The Annals of Statistics
1978, Vol. 6, No. 4, 828–835

REDUCED U-STATISTICS AND THE HODGES–LEHMANN ESTIMATOR

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A reduced U-statistic (of order 2) is defined as the sum of terms $f(X_i, X_j)$, where f is a symmetric function, $(X_1, \cdots, X_N)$ are independent and identically distributed (i.i.d.) random variables (rv's), and (i, j) are drawn from a restricted, though balanced, set of pairs. (A U-statistic corresponds to summation over *all* (i, j) pairs.) A limit normal distribution is found for the reduced U-statistic, and it follows that estimates based on reduced U-statistics can have asymptotic efficiencies comparable with those based on U-statistics, even though the number of steps in computing a reduced U-statistic becomes asymptotically negligible in comparison with the number required for the corresponding U-statistic. As an illustration, a short-cut version of the Hodges–Lehmann estimator is defined, and its asymptotic properties derived, from a corresponding reduced U-statistic. A multivariate limit theorem is proved for a vector of reduced U-statistics, plus another result obtaining asymptotic normality even when (i, j) are drawn from an unbalanced set of pairs. The present results are related to those of Blom.

1. Introduction. Let $X_1, \cdots, X_N, \cdots$ be i.i.d. rv's, let $f(\cdot, \cdot)$ be a symmetric function, and C_K be a set of pairs (i, j), with $1 \leqq i < j \leqq N$, such that each positive integer $\leqq N$ is present in exactly $2K$ pairs of C_K. Thus, C_K contains exactly NK pairs, every one of which shares a common index with $2(2K-1)$ other pairs. (Values of $K = \frac{1}{2}, \frac{3}{2}, \cdots$ are possible when N is even, but we do not consider this possibility. Strictly speaking, C_K should be denoted by $C_{N,K}$, but for notational simplicity we suppress the dependence upon N.) Let

$$S_N = \sum_{C_K} f(X_i, X_j) .$$

If the summation were over all (i, j) pairs $(1 \leqq i < j \leqq N)$ rather than just C_K, S_N would be a U-statistic ([6]), say T_N. As it is, S_N could well be called something like a balanced incomplete U-statistic, but we prefer the simpler term *reduced* U-statistic. The computation of S_N involves a number of steps which as $N \to \infty$ becomes negligible in comparison with the number required to compute T_N; while $(NK)^{-1}S_N$ will be an unbiased estimator, as is $\{\frac{1}{2}N(N-1)\}^{-1}T_N$, for $\theta = E\{f(X_1, X_2)\}$.

In Theorem 1, we find a limit normal distribution, as $N \to \infty$, for S_N. This limit distribution depends upon a constant $\rho \geqq 0$ (to be defined in Section 2), and for the nonsingular case $\rho > 0$, the limit distribution has a variance which shows that $(NK)^{-1}S_N$, as an estimator of θ, has efficiency comparable to that of

Received August 1976; revised August 1977.

AMS 1970 *subject classifications*. Primary 60F05, 60G05; Secondary 60G20, 60G25.

Key words and phrases. U-statistics, Hodges–Lehmann estimator, asymptotic efficiency, convergence of moments.

the corresponding U-statistic estimator $\{\frac{1}{2}N(N-1)\}^{-1}T_N$, while involving a far smaller number of computations. This efficiency may be reasonable even for the simple estimator when $K=1$, while, in any case, choice of K suitably large ensures efficiency arbitrarily close to one, as long as $\rho > 0$.

Also, for the case $\rho\sigma^2 > 0$, it may be of interest to note that the efficiency is one if K is allowed to $\to \infty$ as $N \to \infty$. This can be seen by applying Hájek's projection method, which is the customary method of proving asymptotic normality of U-statistics, to show that in this case the reduced U-statistic and the (ordinary) U-statistic are asymptotically equivalent as $N \to \infty$.

Section 1 contains the statements of, and corollaries to, Theorems 1 and 2, the latter being a multivariate version of the former. Proofs are given in Section 3, while Section 4 contains a result (Theorem 3) under which S_N is still asymptotically normal even if the requirements of balance, on the sets C_K, are somewhat relaxed. Section 5 discusses, as an application of reduced U-statistics, a short-cut version of the Hodges–Lehmann (H–L) estimator.

