REDUCIBILITIES AMONG DECISION PROBLEMS
FOR HNN GROUPS, VECTOR ADDITION SYSTEMS
AND SUBSYSTEMS OF PEANO ARITHMETIC

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ABSTRACT. Our purpose is to exhibit reducibilities among decision problems for conjugate powers in HNN groups, reachability sets of vector addition systems and sentences in subsystems of Peano arithmetic, and show that although these problems are not primitive recursively decidable, they do admit decision procedures which are primitive recursive in the Ackermann function.

By the class of *vector groups* VA we understand the HNN groups $G(p_1, q_1, \ldots, p_k, q_k)$ given by

$$(1) \qquad \left(a_1, \ldots, a_k, b; \ a_1^{-1} b^{p_1} a_1 = b^{q_1}, \ldots, a_k^{-1} b^{p_k} a_k = b^{q_k}\right),$$

where the exponent pairs p_i, q_i occurring in (1) are positive and relatively prime. (For concepts and results of a group-theoretic nature not explicitly discussed here the reader should consult Lyndon and Schupp [5].)

Let G be a vector group, m a *positive conjugate* power of l in G when $b^m = xb^l x^{-1}$, where x in G is given by a positive word in the generators $a_1, \ldots, a_k$, b of G (i.e. one which involves no negative exponents). The set of positive conjugate powers of l in G, or *positive conjugate power set* is denoted PCP(l, G).

By the *equality problem* for positive conjugate power sets, we mean the question of determining for any integers l_1, l_2 and vector groups G_1, G_2, whether PCP(l_1, G_1) = PCP(l_2, G_2). The *special equality problem* is to decide the equality problem in those cases where $l_1 = l_2$ and G_2 arises from G_1 by removing a particular generating symbol a_i and its corresponding defining relation $a_i^{-1} b^{p_i} a_i = b^{q_i}$ from the presentation of G_1 as in (1). The *finite special equality problem* is to decide the special equality problem in those cases where PCP(l_1, G) is finite.

We identify a decision problem with the set of Gödel numbers of its instances and use this identification to discuss the complexity of the problem.

A function g is *primitive recursive* in a function f iff g is in the class obtained by primitive recursion and composition from f together with the usual initial functions.

It is shown in Anshel [1] that the special equality problem for vector groups is undecidable. In contrast, we will prove

THEOREM 1. *The finite special equality problem for vector groups is*
(i) *decidable but not primitive recursive,*
(ii) *primitive recursive in the Ackermann function.*

By a *vector addition system* (*of dimension n*) we mean a pair (d, W), d in N^n, W a finite subset of Z^n (where N and Z denote the nonnegative integers and integers respectively). Let us call d' *reachable* in (d, W) if $d = d'$ or $d' = d + w_1 + \cdots + w_t$, w_i in W and $d + w_1 + \cdots + w_s$ in N^n, $s = 1, \ldots, t$.

Let $R(d, W)$ denote the reachable vectors in (d, W) or *reachability set* of (d, W). Let K be a class of vector addition systems. By the *equality problem* for K we mean the problem of deciding for arbitrary (d_1, W_1), (d_2, W_2) in K whether $R(d_1, W_1) = R(d_2, W_2)$. The *special equality problem* for K is to decide the equality problem in those cases where $d_1 = d_2$ and W_2 arises from W_1 by removal of a vector.

Let VAS denote the class of vector addition systems and BVAS denote the *bounded* vector addition systems (i.e. (d, W) is in BVAS iff $R(d, W)$ is finite). (For a general introduction to vector addition systems via their graph-theoretic counterparts, Petri nets, the reader should consult Peterson [9].)

We say that a function f is *Kalmar-elementary* or simply *elementary*, if f can be obtained by composition, explicit transformation and limited recursion from the exponentiation and successor functions. Thus f is elementary if f is in the Grzegorcyck class $\mathcal{E}_3$; in machine theoretic terms f is elementary if f can be computed by a Turing machine bounded in space or time by an iterated exponential function of the input length, that is, *predictably computable* in the sense of Ritchie. For an exposition of these results the reader may consult Yasuhara [12].

We say that a decision problem A is *elementarily reducible* to B provided there is an elementary function $g: A \to B$ such that each instance x of A may be decided by computing the instance $g(x)$ of B and then deciding $g(x)$. We say that A and B are *elementarily bireducible* whenever each problem is elementarily reducible to the other.

We observe that if A and B are elementarily bireducible, then A is primitive recursively decidable (in the Ackermann function) precisely when B is primitive recursively decidable (in the Ackermann function).

