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Reduced order linear anti-windup augmentation for stable linear systems

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In this paper we provide algorithms based on Linear Matrix Inequalities for the design of full-authority and external linear anti-windup compensators of reduced order guaranteeing finite global $\mathcal{L}_2$ gain. Previous results showed that the reduced order anti-windup design problem is non-convex. We propose here a convex approximation of the non-convex constraints leading to constructive design algorithms. The proposed algorithms are successfully tested on two simulation examples taken from the literature.

Keywords: Anti-windup; Linear Matrix Inequalities (LMI); Reduced order design; Convex relation

1. Introduction

1.1. *Saturation and anti-windup*

Anti-windup constructions arose more than fifty years ago, when analog control circuits were often leading to undesirable behaviour upon reach of the maximum allowable output voltages and consequent controller saturation. In the analog controllers setting, this phenomenon was typically caused by overly large charges accumulating in capacitors, or other energy storing elements of the analog control devices. The term "windup," and the consequent "anti-windup" terminology for solutions to these problems, arose between the 1950s and the 1960s (see, e.g., Lozier (1956), Fertik and Ross (1967)), during the migration of control systems from simple analog implementations (such as PID controllers) to more complicated digital ones. Already in those early years, "anti-windup designs" were characterized as augmentations to a prespecified linear controller (which induces a highly desirable closed-loop behaviour when interconnected to the plant without saturation) with the goal of guaranteeing two properties from the arising augmented nonlinear closed-loop system: (1) as long as the actuators do not saturate, the response coincides with the linear, unconstrained response; (2) if the actuators saturate, stability is preserved and performance is recovered as much as possible (within the limits imposed by the saturation constraint).

At the early stages of anti-windup research, roughly corresponding to the 1970–1990 period, most of the available tools were generalizations of ad-hoc techniques developed by industrialists when facing the input saturation problem. Nevertheless, already in those years, the anti-windup problem started to be addressed in a more systematic way and solutions that were applicable to classes of control systems became available (see, e.g., Hanus (1988), Åström and Rundqwist (1989) for surveys of these approaches). More recently, the anti-windup research strand has been recognized as a fully nonlinear theoretically challenging problem, where the bounded stabilization goal has to be accomplished while guaranteeing a local preservation property imposing additional constraints on the type of control task to be performed. Indeed, in light of the great advances in nonlinear control design characterizing the last two decades, several constructive anti-windup techniques that apply to large classes of

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systems have been proposed and shown to be successful in several control applications.

While the first anti-windup schemes mainly consisted of static gains feeding back to the controller a weighted version of the mismatch between controller output and plant input (so that when saturation did not occur, no modification was enforced), more effective and systematic generalizations of those approaches involve the design of linear dynamic anti-windup, and up to fully nonlinear anti-windup compensation schemes, where even the plant under consideration may be nonlinear (before saturation). In particular, according to figure 1 and 2, "linear anti-windup augmentation" denotes cases where the filter $\mathcal{AW}$ driven by the excess of saturation is linear. Moreover, "linear static" anti-windup denotes the case where $\mathcal{AW}$ is simply a linear gain, while "linear dynamic" denotes the case where $\mathcal{AW}$ is a linear dynamic filter (namely, it has an internal state). Nonlinear static and dynamic anti-windup denotes generalizations of these schemes where the filter $\mathcal{AW}$ is nonlinear. Linear anti-windup with formal stability (and, sometimes, performance) guarantees was recently addressed, e.g., in Kothare *et al.* (1994), Zheng *et al.* (1994), Park and Choi (1995), Edwards and Postlethwaite (1999), Mulder *et al.* (2000), Zaccarian and Teel (2003), Cao *et al.* (2002a, b) Grimm *et al.* (2003a, b, 2004a, b), Wu and Lu (2004) whereas nonlinear anti-windup was addressed, e.g., in Teel and Kapoor (1997), Teel (1999), Zaccarian and Teel (2004), Bemporad *et al.* (2004), Galeani *et al.* (2004). Many anti-windup approaches were also proposed in the context of the so-called reference governor framework (see, e.g., Kapasouris *et al.* (1988), Gilbert *et al.* (1995), Bemporad *et al.* (1997), Angeli and Mosca (1999), Gilbert and Kolmanovsky (1999), Shamma (2000). Finally, some schemes for nonlinear anti-windup for nonlinear plants were also proposed recently in, e.g., Kendi and Doyle III (1997), Bemporad (1998), Kapoor and Daoutidis (1999), Angeli and Mosca (1999), Hu and Rangaiah (2000), Morabito *et al.* (2004).

1.2. *Full-authority and external linear anti-windup*

When restricting the attention to linear anti-windup schemes, it is useful to characterize the degrees of freedom available to the anti-windup compensator for injecting modifications in the pre-designed controller dynamic equations. In particular, in the so-called "full-authority" case represented in figure 1, the anti-windup compensator $\mathcal{AW}$ can inject signals at each state equation (through the signal v_1) and at the output equation (through the signal v_2) of the pre-designed controller $\mathcal{C}$. When considering exponentially

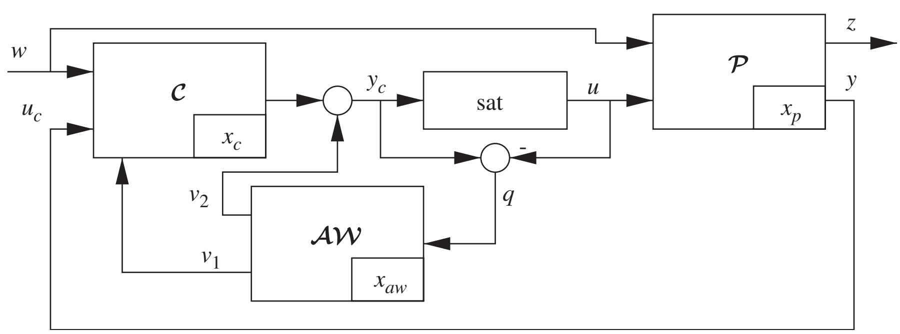

Figure 1. The full-authority linear anti-windup augmentation scheme.

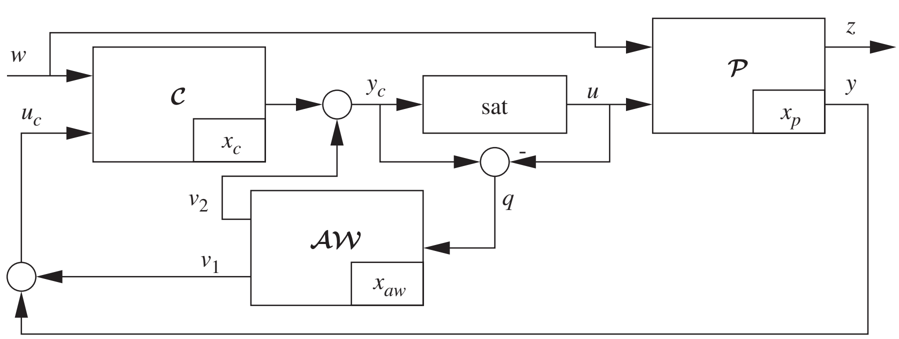

Figure 2. The external linear anti-windup augmentation scheme.

stable plants (so that global finite $\mathcal{L}_2$ gain of the closed-loop could be achieved) the linear full-authority anti-windup augmentation problem has been thoroughly addressed and characterized in Grimm *et al*. (2003a).