Since the original version of the present paper was prepared, the paper of Blom [3] has appeared, and in it reduced U-statistics (termed *incomplete U-statistics* there) of orders $r \geqq 2$ are discussed. Variances are computed, several examples discussed, and asymptotic normality stated to hold under conditions similar to ours of Section 4 for $r=2$. It seems worth pointing out that the methods of proof used herein will work also for reduced U-statistics of orders $r > 2$; in the graph-theoretic language we employ, the structure of 2 vertices joined by an edge must be replaced by a structure of r vertices, every pair of which is connected by an edge. The language of graph theory is only a convenient way of handling counting problems; it is well suited to the case $r=2$ but becomes more unwieldy for $r > 2$.

2. Notation and results. In some applications it is desirable to replace the fixed function f by a sequence of symmetric functions $\{f_N, N \geqq 1\}$, in the definition of S_N. To include this case, let

$$S_N = \textstyle\sum_{C_K} f_N(X_i, X_j)\,,$$
$$\theta_N = Ef_N(X_1, X_2)\,,$$
$$\sigma_N^2 = \operatorname{Var} f_N(X_1, X_2)\,,$$

and

$$\rho_N \sigma_N^2 = \operatorname{Cov}\{f_N(X_1, X_2), f_N(X_1, X_3)\}\,.$$

It then follows easily that

$$\operatorname{Var}(S_N) = NK\sigma_N^2(1 + 2(2K-1)\rho_N)\,. \tag{1}$$

Our main result is

THEOREM 1. *If the finite limits* $\sigma^2 = \lim_{N\to\infty} \sigma_N^2$ *and* $\rho\sigma^2 = \lim_{N\to\infty} \rho_N \sigma_N^2$ *both exist, if* $\sigma^2 > 0$, *and if*

$$\{f_N(X_1, X_2) - \theta_N, N \geqq 1\} \quad \textit{is uniformly square integrable,} \tag{2}$$

then $(NK)^{-\frac{1}{2}}(S_N - NK\theta_N)$ *converges in distribution as* $N \to \infty$ *to a normal law with mean zero and variance* $\sigma^2(1 + 2(2K - 1)\rho)$.

$\{\frac{1}{2}N(N-1)\}^{-1}T_N$ is the U-statistic estimator of θ_N corresponding to $(NK)^{-1}S_N$, and has variance $2\sigma_N^2\{N(N-1)\}^{-1}\{1 + 2\rho_N(N-2)\}$, so an immediate consequence of Theorem 1 is

COROLLARY 1. *When* $\rho\sigma^2 > 0$, *the estimators* $\{(NK)^{-1}S_N, N \geqq 1\}$ *of* $\{\theta_N, N \geqq 1\}$ *have asymptotic efficiency* $2K\rho\{\frac{1}{2} + (2K-1)\rho\}^{-1}$, *relative to the corresponding U-statistic estimators* $[\{\frac{1}{2}N(N-1)\}^{-1}T_N, N \geqq 1]$, *as* $N \to \infty$.

The above expression for U-statistic variance shows that $\rho_N \geqq -(N-2)^{-1}$ for all N, and hence that $\rho \geqq 0$. On the other hand, by letting

$$Z = f_N(X_1, X_2) + f_N(X_1, X_3) + f_N(X_2, X_4) + f_N(X_3, X_4) - 2\{f_N(X_2, X_3) + f_N(X_1, X_4)\},$$

and simplifying the equation $0 \leqq E(Z^2)$, we find that ρ_N, and hence ρ, is $\leqq \frac{1}{2}$. Thus $0 \leqq \rho \leqq \frac{1}{2}$, and the efficiency given in the corollary $\in [0, 1]$. However, for any fixed $\rho > 0$, the efficiency can be made arbitrarily close to one by taking K large enough, and it may even be possible that the extremely simple estimator when $K = 1$ yields a reasonable efficiency. For example, in Section 5, reduced U-statistics lead to a simple version of the H–L estimator. In this case, $\rho = \frac{1}{3}$ and we obtain efficiency $4K(4K+1)^{-1}$, which is already $\frac{4}{5}$ for $K = 1$.

