we have the same results on f_i:

$$[V_{-1}\oplus V_0\oplus V_1, e_i] = [V_{-1}\oplus V_0\oplus V_1, f_i] = \{0\}.$$

Since $L(r,n;A,D,\Gamma)$ is generated by $V_{-1}\oplus V_0\oplus V_1$, we have our result. □

From Lemma 3.5, we have the following claim immediately.

Lemma 3.6. *Let $P(r,n;A,D,\Gamma)$ be a pentad of Cartan type and assume that D and Γ are of the forms*

$$D = \left(\begin{array}{c|c} D' & O \end{array}\right),\quad \Gamma = \left(\begin{array}{c|c} \Gamma' & O \\ \hline O & \Gamma'' \end{array}\right)$$

for some $D' \in \mathrm{M}(r,n';\mathbb{C})$, $\Gamma' \in \mathrm{M}(n',n';\mathbb{C})$, $\Gamma'' \in \mathrm{M}(n-n',n-n';\mathbb{C})$. Then we have an isomorphism of Lie algebras:

$$L(r,n;A,D,\Gamma) \simeq \mathfrak{gl}_1^{2(n-n')} \oplus L(r,n';A,D',\Gamma')$$

up to grading.

From Lemma 3.6, to simplify the calculation, we can assume that D does not have zero-column vectors without loss of generality.

EXAMPLE 3.7. We retain to use the notations in Example 2.2. Let us find the structure of $L(\mathfrak{sl}_2, m\Lambda_1, V(m+1), \mathrm{Hom}(V(m+1),\mathbb{C}), K_{\mathfrak{sl}_2})$ from the pentad (2.2). For this, we need to calculate the Cartan matrix of (2.2):

$$C\left(\begin{pmatrix}1/8\end{pmatrix}, \begin{pmatrix}2 & -m\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}\right) = \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}\cdot\begin{pmatrix}2\\ -m\end{pmatrix}\cdot\begin{pmatrix}1/8\end{pmatrix}\cdot\begin{pmatrix}2 & -m\end{pmatrix} = \begin{pmatrix}2 & -m\\ -m & m^2/2\end{pmatrix}.$$

Under the notations of Theorem 3.2, we have equations

$$\dim Z = \operatorname{rank}\begin{pmatrix} 2 & -m \\ -m & m^2/2 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} 2 & -m \end{pmatrix} = 1 - 1 = 0,$$
$$\dim \Delta = 1 - \operatorname{rank}\begin{pmatrix} 2 & -m \end{pmatrix} = 1 - 1 = 0$$

for any integer $m \geq 0$. Thus, we have an isomorphism of Lie algebras:

$$L(\mathfrak{sl}_2, m\Lambda_1, V(m+1), \mathrm{Hom}(V(m+1), \mathbb{C}), K_{\mathfrak{sl}_2}) \simeq G'\left(\begin{pmatrix} 2 & -m \\ -m & m^2/2 \end{pmatrix}\right).$$

That is, the representation $(m\Lambda_1, V(m+1))$ of $\mathfrak{sl}_2$ can be embedded into the reduced contragredient Lie algebra:

$$G'\left(\begin{pmatrix} 2 & -m \\ -m & m^2/2 \end{pmatrix}\right).$$

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