On the other hand, the so-called "external" case corresponds to the situation where only the input and output signals of the pre-designed controller are accessible for anti-windup modifications and the full authority over all the controller states is not allowed anymore to the anti-windup signals. This situation is depicted in figure 2, where v_1 is now only allowed to modify the controller input. When considering exponentially stable plants (so that global finite $\mathcal{L}_2$ gain of the closed-loop could be achieved) the linear external anti-windup augmentation problem has been thoroughly addressed and characterized in Grimm *et al*. (2004a).

According to the results reported in Grimm *et al*. (2003a, 2004a), absolute stability tools may be employed to design linear anti-windup compensators with stability and performance guarantees. In particular, based on the sector and incremental sector properties of the saturation function, quadratic stability of the compensated closed-loop system can be guaranteed by writing suitable matrix conditions involving the matrices characterizing the state-space representation of the plant $\mathcal{P}$, of the controller $\mathcal{C}$ and of the anti-windup compensator $\mathcal{AW}$. In Grimm *et al*. (2003a, 2004a) it is shown that for a given anti-windup compensator (namely if all the systems in figure 1 and 2 are known), the closed-loop quadratic stability property and an upper bound on the $\mathcal{L}_2$ gain from w to z can be determined by solving a convex optimization problem expressed in terms of Linear Matrix Inequalities (LMIs). (The characterization in terms of LMIs is useful because LMI solvers such as Hu *et al*. (2005) are very efficient in providing solutions to the corresponding problems.) It is also shown in Grimm *et al*. (2003a, 2004a) that the quadratic stability and performance conditions remain convex also in the case where the matrices of the anti-windup filter $\mathcal{AW}$ are unknown, as long as the order of the filter is restricted to being either zero (thereby characterizing a static compensation scheme) or equal to the order of the plant $\mathcal{P}$. Therefore, constructive tools for linear full-authority and external anti-windup design are given in Grimm *et al*. (2003a, 2004a), respectively, which only cover the static and the plant order case. On the other hand, it is established in Grimm *et al*. (2003a, 2004a) that anti-windup design of reduced order (namely, the design of a dynamic compensator of order smaller than the order of the plant) leads to non-convex constraints involving a rank condition. It is worth mentioning additional recent work on LMI-based static and plant-order regional anti-windup designs. These recent approaches are aimed at improving the anti-windup performance in a prescribed region, at the cost of giving up the global properties of the above mentioned works. In general, this goal is achieved by relying on absolute stability concepts combined with generalized sector conditions such as those proposed in Hu *et al*. (2002a, b). Applications of these approaches can be found in Cao *et al*. (2002a, b), Gomes da Silva Jr and Tarbouriech (2003), Fang *et al*. (2004), Tarbouriech *et al*. (2004) and Hu *et al*. (2005). Although we only focus here on the global designs of Grimm *et al*. (2003a, 2004a), it is quite straightforward to extend the proposed methods to the regional anti-windup approaches of Hu *et al*. (2005).

1.3. *Contribution*

In this paper we address the reduced order linear anti-windup design problem for exponentially stable plants, in both the full-authority and external cases. We use a convex relaxation of the nonconvex constraints given in Grimm *et al*. (2003a, 2004a), based on standard approaches for the design of reduced order H_2/H_∞ compensators.

Reduced order linear anti-windup design is of great importance because it often represents the most desirable trade-off between closed-loop performance and/or stability guarantees and implementation complexity. Indeed, it is shown in Grimm *et al*. (2003a, 2004a) that the convex tools for static anti-windup design are only applicable to a restricted set of cases, where suitable system theoretic conditions are satisfied. If these conditions do not hold, dynamic anti-windup compensation is necessary, and for high order plants, the convex tools for plant-order anti-windup may lead to overly complex solutions. Therefore, tools for reduced order dynamic anti-windup design are mandatory to address all of these cases.

The main contribution of this paper is to provide constructive algorithms for reduced order anti-windup design. These algorithms are mostly relevant for all the cases where static anti-windup design is not feasible (or yields unacceptable $\mathcal{L}_2$ performance). Then, the algorithms can be used to compute a low order anti-windup compensator guaranteeing stability and a prescribed level of quadratic $\mathcal{L}_2$ performance (whenever the corresponding problem is feasible). These algorithms are conveniently immersed in a context where the performance levels induced by static and plant order compensators can be exactly computed with the convex tools of Grimm *et al*. (2003a, 2004a). Then the performance level imposed to the convex relaxations used here could range between these two limit values (the first one possibly being infinity). By spanning the whole set of possible performance values, a conservative estimate of the best performance achievable by a fixed (reduced) order compensator could be determined. This approach gave interesting results on a few low order simulation examples studied here. Checking the potentials of these algorithms on larger order examples would also be an interesting problem to study.

The paper is structured as follows: in section 2 we introduce the problem data and clarify the anti-windup construction problem; in section 3 we summarize the non-convex design algorithms given in Grimm *et al.* (2003a, 2004a) and in section 4 we give the relaxations of these algorithms leading to convex constructions. Finally, in section 5 we test the reduced order construction on two simulation examples.

Notation: Given a generic matrix B, $B_\perp$ will denote a matrix of full column rank that spans the null space of B. For a square matrix X, $\mathrm{He}(X) := X + X^T$.

2. Problem definition

2.1. *The unconstrained closed-loop system*

Consider a linear stable *plant* given by

$$\mathcal{P}\begin{cases} \dot{x}_p = A_p x_p + B_{p,u} u + B_{p,w} w \\ y = C_{p,y} x_p + D_{p,yu} u + D_{p,yw} w \\ z = C_{p,z} x_p + D_{p,zu} u + D_{p,zw} w, \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the exogenous input (possibly containing disturbance, reference and measurement noise), $y \in \mathbb{R}^{n_y}$ is the plant output available for measurement, $z \in \mathbb{R}^{n_z}$ is the performance output (possibly corresponding to a weighted tracking error) and A_p, $B_{p,u}$, $B_{p,w}$, $C_{p,y}$, $D_{p,yu}$, $D_{p,yw}$, $C_{p,z}$, $D_{p,zu}$ and $D_{p,zw}$ are matrices of suitable dimensions.

Assume also that an unconstrained controller of the form

$$\mathcal{C}\begin{cases} \dot{x}_c = A_c x_c + B_{c,w} w + B_{c,y} u_c + v_1 \\ y_c = C_c x_c + D_{c,w} w + D_{c,y} u_c + v_2 \end{cases} \tag{2}$$

is given (where $x_c \in \mathbb{R}^{n_c}$ is the controller state, $y_c \in \mathbb{R}^{n_u}$ is the controller output and A_c, $B_{c,w}$, $B_{c,y}$, C_c, $D_{c,w}$ and $D_{c,y}$ are matrices of suitable dimensions). Moreover, v_1 and v_2 are additional inputs that will be used for anti-windup augmentation. Also assume that the controller interconnection to the linear plant through the equations

$$u = y_c, \quad u_c = y, \qquad v_1 = 0, \quad v_2 = 0, \tag{3}$$

is well-posed and guarantees internal stability of the arising unconstrained closed-loop system (1), (2), (3).

