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Reduced Models in the Medium Frequency Range for General Dissipative Structural-Dynamics Systems

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Abstract. This paper presents a theoretical approach for constructing a reduced model in the medium frequency range in the area of structural dynamics for a general three-dimensional anisotropic and inhomogeneous viscoelastic bounded medium. All the results presented can be used for beams, plates and shells. The boundary value problem in the frequency domain and its variational formulation are presented. For a given medium frequency band, an energy operator which is intrinsic to the dynamical system is introduced and mathematically studied. This energy operator depends on the dissipative part of the dynamical system. It is proved that this operator is a positive-definite symmetric trace operator in a Hilbert space and that its dominant eigensubspace allows a reduced model to be constructed using the Ritz-Galerkin method. A finite dimension approximation of the continuous case is presented (for instance using the finite element method). An effective construction of the dominant subspace using the subspace iteration method is developed. Finally, an example is given to validate the concepts and the algorithms.

1. Introduction

This paper is the continuation of initial papers published by the author (Soize, 1982a and 1982b) in the area of modeling and associated solving methods for linear dissipative structural-dynamics problems in the medium-frequency (MF) range. In these papers, we introduced the notion of a narrow MF band B, and any broad MF band was written as a finite union of narrow MF bands. For such a given band B, we proposed a solving method for constructing the operator-valued frequency response function. This solving method, combined with the finite element method for spatial discretization, has allowed a number of complex three-dimensional structural-acoustics and vibration problems to be effectively solved in the MF range. In addition, we introduced a positive-definite symmetric operator $\mathbf{E}_B$ related to band B, called the energy operator, whose spectral theory allowed extremum vibratory states of the structure to be characterized. Concerning the developments, extensions and applications of this initial work, we refer the reader to Soize *et al.*, 1986 and 1992; Chabas *et al.*, 1986; Soize, 1986, 1993a, 1995 and 1997; and for a general overview of these questions, to Ohayon and Soize, 1997. Nevertheless, these developments did not propose the construction of a reduced model in the medium-frequency range. However, it is well known (see for instance Argyris and Mlejnek, 1991; Clough and Penzien, 1975; Leung, 1993; Meirovitch, 1980; Morand and Ohayon, 1995; Roseau, 1980) that, for low-frequency dynamic analysis in structural dynamics, reduced models are a very efficient tool for constructing the solution. These techniques correspond to a Ritz-Galerkin reduction of the structural-dynamics model using the normal modes corresponding to the lowest eigenfrequencies of the associated conservative structure. The efficiency of this kind of reduced model is due to the small number of generalized dynamical degrees of freedom used in the representation and in addition, is obtained by solving a well-stated generalized symmetric eigenvalue problem for which only the first eigenvalues and the corresponding eigenfunctions have to be calculated. In addition, when such a reduced model is obtained, responses to deterministic or random excitations (see for instance Soize, 1994, Chapters 3 to 6) can be calculated for no significant additional numerical cost, and the reduced model can be used directly for solving various structural-acoustics problems in the low-frequency range (see for instance Ohayon and Soize, 1997). Unfortunately, this modal method which is very efficient in the low-frequency domain cannot be used in the medium-frequency domain (Soize, 1982b) for general three-dimensional structures. The fundamental problem related to the construction of a reduced model in the medium-frequency range for general dissipative structural-dynamics systems has not yet received any solution. A modal hybridization method was proposed (Morand, 1992), but this method is based on the use of the normal modes which cannot be calculated in the medium-frequency range for a general three-dimensional structure. A modal sampling method was also proposed (Guyader, 1990) but this method is developped in a context of an analytical theory, i.e. uses the normal modes in the medium- and high-frequency ranges; this method can only be used for simple shape structures (rectangular thin plate in bending mode, circular cylindrical shell with a constant thickness,etc.) Hereinafter, we propose an efficient solution for constructing a reduced model. These theoretical developments are presented in the context of three-dimensional viscoelasticity for an arbitrary geometry of the domain and for an anisotropic and inhomogeneous material. Extension of the results presented to beams, plates and shells is straightforward. It should be noted that the reduced model is constructed for each narrow MF band B and allows the damping and stiffness operators to depend on the frequency (viscoelastic material). Consequently, the reduced model proposed is adapted to each narrow MF band B and also to the structural damping model. In Section 2, we present the boundary value problem to be solved in the frequency domain and we establish its variational formulation. We show that there is a unique solution and we introduce the operator-valued frequency response function of the dynamical system. Section 3 is devoted to construction of the reduced model. For that, we introduce an energy operator $\mathbf{E}_B$ and we prove that $\mathbf{E}_B$ is a positive-definite symmetric trace operator in a Hilbert space. The spectral theory of operator $\mathbf{E}_B$ gives a complete family in the set of admissible displacement fields constituted by the eigenfunctions of operator $\mathbf{E}_B$. The reduced model is then introduced using the Ritz-Galerkin projection of the variational formulation on the dominant eigensubspace of operator $\mathbf{E}_B$, spanned by the eigenfunctions which correspond to the highest eigenvalues of operator $\mathbf{E}_B$. In Section 4, we present the finite dimension approximation of the countinuous case allowing the calculation to be carried out for the general case. In Section 5, we give an efficient procedure for construction of the dominant subspace using the subspace iteration method. Finally, an example is presented in order to validate the concepts and the algorithms.

2. Boundary Value Problem and Its Weak Solution

2.1. Definition of the boundary value problem

We consider linear vibrations (formulated in the frequency domain ω) of a three-dimensional structure around a static equilibrium configuration considered as a natural state. Let Ω be a bounded open domain of $\mathbb{R}^3$, occupied by the structure at static equilibrium and made of viscoelastic material. Let $\partial\Omega = \Gamma_0 \cup \Gamma$ be the boundary which is assumed to be sufficiently smooth and such that $\Gamma_0 \cap \Gamma = \emptyset$. Let $\mathbf{n}$ be its outward unit normal. Let $\mathbf{u} = (u_1, u_2, u_3)$ be the displacement field at each point $\mathbf{x} = (x_1, x_2, x_3)$ in cartesian coordinates. On part Γ_0 of the boundary, the structure is fixed ($\mathbf{u} = \mathbf{0}$) while on part Γ it is free. We introduce a narrow MF band which is defined as the compact interval of $\mathbb{R}^+$,

$$B = [\,\omega_B - \Delta\omega/2\,, \omega_B + \Delta\omega/2\,] \quad , \tag{1}$$

in which ω_B is the central frequency of the band and $\Delta\omega$ is the bandwidth such that

$$\Delta\omega/\omega_B \ll 1 \quad , \quad \omega_B - \Delta\omega/2 > 0 \quad . \tag{2}$$

With B we associate interval $\widetilde{B}$ such that

$$\widetilde{B} = [\,-\omega_B - \Delta\omega/2\,, -\omega_B + \Delta\omega/2\,] \quad . \tag{3}$$

There are external prescribed volumetric and surface force fields applied to Ω and Γ, written as $\eta(\omega)\mathbf{g}_{\text{vol}}(\mathbf{x},\omega)$ and $\eta(\omega)\mathbf{g}_{\text{surf}}(\mathbf{x},\omega)$ respectively, in which $\omega \mapsto \eta(\omega)$ is a function from $\mathbb{R}$ into $\mathbb{C}$, having a compact support $B \cup \widetilde{B}$, continuous on B, verifying $|\eta(-\omega)| = |\eta(\omega)|$ and such that $|\eta(\omega)| \neq 0$ for all ω in B. For all ω in $B \cup \widetilde{B}$, the boundary value problem in $\mathbf{u}(\mathbf{x},\omega)$ is written as follows (the convention for the Fourier transform being $\mathbf{u}(\mathbf{x},\omega) = \int_{\mathbb{R}} e^{-i\omega t}\mathbf{u}(\mathbf{x},t)\,dt$),

$$-\omega^2 \rho\, u_i - \sigma_{ij,j} = \eta\, g_{\text{vol},i} \quad \text{in} \quad \Omega \quad , \quad i = 1,2,3 \quad , \tag{4}$$

$$\sigma_{ij} n_j = \eta\, g_{\text{surf},i} \quad \text{on} \quad \Gamma \quad , \quad i = 1,2,3 \quad , \tag{5}$$

$$u_i = 0 \quad \text{on} \quad \Gamma_0 \quad , \quad i = 1,2,3 \quad , \tag{6}$$

in which the summation convention over repeated latin indices is used, $\rho(\mathbf{x}) > 0$ is the mass density which is assumed to be a bounded function on Ω and $\sigma_{ij,j} = \sum_{j=1}^{3} \partial\sigma_{ij}/\partial x_j$. For a linear viscoelastic material, stress tensor σ_{ij} is written as

$$\sigma_{ij} = \mathbb{a}_{ijkh}(\mathbf{x},\omega)\,\varepsilon_{kh}(\mathbf{u}) + \mathbb{b}_{ijkh}(\mathbf{x},\omega)\,\varepsilon_{kh}(i\omega\,\mathbf{u}) \quad , \tag{7}$$

in which $\varepsilon_{kh}(\mathbf{u}) = (\partial u_k/\partial x_h + \partial u_h/\partial x_k)/2$ is the linearized strain tensor. The mechanical coefficients $\mathbb{a}_{ijkh}(\mathbf{x},\omega)$ and $\mathbb{b}_{ijkh}(\mathbf{x},\omega)$ are real, depend on $\mathbf{x}$ and ω, verify the usual properties of symmetry and positiveness (see for instance Mandel, 1966; Fung, 1968; Marsden and Hughes, 1983; Truesdell, 1984; Germain, 1986; Ciarlet, 1988) and are such that

$$\mathbb{a}_{ijkh}(\mathbf{x},-\omega) = \mathbb{a}_{ijkh}(\mathbf{x},\omega) \quad , \quad \mathbb{b}_{ijkh}(\mathbf{x},-\omega) = \mathbb{b}_{ijkh}(\mathbf{x},\omega) \quad . \tag{8}$$

Since mechanical coefficients $\mathbb{a}_{ijkh}(\mathbf{x},\omega)$ and $\mathbb{b}_{ijkh}(\mathbf{x},\omega)$ depend on ω, Eq. (7) corresponds to a general linear viscoelastic medium (Truesdell, 1984). In particular, if the mechanical coefficients are independent of ω, then Eq. (7) corresponds to a Kelvin-Voight medium (Germain, 1986).

2.2. Variational formulation

The variational formulation of the boundary value problem is constructed in this subsection (for the general methodology of constructing a variational formulation of a boundary value problem, we refer the reader to Brezis, 1987; Dautray and Lions, 1992; Duvaut and Lions, 1976; Oden and Reddy, 1983; Raviart and Thomas, 1983).

A- Set of admissible displacement fields and related vector spaces

We introduce the Hilbert space

$$H = \{ \mathbf{u} = (u_1, u_2, u_3) \, , \, u_j \in L^2(\Omega) \} \quad , \tag{9}$$

equipped with the inner product

$$(\mathbf{u} \, , \mathbf{v})_H = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \overline{\mathbf{v}(\mathbf{x})} \, d\mathbf{x} \quad , \tag{10}$$

and the associated norm $\|\mathbf{u}\|_H = (\mathbf{u} \, , \mathbf{u})_H^{1/2}$, in which $\mathbf{u} \cdot \overline{\mathbf{v}} = \sum_{j=1}^3 u_j \overline{v_j}$ with $\overline{v_j}$ the conjuguate of v_j and where $L^2(\Omega)$ denotes the set of all the square integrable functions from Ω into $\mathbb{C}$. Let V be the Hilbert space representing the set of admissible displacement fields with values in $\mathbb{C}^3$,

$$V = \{ \mathbf{u} = (u_1, u_2, u_3) \, , \, u_j \in H^1(\Omega) \, , \, \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0 \} \quad , \tag{11}$$

equipped with the inner product

$$(\mathbf{u} \, , \mathbf{v})_V = (\mathbf{u} \, , \mathbf{v})_H + \sum_{j=1}^3 \left(\frac{\partial \mathbf{u}}{\partial x_j} \, , \frac{\partial \mathbf{v}}{\partial x_j} \right)_H \quad , \tag{12}$$

and the associated norm $\|\mathbf{u}\|_V = (\mathbf{u} \, , \mathbf{u})_V^{1/2}$ where $H^1(\Omega)$ is the set of all the functions $w \in L^2(\Omega)$ such that, for all $j = 1, 2, 3$, each partial derivative $\partial w / \partial x_j$ belongs to $L^2(\Omega)$. Let V' be the continuous antidual space of V and $<\mathbf{f} \, , \mathbf{u}>_{V',V}$ be the antiduality bracket between $\mathbf{f} \in V'$ and $\mathbf{u} \in V$ which is linear with respect to $\mathbf{f}$ and antilinear with respect to $\mathbf{u}$. Hilbert space H being identified to its continuous antidual space H', we have the classical diagram

$$V \overset{\mathbf{j}_c}{\hookrightarrow} H = H' \overset{{}^t\mathbf{j}_c}{\hookrightarrow} V' \quad , \tag{13}$$

in which $\mathbf{j}_c$ and ${}^t\mathbf{j}_c$ are continuous and compact injections, V being dense in H and H' dense in V' (because Ω is a bounded domain of $\mathbb{R}^3$, see for instance Dautray and Lions, 1992; Treves, 1975).

B- Antilinear form representing the prescribed external forces

For all $i = 1, 2, 3$ and for all ω in $\mathbb{R}$, we assume that $g_{\text{vol},i}(\omega) = \{\mathbf{x} \mapsto g_{\text{vol},i}(\mathbf{x}, \omega)\} \in L^2(\Omega)$ and $g_{\text{surf},i}(\omega) = \{\mathbf{x} \mapsto g_{\text{surf},i}(\mathbf{x}, \omega)\} \in L^2(\Gamma)$ where $L^2(\Gamma)$ is the set of all the ds-square integrable functions from Γ into $\mathbb{C}$. For all ω in $B \cup \widetilde{B}$, the antilinear form $\ell(\mathbf{v} \, ; \omega)$ on V (representing the prescribed external forces) defined by

$$\ell(\mathbf{v} \, ; \omega) = \eta(\omega) \left\{ \int_\Gamma \mathbf{g}_{\text{surf}}(\mathbf{x}, \omega) \cdot \overline{\mathbf{v}(\mathbf{x})} \, ds(\mathbf{x}) + \int_\Omega \mathbf{g}_{\text{vol}}(\mathbf{x}, \omega) \cdot \overline{\mathbf{v}(\mathbf{x})} \, d\mathbf{x} \right\} \quad , \tag{14}$$

is then continuous on V. Consequently, there exists a unique element $\mathbf{f}_\ell(\omega)$ in V' such that

$$\ell(\mathbf{v} \, ; \omega) = \eta(\omega) <\mathbf{f}_\ell(\omega) \, , \mathbf{v}>_{V',V} \quad , \quad \forall \mathbf{v} \in V \quad . \tag{15}$$

It should be noted that if $\mathbf{g}_{\text{surf}} = \mathbf{0}$, then $\mathbf{f}_\ell(\omega) \in H \subset V'$; in general, $\mathbf{g}_{\text{surf}} \neq \mathbf{0}$ and then $\mathbf{f}_\ell(\omega)$ is not in H.