LEMMA 1. *The* (*finite*) *special equality problem for vector groups is elementarily bireducible to the special equality problem for* (*bounded*) *vector addition systems.*

PROOF. The following reduction of the (finite) special equality problem for vector groups to the special equality problem for (BVAS) VAS is based on one in [1]. We may assume w.l.o.g. that l is a nonzero integer, G_1 is given as in (I) and G_2 arises from G_1 by removal of a_{i_0} and $a_{i_0}^{-1} b^{p_{i_0}} a_{i_0} = b^{q_{i_0}}$.

Reduction Algorithm A.

Step 1. Input l, G_1, G_2.

Step 2. Create a list $c_1, \ldots, c_n$ of the positive prime divisors of the exponents of G_1.

Step 3. For $i = 1, \ldots, k$, create the exponent lists $e_{1i}, \ldots, e_{ni}$ and $e'_{1i}, \ldots, e'_{ni}$ where $p_i = c_1^{e_{ni}} \cdots c_n^{e_{ni}}$ and $q_i = c^{e'_{1i}} \cdots c_n^{e'_{ni}}$.

Step 4. Compute the list $s_1, \ldots, s_n$ of nonnegative integers uniquely determined by the equation $l = c_1^{s_1} \cdots c_n^{s_n} l'$ where l' is relatively prime to each c_i, $i = 1, \ldots, n$.

Step 5. Create the vector addition systems (d_1, W_1), (d_2, W_2) where $d = (s_1, \ldots, s_n)$, W_1 consists of $w_1, \ldots, w_k$ and $W_2 = W_1 - \langle w_{i_0} \rangle$, such that $w_j = (u_{j1}, \ldots, u_{jn})$ with

$u_{jt}, t = 1, \ldots, n,$ given as in the following manner:

$$u_{jt} = \begin{cases} e_{jt}, & \text{if } e_{jt} \neq 0, \\ -e'_{jt}, & \text{otherwise.} \end{cases}$$

Step 6. Output $(d_1, W_1), (d_2, W_2)$.

From Lemma 1 of [1] we have that $\mathrm{PCP}(l, G_1) = \mathrm{PCP}(l, G_2)$ iff $R(d, W_1) = R(d, W_2)$. Moreover, $\mathrm{PCP}(l, G_1)$ is finite iff $R(d, W_1)$ is finite. That algorithm A above defines an elementary function follows from the fact that the lists created in Steps 2–4 are computable by elementary functions.

The required reduction of the special equality problem for (B)VAS to the (finite) special equality problem for vector groups is exhibited below. Let (d, W_1), (d, W_2) be vector addition systems where $d = (s_1, \ldots, s_n)$, $W_1 = \langle w_1, \ldots, w_k \rangle$, $W_2 = W_1 - \langle w_{i_0} \rangle$ and $w_i = (u_{1i}, \ldots, u_{ni})$, $i = 1, \ldots, k$. Also let c_j denote the jth positive prime, $j = 1, \ldots, n$.

Reduction Algorithm B.

Step 1. Input $(d, W_1), (d, W_2)$.

Step 2. Compute $l = c_1^{s_1} \cdots c_n^{s_n}$.

Step 3. For $i = 1, \ldots, k$, compute $p_i = c_1^{e_{i1}} \cdots c_n^{e_{in}}$, $q_i = c_1^{e'_{1i}} \cdots c_n^{e'_{ni}}$, such that $j = 1, \ldots, n,$

$$e_{ij} = \begin{cases} u_{ij}, & \text{if } u_{ij} \geqslant 0, \\ 0, & \text{otherwise.} \end{cases} \qquad e'_{ij} = \begin{cases} -u_{ij}, & \text{if } u_i \leqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 4. Create the vector groups G_1, G_2; $G_1 = G(p_1, q_1, \ldots, p_k, q_k)$ and G_2 obtained from G_1 by omitting a_{i_0} and the relation $a_{i_0}^- b^{p_{i_0}} a_{i_0} = b^{q_{i_0}}$ from the presentation of G_1.

Step 5. Output l, G_1, G_2.

Now $R(d, W_1) = R(d, W_2)$ iff $\mathrm{PCP}(l, G_1) = \mathrm{PCP}(l, G_2)$, and $R(d, W_1)$ is finite iff $\mathrm{PCP}(l, G_1)$ is finite. That algorithm B defines an elementary function follows from the fact that $l, p_i, q_i, i = 1, \ldots, k$, are computable by elementary functions. $\square$

The *boundedness problem* for a class K of vector addition systems (resp. vector groups) is to decide for any (d, W) in K (resp. G in K and integer l) whether $R(d, W)$ is finite (resp. $\mathrm{PCP}(l, G)$ is finite).