2.2. *Input saturation and anti-windup augmentation*

Assume that a nonlinearity is present at the input u of the plant. In particular, in this paper we will restrict the attention to the decentralized saturation $\mathrm{sat}(\cdot)\colon \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ defined as $\mathrm{sat}(u) = [\sigma_1(u_1) \ \cdots \ \sigma_{n_u}(u_{n_u})]^T$, with

$$\sigma_i(u_i) := \begin{cases} u_{Mi}, & \text{if } u_i > u_{Mi}, \\ u_{mi}, & \text{if } u_i < u_{mi}, \\ u_i, & \text{otherwise}, \end{cases} \tag{4}$$

where $u_{mi} < 0 < u_{Mi}$, $i = 1, \ldots, n_u$ are the saturation levels of each input channel. Note that in Grimm *et al.* (2003a, 2004a), the saturation function is assumed to belong to a larger class on nonlinearities (including the decentralized saturations), so that necessary and sufficient conditions can be established in the feasibility results. We restrict the attention to decentralized saturations here, because they represent the most common case encountered in practice and, as commented in Grimm *et al.* (2003a, Remark 5), the decentralized property can be exploited to gain extra degrees of freedom when selecting the anti-windup compensator matrices.

Suppose the control input of the plant is subject to the above defined decentralized saturation. Then, given an integer $n_{aw} \geq 0$, the full-authority anti-windup compensation problem deals with the design of an order n_{aw} linear filter called full-authority anti-windup compensator

$$\mathcal{AW}_{fa}\begin{cases} \dot{x}_{aw} = A_{aw} x_{aw} + B_{aw}(y_c - u) \\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C_{aw} x_{aw} + D_{aw}(y_c - u), \end{cases} \tag{5}$$

that guarantees closed-loop quadratic stability and a desirable $\mathcal{L}_2$ bound on the norm of the performance output z, based on the $\mathcal{L}_2$ norm of the exogenous input w when interconnected to the closed-loop (1), (2) through the full-authority anti-windup interconnection

$$u = \mathrm{sat}(y_c), \quad u_c = y. \tag{6}$$

Similarly, given an integer $n_{aw} \geq 0$, the external anti-windup compensation problem deals with the design of an order n_{aw} linear filter called external anti-windup compensator

$$\mathcal{AW}_{e}\begin{cases} \dot{x}_{aw} = A_{aw} x_{aw} + B_{aw}(y_c - u) \\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C_{aw} x_{aw} + D_{aw}(y_c - u), \end{cases} \tag{7}$$

that guarantees closed-loop quadratic stability and a desirable $\mathcal{L}_2$ bound on the norm of the performance output z, based on the $\mathcal{L}_2$ norm of the exogenous input w when interconnected to the closed-loop (1), (2) through the external anti-windup interconnection

$$u = \mathrm{sat}(y_c), \quad u_c = y, \qquad v_1 = B_{c,y}\nu_1, \quad v_2 = \nu_2. \qquad (8)$$

3. Non-convex algorithms for anti-windup synthesis

In this section we report on the nonconvex algorithms for reduced order anti-windup design given in Grimm *et al.* (2003a) for the full-authority case and mentioned in Grimm *et al.* (2004a, Remark 2) for the external case. The convex formulations that we report in the following section 4 are based on an approximation of these algorithms.

3.1. *Aggregated systems and compact closed-loop*

Similar to Grimm *et al.* (2004a), we need to define the two aggregate systems corresponding to two different interconnections between the plant (1) and the unconstrained controller (2). The first aggregate system corresponds to the unconstrained closed-loop system and is given by

$$\begin{aligned} \dot{x}_{c\ell} &= A_{cl}x_{c\ell} + B_{cl,w}w \\ z &= C_{cl,z}x_{c\ell} + D_{cl,zw}w, \end{aligned} \qquad (9)$$

where $x_{c\ell} := \left[x_p^T \ x_c^T \right]^T \in \mathbb{R}^{n_{cl}}$ represents the unconstrained closed-loop state (with $n_{cl} := n_p + n_c$) and A_{cl}, $B_{cl,w}$, $C_{cl,z}$ and $D_{cl,zw}$ are uniquely determined by the matrices in (1) and (2). For completeness, we report next the explicit values of these matrices

The second aggregate system corresponds once again to an interconnection between (1) and (2). However, here the saturation function is replaced by the zero function, the plant output is disconnected from the controller input and an additional input signal $\tilde{\nu}_1$ is considered for the closed-loop

$$\begin{aligned} \dot{x}_{o\ell} &= A_{ol}x_{o\ell} + B_{ol,w}w + B_{ol}\tilde{\nu}_1 \\ z &= C_{ol,z}x_{o\ell} + D_{ol,zw}w. \end{aligned} \qquad (11)$$

In (11), the state $x_{o\ell} := \left[x_p^T \ x_c^T \right]^T \in \mathbb{R}^{n_{cl}}$ corresponds to the aggregated state and the matrices A_{ol}, B_{ol}, $B_{ol,w}$, $C_{ol,z}$ and $D_{ol,zw}$ are uniquely determined by the matrices in (1) and (2). For completeness, explicit expressions for these matrices are given in the following equations:

$$\left[\begin{array}{c|c|c} A_{ol} & B_{ol,w} & B_{ol} \\ \hline C_{ol,z} & D_{ol,zw} & \end{array}\right] = \left[\begin{array}{cc|c|c} A_p & 0 & B_{p,w} & 0 \\ 0 & A_c & B_{c,w} & B_{aw} \\ \hline C_{p,z} & 0 & D_{p,zw} & \end{array}\right]. \qquad (12)$$

It will also be useful to represent the closed-loop system (1), (2), (6) in a compact form, where the anti-windup filter (to be designed) is disconnected (so u in (1) is replaced by $y_c - q$) so that the signals $q := y_c - \mathrm{sat}(y_c)$ and

$$v := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

appear as external inputs

$$\begin{aligned} \dot{x}_{cl} &= A_{cl}x_{cl} + B_{cl,w}w + B_{cl,q}q + B_{cl,v}v \\ z &= C_{cl,z}x_{cl} + D_{cl,zw}w + D_{cl,zq}q + D_{cl,zv}v \\ y_c &= C_{cl,y}x_{cl} + D_{cl,yw}w + D_{cl,yq}q + D_{cl,yv}v, \end{aligned} \qquad (13)$$

$$\left[\begin{array}{c|c} A_{cl} & B_{cl,w} \\ \hline C_{cl,z} & D_{cl,zw} \end{array}\right] = \left[\begin{array}{cc|c} A_p + B_{p,u}\Delta_u D_{c,y} C_{p,y} & B_{p,u}\Delta_u C_c & B_{p,w} + B_{p,u}\Delta_u(D_{c,y}D_{p,yw} + D_{c,w}) \\ B_{c,y}\Delta_y C_{p,y} & A_c + B_{c,y}\Delta_y D_{p,yu} C_c & B_{c,w} + B_{c,y}\Delta_y(D_{p,yu}D_{c,w} + D_{p,yw}) \\ \hline D_{p,zu}\Delta_u D_{c,y} C_{p,y} + C_{p,z} & D_{p,zu}\Delta_u C_c & D_{p,zw} + D_{p,zu}\Delta_u(D_{c,y}D_{p,yw} + D_{c,w}) \end{array}\right], \qquad (10)$$

where $\Delta_y := (I - D_{p,yu}D_{c,y})^{-1}$ and $\Delta_u := (I - D_{c,y} D_{p,yu})^{-1}$ are always well defined (namely, the matrices in parentheses are invertible) if the unconstrained closed-loop system is well-posed (see section 2).