C- Sesquilinear form representing the dynamic-stiffness operator

For all ω in $B \cup \widetilde{B}$, the sesquilinear form $a(\mathbf{u}, \mathbf{v} \, ; \omega)$ on $V \times V$ defined by

$$a(\mathbf{u}, \mathbf{v} \, ; \omega) = -\omega^2 \, m(\mathbf{u}, \mathbf{v}) + i\omega \, d(\mathbf{u}, \mathbf{v} \, ; \omega) + k(\mathbf{u}, \mathbf{v} \, ; \omega) \quad , \tag{16}$$

represents the dynamic-stiffness operator in which the right-hand side is defined hereinafter. Sesquilinear form $m(\mathbf{u}, \mathbf{v})$ (mass term) is defined by

$$m(\mathbf{u}, \mathbf{v}) = \int_\Omega \rho(\mathbf{x}) \, \mathbf{u}(\mathbf{x}) \cdot \overline{\mathbf{v}(\mathbf{x})} \, d\mathbf{x} \quad , \tag{17}$$

and is Hermitian, positive definite, continuous on $H \times H$ and consequently, continuous on $V \times V$. Then, sesquilinear form $m(\mathbf{u}, \mathbf{v})$ can define either an operator $\mathbf{M} \in \mathcal{L}(H)$ (set of all the bounded operators in H) such that, for all $\mathbf{u}$ and $\mathbf{v}$ in H,

$$(\mathbf{M} \, \mathbf{u}, \mathbf{v})_H = m(\mathbf{u}, \mathbf{v}) \quad , \tag{18 - 1}$$

or an operator $\mathbf{M} \in \mathcal{L}(V, V')$ (set of all the bounded operators from V into V') such that, for all **u** and **v** in V,

$$< \mathbf{M}\,\mathbf{u}\,,\mathbf{v} >_{V',V} = m(\mathbf{u}, \mathbf{v}) \quad . \tag{18-2}$$

The usual properties of mechanical coefficients $\mathbb{b}_{ijkh}(\mathbf{x}, \omega)$ and $\mathbb{a}_{ijkh}(\mathbf{x}, \omega)$ are such that sesquilinear forms $d(\mathbf{u}, \mathbf{v}\,; \omega)$ (damping term) and $k(\mathbf{u}, \mathbf{v}\,; \omega)$ (stiffness term) which are defined by

$$d(\mathbf{u}, \mathbf{v}\,; \omega) = \int_\Omega \mathbb{b}_{ijkh}(\mathbf{x}, \omega)\, \varepsilon_{kh}(\mathbf{u})\, \varepsilon_{ij}(\overline{\mathbf{v}})\, d\mathbf{x} \quad , \tag{19}$$

$$k(\mathbf{u}, \mathbf{v}\,; \omega) = \int_\Omega \mathbb{a}_{ijkh}(\mathbf{x}, \omega)\, \varepsilon_{kh}(\mathbf{u})\, \varepsilon_{ij}(\overline{\mathbf{v}})\, d\mathbf{x} \quad , \tag{20}$$

are Hermitian, positive definite, continuous on $V \times V$ and V-coercive (V-elliptic). Then, for all ω in $B \cup \widetilde{B}$, sesquilinear forms $d(\mathbf{u}, \mathbf{v}\,; \omega)$ and $k(\mathbf{u}, \mathbf{v}\,; \omega)$ define linear operators $\mathbf{D}(\omega)$ and $\mathbf{K}(\omega)$ belonging to $\mathcal{L}(V, V')$ such that, for all **u** and **v** in V,

$$< \mathbf{D}(\omega)\,\mathbf{u}, \mathbf{v} >_{V',V} = d(\mathbf{u}, \mathbf{v}\,; \omega) \quad , \tag{21}$$

$$< \mathbf{K}(\omega)\,\mathbf{u}, \mathbf{v} >_{V',V} = k(\mathbf{u}, \mathbf{v}\,; \omega) \quad , \tag{22}$$

and Eq. (8) yields

$$\mathbf{D}(-\omega) = \mathbf{D}(\omega) \quad , \quad \mathbf{K}(-\omega) = \mathbf{K}(\omega) \quad . \tag{23}$$

For all ω in $B \cup \widetilde{B}$, we then deduce that sesquilinear form $a(\mathbf{u}, \mathbf{v}\,; \omega)$ is continuous on $V \times V$ and consequently, defines a linear operator $\mathbf{A}(\omega) \in \mathcal{L}(V, V')$ such that, for all **u** and **v** in V,

$$< \mathbf{A}(\omega)\,\mathbf{u}, \mathbf{v} >_{V',V} = a(\mathbf{u}, \mathbf{v}\,; \omega) \quad . \tag{24}$$

From Eqs. (16), (18-2), (21), (22) and (24), we deduce the expression of the dynamic-stiffness operator

$$\mathbf{A}(\omega) = -\omega^2 \mathbf{M} + i\omega\, \mathbf{D}(\omega) + \mathbf{K}(\omega) \quad , \tag{25}$$

in which $\mathbf{M}$, $\mathbf{D}(\omega)$ and $\mathbf{K}(\omega)$ are the mass, damping and stiffness operators. We will assume that mechanical coefficients $\mathbb{b}_{ijkh}(\mathbf{x}, \omega)$ and $\mathbb{a}_{ijkh}(\mathbf{x}, \omega)$ are such that $\omega \mapsto \mathbf{A}(\omega)$ is a continuous function from B into $\mathcal{L}(V, V')$,

$$\{\omega \mapsto \mathbf{A}(\omega)\} \in \mathcal{C}^0(B, \mathcal{L}(V, V')) \quad . \tag{26}$$

D- Variational formulation of the boundary value problem

It can easily be proved that the variational formulation of the boundary value problem defined by Eqs. (4) to (6) can be expressed as follows. For all ω in $B \cup \widetilde{B}$, find $\mathbf{u}(\omega) = \{\mathbf{x} \mapsto \mathbf{u}(\mathbf{x}, \omega)\}$ in V such that

$$a(\mathbf{u}, \mathbf{v}\,; \omega) = \ell(\mathbf{v}\,; \omega) \quad , \quad \forall \mathbf{v} \in V \quad . \tag{27}$$

The operator equation corresponding to Eq. (27) is

$$\mathbf{A}(\omega)\, \mathbf{u}(\omega) = \eta(\omega)\, \mathbf{f}_\ell(\omega) \quad , \tag{28}$$

in which $\mathbf{A}(\omega)$ and $\mathbf{f}_\ell(\omega)$ are defined by Eqs. (25) and (15) respectively.

2.3. Operator-valued frequency response function

A- Existence and uniqueness of the solution

For all ω in $B \cup \widetilde{B}$, Eq. (27) has a unique solution $\mathbf{u}(\omega)$ in V. The proof is obtained using the Lax-Milgram theorem; for $\omega \in B \cup \widetilde{B}$, we introduce $\widetilde{a}(\mathbf{u}, \mathbf{v}\,;\omega) = -i\varepsilon\, a(\mathbf{u}, \mathbf{v}\,;\omega)$ and $\widetilde{\ell}(\mathbf{v}\,;\omega) = -i\varepsilon\, \ell(\mathbf{v}\,;\omega)$ with $\varepsilon = +1$ if $\omega > 0$ and $\varepsilon = -1$ if $\omega < 0$; then $\Re e\{\widetilde{a}(\mathbf{u}, \mathbf{u}\,;\omega)\} = |\omega|\, d(\mathbf{u}, \mathbf{u}\,;\omega) \geq c(\omega)\, \|\mathbf{u}\|_V^2$ with $c(\omega) > 0$.

B- Operator-valued frequency response function $\mathbf{T}(\omega)$

From Section 2.3.A, we deduce that, for all ω in $B \cup \widetilde{B}$, $\mathbf{A}(\omega) \in \mathcal{L}(V, V')$ is invertible,

$$\mathbf{T}(\omega) = \mathbf{A}(\omega)^{-1} \in \mathcal{L}(V', V) \quad . \tag{29}$$

Function $\omega \mapsto \mathbf{T}(\omega)$ from $B \cup \widetilde{B}$ into $\mathcal{L}(V', V)$ is called the operator-valued frequency response function. From Eq. (28), we deduce that solution $\mathbf{u}(\omega)$ in V can be written as

$$\mathbf{u}(\omega) = \eta(\omega)\, \mathbf{T}(\omega)\, \mathbf{f}_\ell(\omega) \quad . \tag{30}$$

Solution $\mathbf{u}(\omega) \in V$ given by Eq. (30) will be called the vibration induced by excitation $\mathbf{f}_\ell(\omega) \in V'$.

3. Construction of a Reduced Model

3.1. Operator-valued frequency response function $\mathbf{T}_H(\omega)$

For all ω in $B \cup \widetilde{B}$, $\mathbf{A}(\omega)$ is an unbounded operator in H whose domain $Dom = \{\mathbf{u} \in V\,, \mathbf{A}(\omega)\,\mathbf{u} \in H\}$ is such that

$$Dom = \left(H^2(\Omega)\right)^3 \cap V \quad . \tag{31}$$

Let $\mathbf{A}_H(\omega)$ be the restriction of operator $\mathbf{A}(\omega) \in \mathcal{L}(V, V')$ to Dom. Then $\mathbf{A}_H(\omega)$ belongs to $\mathcal{L}(Dom, H)$ and is invertible,

$$\mathbf{A}_H(\omega) \in \mathcal{L}(Dom, H) \quad , \quad \mathbf{A}_H(\omega)^{-1} \in \mathcal{L}(H, Dom) \quad . \tag{32}$$

Proposition 1. *The injection $\mathbf{j}_{\text{HS}}$ from Dom into H is a Hilbert-Schmidt operator and for all ω in $B \cup \widetilde{B}$, unbounded operator $\mathbf{A}(\omega)$ in H with domain Dom has a bounded inverse $\mathbf{T}_H(\omega)$ which is a Hilbert-Schmidt operator in H,*

$$\mathbf{T}_H(\omega) = \mathbf{j}_{\text{HS}} \circ \mathbf{A}_H(\omega)^{-1} \in \mathcal{L}_2(H) \quad , \tag{33}$$

whose range space is

$$R\{\mathbf{T}_H(\omega)\} = Dom \subset V \quad , \tag{34}$$

and where $\mathcal{L}_2(H)$ denotes the set of all the Hilbert-Schmidt operators in H.

Proof. We use the following known result concerning Sobolev spaces H^s. Let Ω be a bounded open set of $\mathbb{R}^n$ and $s \in \mathbb{R}$. If $k > n/2$, then the injection $H^{s+k}(\Omega) \to H^s(\Omega)$ is a Hilbert-Schmidt operator. Considering Eqs. (9) and (31) and taking $n = 3$ and $s = 0$ ($H^0(\Omega) = L^2(\Omega)$), we deduce that $\mathbf{j}_{\text{HS}} \in \mathcal{L}_2(Dom, H)$. Since $\mathbf{A}_H(\omega)^{-1} \in \mathcal{L}(H, Dom)$ and $\mathbf{j}_{\text{HS}} \in \mathcal{L}_2(Dom, H)$, we deduce Eq. (33) because $\mathcal{L}_2 \circ \mathcal{L} = \mathcal{L}_2$ (see for instance Kato, 1966; Reed and Simon, 1980; Dautray and Lions, 1992).

3.2. Energy operator as a trace operator in H

Let $\mathbf{f}$ be in H and be independent of ω. Let $\mathbf{u}^{\mathbf{f}}$ be the vibration due to excitation $\eta(\omega)\,\mathbf{f}$. We then have

$$\mathbf{u}^{\mathbf{f}}(\omega) = \eta(\omega)\, \mathbf{T}_H(\omega)\, \mathbf{f} \quad , \tag{35}$$

in which $\mathbf{T}_H(\omega) \in \mathcal{L}_2(H)$. We then define the energy $\varepsilon_B(\mathbf{u}^{\mathbf{f}})$ of vibration $\mathbf{u}^{\mathbf{f}}$ as twice the value of the total kinetic energy, i.e. using Plancherel's formula (see for instance Dautray and Lions, 1992; Soize, 1993b),

$$\varepsilon_B(\mathbf{u}^{\mathbf{f}}) = \frac{1}{2\pi} \int_{B \cup \widetilde{B}} \omega^2\, (\mathbf{M}\, \mathbf{u}^{\mathbf{f}}(\omega), \mathbf{u}^{\mathbf{f}}(\omega))_H \, d\omega \quad , \tag{36}$$

in which $\mathbf{M} \in \mathcal{L}(H)$ is defined by Eq. (18-1). Therefore, it is coherent to introduce the following definition of the energy operator related to band B.

Definition 1. (Energy operator). Let $\mathbf{u}^{\mathbf{f}}(\omega) = \eta(\omega)\,\mathbf{T}_{_H}(\omega)\,\mathbf{f}$ and $\mathbf{u}^{\mathbf{g}}(\omega) = \eta(\omega)\,\mathbf{T}_{_H}(\omega)\,\mathbf{g}$ be the vibrations due to excitations $\eta(\omega)\,\mathbf{f}$ and $\eta(\omega)\,\mathbf{g}$ respectively, where $\mathbf{f}$ and $\mathbf{g}$ are in H and are independent of ω. The energy operator $\mathbf{E}_B$ related to band B is defined by

$$(\mathbf{E}_B\,\mathbf{f},\mathbf{g})_{_H} = \frac{1}{2\pi}\int_{B\cup\widetilde{B}} \omega^2\,(\mathbf{M}\,\mathbf{u}^{\mathbf{f}}(\omega),\mathbf{u}^{\mathbf{g}}(\omega))_{_H}\,d\omega \quad . \tag{37}$$

Remarks. From Eqs. (36) and (37), we deduce that

$$\varepsilon_B(\mathbf{u}^{\mathbf{f}}) = (\mathbf{E}_B\,\mathbf{f},\mathbf{f})_{_H} \quad . \tag{38}$$

It can be seen that operator $\mathbf{E}_B$ is an intrinsic operator which depends on B and η, but does not depend on the spatial parts $\mathbf{f}$ and $\mathbf{g}$ of the excitations.

Proposition 2. (Characterization of the energy operator). *Energy operator $\mathbf{E}_B$ is a positive-definite symmetric trace operator in H whose range space is Dom,*

$$R\{\mathbf{E}_B\} = Dom \subset V \quad , \quad \mathbf{E}_B \in \mathcal{L}_1(H) \quad , \tag{39}$$

in which $\mathcal{L}_1(H)$ denotes the set of all the trace operators in H. This operator is written as

$$\mathbf{E}_B = \frac{1}{2\pi}\int_{B\cup\widetilde{B}} \omega^2\,|\eta(\omega)|^2\,\mathbf{T}_{_H}(\omega)^*\,\mathbf{M}\,\mathbf{T}_{_H}(\omega)\,d\omega \quad , \tag{40}$$

in which $\mathbf{T}_{_H}(\omega)^* \in \mathcal{L}_2(H)$ is the adjoint of $\mathbf{T}_{_H}(\omega) \in \mathcal{L}_2(H)$ and $\mathbf{M} \in \mathcal{L}(H)$ is defined by Eq. (18-1). Operator $\mathbf{E}_B$ can also be written as

$$\mathbf{E}_B = \frac{1}{\pi}\int_B \omega^2\,|\eta(\omega)|^2\,\Re e\,\{\mathbf{T}_{_H}(\omega)^*\,\mathbf{M}\,\mathbf{T}_{_H}(\omega)\}\,d\omega \quad , \tag{41}$$

in which $\Re e$ denotes the real part.