A multivariate version of Theorem 1 is

THEOREM 2. *Under the conditions and notation of Theorem* 1, *let* $S_N^{(1)}, \cdots, S_N^{(p)}$ *be reduced U-statistics corresponding to sets (of pairs)* $C_{K_1}^{(1)}, \cdots, C_{K_p}^{(p)}$. *Then* $\{S_N^{(1)}, \cdots, S_N^{(p)}\}$, *when suitably normalized, converges in distribution as* $N \to \infty$ *to a multivariate normal distribution.*

The covariance structure of the limit multinormal distribution is determined by the limiting form of the covariances between $S_N^{(\alpha)}, S_N^{(\beta)}$. These however are not easy to specify unless the $\{C_{K_\alpha}^{(\alpha)}, 1 \leqq \alpha \leqq p\}$ are disjoint as in

COROLLARY 2. *Let* $\{C_{K_\alpha}^{(\alpha)}, 1 \leqq \alpha \leqq p\}$ *be disjoint. Then for* $\alpha \neq \beta$,

$$\operatorname{Cov}(S_N^{(\alpha)}, S_N^{(\beta)}) = 4NK_\alpha K_\beta \rho_N \sigma_N^2 \tag{3}$$

and the covariance structure of the limit distribution in Theorem 2 *is determined.*

PROOF. Use (1), Theorem 1, and the fact that $C_{K_\alpha}^{(\alpha)} \cup C_{K_\beta}^{(\beta)}$ is a set of type $C_{K_\alpha + K_\beta}$, to evaluate the limit distribution and variance of $S_N^{(\alpha)} + S_N^{(\beta)}$.

3. Proofs.

PROOF OF THEOREM 1. The proof is divided into a preliminary section (A), and a main section (B) in which the moments of $(NK)^{-\frac{1}{2}}(S_N - NK\theta_N)$ are shown to converge as $N \to \infty$ to those of the limit normal distribution. For notational simplicity, the suffixes N belonging in f_N, θ_N, ρ_N, and σ_N^2 are suppressed.

(A) We may assume without loss of generality that $|f| \leqq M$, for otherwise we could set $g = fI_{[|f| \leqq M]}$ and $h = fI_{[|f| > M]}$, with $h(X_1, X_2)$ having mean μ_h, variance σ_h^2, and with $\text{Cov}\,\{h(X_1, X_2), h(X_1, X_3)\} = \rho_h \sigma_h^2$; then write

(4) $$N^{-\frac{1}{2}}(S_N - NK\theta) = N^{-\frac{1}{2}} \sum_{C_K} \{g(X_i, X_j) - \mu_g\} + N^{-\frac{1}{2}} \sum_{C_K} \{h(X_i, X_j) - \mu_h\} .$$

By applying the formula (1), with h replacing f, we see that the second term on the right-hand side of (2) has variance $K\sigma_h^2(1 + 2(2K - 1)\rho_h)$, which is made arbitrarily small by taking M large, since $\lim_{M\to\infty} \sigma_h^2 = 0$, from (2). Thus the right-hand term of (2) is made stochastically small by taking M large, and attention may by confined to the term involving $\sum g(X_i, X_j)$, where $|g| \leqq M$. Equivalently, we may and do assume at the outset that $|f| \leqq M$.

(B) Assume without loss of generality that $\theta = 0$ (or else replace f by $f - \theta$). For $r \geqq 2$,

(5) $$ES_N^r = \sum E \prod_{\nu=1}^r f(X_{i_\nu}, X_{j_\nu}) ,$$

where the summation is over all pairs $(i_1, j_1), \cdots, (i_r, j_r) \in C_K$. To every term in this sum there corresponds an undirected multigraph (henceforth called a *graph*) with vertices $i_1, j_1, \cdots, i_r, j_r$ and r edges, joining vertices i_ν and j_ν for $\nu = 1, 2, \cdots, r$.

Firstly,

(6) the number of terms of (5) having graphs with m connected components is $O(N^m)$ as $N \to \infty$.