where A_{cl} $B_{cl,w}$, $C_{cl,z}$ and $D_{cl,zw}$ have been defined above in (10), and the remaining matrices (plus matrices $B_{cl,v}$, $D_{cl,zv}$ and $D_{cl,yv}$, which will be used in the sequel) are given by

$$\left.\begin{array}{l} \left[\begin{array}{c|c|c} B_{cl,q} & B_{cl,v} & B_{cl,v} \\ \hline D_{cl,zq} & D_{cl,zv} & D_{cl,zv} \\ \hline D_{cl,yq} & D_{cl,yv} & D_{cl,yv} \end{array}\right] = \left[\begin{array}{c|cc|cc} -B_{p,u}\Delta_u & 0 & B_{p,u}\Delta_u & -B_{p,u}\Delta_u D_{c,y} & B_{p,u}\Delta_u \\ -B_{c,y}\Delta_y D_{p,yu} & I & B_{c,y}\Delta_y D_{p,yu} & B_{c,y}\Delta_y & \Delta_y D_{p,yu} \\ \hline -D_{p,zu}\Delta_u & 0 & D_{p,zu}\Delta_u & D_{p,zu}\Delta_u D_{c,y} & D_{p,zu}\Delta_u \\ \hline I - \Delta_u & 0 & \Delta_u & \Delta_u D_{c,y} & \Delta_u \end{array}\right] \\ \left[\, C_{cl,y} \mid D_{cl,yw} \right] = \left[\Delta_u D_{c,y} C_{p,y} \quad \Delta_u C_c \mid \ \Delta_u(D_{c,w} + D_{c,y}D_{p,yw}) \right]. \end{array}\right\} \qquad (14)$$

3.2. *Design algorithms*

We report in this section the generic-order anti-windup construction algorithms given in Grimm *et al.* (2003a, Procedure 1) (for the full-authority case) and suggested in Grimma *et al.* (2004a, Remark 2) (for the external case). For completeness, we report here the explicit expressions of all construction matrices (which wasn't done in Grimm *et al.* (2003a, 2004a)). Note that the corresponding conditions (and, in particular, the feasibility conditions at Step 1 of both algorithms) are non-convex, so that the algorithms cannot be directly applied. In the next section we will comment on how to relax the non-convex feasibility conditions to provide a constructive design tool.

Procedure 1 (Full-authority anti-windup design)**:**

Step 1 (Solve the feasibility LMIs)**:** Find a solution (R_{11}, S, γ) to the following set of nonlinear matrix inequalities (possibly minimizing γ):

$$\begin{bmatrix} R_{11}A_p^T + A_p R_{11} & B_{p,w} & R_{11}C_{p,z}^T \\ B_{p,w}^T & -\gamma I & D_{p,zw}^T \\ C_{p,z}R_{11} & D_{p,zw} & -\gamma I \end{bmatrix} < 0 \tag{15a}$$

$$\begin{bmatrix} SA_{cl}^T + A_{cl}S & B_{cl,w} & SC_{cl,z}^T \\ B_{cl,w}^T & -\gamma I & D_{cl,zw}^T \\ C_{cl,z}S & D_{cl,zw} & -\gamma I \end{bmatrix} < 0 \tag{15b}$$

$$R_{11} = R_{11}^T > 0 \tag{15c}$$

$$S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} > 0 \tag{15d}$$

$$R_{11} - S_{11} \geq 0 \tag{15e}$$

$$\mathrm{rank}(R_{11} - S_{11}) \leq n_{aw}. \tag{15f}$$

Step 2 (Construct the matrix Q)**:** Using the solution (R_{11}, S, γ) from Step 1, define the matrices

$$R = \begin{bmatrix} R_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

and $N \in \mathbb{R}^{n_{cl} \times n_{aw}}$ as a solution of the following equation:

$$R\,S^{-1}\,R - R = N\,N^T. \tag{16}$$

Since R and S are invertible and Conditions (15e) and (15f) are satisfied, then $R\,S^{-1}\,R - R$ is positive semidefinite and of rank n_{aw}, so there always exists a matrix N satisfying equation (16). Define the matrix $M \in \mathbb{R}^{n_{aw} \times n_{aw}}$ as

$$M := I + N^T\,R^{-1}\,N. \tag{17}$$

Finally, define the matrix $Q \in \mathbb{R}^{(n_{cl}+n_{aw}) \times (n_{cl}+n_{aw})}$ as

$$Q := \begin{bmatrix} R & N \\ N^T & M \end{bmatrix}. \tag{18}$$

Step 3 (Build required matrices)**:** Set $n = n_p + n_c + n_{aw}$. Construct the matrices $A_0 \in \mathbb{R}^{n \times n}$, $B_{q0} \in \mathbb{R}^{n \times n_u}$, $C_{y0} \in \mathbb{R}^{n_u \times n}$, $D_{yq0} \in \mathbb{R}^{n_u \times n_u}$, $C_{z0} \in \mathbb{R}^{n_z \times n}$, $D_{zq0} \in \mathbb{R}^{n_z \times n_u}$, $B_w \in \mathbb{R}^{n \times n_w}$, $D_{zw} \in \mathbb{R}^{n_z \times n_w}$ and $D_{yw} \in \mathbb{R}^{n_u \times n_w}$ as

$$\left[\begin{array}{c|c|c} A_0 & B_{q0} & B_w \\ \hline C_{y0} & D_{yq0} & D_{yw} \\ \hline C_{z0} & D_{zq0} & D_{zw} \end{array}\right] = \left[\begin{array}{cc|c|c} A_{cl} & 0 & B_{cl,q} & B_{cl,w} \\ 0 & 0 & 0 & 0 \\ \hline C_{cl,y} & 0 & D_{cl,yq} & D_{cl,yw} \\ \hline C_{cl,z} & 0 & D_{cl,zq} & D_{cl,zw} \end{array}\right].$$