Proof. 1)- Eq. (40) is directly deduced from Eq. (37) knowing that if $\mathbf{T}_{_H}(\omega) \in \mathcal{L}_2(H)$, then $\mathbf{T}_{_H}(\omega)^* \in \mathcal{L}_2(H)$. 2)- Eq. (41) is deduced from Eq. (40) using Eq. (23) and the fact that $|\eta(-\omega)| = |\eta(\omega)|$. 3)- For all ω in B, $\mathbf{T}_{_H}(\omega)^* \in \mathcal{L}_2(H)$, $\mathbf{M} \in \mathcal{L}(H)$ and $\mathbf{T}_{_H}(\omega) \in \mathcal{L}_2(H)$; then (see for instance Kato, 1966; Reed and Simon, 1980) $\mathbf{T}_{_H}(\omega)^*\,\mathbf{M}\,\mathbf{T}_{_H}(\omega) \in \mathcal{L}_1(H)$. 4)- From Eq. (26), $\omega \mapsto \mathbf{A}_{_H}(\omega)$ is continuous from B into $\mathcal{L}(Dom, H)$. Since $\mathbf{A}_{_H}(\omega) \mapsto \mathbf{A}_{_H}(\omega)^{-1}$ is continuous from $\mathcal{L}(Dom, H)$ into $\mathcal{L}(H, Dom)$, we deduce that $\omega \mapsto \mathbf{A}_{_H}(\omega)^{-1}$ is continuous from B into $\mathcal{L}(H, Dom)$. 5)- Let $\|\,.\,\|_2$ be the Hilbert-Schmidt norm. For ω and ω' in B, we have $\|\mathbf{T}_{_H}(\omega) - \mathbf{T}_{_H}(\omega')\|_2 = \|\mathbf{j}_{\text{HS}} \circ (\mathbf{A}_{_H}(\omega)^{-1} - \mathbf{A}_{_H}(\omega')^{-1})\|_2 \leq \|\mathbf{j}_{\text{HS}}\|_2\,\|\mathbf{A}_{_H}(\omega)^{-1} - \mathbf{A}_{_H}(\omega')^{-1}\|_{\mathcal{L}(H,Dom)}$. Using 4), we deduce that $\omega \mapsto \mathbf{T}_{_H}(\omega)$ is continuous from B into $\mathcal{L}_2(H)$. 6)- Since $\omega \mapsto \eta(\omega)$ is continuous on B (see Section 2.1) and from 3) and 5), we deduce (see for instance Reed and Simon, 1980) that $\omega \mapsto \omega^2|\eta(\omega)|^2\Re e\,\{\mathbf{T}_{_H}(\omega)^*\,\mathbf{M}\,\mathbf{T}_{_H}(\omega)\}$ is continuous from B into $\mathcal{L}_1(H)$ and consequently, $\mathbf{E}_B \in \mathcal{L}_1(H)$. 7)- From Section 2.2.C, we deduce that, for all ω in B, $\mathbf{T}_{_H}(\omega)^* = \overline{\mathbf{T}_{_H}(\omega)}$. Therefore, using Eq. (34), the range space of $\mathbf{T}_{_H}(\omega)^*$ is Dom and then, the range space of trace operator $\omega^2|\eta(\omega)|^2\Re e\,\{\mathbf{T}_{_H}(\omega)^*\,\mathbf{M}\,\mathbf{T}_{_H}(\omega)\}$ is Dom. We then conclude that the range space of trace operator $\mathbf{E}_B$ is Dom. 8)- It can easily be verified that $\mathbf{E}_B^* = \mathbf{E}_B$ and consequently, $\mathbf{E}_B$ is a symmetric operator in H. 9)- Finally, from Eq. (37), we deduce that $\mathbf{E}_B$ is positive definite because $\mathbf{M}$ is positive definite.

3.3. Spectral theory of energy operator $\mathbf{E}_B$

Since a trace operator is a compact operator, the spectral theory of operator $\mathbf{E}_B \in \mathcal{L}_1(H)$ is directly deduced from the spectral theory of symmetric compact operators in Hilbert spaces. Consequently, $\mathbf{E}_B$ has a countable number of positive eigenvalues with finite multiplicity, possibly excepting zero,

$$\lambda_1 \geq \lambda_2 \geq \ldots \to 0 \quad , \tag{42}$$

in which the λ_ν's are the repeated eigenvalues of $\mathbf{E}_B$ (each eigenvalue λ_ν counted repeatedly according to its multiplicity). The corresponding eigenfunctions $\{\mathbf{e}_\nu\}_{\nu\geq 1}$, such that

$$\mathbf{E}_B\,\mathbf{e}_\nu = \lambda_\nu\,\mathbf{e}_\nu \quad , \tag{43}$$

are functions $\mathbf{x} \mapsto \mathbf{e}_\nu(\mathbf{x})$ from Ω into $\mathbb{R}^3$ and form a complete orthonormal family in H,

$$(\mathbf{e}_\nu\,,\mathbf{e}_{\nu'})_{_H} = \delta_{\nu\nu'} \quad . \tag{44}$$

Since $\mathbf{E}_B$ is a positive-definite symmetric trace operator, we have $\sum_{\nu=1}^{+\infty} |\lambda_\nu| = \sum_{\nu=1}^{+\infty} \lambda_\nu < +\infty$. The trace norm of $\mathbf{E}_B$ is such that

$$\|\mathbf{E}_B\|_1 = \text{tr } \mathbf{E}_B = \sum_{\nu=1}^{+\infty} \lambda_\nu < +\infty \quad , \tag{45}$$

and $\mathbf{E}_B$ can be written as

$$\mathbf{E}_B = \sum_{\nu=1}^{+\infty} \lambda_\nu \; (\; . \; , \mathbf{e}_\nu)_H \, \mathbf{e}_\nu \quad . \tag{46}$$

Proposition 3. (Characterization of the eigenfunctions of operator $\mathbf{E}_B$). *The set $\{\mathbf{e}_\nu, \nu \geq 1\}$ of eigenfunctions of operator $\mathbf{E}_B$ is a complete family in $V \subset H$, orthonormal for the inner product of H, and each eigenfunction $\mathbf{e}_\nu$ is a continuous function from Ω into $\mathbb{R}^3$.*

Proof. Since the range space of $\mathbf{E}_B$ is Dom (see Proposition 2), we have $\mathbf{E}_B \, \mathbf{e}_\nu = \lambda_\nu \, \mathbf{e}_\nu \in Dom$ and therefore, for all $\nu \geq 1$, $\mathbf{e}_\nu \in Dom \subset V$. We then deduce that $\{\mathbf{e}_\nu, \nu \geq 1\}$ is a complete family in Dom and therefore, in V. Since $Dom \subset (H^2(\Omega))^3$ (see Eq. (31)) and since, for any open domain Ω of $\mathbb{R}^n$ having a sufficiently smooth boundary, the injection $H^m(\Omega) \hookrightarrow \mathcal{C}^0(\overline{\Omega})$ is continuous for $m > n/2$, we deduce that $\mathbf{x} \mapsto \mathbf{e}_\nu(\mathbf{x})$ is a continuous function.

3.4. Reduced model adapted to frequency band B

The reduced model adapted to frequency band B is obtained using the Ritz-Galerkin projection of the variational formulation on the subspace V^N of V spanned by the eigenfunctions $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ which correspond to the N highest eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$ of energy operator $\mathbf{E}_B$. From Proposition 3, we deduce that the sequence $\{V^N\}_N$ is dense in V. Let $\mathbf{u}(\omega) \in V$ be the unique solution of Eq. (27), given by Eq. (30), and let $\mathbf{u}^N(\omega) = \{\mathbf{x} \mapsto \mathbf{u}^N(\mathbf{x}, \omega)\}$ be the projection of $\mathbf{u}(\omega)$ on V^N,

$$\mathbf{u}^N(\mathbf{x}, \omega) = \sum_{\nu=1}^{N} \theta_\nu(\omega) \, \mathbf{e}_\nu(\mathbf{x}) \quad , \tag{47}$$

in which $\theta_\nu(\omega) \in \mathbb{C}$. From Eqs. (15) and (27), we deduce that, for all ω in $B \cup \widetilde{B}$, $\boldsymbol{\Theta}(\omega) = (\theta_1(\omega), \ldots, \theta_N(\omega)) \in \mathbb{C}^N$ is the solution of the linear equation

$$[\mathcal{A}_N(\omega)] \, \boldsymbol{\Theta}(\omega) = \eta(\omega) \, \boldsymbol{\mathcal{F}}(\omega) \quad , \tag{48}$$

in which $[\mathcal{A}_N(\omega)]$ is the symmetric $(N \times N)$ complex matrix defined by

$$[\mathcal{A}_N(\omega)]_{\nu\nu'} = a(\mathbf{e}_{\nu'}, \mathbf{e}_\nu \, ; \omega) \quad , \tag{49}$$

and where $\boldsymbol{\mathcal{F}}(\omega) = (\mathcal{F}_1(\omega), \ldots, \mathcal{F}_N(\omega)) \in \mathbb{C}^N$ is such that

$$\mathcal{F}_\nu(\omega) = < \mathbf{f}_\ell(\omega) \, , \mathbf{e}_\nu >_{V',V} \quad . \tag{50}$$

Definition 2. (Reduced model). The set of Eqs. (47) to (50) constitutes the so-called reduced model adapted to frequency band B of the dynamical system described by Eq. (27) or (28).

For all ω in $B \cup \widetilde{B}$, matrix $[\mathcal{A}_N(\omega)]$ is invertible,

$$[\mathcal{T}_N(\omega)] = [\mathcal{A}_N(\omega)]^{-1} \quad ; \quad [\mathcal{T}_N(\omega)]^T = [\mathcal{T}_N(\omega)] \quad , \tag{51}$$

and the solution of Eq. (48) is written as

$$\boldsymbol{\Theta}(\omega) = \eta(\omega) \, [\mathcal{T}_N(\omega)] \, \boldsymbol{\mathcal{F}}(\omega) \quad . \tag{52}$$

3.5. Upper bound of energy of the response when $\mathbf{f}_\ell$ is independent of ω

In this section, we assume that $\mathbf{f}_\ell(\omega) = \mathbf{f}_\ell$ is independent of ω. From Eq. (50), we then deduce that $\mathcal{F}_\nu(\omega) = \mathcal{F}_\nu$ is independent of ω. Let $\mathbf{E}_B^N$ be the projection of energy operator $\mathbf{E}_B \in \mathcal{L}_1(H)$ on subspace $V^N \subset V \subset H$. From Eq. (46), we deduce that

$$\mathbf{E}_B^N = \sum_{\nu=1}^{N} \lambda_\nu \; (\; . \; , \mathbf{e}_\nu)_H \, \mathbf{e}_\nu \quad , \tag{53}$$

and its trace norm is written as

$$\|\mathbf{E}_B^N\|_1 = \text{tr}\ \mathbf{E}_B^N = \sum_{\nu=1}^{N} \lambda_\nu \quad . \tag{54}$$

Since $\mathbf{e}_\nu \in V$, the right-hand side of Eq. (53) can be extended to V' by replacing $(\ .\ ,\mathbf{e}_\nu)_H$ by $<\ .\ ,\mathbf{e}_\nu>_{V',V}$. We then deduce that energy $\varepsilon_B(\mathbf{u}^N)$ of vibration $\mathbf{u}^N$ is written as

$$\varepsilon_B(\mathbf{u}^N) = \sum_{\nu=1}^{N} \lambda_\nu\, |\mathcal{F}_\nu|^2 \quad . \tag{55}$$

Equation (55) yields the energy upper bound (Rayleigh quotient) of vibration $\mathbf{u}^N$ which is such that

$$\frac{\varepsilon_B(\mathbf{u}^N)}{\|\boldsymbol{\mathcal{F}}\|^2} \leq \lambda_1 \quad . \tag{56}$$

4. Finite Dimension Approximation

An explicit construction of eigenfunctions $\{\mathbf{e}_1,\ldots,\mathbf{e}_N\}$ of energy operator $\mathbf{E}_B$ cannot be obtained in the general case. A finite dimension approximation $\mathbf{E}_{B,n}$ of $\mathbf{E}_B$ must be introduced and the eigenfunctions $\{\mathbf{e}_1^n\ldots,\mathbf{e}_N^n\}$ of $\mathbf{E}_{B,n}$ (associated with the N highest eigenvalues) constitute the approximation of $\{\mathbf{e}_1\ldots,\mathbf{e}_N\}$. This finite approximation is obtained using the Ritz-Galerkin method (whose finite element method is a particular case; see for instance Argyris and Mlejnek, 1991; Bathe and Wilson, 1976; Dautray and Lions, 1992; Zienkiewicz and Taylor, 1989). We then consider a subspace $V_n \subset V$ of finite dimension $n \geq 1$ spanned by a family $\{\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_n\}$ of independent $\mathbb{R}^3$-valued functions $\{\boldsymbol{\psi}_\alpha\}_\alpha$ in V.