To see this, let the numbers of edges of the m connected components be $r_1, \cdots, r_m$, with $\sum_1^m r_i = r$. From the structure of C_K, the number of ways of achieving this is

$$\leqq \prod_{i=1}^m (NK)(2K)^{r_i-1} r_i! = O(N^m)$$

as $N \to \infty$, and summing over all $(r_1, \cdots, r_m)$ still leaves $O(N^m)$ possibilities.

Next, any term of (5), whose graph contains a connected component with only one edge, equals zero, since $Ef(X_i, X_j) = 0$ for $i \neq j$. It follows immediately from (6) and the boundedness of f that

$$ES_N^r = O(N^{\frac{1}{2}(r-1)}) = o(N^{\frac{1}{2}r})$$

as $N \to \infty$, when r is odd.

Similarly, when r is even

(7) $$ES_N^r = \sum{}^* E \prod_{\nu=1}^r f(X_{i_\nu}, X_{j_\nu}) + O(N^{\frac{1}{2}r-1}) ,$$

where $\sum^*$ denotes summation over only those terms whose graphs have exactly $\frac{1}{2}r$ connected components, each with 2 edges.

Now the derivation of (7) also holds if the $\{f(X_i, X_j)\}$ are replaced by jointly normal rv's $\{Y_{ij}\}$ with $EY_{ij} = 0$, $\text{Var}\,(Y_{ij}) = \sigma^2$, $\text{Cov}\,(Y_{ij}, Y_{ik}) = \rho\sigma^2$ for $j \neq k$ and $\text{Cov}\,(Y_{ij}, Y_{kl}) = 0$ for i, j, k, l all different. (The $\{Y_{ij}\}$ are not bounded but the normal distribution implies all appropriate moments finite.) However, the sum $\sum^*$ involves only expectations of products of two factors f, and so is unchanged by substitution of $\{Y_{ij}\}$ for $\{f(X_i, X_j)\}$. But then S_N has a zero-mean normal distribution and $ES_N{}^r = r!\, 2^{-\frac{1}{2}r}\{E(S_N{}^2)\}^{\frac{1}{2}r}/(\frac{1}{2}r)!$ for r even. These considerations imply that for r even,

$$ES_N{}^r = \frac{r!\, 2^{-\frac{1}{2}r}}{(\frac{1}{2}r)!}\{E(S_N{}^2)\}^{\frac{1}{2}r} + o(N^{\frac{1}{2}r}) \qquad \text{as} \quad N \to \infty\,,$$

and the proof is complete.

PROOF OF THEOREM 2. To find the limit moments of $\sum_{\alpha=1}^{p} \lambda_\alpha S_N{}^{(\alpha)}$, for arbitrary $\{\lambda_\alpha\}$, reason as in the proof of Theorem 1 to show the odd moments to be $o(N^{\frac{1}{2}r})$, and for even r

$$E\{\textstyle\sum_\alpha \lambda_\alpha S_N{}^{(\alpha)}\}^r = \sum^0 E \prod_{\nu=1}^{r} \tau(i_\nu, j_\nu) f(X_{i_\nu}, X_{j_\nu}) + O(N^{\frac{1}{2}r-1})$$

as $N \to \infty$, where $\sum^0$ denotes summation over those $(i_1, j_1), \cdots, (i_r, i_r) \in \bigcup_{\alpha=1}^{p} C_{K_\alpha}^{(\alpha)}$ whose graphs have $\frac{1}{2}r$ connected components, each of 2 edges, and where

$$\tau(i, j) = \textstyle\sum_{\{\alpha:(i,j)\in C_{K_\alpha}^{(\alpha)}\}} \lambda_\alpha\,.$$

The reasoning to complete the proof is as in the proof of Theorem 1.

4. A modification. In this section it is shown that the asymptotic normality of S_N may hold even if the sets C_K are replaced by more general sets.

THEOREM 3. *Let $C^{(N)}$ be a set of pairs (i, j), $1 \leqq i < j \leqq N$, such that the index i occurs exactly $\nu_i = \nu_{N,i}$ times in $C^{(N)}$, and let*

$$Q_j = Q_{N,j} = \textstyle\sum_{i=1}^{N} \nu_{N,i}^j\,.$$

Also let $S_N = \sum_{\{(i,j)\in C^{(N)}\}} f_N(X_i, X_j)$, and let θ_N, $\sigma_N{}^2$ and $\rho_N \sigma_N{}^2$ be as defined in Section 2.