Step 4 (Anti-windup compensator LMI)**:** Choose $\delta \in \mathbb{R}$, $\delta > 0$ and define $U = \delta W^{-1}$. Based on Steps 2 and 3, construct the matrices $H \in \mathbb{R}^{(n_{aw}+n_v) \times (n+n_u+n_w+n_z)}$, $\Psi \in \mathbb{R}^{(n+n_u+n_w+n_z) \times (n+n_u+n_w+n_z)}$ and $G \in \mathbb{R}^{(n_{aw}+n_u) \times (n+n_u+n_w+n_z)}$ as follows:

$$\Psi = \mathrm{He}\left(\begin{bmatrix} A_0 Q & B_{q0}U + QC_{y0}^T & B_w & QC_{z0}^T \\ 0 & D_{yq0}U - U & D_{yw} & UD_{zq0}^T \\ 0 & 0 & -\frac{\gamma}{2}I & D_{zw}^T \\ 0 & 0 & 0 & -\frac{\gamma}{2}I \end{bmatrix}\right),$$

$$H = \left[\begin{array}{cc|c|c|c} 0 & I_{n_{aw}} & 0 & 0 & 0 \\ B_{cl,v}^T & 0 & D_{cl,yv}^T & 0 & D_{cl,zv}^T \end{array}\right]$$

$$G = \left[\begin{array}{cc|c|c|c} N^T & M & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{array}\right]$$

Finally, solve the LMI

$$\Psi + G^T \Lambda^T H + H^T \Lambda G < 0. \tag{19}$$

in the unknowns $\Lambda \in \mathbb{R}^{(n_{aw}+n_v) \times (n_{aw}+n_u)}$ and $U \in \mathbb{R}^{n_u \times n_u}$, $U > 0$ diagonal, and compute the matrices of the full-authority anti-windup compensator (5) as follows:

$$\begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} = \Lambda \begin{bmatrix} I & 0 \\ 0 & U^{-1} \end{bmatrix}. \tag{20}$$

Remark 1: Due to numerical optimization problems, the anti-windup matrices resulting from solving the LMI (19) at Step 4 of Procedure 1 may have overly large entries. This effect can be mitigated by augmenting that LMI constraint with the following two extra constraints:

$$\begin{bmatrix} M_{aw}I & \Lambda \\ \Lambda^T & M_{aw}I \end{bmatrix} > 0, \quad U > I,$$

where $M_{aw} > 0$ is a prescribed maximum size to be enforced on the anti-windup compensator matrices. It is easily seen that the first constraint above is equivalent to requiring that the maximum singular value of Λ is smaller than M_{aw}. The second constraint, combined with the transformation (20), implies that also the maximum singular value of the anti-windup matrix at the left hand side is upper bounded by M_{aw}.

The procedure for external anti-windup compensation, only slightly differs from the preceding one. It is reported next.

Procedure 2 (External anti-windup design)**:**

Step 1 (Solve the feasibility LMIs)**:** Find a solution (R, S, γ) to the following set of nonlinear matrix inequalities (possibly minimizing γ):

$$\begin{bmatrix} B_{ol\perp}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} RA_{ol}^T + A_{ol}R & B_{ol,w} & RC_{ol,z}^T \\ B_{ol,w}^T & -\gamma I & D_{ol,zw}^T \\ C_{ol,z}R & D_{ol,zw} & -\gamma I \end{bmatrix} \times \begin{bmatrix} B_{ol\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \tag{21a}$$

$$\begin{bmatrix} SA_{cl}^T + A_{cl}S & B_{cl,w} & SC_{cl,z}^T \\ B_{cl,w}^T & -\gamma I & D_{cl,zw}^T \\ C_{cl,z}S & D_{cl,zw} & -\gamma I \end{bmatrix} < 0 \tag{21b}$$

$$R = R^T > 0 \tag{21c}$$

$$S = S^T > 0 \tag{21d}$$

$$R - S \geq 0 \tag{21e}$$

$$\operatorname{rank}(R - S) \leq n_{aw}, \tag{21f}$$

where $B_{ol\perp}$ is any matrix of full column rank that spans the null space of B_{ol}^T (namely $B_{ol}^T B_{ol\perp} = 0$).

Step 2 (Construct the matrix Q)**:** Same as Procedure 1.

Step 3 (Build required matrices)**:** Same as Procedure 1.

Step 4 (Anti-windup compensator LMI)**:** Construct the matrices $\Psi \in \mathbb{R}^{(n+n_u+n_w+n_z)\times(n+n_u+n_w+n_z)}$ and $G \in \mathbb{R}^{(n_{aw}+n_u)\times(n+n_u+n_w+n_z)}$ as in Step 4 of Procedure 1 and the matrix $H \in \mathbb{R}^{(n_{aw}+n_v)\times(n+n_u+n_w+n_z)}$ as follows:

$$H = \left[\begin{array}{cc|c|c|c} 0 & I_{n_{aw}} & 0 & 0 & 0 \\ B_{cl,v}^T & 0 & D_{cl,yv}^T & 0 & D_{cl,zv}^T \end{array}\right].$$

Finally, solve the LMI (19) in the unknowns $\Lambda \in \mathbb{R}^{(n_{aw}+n_v)\times(n_{aw}+n_u)}$ and $U \in \mathbb{R}^{n_u \times n_u}$, $U > 0$ diagonal, and compute the matrices of the external anti-windup compensator (7) using (20).

4. Convex algorithms for anti-windup synthesis

As extensively commented in Grimm *et al.* (2003a, 2004a), the two procedures reported in the previous section become convex when the anti-windup order n_{aw} is either equal to zero (thus leading to static anti-windup compensation) or equal to the order n_p of the plant $\mathcal{P}$. However, for generic values of n_{aw}, the corresponding anti-windup construction cannot be formulated in terms of LMIs and the corresponding anti-windup design could become very complicated.

For a fixed anti-windup compensator order n_{aw} (with $n_{aw} \neq 0$ and $n_{aw} \neq n_p$), what makes Procedures 1 and 2 difficult to apply is determining the solution at the first step. Indeed, conditions (15) and (21) both represent an optimization problem including a nonlinear rank constraint (corresponding, respectively, to (15f) and (21f)) which cannot be incorporated directly in a convex optimization approach. Nevertheless, due to the structure of the optimization problems (15) and (21), it is possible to relax the nonlinear constraints and replace them with linear ones so that reduced order anti-windup design becomes easier to accomplish (at the cost of possible additional conservativeness). In particular, with reference to (15f) (respectively, (21f)), since $R_{11} - S_{11} \geq 0$ (respectively, $R - S \geq 0$), the trace of this matrix (namely, the sum of all its diagonal entries) is proportional to its size. This fact is well known and widely used in trace minimization relaxations to rank conditions (see e.g. Boyd *et al.* (1994), page 117) and references therein, in addition to Fazel *et al.* (2001), where more complex rank minimization problems are considered). Therefore, minimizing the trace of $R_{11} - S_{11}$ (respectively, $R - S$) may lead to a rank deficient result whose rank will be the reduced anti-windup compensator order. The optimization problems (15) and (21) can then be solved by removing the nonlinear rank constraint and minimizing a new cost variable which is trace$(R_{11} - S_{11})$ (respectively, trace$(R - S)$). Note however that during this minimization it is essential to maintain the focus on the performance γ guaranteed by the arising compensator, as this performance quantifies the quality level induced by the arising anti-windup compensator.