4.1. Projection of the variational formulation

The restriction of antilinear form $\ell(\mathbf{v}\,;\omega)$ on V to V_n is represented by $\eta(\omega)\,\mathbf{F}(\omega)$ in which $\mathbf{F}(\omega) = (F_1(\omega),\ldots,F_n(\omega)) \in \mathbb{C}^n$ is such that

$$F_\alpha(\omega) = \int_\Gamma \mathbf{g}_{\text{surf}}(\mathbf{x},\omega)\cdot\boldsymbol{\psi}_\alpha(\mathbf{x})\,ds(\mathbf{x}) + \int_\Omega \mathbf{g}_{\text{vol}}(\mathbf{x},\omega)\cdot\boldsymbol{\psi}_\alpha(\mathbf{x})\,d\mathbf{x} \quad . \tag{57}$$

The restriction of sesquilinear form $a(\mathbf{u},\mathbf{v}\,;\omega)$ on $V\times V$ to $V_n\times V_n$ is represented by the symmetric $(n\times n)$ complex matrix $[A_n(\omega)]$ such that

$$[A_n(\omega)]_{\alpha\beta} = a(\boldsymbol{\psi}_\beta,\boldsymbol{\psi}_\alpha\,;\omega) \quad . \tag{58}$$

Using Eq. (16), matrix $[A_n(\omega)]$ can be written as

$$[A_n(\omega)] = -\omega^2\,[\,M\,] + i\omega\,[D(\omega)] + [K(\omega)] \quad , \tag{59}$$

in which $[\,M\,]$, $[D(\omega)]$ and $[K(\omega)]$ are positive-definite symmetric $(n\times n)$ real matrices such that

$$[\,M\,]_{\alpha\beta} = m(\boldsymbol{\psi}_\beta,\boldsymbol{\psi}_\alpha) \quad , \tag{60}$$

$$[D(\omega)]_{\alpha\beta} = d(\boldsymbol{\psi}_\beta,\boldsymbol{\psi}_\alpha\,;\omega) \quad , \tag{61}$$

$$[K(\omega)]_{\alpha\beta} = k(\boldsymbol{\psi}_\beta,\boldsymbol{\psi}_\alpha\,;\omega) \quad . \tag{62}$$

For all ω in $B\cup\widetilde{B}$, symmetric $(n\times n)$ complex matrix $[A_n(\omega)]$ is invertible and we have

$$[T_n(\omega)] = [A_n(\omega)]^{-1} \quad , \quad [T_n(\omega)]^* = \overline{[T_n(\omega)]} \quad . \tag{63}$$

4.2. Projection of the energy operator

Proposition 4. *Projection* $\mathbf{E}_{B,n}$ *of operator* $\mathbf{E}_B$ *on* $V_n \subset H$ *is written as*

$$\mathbf{E}_{B,n} = \sum_{\alpha,\beta=1}^{n} [E_n]_{\alpha\beta} \ (\ . \ , \boldsymbol{\psi}_\beta)_H \boldsymbol{\psi}_\alpha \quad , \tag{64}$$

in which $[E_n]$ *is a positive-definite symmetric* $(n \times n)$ *real matrix such that*

$$[E_n] = \int_B [e_n(\omega)] \, d\omega \quad , \tag{65 − 1}$$

$$[e_n(\omega)] = \frac{1}{\pi} \omega^2 \, |\eta(\omega)|^2 \, \Re e \, \{[T_n(\omega)]^* \, [\, M \,] \, [T_n(\omega)]\} \quad . \tag{65 − 2}$$

Proof. Let $\mathbf{f}$ and $\mathbf{g}$ be in H and be independent of ω. The projections $\mathbf{u}^{\mathbf{f}}_n(\omega)$ and $\mathbf{u}^{\mathbf{g}}_n(\omega)$ of $\mathbf{u}^{\mathbf{f}}(\omega)$ and $\mathbf{u}^{\mathbf{g}}(\omega)$ on V_n are written as

$$\mathbf{u}^{\mathbf{f}}_n(\mathbf{x}, \omega) = \sum_{\beta'=1}^{n} q^{\mathbf{f}}_{\beta'}(\omega) \ \boldsymbol{\psi}_{\beta'}(\mathbf{x}) \quad , \quad \mathbf{u}^{\mathbf{g}}_n(\mathbf{x}, \omega) = \sum_{\alpha'=1}^{n} q^{\mathbf{g}}_{\alpha'}(\omega) \ \boldsymbol{\psi}_{\alpha'}(\mathbf{x}) \quad , \tag{66}$$

in which $q^{\mathbf{f}}_{\beta'}(\omega)$ and $q^{\mathbf{g}}_{\alpha'}(\omega)$ are such that

$$q^{\mathbf{f}}_{\beta'}(\omega) = \eta(\omega) \sum_{\beta=1}^{n} [T_n(\omega)]_{\beta'\beta} \ (\mathbf{f} \, , \boldsymbol{\psi}_\beta)_H \quad , \quad q^{\mathbf{g}}_{\alpha'}(\omega) = \eta(\omega) \sum_{\alpha=1}^{n} [T_n(\omega)]_{\alpha'\alpha} \ (\mathbf{g} \, , \boldsymbol{\psi}_\alpha)_H \quad . \tag{67}$$

Form Definition 1 and Eqs. (18-1), (37), operator $\mathbf{E}_{B,n}$ is such that

$$(\mathbf{E}_{B,n} \mathbf{f} \, , \mathbf{g})_H = \frac{1}{2\pi} \int_{B \cup \widetilde{B}} \omega^2 \, m(\mathbf{u}^{\mathbf{f}}_n(\omega), \mathbf{u}^{\mathbf{g}}_n(\omega)) \, d\omega \quad . \tag{68}$$

Substituting Eq. (66) with Eq. (67) in the right-hand side of Eq. (68) yields Eq. (64) in which matrix $[E_n]$ is such that

$$[E_n] = \frac{1}{2\pi} \int_{B \cup \widetilde{B}} \omega^2 \, |\eta(\omega)|^2 \, [T_n(\omega)]^* \, [\, M \,] \, [T_n(\omega)] \, d\omega \quad . \tag{69}$$

Since $|\eta(-\omega)| = |\eta(\omega)|$, $[D(-\omega)] = [D(\omega)]$ and $[K(-\omega)] = [K(\omega)]$, from Eq. (69), we deduce Eq. (65). Finally, the properties of matrix $[E_n]$ are directly deduced from Proposition 2.

4.3. Spectral theory of the projected energy operator

Let $\{\lambda^n_1, \ldots, \lambda^n_n\}$ and $\{\mathbf{e}^n_1, \ldots \mathbf{e}^n_n\}$ be the eigenvalues and the corresponding eigenfunctions of $\mathbf{E}_{B,n}$,

$$\mathbf{E}_{B,n} \ \mathbf{e}^n_\nu = \lambda^n_\nu \ \mathbf{e}^n_\nu \quad . \tag{70}$$

Proposition 4 shows that the eigenvalues are positive real numbers,

$$\lambda^n_1 \geq \lambda^n_2 \geq \ldots \geq \lambda^n_n > 0 \quad , \tag{71}$$

and that the corresponding eigenfunctions form an orthonormal basis in V_n for the inner product of H

$$(\mathbf{e}^n_\nu \, , \mathbf{e}^n_{\nu'})_H = \delta_{\nu\nu'} \quad . \tag{72}$$

We have

$$\|\mathbf{E}_{B,n}\|_1 = \mathrm{tr} \ \mathbf{E}_{B,n} = \sum_{\nu=1}^{N} \lambda^n_\nu \quad , \tag{73}$$

and operator $\mathbf{E}_{B,n}$ can be written as

$$\mathbf{E}_{B,n} = \sum_{\nu=1}^{n} \lambda^n_\nu \ (\ . \ , \mathbf{e}^n_\nu)_H \ \mathbf{e}^n_\nu \quad . \tag{74}$$

Proposition 5. (Characterization of the eigenfunctions). *Eigenvalues* $\{\lambda_1^n, \ldots, \lambda_n^n\}$ *of operator* $\mathbf{E}_{B,n}$ *are the solutions of the generalized symmetric eigenvalue problem*

$$[\,H\,]\,\mathbf{P}^\nu = \lambda_\nu^n\,[\,G\,]\,\mathbf{P}^\nu \quad , \tag{75}$$

in which $[\,G\,]$ *and* $[\,H\,]$ *are positive-definite symmetric* $(n\times n)$ *real matrices such that*

$$[\,G\,]_{\beta\alpha} = (\boldsymbol{\psi}_\alpha\,, \boldsymbol{\psi}_\beta)_H \quad , \tag{76}$$

$$[\,H\,] = [\,G\,]\,[E_n]\,[\,G\,] \quad . \tag{77}$$

The eigenvectors $\{\mathbf{P}^1, \ldots, \mathbf{P}^n\}$ *form a basis of* $\mathbb{R}^n$ *and verify the orthogonality properties*

$$([\,G\,]\,\mathbf{P}^\nu, \mathbf{P}^{\nu'})_{\mathbb{R}^n} = \delta_{\nu\nu'} \quad , \tag{78}$$

$$([\,H\,]\,\mathbf{P}^\nu, \mathbf{P}^{\nu'})_{\mathbb{R}^n} = \lambda_\nu^n\,\delta_{\nu\nu'} \quad . \tag{79}$$

Eigenfunctions $\{\mathbf{e}_1^n, \ldots \mathbf{e}_n^n\}$ *of operator* $\mathbf{E}_{B,n}$ *can be written as*

$$\mathbf{e}_\nu^n = \sum_{\alpha=1}^n P_\alpha^\nu\,\boldsymbol{\psi}_\alpha \quad , \quad \nu = 1, \ldots, n \quad , \tag{80}$$

in which $\mathbf{P}^\nu = (P_1^\nu, \ldots, P_n^\nu)$. *We have*

$$(\mathbf{e}_\nu^n\,, \mathbf{e}_{\nu'}^n)_H = ([\,G\,]\,\mathbf{P}^\nu, \mathbf{P}^{\nu'})_{\mathbb{R}^n} = \delta_{\nu\nu'} \quad . \tag{81}$$

Proof. Since $\mathbf{e}_\nu^n \in V_n$ and $\{\boldsymbol{\psi}_\alpha\}_\alpha$ is a basis of V_n, Eq. (80) holds with $P_\alpha^\nu \in \mathbb{R}$ because $\mathbf{e}_\nu^n$ is an $\mathbb{R}^3$-valued function. Eq. (70) is equivalent to

$$(\mathbf{E}_{B,n}\mathbf{e}_\nu^n\,, \mathbf{v})_H = \lambda_\nu^n\;(\mathbf{e}_\nu^n\,, \mathbf{v})_H \quad , \quad \forall \mathbf{v} \in V_n \quad . \tag{82}$$

Taking $\mathbf{v} = \sum_{\alpha=1}^n v_\alpha \boldsymbol{\psi}_\alpha$ and substituting $\mathbf{E}_{B,n}$ given by Eq. (64) and $\mathbf{e}_\nu^n$ given by Eq. (80) into Eq. (82) yields Eq. (75). It is clear that $[\,G\,]$ and $[\,H\,]$ are positive-definite symmetric matrices and consequently, the generalized symmetric eigenvalue problem defined by Eq. (75) leads to Eqs. (78) and (79). Finally, due to Eq. (78), Eq. (81) (i.e. Eq. (72)) holds.

Remarks. 1)- In the particular case for which $\rho(\mathbf{x}) = \rho_0$ is a constant in domain Ω, we have $[\,G\,] = \rho_0^{-1}\,[\,M\,]$. 2)- If the finite element method is used, then $[\,G\,]$ is a sparse matrix which is "similar" to mass matrix $[\,M\,]$.

4.4. Reduced model adapted to frequency band B

A- Definition of the reduced model

We consider the construction introduced in Section 3.4 in which we replace the eigenfunctions of $\mathbf{E}_B$ by the eigenfunctions of $\mathbf{E}_{B,n}$. Let $N < n$ (generally $N \ll n$). Let $\mathbf{u}(\omega) \in V$ be the unique solution of Eq. (27), given by Eq. (30). Then, the reduced model adapted to frequency band B is defined as the projection $\mathbf{u}_n^N(\omega)$ of $\mathbf{u}(\omega)$ on the subspace $V_n^N \subset V_n \subset V$ spanned by the eigenfunctions $\{\mathbf{e}_1^n, \ldots, \mathbf{e}_N^n\}$ which correspond to the N highest eigenvalues $\lambda_1^n \geq \lambda_2^n \geq \ldots \geq \lambda_N^n$ of operator $\mathbf{E}_{B,n}$,

$$V_n^N = \text{span}\,\{\mathbf{e}_1^n, \ldots, \mathbf{e}_N^n\} \quad . \tag{83}$$

For all ω in $B \cup \widetilde{B}$, projection $\mathbf{u}_n^N(\omega)$ is written as

$$\mathbf{u}_n^N(\mathbf{x}, \omega) = \sum_{\nu=1}^N \theta_\nu^n(\omega)\,\mathbf{e}_\nu^n(\mathbf{x}) \quad , \tag{84}$$

in which $\boldsymbol{\Theta}^n(\omega) = (\theta_1^n(\omega), \ldots, \theta_N^n(\omega)) \in \mathbb{C}^N$ is the solution of the linear equation

$$[\mathcal{A}_N^n(\omega)]\,\boldsymbol{\Theta}^n(\omega) = \eta(\omega)\,\boldsymbol{\mathcal{F}}^n(\omega) \quad , \tag{85}$$

in which $[\mathcal{A}_N^n(\omega)]$ is the symmetric $(N\times N)$ complex matrix defined by

$$[\mathcal{A}_N^n(\omega)]_{\nu\nu'} = a(\mathbf{e}_{\nu'}^n, \mathbf{e}_\nu^n\,;\omega) \quad , \tag{86}$$

and where $\boldsymbol{\mathcal{F}}^n(\omega) = (\mathcal{F}_1^n(\omega),\ldots,\mathcal{F}_N^n(\omega)) \in \mathbb{C}^N$ is such that (see Eqs. (14) and (15)),

$$\mathcal{F}_\nu^n(\omega) = \int_\Gamma \mathbf{g}_{\text{surf}}(\mathbf{x},\omega)\cdot\mathbf{e}_\nu^n(\mathbf{x})\,ds(\mathbf{x}) + \int_\Omega \mathbf{g}_{\text{vol}}(\mathbf{x},\omega)\cdot\mathbf{e}_\nu^n(\mathbf{x})\,d\mathbf{x} \quad . \tag{87}$$

The set of Eqs. (84) to (87) constitutes the finite dimension approximation of the reduced model introduced in Definition 2 and adapted to frequency band B. For all ω in $B\cup\widetilde{B}$, matrix $[\mathcal{A}_N^n(\omega)]$ is invertible,

$$[\mathcal{T}_N^n(\omega)] = [\mathcal{A}_N^n(\omega)]^{-1} \quad , \quad [\mathcal{T}_N^n(\omega)]^T = [\mathcal{T}_N^n(\omega)] \quad , \tag{88}$$

and the solution of Eq. (85) is written as

$$\boldsymbol{\Theta}^n(\omega) = \eta(\omega)\,[\mathcal{T}_N^n(\omega)]\,\boldsymbol{\mathcal{F}}^n(\omega) \quad . \tag{89}$$

B- Expression of the matrices of the reduced model

Let $[P_N^n]$ be the $(n\times N)$real matrix whose columns consist of the N eigenvectors $\{\mathbf{P}^1,\ldots,\mathbf{P}^N\}$ corresponding to the N highest eigenvalues $\lambda_1^n \geq \ldots \geq \lambda_N^n$ of the generalized symmetric eigenvalue problem defined by Eq. (75),

$$[P_N^n]_{\alpha\,\nu} = P_\alpha^\nu \quad , \quad \alpha = 1,\ldots,n \quad , \quad \nu = 1,\ldots,N \quad . \tag{90}$$

Substituting Eq. (80) into Eqs. (86) and (87) yields

$$[\mathcal{A}_N^n(\omega)] = [P_N^n]^T\,[A_n(\omega)]\,[P_N^n] \quad , \tag{91}$$

$$\boldsymbol{\mathcal{F}}^n(\omega) = [P_N^n]^T\,\mathbf{F}(\omega) \quad , \tag{92}$$

in which matrix $[A_n(\omega)]$ and vector $\mathbf{F}$ are defined by Eqs. (59) and (57) respectively.