If the conditions of Theorem 1 hold, and if either

$$\lim_{N\to\infty} Q_3 Q_2^{-\frac{3}{2}} = 0 \qquad \textit{for} \quad \rho > 0\,, \tag{8}$$

or

$$\lim_{N\to\infty} Q_3 Q_1^{-\frac{3}{2}} = 0 \qquad \textit{for} \quad \rho = 0\,, \tag{9}$$

then

$$\{(\tfrac{1}{2} - \rho)Q_1 + \rho Q_2\}^{-\frac{1}{2}}\{S_N - \tfrac{1}{2}\theta_N Q_1\} \to_{\mathscr{L}} N(0, \sigma^2)$$

as $N \to \infty$.

REMARK. Let $m = m_N = \max_{i\leqq N} \nu_{N,i}$. By applying the inequality $m^3 \leqq Q_3 \leqq mQ_2$ to the numerator in (8), it is seen that (8) is equivalent to

$$\lim_{N\to\infty} mQ_2^{-\frac{1}{2}} = 0\,. \tag{10}$$

There seems not to be a similar equivalence for (9), although (9) does imply that

$$\lim_{N\to\infty} mQ_1^{-\frac{1}{2}} = 0\,. \tag{11}$$

PROOF OF THEOREM 3. Consider the terms of ES_N^r (cf. (5)) whose graphs have $s + s'$ connected components, of which s have 2 edges and s' have at least 3 edges. A component with $\gamma \geqq 3$ edges must contain either a vertex where three edges meet, or two vertices connected by an edge and at least one other edge at both vertices, so the number of such components is

$$O\{(Q_3 + \sum_{(i,j)\in C} \nu_i\nu_j)m^{\gamma-3}\} = O(Q_3 m^{\gamma-3})$$

since $\sum_C \nu_i\nu_j \leqq \frac{1}{2}\sum_C (\nu_i^2 + \nu_j^2) = \frac{1}{2}Q_3$.

It follows that the number of terms of the above type in ES_N^r is $O(Q_2^s Q_3^{s'} m^{r-2s-3s'})$ if $\rho > 0$ (cf. the proof of Theorem 1), while if $\rho = 0$, then two edged components with three vertices are zero, and the above number of terms in ES_N^r is reduced to $O(Q_1^s Q_3^{s'} m^{r-2s-3s'})$.

If $s < \frac{1}{2}r$, whence $s' > 0$, (8) and (10) for $\rho > 0$, and (9), (11) for $\rho = 0$ imply that the above numbers of terms are $o(Q_2^{\frac{1}{2}r})$ and $o(Q_1^{\frac{1}{2}r})$ respectively, which is $o(ES_N^2)^{\frac{1}{2}r}$ in both cases, since $\operatorname{Var} S_N = \frac{1}{2}\sigma^2 Q_1 + \rho\sigma^2(Q_2 - Q_1)$.

From this point, the argument follows on as in the proof of Theorem 1.

5. A simple Hodges–Lehmann estimator. Suppose for $j = 1, 2, \cdots, N$ that $X_j = \theta + Y_j$, where the $\{Y_j\}$ are i.i.d. rv's, symmetric about zero, with df G and continuous bounded density g. The H–L estimator of θ (see [5]) is the median of $\{\frac{1}{2}(X_i + X_j), 1 \leqq i, j \leqq N\}$, and an asymptotically equivalent estimator is $\hat{\theta}_N$, the median of $\{\frac{1}{2}(X_i + X_j), 1 \leqq i < j \leqq N\}$.

Theorem 1 suggests that a *reduced* H–L estimator of θ be defined as

$$\xi = \operatorname{median}_{(i,j)\in C_K} \{\tfrac{1}{2}(X_i + X_j)\},$$

an estimator whose computation involves a number of steps which as $N \to \infty$ becomes negligible in comparison with the number required to compute the H–L estimator $\hat{\theta}_N$.