In the numerical implementation of this LMI-based algorithm, the nonstrict inequality constraints (15e) and (21e) will actually be replaced by strict inequality constraints and the corresponding solutions will be very close to being rank deficient (this incorporates the numerical errors of the LMI solver). Therefore, it will be useful to use a Singular Value Decomposition (SVD) to determine a new selection of S for which $R_{11} - S_{11} \geq 0$ (respectively, $R - S \geq 0$) actually holds. Since the S matrix is slightly changed by this modification, it is mandatory to verify that the new selection of S still satisfies the LMI condition (15b) (respectively, (21b)). To this aim, anticipating for such a perturbation, it is useful to add a small feasibility margin to that LMI condition to make it robust to these perturbations (at least to a certain extent). The arising constraint will be

$$\begin{bmatrix} SA_{cl}^T + A_{cl}S + \epsilon I & B_{cl,w} & SC_{cl,z}^T \\ B_{cl,w}^T & -\gamma I_{n_w} & D_{cl,zw}^T \\ C_{cl,z}S & D_{cl,zw} & -\gamma I_{n_z} \end{bmatrix} < 0,$$

where $\epsilon > 0$ is a small constant. The complete algorithm, which replaces Step 1 of Procedures 1 and 2, requires some tuning for ϵ but is typically quite straightforward to apply once the required performance level is specified. The two arising algorithms for the full-authority and the anti-windup case are reported next.

Procedure 3 (Reduced order full-authority anti-windup design with guaranteed performance)**:**

Step 1: Given a desired performance $\bar{\gamma}$, find the optimal solution $(R, S) \in \mathbb{R}^{n_p \times n_p} \times \mathbb{R}^{n_{cl} \times n_{cl}}$ to the following LMI eigenvalue problem:

$$\begin{aligned} \min_{R,S} \quad & \text{trace}(R_{11} - S_{11}) \text{ subject to} \\ & R = R^T = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} > 0, \\ & S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} > 0, \\ & R_{11} - S_{11} > 0, \\ & \begin{bmatrix} R_{11}A_p^T + A_pR_{11} & B_{p,w} & R_{11}C_{p,z}^T \\ B_{p,w}^T & -\bar{\gamma} I_{n_w} & D_{p,zw}^T \\ C_{p,z}R_{11} & D_{p,zw} & -\bar{\gamma} I_{n_z} \end{bmatrix} < 0, \\ & \begin{bmatrix} SA_{cl}^T + A_{cl}S + \epsilon I & B_{cl,w} & SC_{cl,z}^T \\ B_{cl,w}^T & -\bar{\gamma} I_{n_w} & D_{cl,zw}^T \\ C_{cl,z}S & D_{cl,zw} & -\bar{\gamma} I_{n_z} \end{bmatrix} < 0. \end{aligned}$$

Step 2: Compute the Singular Value Decomposition (SVD) of the symmetric matrix $R_{11} - S_{11}$ determined from the solution of the previous step, namely compute F, V such that $FVF^T = R_{11} - S_{11}$, where V is a positive semidefinite diagonal matrix whose diagonal entries are non-increasing (note that the SVD has a special structure in this case because of the symmetry of $R_{11} - S_{11}$). Set $n_{aw} = n_p$.

Step 3: Call $v_1, \ldots v_{n_{aw}-1}$ the first $n_{aw} - 1$ entries on the diagonal of V and define $\hat{V} := \text{diag}\{v_1, \ldots v_{n_{aw}-1}, 0, \ldots, 0\}$, namely $\hat{V}$ is a diagonal matrix having its first $n_{aw} - 1$ diagonal entries equal to the entries of V and the remaining entries equal to zero. Also define $\hat{S}_{11} := S_{11} + F(V - \hat{V})F^T$.

Step 4: Given the matrix

$$\hat{S} := \begin{bmatrix} \hat{S}_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

where $\hat{S}_{11}$ was determined at the previous step, minimize the variable $\tilde{\gamma}$ in the following LMI:

$$\begin{bmatrix} \hat{S}A_{cl}^T + A_{cl}\hat{S} & B_{cl,w} & \hat{S}C_{cl,z}^T \\ B_{cl,w}^T & -\tilde{\gamma} I_{n_w} & D_{cl,zw}^T \\ C_{cl,z}\hat{S} & D_{cl,zw} & -\tilde{\gamma} I_{n_z} \end{bmatrix} < 0.$$

If the LMI is feasible, then set $(R^*, S^*, \gamma) := (R, \hat{S}, \tilde{\gamma})$, set $n_{aw} = n_{aw} - 1$ and go to step 3. If the LMI is not feasible, then go to step 5.

Step 5: Select the order of the anti-windup compensator as n_{aw} and the required solution as (R^*, S^*, γ).

Step 6: Follow the remaining Steps 2–4 of Procedure 1.

Procedure 4 (Reduced order external anti-windup design with guaranteed performance)**:**

Step 1: Given a desired performance $\bar{\gamma}$, find the optimal solution $(R, S) \in \mathbb{R}^{n_{cl} \times n_{cl}} \times \mathbb{R}^{n_{cl} \times n_{cl}}$ to the following LMI eigenvalue problem:

$$\begin{aligned} \min_{R,S} \quad & \text{trace}(R - S) \text{ subject to} \\ & R = R^T > 0, \\ & S = S^T > 0, \\ & R - S > 0, \\ & \begin{bmatrix} B_{ol\perp}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} RA_{ol}^T + A_{ol}R & B_{ol,w} & RC_{ol,z}^T \\ B_{ol,w}^T & -\bar{\gamma} I & D_{ol,zw}^T \\ C_{ol,z}R & D_{ol,zw} & -\bar{\gamma} I \end{bmatrix} \\ & \times \begin{bmatrix} B_{ol\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0, \\ & \begin{bmatrix} SA_{cl}^T + A_{cl}S + \epsilon I & B_{cl,w} & SC_{cl,z}^T \\ B_{cl,w}^T & -\bar{\gamma} I_{n_w} & D_{cl,zw}^T \\ C_{cl,z}S & D_{cl,zw} & -\bar{\gamma} I_{n_z} \end{bmatrix} < 0. \end{aligned}$$

Step 2: Compute the Singular Value Decomposition (SVD) of the symmetric matrix $R - S$ determined from the solution of the previous step, namely compute F, V such that $FVF^T = R - S$, where V is a positive semidefinite diagonal matrix whose diagonal entries are non increasing (note that the SVD has a special structure in this case because of the symmetry of $R-S$). Set $n_{aw}=n_{cl}$.

Step 3: Call $v_1, \ldots v_{n_{aw}-1}$ the first $n_{aw}-1$ entries on the diagonal of v and define $\hat{v} := \mathrm{diag}\{v_1, \ldots v_{n_{aw}-1}, 0, \ldots, 0\}$, namely $\hat{V}$ is a diagonal matrix having its first $n_{aw}-1$ diagonal entries equal to the entries of V and the remaining entries equal to zero. Also define $\hat{S} := S + F(V-\hat{V})F^T$.