C- Energy of the response calculated with the reduced model and its upper bound when $\mathbf{f}_\ell$ is independent of ω

In Sections 4.4.C and 4.4.D, we assume that $\mathbf{f}_\ell(\omega) = \mathbf{f}_\ell$ is independent of ω, i.e. $\mathbf{g}_{\text{surf}}(\mathbf{x},\omega) = \mathbf{g}_{\text{surf}}(\mathbf{x})$ and $g_{\text{vol}}(\mathbf{x},\omega) = g_{\text{vol}}(\mathbf{x})$ are independent of ω. Therefore, from Eqs. (57) and (87), we deduce that $\mathbf{F}(\omega) = \mathbf{F}$ and $\boldsymbol{\mathcal{F}}^{\,n}(\omega) = \boldsymbol{\mathcal{F}}^n$ are independent of ω. Similarly to Section 3.5, energy $\varepsilon_B(\mathbf{u}_n^N)$ of $\mathbf{u}_n^N$ is given by

$$\varepsilon_B(\mathbf{u}_n^N) = \sum_{\nu=1}^N \lambda_\nu^n\,|\mathcal{F}_\nu^n|^2 \quad , \tag{93}$$

and the upper bound of the normalized energy is such that

$$\frac{\varepsilon_B(\mathbf{u}_n^N)}{\|\boldsymbol{\mathcal{F}}^n\|^2} \leq \lambda_1^n \quad . \tag{94}$$

It is clear that Eqs. (93) and (94) hold for $N = n$.

Proposition 6. *Energy $\varepsilon_B(\mathbf{u}_n^N)$ of $\mathbf{u}_n^N$, given by Eq. (93), can also be expressed as*

$$\varepsilon_B(\mathbf{u}_n^N) = ([E_{B,n}^N]\,\mathbf{F}\,,\mathbf{F})_{\mathbb{R}^n} \quad , \tag{95}$$

in which $\mathbf{F}$ defined by Eq. (57) is independent of ω and where $[E_{B,n}^{\,N}]$ is a positive symmetric $(n\times n)$ real matrix such that

$$[E_{B,n}^N] = \int_B [e_n^N(\omega)]\,d\omega \quad , \tag{96-1}$$

$$[e_n^N(\omega)] = [P_N^n]\,[\varepsilon_n^N(\omega)]\,[P_N^n]^T \quad , \tag{96-2}$$

in which, for all ω in B, $[\varepsilon_n^N(\omega)]$ is a positive-definite symmetric $(N\times N)$ real matrix such that

$$[\varepsilon_n^N(\omega)] = \frac{1}{\pi}\omega^2\,|\eta(\omega)|^2\,\Re e\,\{[\mathcal{T}_N^n(\omega)]^*\,[\mathcal{M}_N^n]\,[\mathcal{T}_N^n(\omega)]\} \quad , \tag{96-3}$$

where $[\mathcal{M}_N^n] = [P_N^n]^T\,[\,M\,]\,[P_N^n]$.

Proof. Eqs. (36) and (18-1) yield

$$\varepsilon_B(\mathbf{u}_n^N) = \frac{1}{2\pi}\int_{B\cup\widetilde{B}} \omega^2\, m(\mathbf{u}_n^N(\omega), \mathbf{u}_n^N(\omega))\, d\omega \quad , \tag{97}$$

where $\mathbf{u}_n^N(\omega)$ is given by Eq. (84). Substituting Eq. (84) with Eq. (89) into Eq. (97) yields

$$\varepsilon_B(\mathbf{u}_n^N) = ([\mathcal{E}_{B,n}^N]\,\boldsymbol{\mathcal{F}}^n\,, \boldsymbol{\mathcal{F}}^n)_{\mathbb{R}^N} \quad , \tag{98}$$

in which $[\mathcal{E}_{B,n}^N] = \int_B [\varepsilon_n^N(\omega)]\, d\omega$. Substituting Eq. (92) into the right-hand side of Eq. (98) yields Eq. (96).

Remarks. The dimension of the null vector space of $[E_{B,n}^N]$ is $n - N$ and we have

$$\ker\,[E_{B,n}^N] = \text{span}\,\{\mathbf{P}^{N+1}, \ldots, \mathbf{P}^n\} \quad . \tag{99}$$

For all ω in B, we have

$$[e_n^n(\omega)] = [e_n(\omega)] \quad , \tag{100}$$

where $[e_n(\omega)]$ is defined by Eq. (65-2) and $[e_n^n(\omega)]$ by Eq. (96-2) with $N = n$.

D- Dominant eigensubspace of the energy operator and criterion for calculating the order of the reduced model

Taking $N = n$ into Eq. (93) yields

$$\varepsilon_B(\mathbf{u}_n) = \sum_{\nu=1}^{n} \lambda_\nu^n\, |\mathcal{F}_\nu^n|^2 \quad , \tag{101}$$

where $\mathbf{u}_n(\omega)$ is the projection of $\mathbf{u}(\omega)$ on V_n. We then have $\varepsilon_B(\mathbf{u}_n) \le \lambda_1^n\, \|\boldsymbol{\mathcal{F}}^n\|^2$ and since the upper bound is effectively reached,

$$\varepsilon_{\text{max}} = \max_{\boldsymbol{\mathcal{F}}^n\in\mathbb{C}^n} \varepsilon_B(\mathbf{u}_n) = \lambda_1^n\, \|\boldsymbol{\mathcal{F}}^n\|^2 \quad . \tag{102}$$

From Eqs. (93), (101) and (102), we deduce that

$$\frac{\varepsilon_B(\mathbf{u}_n) - \varepsilon_B(\mathbf{u}_n^N)}{\varepsilon_{\text{max}}} \le \frac{\lambda_{N+1}^n}{\lambda_1^n} \quad . \tag{103}$$

Since $\{\lambda_\nu\}_\nu$ is a decreasing sequence of positive numbers as ν approaches $+\infty$, if n is sufficiently large, then there exists $N < n$ such that

$$\frac{\lambda_{N+1}^n}{\lambda_1^n} \ll 1 \quad . \tag{104}$$

Definition 3. (Dominant eigensubspace). If $N < n$ is such that Eq. (104) holds, then subspace V_n^N is called the dominant eigensubspace of operator $\mathbf{E}_{B,n}$ corresponding to the N highest eigenvalues $\lambda_1^n \ge \ldots \ge \lambda_N^n$ and N is the order of the reduced model.

5. Construction of the Dominant Eigensubspace Using the Subspace Iteration Method

The reduced model defined by Eqs. (84) to (87) requires construction of the dominant eigensubspace of $\mathbf{E}_{B,n}$, i.e. calculation of the eigenvectors $\mathbf{P}^1, \ldots, \mathbf{P}^N$ in $\mathbb{R}^n$ corresponding to the highest eigenvalues $\lambda_1^n \ge \ldots \ge \lambda_N^n$ of the generalized symmetric eigenvalue problem defined by Eq. (75). Since n is large and $N \ll n$, the subspace iteration method or the Lanczos method (Bathe and Wilson, 1976; Parlett, 1980; Golub and Van Loan, 1989; Chatelin, 1993) can *a priori* be used. The algebraic structure of matrix $[E_n]$ defined by Eq. (65) shows that the use of the subspace iteration method allows a very efficient solving method to be constructed. Below, we present this approach. Let m be the dimension of the subspace used for the iterations such that $N < m \ll n$ (in practice, $m = \min\{2N\,, N + 8\}$, see Bathe and Wilson, 1976). Then, the generalized symmetric eigenvalue problem defined by Eqs. (75), (78) and (79) is rewritten in matrix form as

$$[\,H\,]\,[\,P\,] = [\,G\,]\,[\,P\,]\,[\,\Lambda\,] \quad , \tag{105}$$

where $[\,P\,]$ is an $(n\times m)$ real matrix such that

$$[\,P\,]^T\,[\,G\,]\,[\,P\,] = [\,I\,] \quad , \quad [\,P\,]^T\,[\,H\,]\,[\,P\,] = [\,\Lambda\,] \quad , \tag{106}$$

and where $[\,\Lambda\,]$ is the $(m\times m)$ real diagonal matrix of the eigenvalues. We then have to calculate the N highest eigenvalues $\lambda_1^n = [\,\Lambda\,]_{11} \geq \ldots \geq \lambda_N^n = [\,\Lambda\,]_{NN}$ and the corresponding eigenvectors $\mathbf{P}^1, \ldots, \mathbf{P}^N$ constituted by the first N columns of matrix $[\,P\,]$. Since the usual formulation of the subspace iteration method is adapted to calculating the lowest eigenvalues, Eqs. (105) and (106) are transformed as follows

$$[\,G\,]\,[\,S\,] = [\,H\,]\,[\,S\,]\,[\,\Gamma\,] \quad , \tag{107}$$

in which $[\,S\,]$ is an $(n\times m)$ real matrix and $[\,\Gamma\,]$ a diagonal $(m\times m)$ real matrix such that

$$[\,S\,]^T\,[\,H\,]\,[\,S\,] = [\,I\,] \quad , \quad [\,S\,]^T\,[\,G\,]\,[\,S\,] = [\,\Gamma\,] \quad . \tag{108}$$

with

$$[\,\Lambda\,] = [\,\Gamma\,]^{-1} \quad , \tag{109}$$

$$[\,P\,] = [\,S\,]\,[\,\Gamma\,]^{-1/2} \quad . \tag{110}$$

We then have to calculate the N lowest eigenvalues and associated eigenvectors of the symmetric eigenvalue problem defined by Eqs. (107) and (108).

5.1. Algorithm

Using Eq. (77) and since $[\,G\,]$ is invertible, the classical algorithm of the subspace iteration vector (Bathe and Wilson, 1976) applied to Eqs. (107) to (110) can be adapted and rewritten as follows.
1- Initialization:

$$[\Lambda_0] = [\,0\,] \quad , \quad [Q_0] = [\,I\,] \quad , \quad [W_0] \quad , \quad r_0 = +\infty \quad , \tag{111}$$

in which $[\,0\,]$ is the $(m\times m)$ null matrix, $[\,I\,]$ is the $(m\times m)$ identity matrix and $[W_0]$ is an $(n\times m)$ real matrix of a selection of the starting iteration vectors (constructed, for instance, using the Lanczos method). The columns of matrix $[W_0]$ must be a set of m algebraically independent vectors in $\mathbb{R}^n$.
2- For $k = 1, 2, \ldots$, iterate from $k-1$ to k:

$$[\widetilde{S}_k] = [W_{k-1}]\,[Q_{k-1}] \quad , \tag{112}$$

in which $[\widetilde{S}_k]$ and $[W_{k-1}]$ are $(n\times m)$ real matrices and $[Q_{k-1}]$ is an $(m\times m)$ real matrix. Calculate the $(m\times m)$ real matrices $[\widetilde{H}_k]$ and $[\widetilde{G}_k]$ such that

$$[X_k] = [\,G\,]\,[\widetilde{S}_k] \quad , \tag{113}$$

$$[W_k] = [E_n]\,[X_k] \quad , \tag{114}$$

$$[\widetilde{H}_k] = [X_k]^T\,[W_k] \quad , \tag{115}$$

$$[\widetilde{G}_k] = [\widetilde{S}_k]^T\,[X_k] \quad , \tag{116}$$

in which $[X_k]$ and $[W_k]$ are $(n\times m)$ real matrices. Solve for the projected symmetric generalized eigenvalue problem of dimension $(m\times m)$,

$$[\widetilde{G}_k]\,[Q_k] = [\widetilde{H}_k]\,[Q_k]\,[\Gamma_k] \quad , \tag{117}$$

with

$$[Q_k]^T\,[\widetilde{H}_k]\,[Q_k] = [\,I\,] \quad , \quad [Q_k]^T\,[\widetilde{G}_k]\,[Q_k] = [\Gamma_k] \quad , \tag{118}$$

and such that $[\Gamma_k]_{11} \leq [\Gamma_k]_{22} \leq \ldots \leq [\Gamma_k]_{mm}$. Calculate $[\Lambda_k] = [\Gamma_k]^{-1}$. Measure the convergence by

$$\sup_{\nu=1,\ldots,N} \left| \frac{[\Lambda_k]_{\nu\nu} - [\Lambda_{k-1}]_{\nu\nu}}{[\Lambda_k]_{\nu\nu}} \right| \leq \varepsilon \quad , \tag{119-1}$$

or by the criteria

$$|r_k - r_{k-1}| \leq \varepsilon \quad , \quad r_k = \frac{\|[W_k^N] - [W_{k-1}^N]\|_2}{\|[W_k^N]\|_2} \quad , \tag{119-2}$$

where $[W_k^N]$ is the $(n\times N)$ real matrix such that $[W_k^N]_{j\nu} = [W_k]_{j\nu}$ for $j = 1, \ldots, n$ and $\nu = 1, \ldots, N$ and where $\|[W_k^N]\|_2^2 = \mathrm{tr}\{[W_k^N]^T [W_k^N]\}$.
3- When convergence is reached, $\mathbf{P}^1, \ldots, \mathbf{P}^N$ are the first N columns of the $(n\times m)$ real matrix $[\,P\,]$ which is calculated by

$$[\,P\,] = [\widetilde{S}_k]\,[Q_k]\,[\Lambda_k]^{1/2} \quad , \tag{120}$$

and $\lambda_1^n, \ldots, \lambda_N^n$ are the first N diagonal elements of the $(m\times m)$ diagonal matrix $[\Lambda_k]$.

5.2. Solving method using a direct or indirect procedure in the frequency domain

A- Direct procedure in the frequency domain

If we look at the algorithm in Section 5.1, it would seem that the calculation of matrix $[E_n]$ is necessary. In fact, Eq. (114) shows that we only need to calculate the $(n\times m)$ real matrix $[\,W\,]$ such that

$$[\,W\,] = [E_n]\,[\,X\,] \quad , \tag{121}$$

in which $[\,X\,]$ is a given $(n\times m)$ real matrix (we omit index k to simplify the notation). Let N_{freq} be the number of frequency points required for calculation of the integral in Eq. (65-1), N_{iter} be the number of iterations necessary to reach convergence (see Eq. (119)) and n_b be the "mean" half-bandwidth of symmetric matrices $[\,M\,]$, $[D(\omega)]$ and $[K(\omega)]$. As suggested above, one possibility would be to calculate $[E_n]$ using Eq. (65) and then calculate $[\,W\,]$ each iteration. Below, we define an operation as a multiplication or an addition of two real or complex numbers. If the calculation of symmetric complex matrix $[T(\omega)]$ is carried out using the Gauss elimination method and if it is assumed that $m\,N_{\text{iter}} \ll n$, then such a direct procedure in the frequency domain would lead to the following estimation n_{dir} of the number of operations

$$n_{\text{dir}} \simeq 2\,n^3\,N_{\text{freq}}\,(1+4n_b/n) \quad . \tag{122}$$

B- Indirect procedure in the frequency domain

The following procedure is more efficient. Since $[\,X\,]$ is a real matrix, it can easily be verified that

$$[\,W\,] = \int_B \Re e\left\{[\widehat{Z}(\omega)]\right\}\,d\omega \quad , \tag{123}$$

where $[\widehat{Z}(\omega)]$ is the $(n\times m)$ complex matrix which is the unique solution of the equations

$$[A_n(\omega)]\,[\widehat{Y}(\omega)] = \widehat{\theta}(\omega)\,[\,X\,] \quad , \tag{124}$$

$$[A_n(\omega)]\,[\widehat{Z}(\omega)] = [\,M\,]\,\overline{[\widehat{Y}(\omega)]} \quad , \tag{125}$$

in which $\omega \mapsto \widehat{\theta}(\omega)$ is the function from $\mathbb{R}$ into $\mathbb{R}^+$ such that, for all ω in $\mathbb{R}$,

$$\widehat{\theta}(\omega) = \frac{1}{\pi}\,\omega^2\,|\eta(\omega)|^2\,\mathbf{1}_B(\omega) \quad . \tag{126}$$

Assuming that the N_{freq} factorizations of matrices $[A_n(\omega)]$ are carried out using the Gauss elimination method outside the iteration subspace loop, this procedure leads to the following estimation n_{ind} of the number of operations

$$n_{\text{ind}} \simeq n\,n_b^2\,N_{\text{freq}}\,(1+12\,N_{\text{iter}}\,m/n_b) \quad . \tag{127}$$

Consequently, the gain with respect to the direct procedure is $n_{\text{dir}}/n_{\text{ind}} = 2(n/n_b)^2(1+4n_b/n)/(1+12N_{\text{iter}}m/n_b)$. For instance, in the context of the use of the finite element method, if $n = 10000$, $n_b = 500$, $m = 30$ and $N_{\text{iter}} = 20$, then the gain is approximatively 60.