We now derive the asymptotic behavior of ξ_N as $N \to \infty$. For fixed x let

$$S_N = \textstyle\sum_{C_K} I\{X_i + X_j \leqq 2\theta + 2xN^{-\frac{1}{2}}\}, \quad = \sum_{C_K} I\{Y_i + Y_j \leqq 2xN^{-\frac{1}{2}}\}.$$

Then

$$ES_N = NKG^{2*}(2xN^{-\frac{1}{2}}) = NK\{\tfrac{1}{2} + 2xN^{-\frac{1}{2}}g_N\},$$

where $\lim_{N\to\infty} g_N = g_0 = \int_{-\infty}^{\infty} g^2(y)\,dy$.

By setting $f_N(Y_1, Y_2) = I(Y_1 + Y_2 \leqq 2xN^{-\frac{1}{2}})$, it is not difficult to show that (2) holds, and that $\sigma^2 = \frac{1}{4}$, $\rho = \frac{1}{3}$ so that Theorem 1 can be applied, giving the limit distribution of $(NK)^{-\frac{1}{2}}\{S_N - ES_N\}$, as $N \to \infty$, to be

$$N(0, (4K + 1)/12).$$

But

$$P[N^{\frac{1}{2}}(\xi_N - \theta) \leqq x] = P[S_N > \tfrac{1}{2}NK], \quad = P[(NK)^{-\frac{1}{2}}(S_N - ES_N) > (NK)^{-\frac{1}{2}}(-2xN^{\frac{1}{2}}Kg_N)], \quad \to \Phi\{2xK^{\frac{1}{2}}g_0(12)^{\frac{1}{2}}(4K + 1)^{-\frac{1}{2}}\}, \quad N \to \infty, \tag{12}$$

identifying the limit distribution of $\{N^{\frac{1}{2}}(\xi_N - \theta)\}$ as

$$N\left(0, \frac{4K+1}{48Kg_0^2}\right).$$

This should be compared (see [4]) with the asymptotic distribution $N(0, \{12g_0^2\}^{-1})$ for $N^{\frac{1}{2}}(\hat{\theta}_N - \theta)$, as $N \to \infty$. The efficiency of the reduced H–L estimators $\{\xi_N\}$ relative to the H–L estimators $\{\theta_N\}$ is therefore $4K(4K+1)^{-1}$, which is $\frac{4}{5}$ for $K = 1$, and is made arbitrarily close to one by taking K suitably large.

The efficiency of reduced H–L estimators should also be compared with that of the short-cut H–L estimator of [2], where a simple procedure has high efficiency, but not an asymptotically normal distribution. Antille [1] has a one-step method of evaluating an asymptotic equivalent of the H–L estimator, with the number of steps of computation of the same order as for the reduced H–L estimator described herein. His procedure is therefore certainly superior to ours in an asymptotic sense, although whether it remains so for moderate sample sizes is another question.

In the case $K > 1$, Theorem 2 suggests an estimator asymptotically equivalent to $\{\xi_N\}$, but involving still less computation because of a reduction in the median-finding operation. In this case, choose a C_K consisting of the union of K disjoint sets $C_1^{(1)}, \cdots, C_1^{(K)}$, each obeying the requirements on C_1, then form the corresponding reduced H–L estimators $\xi_N^{(1)}, \cdots, \xi_N^{(K)}$. It follows easily from Theorem 2 and its corollary that the estimator

$$\xi_N' = K^{-1} \sum_{j=1}^{K} \xi_N^{(j)}$$

is asymptotically as efficient as ξ_N. Moreover, by using Theorem 2 and its corollary in conjunction with a multivariate version of the inversion equation (12), it is possible to verify that ξ_N and ξ_N' are asymptotically equivalent, in the sense that

$$N^{\frac{1}{2}}(\xi_N - \xi_N') \to_p 0 \qquad \text{as} \quad N \to \infty.$$

Acknowledgment. The Associate Editor made a very great contribution to the clarity and scope of this paper. In particular, the method of proof of Theorem 1 (which enables the extensions of Theorems 2 and 3 to be made) is due to him.

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