Step 4: Given the matrix $\hat{S}$ determined at the previous step, minimize the variable $\tilde{\gamma}$ in the following LMI:

$$\begin{bmatrix} \hat{S}A_{cl}^T + A_{cl}\hat{S} & B_{cl,w} & \hat{S}C_{cl,z}^T \\ B_{cl,w}^T & -\tilde{\gamma}I_{n_w} & D_{cl,zw}^T \\ C_{cl,z}\hat{S} & D_{cl,zw} & -\tilde{\gamma}I_{n_z} \end{bmatrix} < 0 \ .$$

If the LMI is feasible, then set $(R^*, S^*, \gamma) := (R, \hat{S}, \tilde{\gamma})$, set $n_{aw}=n_{aw}-1$ and go to step 3. If the LMI is not feasible, then go to step 5.

Step 5: Select the order of the anti-windup compensator as n_{aw} and the required solution as (R^*, S^*, γ).

Step 6: Follow the remaining Steps 2–4 of Procedure 2.

Remark 2: Note that Procedures 3 and 4 allow us to determine a ''minimal order'' anti-windup compensator (minimal within the convex relaxation of the nonconvex rank constraints) guaranteeing a prescribed performance level. This feature becomes quite useful within the general context of the anti-windup approaches under consideration. Indeed, by first noticing that static and plant-order anti-windup designs both correspond to convex algorithms (because the rank constraints transform into linear constraints in those two special cases), it is possible to compute a lower bound via the performance induced by plant-order compensation, and an upper bound via the performance induced by static compensation on the anti-windup performance achievable by reduced order compensators (this upper bound may actural be infinity, in cases where static anti-windup design is not feasible). Indeed, by applying Procedure 3 (respectively, Procedure 4 for the external case) for values of $\bar{\gamma}$ within the range defined by these upper and lower bounds, it is possible to compute a ''minimum order'' curve which represents the minimum order compensator guaranteeing a prescribed performance level. The curve will evidently be a piecewise constant nonincreasing curve and may be a useful tool for the designer to determine the most desirable trade-off between guaranteed performance and compensation complexity (namely, the order of the anti-windup compensator).

Remark 3: When adopting the convex relaxations of Procedures 3 and 4, the resulting R^* and S^* matrices may in some cases be prone to numerical errors. In particular, when running the last step of Procedures 1 and 2, corresponding to solving the large LMI (19), numerical problems may make that LMI only feasible for a larger performance level γ (or even infeasible for any γ). This fact has been noticed in several other papers related to the use of LMIs for robust control design and motivated interesting studies on the use of explicit formulas that replace the LMI-based strategy of Step 4 of Procedures 1 and 2 with an explicit computation of the so-called ''central controller'', which is far more robust from a numerical viewpoint (see, e.g., Gahinet (1996) and aslo Wu and Lu (2004)) for an application of these formulas in the anti-windup design context). We do not pursue this type of approach here although we recognize that it may be useful to follow it for an increased robustness of the design algorithm.

5. Examples

In this section we will test the reduced order anti-windup construction on two simulation examples. The first example is a SISO system for which both static full authority and static external anti-windup augmentation are not feasible. The second example is a MIMO system for which static external anti-windup is not feasible and static full-authority anti-windup is feasible. The proposed constructions lead to successful reduced order anti-windup compensation for both examples.

5.1. *An academic SISO example*

This example has been introduced in Mulder and Kothare (2000) where it was shown that quadratically stabilizing static anti-windup compensation is not feasible for it, and revisited in Grimm *et al.* (2003b) where it was shown that plant order anti-windup compensation (which is always feasible for exponentially stable plants) leads to a performance level $\gamma = 4.4766$.

The problem data correspond to the following matrices:

$$\left[\begin{array}{c|c|c} A_p & B_{pu} & B_{pw} \\ \hline C_{py} & D_{pyu} & D_{pyw} \\ \hline C_{pz} & D_{pzu} & D_{pzw} \end{array}\right] = \left[\begin{array}{cc|c|c} -0.2 & -0.2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline -0.4 & -0.9 & -0.5 & 0 \\ \hline 0.4 & 0.9 & 0.5 & I \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_c & B_{cy} & B_{cw} \\ \hline C_c & D_{cy} & D_{cw} \end{array}\right] = \left[\begin{array}{c|c|c} 0 & 1 & -1 \\ \hline 2 & 2 & -2 \end{array}\right].$$

For this example, since the controller state has dimension 1, external anti-windup compensation has full authority, therefore both full authority and external compensation lead to the same results. We will therefore only apply our full authority anti-windup design algorithms. A plant order full authority anti-windup compensator can be designed for this example (see also Grimm *et al.* (2003b) for the same construction) thus obtaining the following second order compensator inducing a finite gain $\gamma = 4.4766$:

$$\left[\begin{array}{c|c} A_{aw} & B_{aw} \\ \hline C_{aw} & D_{aw} \end{array}\right] = \left[\begin{array}{cc|c} -64.125 & -14.075 & -8.4772 \\ -11.784 & -2.7914 & -7.0104 \\ \hline -0.68354 & 0.24182 & -1.5125 \\ -3.0759 & 0.18258 & -1 \end{array}\right].$$

When applying Procedure 3 and requiring a guaranteed performance level $\bar{\gamma} = 4.48$, the following reduced order full authority anti-windup compensator is obtained (recall that quadratically stabilizing static anti-windup compensation is not feasible for this example), leading to the performance level $\gamma = 4.4825$:

$$\left[\begin{array}{c|c} A_{aw} & B_{aw} \\ \hline C_{aw} & D_{aw} \end{array}\right] = \left[\begin{array}{c|c} -0.19595 & -9.7724 \\ \hline 0.26947 & -1.83 \\ 0.59266 & -1 \end{array}\right].$$

was first introduced in Kapasouris *et al.* (1988) and then revisited in Grimm *et al.* (2003a). The plant consists in a fourth order model with two inputs and two outputs and the unconstrained controller is an 8th order controller designed to guarantee desirable linear closed-loop response. The reader is referred to Kapasouris *et al.* (1988) for the numerical entries of the plant and controller matrices.

For this example, static full authority anti-windup is feasible and leads to the performance level $\gamma = 22.19$, while plant-order full authority anti-windup design (which is always feasible) guarantees the improved performance level $\gamma = 19.39$. Simulation results corresponding to these two full authority constructions can be found in Grimm *et al.* (2003a).

When using external anti-windup compensation, static anti-windup design is no longer feasible, so that it is of interest to seek for a reduced order anti-windup construction. When applying the plant-order external anti-windup algorithm of Grimm *et al.* (2004a), the following anti-windup compensator is obtained, which guarantees a performance level $\gamma = 55.541$:

$$\left[\begin{array}{c|c} A_{aw} & B_{aw} \\ \hline C_{aw} & D_{aw} \end{array}\right] = \left[\begin{array}{cccc|cc} -794.86 & -0.074907 & 165.32 & -218.08 & 7.0708 & 3.0414 \\ -107.82 & -0.027143 & -34.201 & -30.791 & -1.743 & 0.74299 \\ 146.04 & -0.037367 & -386.14 & 32.517 & -18.195 & 1.5151 \\ 45.389 & 0.043971 & 126.84 & 15.285 & 6.0989 & -0.96823 \\ \hline -0.061221 & 0.14168 & -0.14379 & -0.31125 & -0.018234 & -5.4325 \cdot 10^{-6} \\ 0.068187 & 0.1233 & 0.11505 & -0.25055 & -0.0071096 & 0.0013914 \\ 20.044 & -0.0017507 & -32.205 & 4.9089 & -0.54445 & 0.086068 \\ 154.19 & 0.031557 & 130.37 & 45.651 & 6.4992 & -0.54433 \end{array}\right].$$

Figure 3 shows the simulation results for this example when the reference input w is selected as a step input switching from 0 to 2 at time $t = 1$. The saturation levels for the plant input are selected as ± 0.5. The dramatic difference between the unconstrained response and the saturated response in figure 3 motivates the introduction of anti-windup compensation for this example. In accordance with the fact that the performance level obtained by the reduced order anti-windup compensator is essentially equal to that achieved by the full order one (4.4825 versus 4.4766), in the considered simulation the reduced order anti-windup response and the plant-order anti-windup response are almost coincident (the corresponding curves in figure 3 are perfectly overlapped).