5.3. Procedure based on the use of the MF solution method in the time domain

If, for all ω in B, the approximation $[D(\omega)] \simeq [D(\omega_B)]$ and $[K(\omega)] \simeq [K(\omega_B)]$ can be used, then a more efficient method than the method presented in Section 5.2.B can be used (if not, then the method of Section 5.2.B is required). This method is based on the use of the MF solving method (Soize, 1982b) and requires factorization of only one symmetric $(n\times n)$ complex matrix whose "mean" half-bandwidth is n_b. Consequently, the core memory necessary for this procedure is much smaller than for the indirect procedure in the frequency domain (Section 5.2.B) for which N_{freq} factorizations are simultaneously present in the memory.

A- Definition of the input signal

Let $\omega \mapsto \widehat{\theta}(\omega)$ be the function defined by Eq. (126). Since $\omega \mapsto \eta(\omega)$ is continuous on B (see Section 2.1),

$$\widehat{\theta} \in L^2(\mathbb{R}) \quad , \quad \operatorname{supp}\widehat{\theta} = B \quad , \tag{128}$$

in which $\operatorname{supp}\widehat{\theta}$ denotes the support of function $\widehat{\theta}$. Consequently, its inverse Fourier transform

$$\theta(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\,\widehat{\theta}(\omega)\,d\omega = \frac{1}{2\pi}\int_B e^{i\omega t}\,\widehat{\theta}(\omega)\,d\omega \quad , \quad t\in\mathbb{R} \quad , \tag{129}$$

belongs to $L^2(\mathbb{R})$ and can be written as

$$\theta(t) = e^{i\omega_B t}\,\theta_0(t) \quad , \quad \forall t \in \mathbb{R} \quad . \tag{130}$$

Function $t \mapsto \theta_0(t)$ belongs to $L^2(\mathbb{R})$ and has a Fourier transform

$$\widehat{\theta}_0(\omega) = \int_{\mathbb{R}} e^{-i\omega t}\,\theta_0(t)\,dt \quad , \quad \omega \in \mathbb{R} \quad , \tag{131}$$

which is such that

$$\widehat{\theta}_0 \in L^2(\mathbb{R}) \quad , \quad \text{supp}\,\widehat{\theta}_0 = B_0 \quad , \tag{132}$$

where B_0 denotes the compact interval of $\mathbb{R}$,

$$B_0 = [\,-\Delta\omega/2\,,\Delta\omega/2\,] \quad . \tag{133}$$

Function $t \mapsto \theta_0(t)$ is the LF signal associated with the MF narrow-band signal $t \mapsto \theta(t)$ (see Soize, 1982b). If for all ω in B, $|\eta(\omega)|^2 = 1$, then θ_0 is written as $\theta_0(0) = (\beta^3 - \alpha^3)/(6\pi^2)$ and for $t \neq 0$,

$$\theta_0(t) = \frac{1}{2\pi^2}\left\{\frac{(\beta^2\gamma - \alpha^2\overline{\gamma})}{it} - 2\frac{(\beta\gamma - \alpha\overline{\gamma})}{(it)^2} + 2\frac{(\gamma - \overline{\gamma})}{(it)^3}\right\} \quad , \tag{134}$$

in which $\alpha = \omega_B - \Delta\omega/2$, $\beta = \omega_B + \Delta\omega/2$ and $\gamma = \exp\{it\Delta\omega/2\}$.

B- Introduction of the approximation related to band B

Let us assume that for all ω in B, we can write $[A_n(\omega)] \simeq [A_B(\omega)]$ where $[A_B(\omega)]$ is the invertible symmetric $(n\times n)$ complex matrix

$$[A_B(\omega)] = -\omega^2\,[\,M\,] + i\omega\,[D_B] + [K_B] \quad , \tag{135}$$

in which $[D_B]$ and $[K_B]$ are the symmetric $(n\times n)$ real matrices independent of the frequency,

$$[D_B] = [D(\omega_B)] \quad , \quad [K_B] = [K(\omega_B)] \quad . \tag{136}$$

Consequently, for all ω in B, we have $[T_n(\omega)] \simeq [T_B(\omega)]$ with

$$[T_B(\omega)] = [A_B(\omega)]^{-1} \quad , \tag{137}$$

and Eqs. (63), (65-1) and (65-2) yield

$$[E_n] \simeq \int_B \Re e\left\{\widehat{\theta}(\omega)\,\overline{[T_B(\omega)]}\,[\,M\,]\,[T_B(\omega)]\right\}\,d\omega \quad . \tag{138}$$

It should be noted that the introduction of this assumption is justified because of the continuity of functions $\omega \mapsto \widehat{\theta}(\omega)$ and $\omega \mapsto [T_n(\omega)]$ on B (there exists $\Delta\omega$ sufficiently small such that this approximation can be used).

C- Expression of $[\,W\,]$ using the equations in the frequency domain

In the context of the approximation defined in Section 5.2.B, $[\,W\,]$ is given by Eq. (123) but Eqs. (124) and (125) must be replaced by

$$[A_B(\omega)]\,[\widehat{Y}(\omega)] = \widehat{\theta}(\omega)\,[\,X\,] \quad , \tag{139}$$

$$[A_B(\omega)]\,[\widehat{Z}(\omega)] = [\,M\,]\,\overline{[\widehat{Y}(\omega)]} \quad . \tag{140}$$

From Eq. (128) and since $\omega \mapsto [T_B(\omega)]$ is continuous on B, we deduce that $\omega \mapsto [\widehat{Y}(\omega)]$ and $\omega \mapsto [\widehat{Z}(\omega)]$ are square integrable functions from $\mathbb{R}$ into $Mat_{\mathbb{C}}(n,m)$ (set of all the $(n\times m)$ complex matrices) and that their supports are such that

$$\text{supp}\,[\,\widehat{Y}\,] = \text{supp}\,[\,\widehat{Z}\,] = B \quad . \tag{141}$$

D- MF equation in the time domain

Let $t \mapsto [Y(t)]$ and $t \mapsto [Z(t)]$ be the square integrable functions from $\mathbb{R}$ into $Mat_{\mathbb{C}}(n,m)$ such that (inverse Fourier transforms)

$$[Y(t)] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\,[\widehat{Y}(\omega)]\,d\omega \quad , \quad [Z(t)] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\,[\widehat{Z}(\omega)]\,d\omega \quad . \tag{142}$$

From Eqs. (129), (135), (139) and (140), we deduce that $[Y(t)]$ and $[Z(t)]$ verify the MF equations in the time domain

$$[\,M\,]\,[\ddot{Y}(t)] + [D_B]\,[\dot{Y}(t)] + [K_B]\,[Y(t)] = \theta(t)\,[\,X\,] \quad , \tag{143}$$

$$[\,M\,]\,[\ddot{Z}(t)] + [D_B]\,[\dot{Z}(t)] + [K_B]\,[Z(t)] = [\,M\,]\,\overline{[Y(-t)]} \quad . \tag{144}$$

E- Associated LF equation in the time domain

Let $[Y_0]$ and $[Z_0]$ be the LF signals associated with the MF signals $[\,Y\,]$ and $[\,Z\,]$ respectively, such that

$$[Y(t)] = e^{i\omega_B t}\,[Y_0(t)] \quad , \quad [Z(t)] = e^{i\omega_B t}\,[Z_0(t)] \quad , \quad \forall t \in \mathbb{R} \quad . \tag{145}$$

Therefore, $t \mapsto [Y_0(t)]$ and $t \mapsto [Z_0(t)]$ are square integrable functions from $\mathbb{R}$ into $Mat_{\mathbb{C}}(n,m)$ and their Fourier transforms

$$[\widehat{Y}_0(\omega)] = \int_{\mathbb{R}} e^{-i\omega t}\,[Y_0(t)]\,dt \quad , \quad [\widehat{Z}_0(\omega)] = \int_{\mathbb{R}} e^{-i\omega t}\,[Z_0(t)]\,dt \quad , \tag{146}$$

are square integrable functions from $\mathbb{R}$ into $Mat_{\mathbb{C}}(n,m)$ such that

$$\text{supp}\,[\,\widehat{Y}_0\,] = \text{supp}\,[\,\widehat{Z}_0\,] = B_0 \quad , \tag{147}$$

in which B_0 is defined by Eq. (133). Substituting Eq. (145) into Eqs. (143) and (144) and using Eq. (130) yields the LF equations in the time domain associated with the MF equations (143) and (144),

$$[\,M\,]\,[\ddot{Y}_0(t)] + [\widetilde{D}_B]\,[\dot{Y}_0(t)] + [\widetilde{K}_B]\,[Y_0(t)] = \theta_0(t)\,[\,X\,] \quad , \tag{148}$$

$$[\,M\,]\,[\ddot{Z}_0(t)] + [\widetilde{D}_B]\,[\dot{Z}_0(t)] + [\widetilde{K}_B]\,[Z_0(t)] = [\,M\,]\,\overline{[Y_0(-t)]} \quad , \tag{149}$$

in which the symmetric $(n\times n)$ complex matrices $[\widetilde{D}_B]$ and $[\widetilde{K}_B]$ are written as

$$[\widetilde{D}_B] = [D_B] + 2\,i\,\omega_B\,[\,M\,] \quad , \tag{150}$$

$$[\widetilde{K}_B] = -\omega_B^2\,[\,M\,] + i\,\omega_B\,[D_B] + [K_B] \quad , \tag{151}$$

F- Expression of $[\,W\,]$ using the solution of the associated LF equation in the time domain

Let $t \mapsto [Y_0(t)]$ and $t \mapsto [Z_0(t)]$ be the functions verifying Eqs. (148) and (149). From Eqs. (123) and (142), we deduce that $[\,W\,] = 2\pi\,\Re e\,\{[Z(0)]\}$. Using Eq. (145) for $t = 0$ yields

$$[\,W\,] = 2\pi\,\Re e\,\{[Z_0(0)]\} \quad . \tag{152}$$

G- Solving procedure

The associated LF equations (148) and (149) are solved using an unconditionally stable implicit step-by-step integration method such as the Newmark method or the Wilson θ method (see for instance, Bathe and Wilson, 1976). Since Eqs. (148) and (149) have the same differential operator $[\,M\,]\,d^2/dt^2 + [\widetilde{D}_B]\,d/dt + [\widetilde{K}_B]$, only one symmetric $(n\times n)$ complex matrix has to be factorized. The basic sampling time step denoted τ is given by Shannon's theorem (see for instance Soize, 1993b). From Eqs. (133) and (147), we deduce that $\tau = 2\pi/\Delta\omega$. The integration time step of the step-by-step integration method is then written as $\Delta t = \tau/\mu$ where $\mu > 1$ is an integer. Since θ_0 and $[Y_0]$ are square integrable functions, for any $\varepsilon > 0$, there exists an initial time $t_I = -I_0 \times \tau$ where $I_0 > 1$ is an integer and a final time $t_F = J_0 \times \tau$ where $J_0 > 1$ is another integer, such that

$$\int_{-\infty}^{t_I} |\theta_0(t)|^2\,dt \leq \varepsilon \|\theta_0\|^2_{L^2} \quad , \quad \int_{t_F}^{+\infty} \|[Y_0(t)]\|^2\,dt \leq \varepsilon \int_{t_I}^{t_F} \|[Y_0(t)]\|^2\,dt \quad . \tag{153}$$

We have the following procedure.
Step 1. Construction of the sequence $[Y_0(j\times\Delta t)]$ for $j = -I,\ldots,J$ with $I = I_0\times\mu$ and $J = J_0\times\mu$ by solving Eq. (148) for $t \in]\,t_I\,,\,t_F]$ with the initial conditions $[Y_0(t_I)] = [\dot{Y}_0(t_I)] = [\,0\,]$.
Step 2. Construction of the sequence $[Z_0(k\times\Delta t)]$ for $k = -J,\ldots,0$ by solving Eq. (149) for $t \in]-t_F\,,\,0\,]$ with the initial conditions $[Z_0(-t_F)] = [\dot{Z}_0(-t_F)] = [\,0\,]$.
For details on the MF solving method, such as the usual values of μ, I, J, we refer the reader to Soize, 1982b and Soize *et al.* 1986 and 1992.

H- Number of operations

We use the notations introduced in Section 5.2. Based on the use of the Newmark method, this procedure leads to the following estimation of the number of operations,

$$n_{\text{MF}} \simeq n\, n_b^2 + 4\, n\, n_b\, m\, N_{\text{iter}}\, (3I + 7J) \quad . \tag{154}$$

Consequently, the gain with respect to the indirect procedure in the frequency domain is $n_{\text{ind}}/n_{\text{MF}} = N_{\text{freq}}(1 + 12N_{\text{iter}}m/n_b)/(1 + 4(3I + 7J)N_{\text{iter}}m/n_b))$. For instance, in the context of the use of the finite element method, if $n = 10000$, $n_b = 500$, $m = 30$ and $N_{\text{iter}} = 20$, then the gain is approximatively $N_{\text{freq}}/(I + 2.2J)$.

6. Example

The example concerns an inhomogeneous continuous dynamical system which has a nondiagonal generalized damping matrix with respect to the normal eigenmodes of the associated conservative dynamical system.