COROLLARY 1. *The boundedness problem for vector groups is elementary recursive.*

PROOF. Rackoff [10] has shown that the boundedness problem for VAS is elementary recursive. The result is then immediate from Lemma 1 above. $\square$

LEMMA 2. *The special equality problem for bounded vector addition systems is* (i) *decidable,* (ii) *not primitive recursive but* (iii) *primitive recursive in the Ackermann function.*

PROOF. (i) With each (d, W) in VAS we can recursively associate its Karp-Miller coverability tree T which is finite (cf. Karp and Miller [3]). Moreover, if $R(d, W)$ is finite, then the nodes of T consist precisely of the reachable vector of (d, W). Thus

given (d_1, W_1) and (d_2, W_2) in BVAS, to compare their reachability sets, it is enough to compare their associated Karp-Miller coverability trees.

(ii) It follows from the proof of the reduction of the reachability set inclusion problem (which asks if $R(d_1, W_1)$ is contained in $R(d_2, W_2)$) to the equality problem, as given in Hack [2], that the inclusion problem for (B)VAS is elementarily bireducible to the special equality problem for (B)VAS (a fact also understood by Mayr [6]). It is a consequence of the Corollary to Theorem 7 of Mayr [6] that the inclusion problem for BVAS is not primitive recursively solvable, and neither is the special equality problem for BVAS.

(iii) By (i) it is enough to give an upper bound for the map $KM\colon (d, W) \mapsto T$ which associates with each (d, W) in VAS its Karp-Miller coverability tree T. The proof that KM is primitive recursive in the Ackermann function is carried out in McAloon [7]. The main steps are the following: Let $I\Sigma_1$ be the subsystem of Peano Arithmetic obtained by restricting the induction scheme to Σ_1^0-formulas (i.e. to those of the form $(\exists \vec{x})\Phi(\vec{x}, \vec{n})$ where Φ contains only bounded quantifiers). Now given (d, W), one has

$$(*) \qquad\qquad I\Sigma_1 \vdash \exists T \big(T = KM(\bar{d}, \bar{w})\big).$$

To obtain $(*)$ one has to analyze the proof that the tree T associated with (d, W) is, in fact, finite. One then has

$$(**) \qquad I\Sigma_1 + 1 - \text{Consistency } (I\Sigma_1) \vdash KM \text{ is a total function}$$

(where $1 - \text{Consistency } (I\Sigma_1)$ asserts that every Σ_1^0-sentence provable in $I\Sigma_1$ is true). By applying results of Kirby and Paris [4] and Paris [8], all recursive functions provably total in $I\Sigma_1 + 1 - \text{Consistency } (I\Sigma_1)$ are primitive recursive in the Ackermann function. $\square$

Observe that Theorem 1 follows directly from Lemmas 1 and 2 above.

Let $(B)VAS_n$ denote the class of (bounded) vector addition systems of dimension n. For each of the decision problems discussed above, there is naturally a version at dimension n. We have the following result.

THEOREM 2. *Let $n \geqslant 0$; the inclusion, equality and special equality problem for $BVAS_n$ are all primitive recursive.*

IDEA OF PROOF. For each dimension n, we have again by an analysis of the proof that the Karp-Miller coverability tree associated with a system is finite

$$I\Sigma_1 \vdash \forall(d, W) \big[(d, W) \text{ of dimension } n \to \exists T \big(T = KM(\bar{d}, \overline{W})\big)\big].$$

Hence, by the methods of Proof Theory (cf. Takeuti [11]) or by the methods of Model Theory (cf. Kirby and Paris [4], Paris [8]) the function KM restricted to systems of dimension n is primitive recursive.

REMARK. The above result shows that the complexity of the special equality problem for BVAS comes from the fact that the arbitrary dimensions are considered.

REMARK. The above result also leads to some open questions as to what reducibilities exist among the various decision problems at dimension n. From the work of

Hack [2] we do know that the inclusion problem for (B)VAS at dimension n is polynomial time reducible to the special equality problem for (B)VAS at dimension $n + 5$.

Our reductions imply that there should be a notion of dimension of vector groups. The one which emerges is the following: define the dimension of a vector group $G(p_1, q_1, \ldots, p_k, q_k)$ as the number of distinct positive prime divisors of the exponents p_i, q_i, $i = 1, \ldots, k$. We then have

COROLLARY 2. *Let $n \geqslant 0$; the finite special equality problem for VA_n, the class of vector groups of dimension n, admits a primitive recursive decision procedure.*

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