5.2. *The longitudinal dynamics of an F8 aircraft*

This example corresponds to the linearized model of the longitudinal dynamics of an F8 aircraft. This example

Applying Procedure 4 and requiring a guaranteed performance level $\bar{\gamma} = 70$, it is found that a second order anti-windup compensator exists yielding the required performance level. When the final LMI is solved in order to determine the reduced order anti-windup compensator matrices, the following state-space representation (inducing a slightly larger performance level $\gamma = 78.016$) is obtained:

$$\left[\begin{array}{c|c} A_{aw} & B_{aw} \\ \hline C_{aw} & D_{aw} \end{array}\right] = \left[\begin{array}{cc|cc} -173.02 & -70.027 & -89.511 & 22.212 \\ -28.136 & -13.009 & -14.158 & 3.7608 \\ \hline -1.5522 & -0.7855 & -1.0782 & 0.22587 \\ -0.32151 & -0.30699 & -0.39495 & 0.11759 \\ -2.8332 & -1.1045 & -0.53272 & 0.35068 \\ 9.4714 & 4.7319 & 4.0364 & -0.53198 \end{array}\right].$$

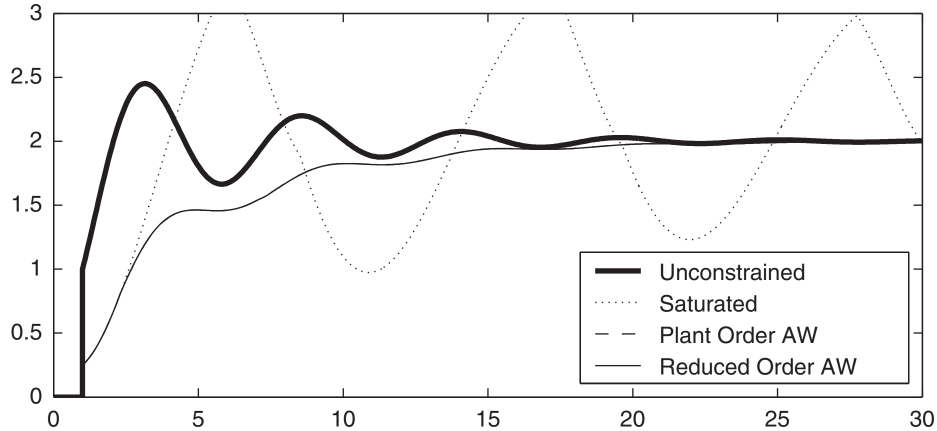

Figure 3. Unconstrained, saturated and anti-windup responses for the Example of section 5.1.

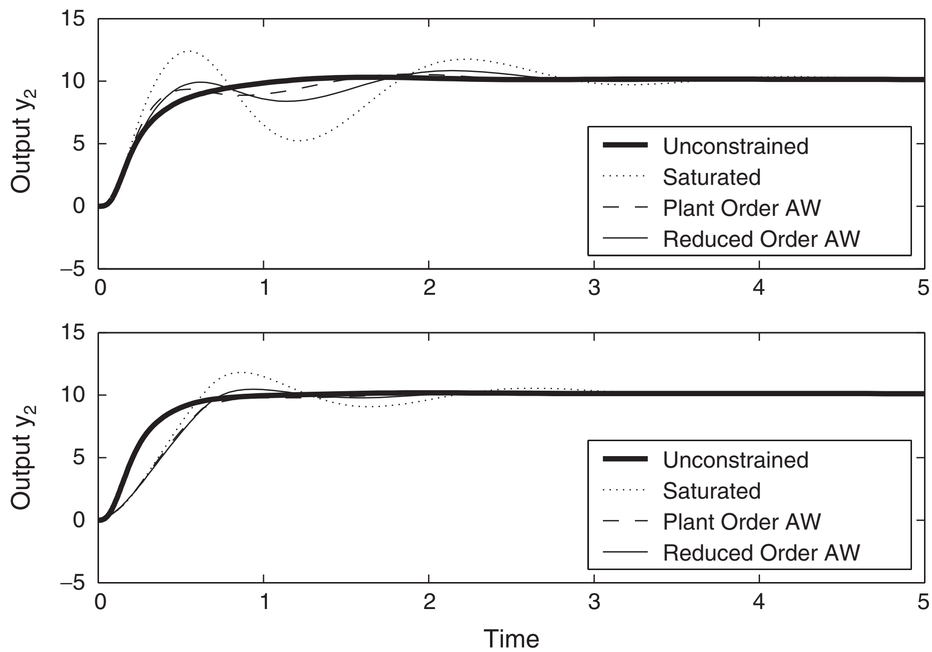

Figure 4. Unconstrained, saturated and anti-windup responses for the Example of section 5.2.

Figure 4 shows the simulation results for this example when the reference input w is selected as a step input switching from $[0\ 0]^T$ a $[0.1745\ 0.1745]^T$ at time $t=0$. The saturation levels for both the plant input are selected as ± 0.4363 (corresponding to a limitation of 25 degrees for the elevator and flaperon angles). The response deterioration arising when saturation is introduced in the unconstrained closed loop, as well as the performance recovery obtained by using either the plant order or the reduced order anti-windup is evident from figure 4. In accordance with the difference in the performance levels guaranteed by the two different anti-windup compensators (55.541 versus 78.016), in the considered simulation the reduced order anti-windup response exhibits larger oscillations than the plant order one. Nevertheless, it still guarantees a better response than the saturated closed-loop system.

6. Conclusions

Detailed constructive algorithms for reduced order anti-windup design have been provided in this paper, using an iterative convex relaxation of a non convex rank constraint arising in LMI based synthesis of reduced order anti-windup compensators. The proposed algorithms are especially relevant for all the cases where static anti-windup design is not feasible, or yields unacceptable performance. In such cases, the algorithms can be used either to design a low order anti-windup compensator guaranteeing stability and a minimized level of quadratic $\mathcal{L}_2$ performance, or to compute (whenever feasible) a low order anti-windup compensator guaranteeing a prescribed level of performance, thus giving the control engineer the possibility to trade-off between performance level and controller complexity (namely, anti-windup compensator order).

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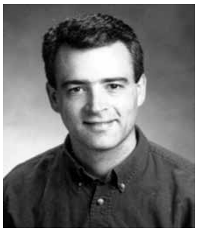

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