6.1. Description of the dynamical system

We consider the MF response of the dynamical system consisting of a rectangular thin plate in bending mode coupled with springs and dashpots as shown in Fig. 1, on the narrow MF band $B = 2\pi\times$[550,600] rad/s defined by Eq. (1), i.e. on the [550,600] Hz frequency interval. We then have (see Eq. (2)) $\Delta\omega/\omega_B = 0.087$ and, for all ω in B, function η is such that $|\eta(\omega)| = 1$. The plate is homogeneous and isotropic, simply supported, rectangular, with a constant thickness, width $L_1 = 0.5$ m, length $L_2 = 1.0$ m, surface-mass density $\rho_s = 40$ kg/m^2, constant damping rate $\xi = 0.001$. We assume that the usual thin plate theory can be used. The lowest eigenfrequency of the associated conservative uncoupled plate is 5 Hz. The plate is coupled with 3 springs having the same stiffness coefficient $k = \varepsilon_k\, \mu\, \omega_B^2$ with $\varepsilon_k = 0.1$, $\mu = \rho_s L_1 L_2 = 20$ kg (total mass of the plate) and with 5 dashpots having the same damping coefficient $d = 2\varepsilon_d\, \mu\, \xi\, \omega_B$ with $\varepsilon_d = 0.1$. The family $\psi_1, \ldots, \psi_n$ of functions introduced in Section 4 is chosen as the first n eigenmodes of the associated conservative uncoupled plate whose corresponding eigenfrequencies are $f_1 < f_2 < \ldots < f_n$. The value retained for n is 407 and has been deduced from the convergence study of the dynamical system response over the [0,700] Hz frequency band. The values of $f_1, \ldots, f_n$ are such that $f_1 = 5$ Hz, $f_{201} = 549$ Hz, $f_{202} = 557$ Hz, $f_{219} = 596$ Hz, $f_{220} = 601$ Hz and $f_{407} = 1097$ Hz. Consequently, there are 201 plate eigenmodes whose eigenfrequencies are below 550 Hz, 18 plate eigenmodes whose eigenfrequencies lie inside the narrow MF band [550,600] Hz, and 188 plate eigenmodes whose eigenfrequencies are between 600 Hz and 1100 Hz. The normalization of the plate eigenmodes is such that, for all α, $\int_0^{L_1}\int_0^{L_2} |\psi_\alpha(x_1, x_2)|^2\, \rho_s\, dx_1\, dx_2 = \mu$. Concerning the finite dimension approximation of the dynamical system operators, matrices $[\,M\,]$ and $[\,G\,]$ defined by Eqs. (60) and (76) are diagonal, and matrices $[\,D\,]$ and $[\,K\,]$ defined by Eqs. (61) and (62) are dense and independent of the frequency. Let $f_1^{\text{DS}} < f_2^{\text{DS}} < \ldots$ be the eigenfrequencies of the associated conservative dynamical system (estimated using the finite dimension approximation introduced in Section 4). These eigenfrequencies are such that $f_1^{\text{DS}} = 10.25$ Hz, $f_{198}^{\text{DS}} = 548.99$ Hz, $f_{199}^{\text{DS}} = 552.99$ Hz, $f_{217}^{\text{DS}} = 598.45$ Hz and $f_{218}^{\text{DS}} = 604.59$ Hz. It can then be seen that the spectrum of the associated conservative dynamical system is such that there are 198 eigenmodes whose eigenfrequencies are below 550 Hz and 19 eigenmodes whose eigenfrequencies lie inside the narrow MF band [550,600] Hz.

6.2. Constructing the reference solution on a broad frequency band

The reference solution is obtained by constructing the mapping $\omega \mapsto \text{tr}[e_n(\omega)]$ using Eq. (65-2), with a sampling frequency step $\delta\omega = 2\pi\times 0.46$. In Eq. (65-2), $[T_n(\omega)]$ is calculated using Eq. (63) in which matrix $[A_n(\omega)]$ is calculated by Eq. (59) with matrices $[\,M\,]$, $[\,D\,]$ and $[\,K\,]$ given by Eqs. (60), (61) and (62). Fig. 2 shows the mapping $\omega \mapsto 10\times\log_{10}(\text{tr}[e_n(\omega)])$ from the broad frequency band $2\pi\times$[0,700] rad/s into $\mathbb{R}^+$ (it should be noted that this mapping does not depend on the spatial excitation).

6.3. Constructing the reference solution on the narrow MF band

Fig. 3 (which is a close-up of Fig. 2) shows the mapping $\omega \mapsto 10\times\log_{10}(\text{tr}[e_n(\omega)])$ on narrow MF band B. This function is constructed with a frequency resolution $\delta\omega = \Delta\omega/N_{\text{freq}}$ in which $N_{\text{freq}} = 300$. This graph is used below as reference solution to evaluate the accuracy of the response constructed using the reduced model. For this reference solution related to frequency band B, matrix $[E_n]$ defined by Eq. (65-1) is calculated using the approximation $[E_n] \simeq \delta\omega \sum_{j=1}^{N_{\text{freq}}} [e_n(\omega_j)]$ in which ω_j's are the sampling frequencies of band B. For the reference solution, the eigenvalues $\lambda_1^n \geq \lambda_2^n \geq \ldots \geq \lambda_n^n > 0$ and the corresponding eigenfuctions $\{\mathbf{e}_1^n, \ldots \mathbf{e}_n^n\}$ of energy operator $\mathbf{E}_{B,n}$ (see Eq. (70)) are calculated using Proposition 5 (Eqs. (75) to (81)). Fig. 4-a shows the graph of the function $\nu \mapsto \lambda_\nu^n$ for $\nu \in \{1, 2, \ldots, 40\}$ and Fig. 4-b shows the graph of the function $\nu \mapsto 10\times\log_{10}(\lambda_\nu^n)$ for $\nu \in \{1, 2, \ldots, 400\}$. There is a strong decrease of the eigenvalues in the interval $[18, 23]$ which means there exists a possibility of constructing an efficient reduced model independent of the spatial excitation of the dynamical system.

6.4. Reduced model adapted to the narrow MF band

In this section, we present a comparison of the reference solution constructed in Section 6.C (see Fig. 3) with the solution obtained by the reduced model constructed using the results of Section 4.4 in which eigenfunctions $\{\mathbf{e}_1^n, \ldots, \mathbf{e}_N^n\}$ are those calculated in Section 6.C. For the three values 17, 20 and 30 of the reduced model of order N, Fig. 5 shows the comparison of function $\omega \mapsto 10 \times \log_{10}(\text{tr}[e_n(\omega)])$ (reference solution) with function $\omega \mapsto 10 \times \log_{10}(\text{tr}[e_n^N(\omega)])$ (reduced model) calculated for each value of order N using Proposition 6.

6.5. Construction of the dominant eigensubspace using the subspace iteration method

For construction of the reference solution (Section 6.C), all the eigenvalues and the corresponding eigenfunctions of matrix $[E_n]$ were calculated (in practice, calculation of all the eigenvalues is never done and cannot be carried out for large finite element models). In this section, we consider the construction of the dominant eigensubspace using the subspace iteration method (this is the method proposed for large models).

A- Solution method using the direct procedure in the frequency domain

For the reduced model of order $N = 20$, the subspace iteration algorithm defined by Eqs. (111) to (120) is used with $m = 28$ and is initialized with an $(n \times m)$ real matrix of $+1$s and -1s whose columns are orthogonal (Hadamard matrix). Each iteration, matrix $[W_k]$ defined by Eq. (114) is calculated using the direct procedure in the frequency domain described in Section 5.2.A. It should be noted that, for large models, this direct procedure is tricky and must be replaced by the indirect procedure described in Section 5.2.B as was explained (for the present validation of the subspace iteration algorithm, this point is not crucial, and in addition, validation of the indirect procedure is straightforward). Convergence of the subspace iteration method is measured using Eq. (119-1) with $\varepsilon = 10^{-4}$. Convergence is obtained for $N_{\text{iter}} = 4$ iterations. Fig. 6 shows the comparison of function $\omega \mapsto 10 \times \log_{10}(\text{tr}[e_n(\omega)])$ (reference solution) with function $\omega \mapsto 10 \times \log_{10}(\text{tr}[e_n^N(\omega)])$ (reduced model) for $N = 20$ using Proposition 6. This result validates the procedure.

B- Procedure using the MF solution method in the time domain

For the reduced model of order $N = 20$, the subspace iteration algorithm defined by Eqs. (111) to (120) is used with $m = 28$ and is initialized as in Section 6.5.A. Each iteration, matrix $[W_k]$ defined by Eq. (114) is calculated by using the the MF solution method in the time domain described in Section 5.3. The Newmark step-by-step integration method is used (Bathe and Wilson, 1976, with scheme parameters $\alpha = 1/4$ and $\delta = 1/2$). Function $\theta_0(t)$ is generated using Eq. (134). The parameters introduced in Section 5.3.G for the time-solution procedure are $\mu = 6$, $I_0 = 4$ and $J_0 = 26$, i.e. $I = 24$ and $J = 156$. Convergence of the subspace iteration method is measured using Eq. (119-2) with $\varepsilon = 0.5$. Convergence is obtained for $N_{\text{iter}} = 4$ iterations. Fig. 7 shows the graph of function $j \mapsto \text{tr}\{[Y_0(j \times \Delta t)]^* \, [Y_0(j \times \Delta t)]\}$ for $j = -I, \ldots, J$ and Fig. 8 shows the graph of function $k \mapsto \text{tr}\{[Z_0(k \times \Delta t)]^* \, [Z_0(k \times \Delta t)]\}$ for $k = -J, \ldots, 0$, corresponding to the last iteration of the subspace iteration algorithm (see Step 1 and Step 2 described in Section 5.3.G). These two figures show that the values of parameters I_0 and J_0 are correctly chosen (a similar result is obtained for each iteration, and not only for the last one). Fig. 9 shows the comparison of function $\omega \mapsto 10 \times \log_{10}(\text{tr}[e_n(\omega)])$ (reference solution) with function $\omega \mapsto 10 \times \log_{10}(\text{tr}[e_n^N(\omega)])$ (reduced model) for $N = 20$ using Proposition 6. This result validates the procedure.

7. Conclusion

A theoretical approach is presented for the construction of a reduced model in the medium frequency range in the area of structural dynamics for a general three-dimensional anisotropic and inhomogeneous viscoelastic bounded medium with an arbitrary geometry. The boundary value problem in the frequency domain and its variational formulation are presented. For a given medium frequency band, the energy operator which is intrinsic to the dynamical system is a positive-definite symmetric trace operator in a Hilbert space which depends on the conservative and dissipative parts of the dynamical system. The eigenfunctions corresponding to the highest eigenvalues (dominant eigensubspace) of the energy operator allow a reduced model to be constructed using the Ritz-Galerkin method. A finite dimension approximation of the continuous case is introduced in a general context (for instance using the finite element method). An effective construction of the dominant subspace of the energy operator is proposed using the subspace iteration method with the introduction of two procedures, one based on the use of an indirect procedure in the frequency domain and the other on the use of the MF solution method in the time domain. We then obtain an efficient method for constructing a reduced model in the MF range. In addition, it can easily be seen that all the results presented can be extended straightforwardly to beams, plates and shells. We present a simple example to validate the concepts and the algorithms.

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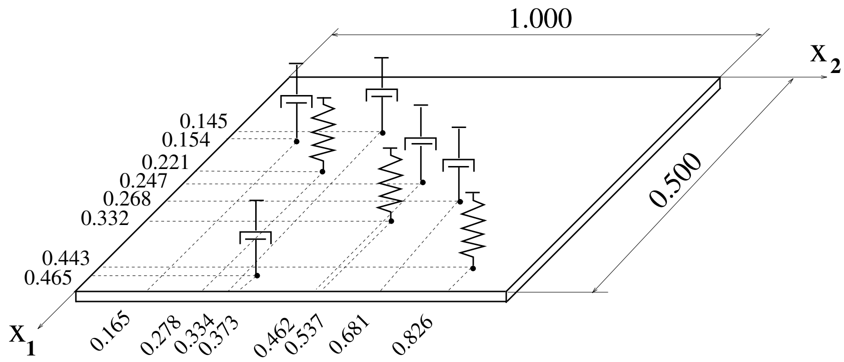

Fig. 1. Geometrical configuration of the dynamical system constituted by a homogeneous isotropic rectangular simply supported thin plate coupled with springs and dashpots.

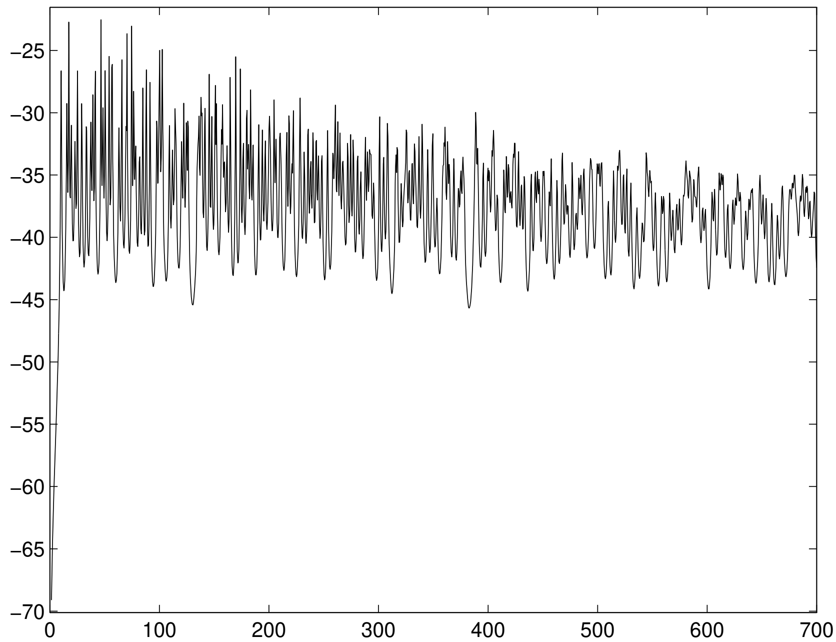

Fig. 2. Reference solution: graph of function $\omega \mapsto 10 \times \log_{10}(\mathrm{tr}[e_n(\omega)])$ over band $B = [0, 700]$ expressed in Hz.

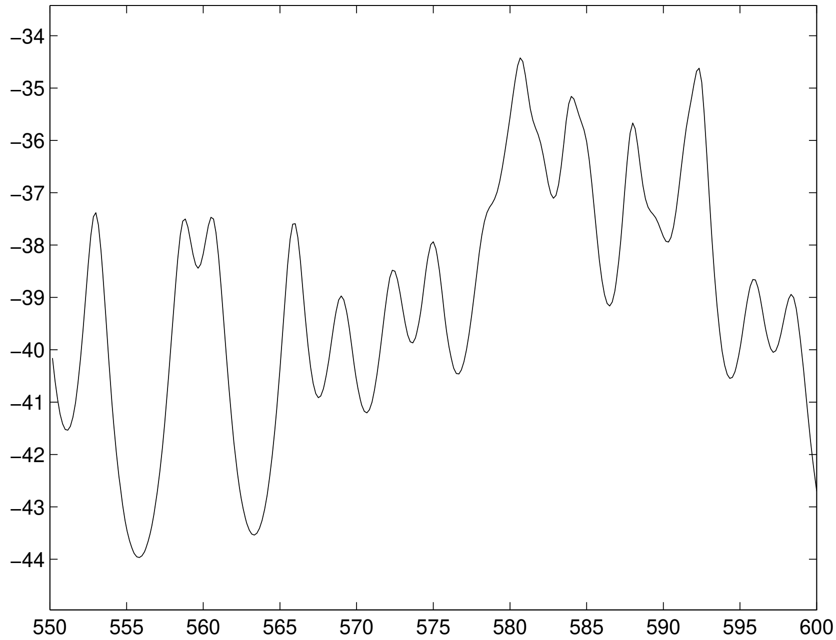

Fig. 3. Reference solution: graph of function $\omega \mapsto 10 \times \log_{10}(\mathrm{tr}[e_n(\omega)])$ over band $B = [550, 600]$ expressed in Hz.

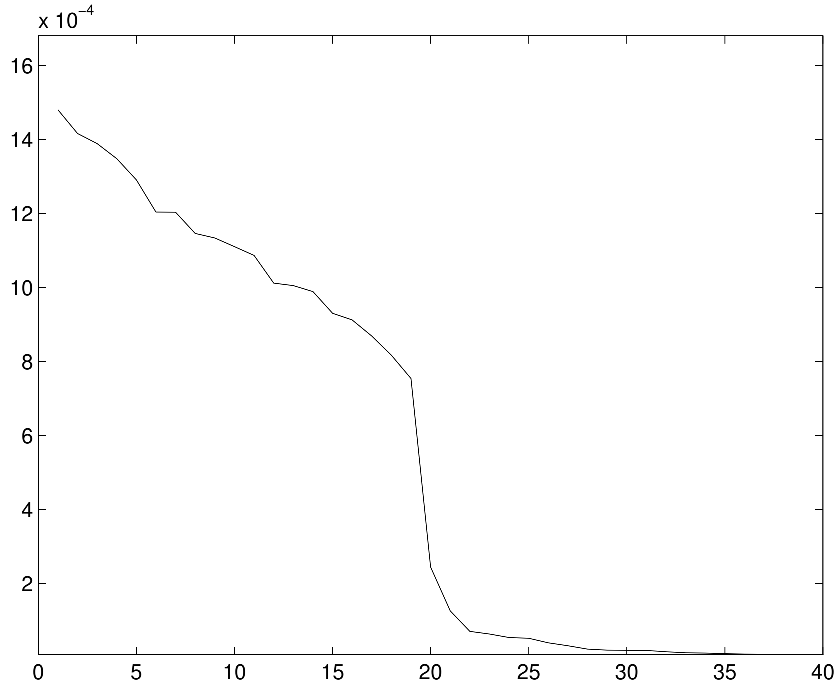

Fig. 4-a. Reference solution: graph of function $\nu \mapsto \lambda_\nu^n$ for $\nu = 1, \ldots, 40$, showing the distribution of eigenvalues λ_ν^n of energy operator $\mathbf{E}_{B,n}$ (see Eq. (70)).

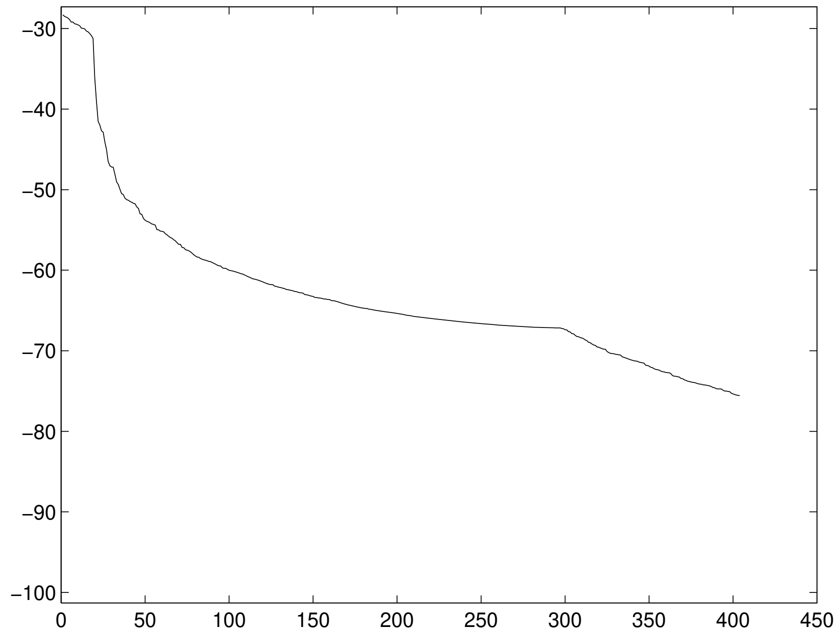

Fig. 4-b. Reference solution: graph of function $\nu \mapsto 10 \times \log_{10}(\lambda_\nu^n)$ for $\nu = 1, \ldots, 400$, showing the distribution of eigenvalues λ_ν^n of energy operator $\mathbf{E}_{B,n}$ (see Eq. (70)).

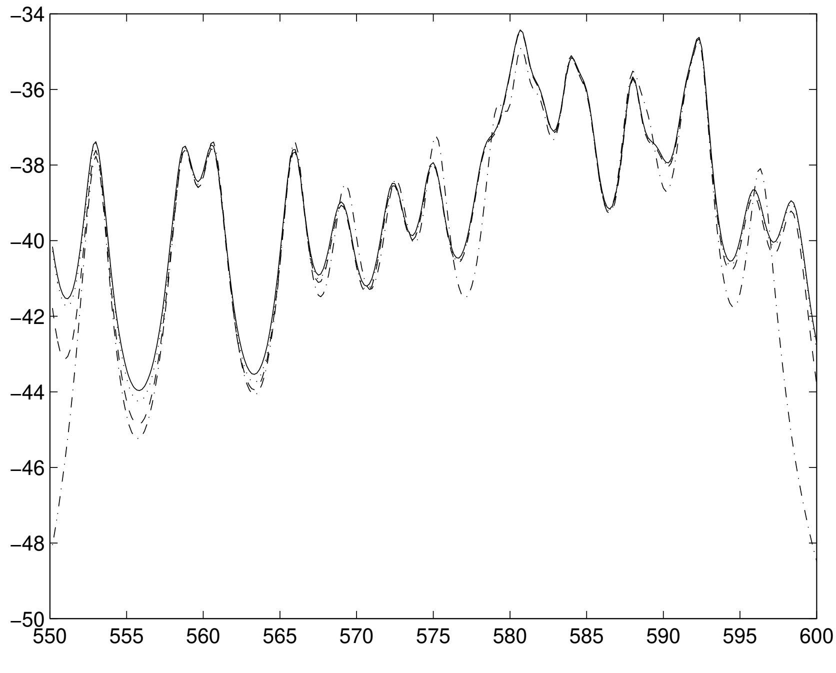

Fig. 5. Reduced model: comparison between function $\omega \mapsto 10 \times \log_{10}(\text{tr}[e_n(\omega)])$ (reference solution (solid line)) and functions $\omega \mapsto 10 \times \log_{10}(\text{tr}[e_n^N(\omega)])$ (reduced model for $N = 17$ (dashdot line), for $N = 20$ (dashed line) and for $N = 30$ (dotted line)), over band $B = [550, 600]$ expressed in Hz.

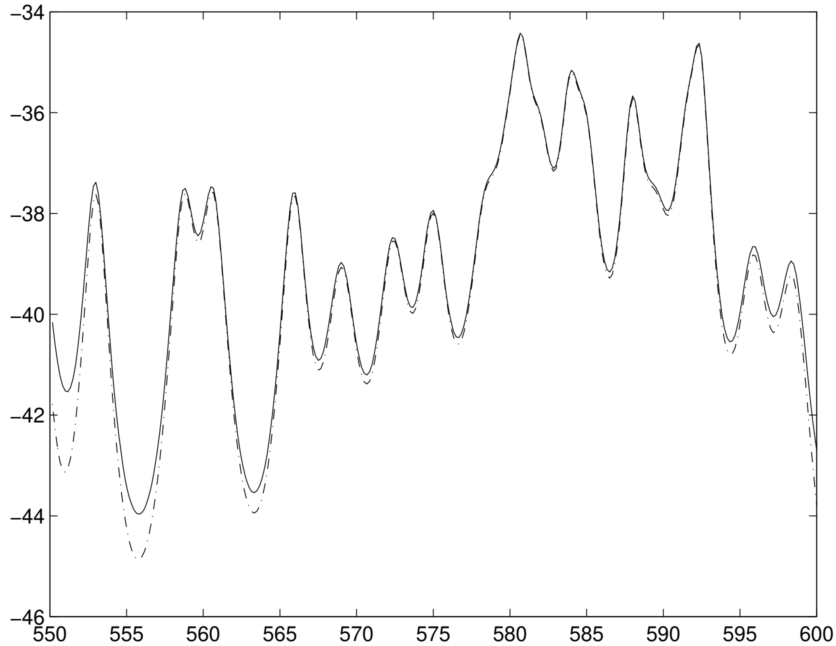

Fig. 6. Reduced model constructed using the subspace iteration method and the direct procedure in the frequency domain: comparison between function $\omega \mapsto 10\times\log_{10}(\text{tr}[e_n(\omega)])$ (reference solution (solid line)) and function $\omega \mapsto 10\times\log_{10}(\text{tr}[e_n^N(\omega)])$ (reduced model for $N = 20$ (dashdot line)), over band $B = [550, 600]$ expressed in Hz.

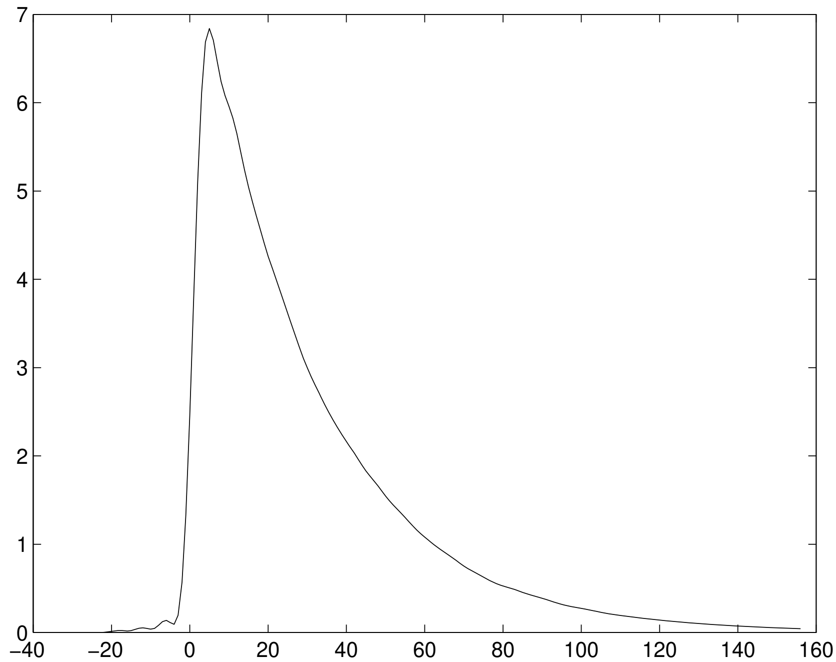

Fig. 7. Construction of $[W_k]$ in the subspace iteration method using the MF solution method in the time domain: graph of function $j \mapsto \text{tr}\{[Y_0(j\times\Delta t)]^*\,[Y_0(j\times\Delta t)]\}$ for $j = -I, \ldots, J$ (see Step 1 of Section 5.3.G).

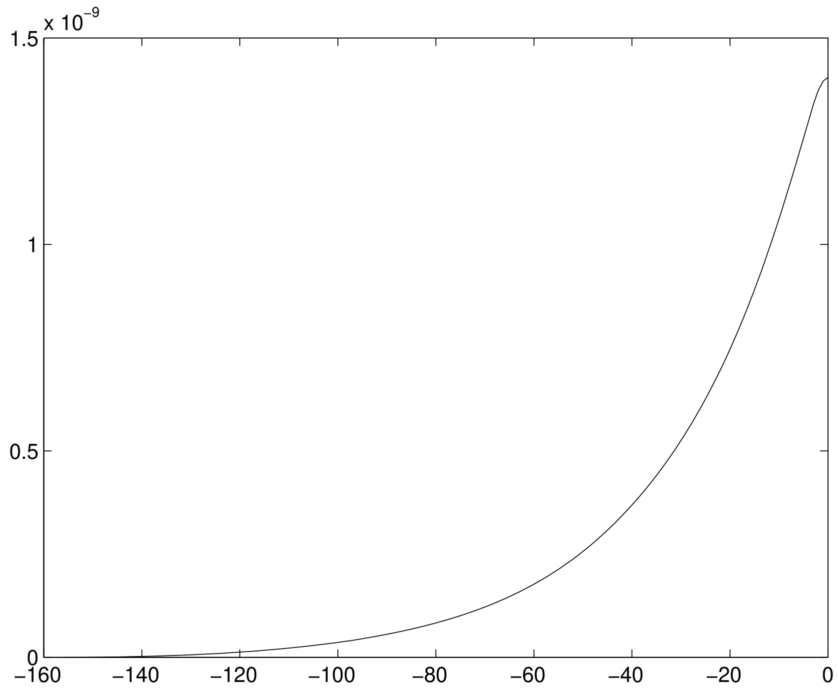

Fig. 8. Construction of $[W_k]$ in the subspace iteration method using the MF solution method in the time domain: graph of function $k \mapsto$ tr$\{[Z_0(k \times \Delta t)]^* \, [Z_0(k \times \Delta t)]\}$ for $k = -J, \ldots, 0$ (see Step 2 of Section 5.3.G).

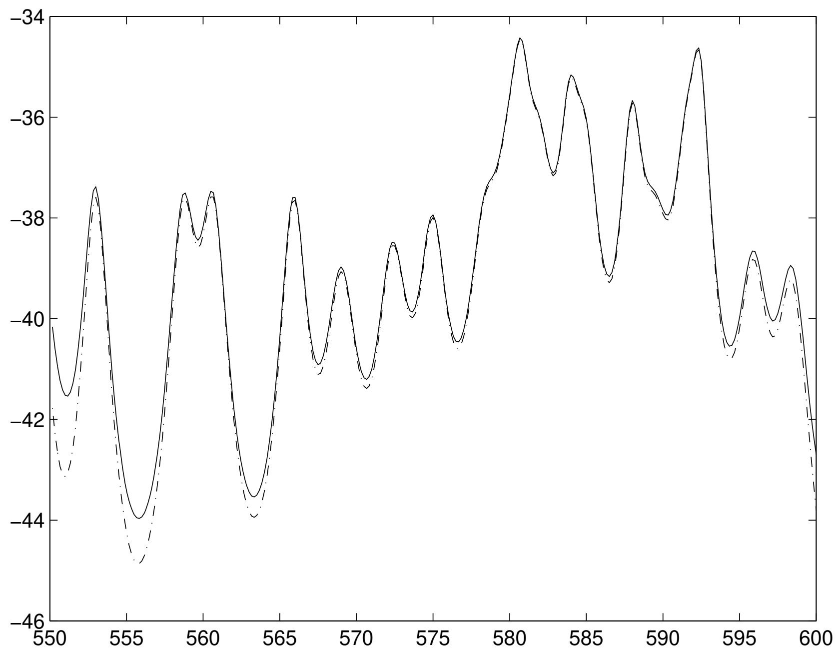

Fig. 9. Reduced model constructed using the subspace iteration method and the MF solution method in the time domain: comparison between function $\omega \mapsto 10 \times \log_{10}(\mathrm{tr}[e_n(\omega)])$ (reference solution (solid line)) and function $\omega \mapsto 10 \times \log_{10}(\mathrm{tr}[e_n^N(\omega)])$ (reduced model for $N = 20$ (dashdot line), over band $B = [550, 600]$ expressed in Hz.