# Reducibility among Geometric Location-Allocation Optimization Problems*

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*ABSTRACT*

Three different classes of multiple points location-allocation problems in the Euclidean plane are considered under a discrete optimization criterion which minimizes the maximum cost based on certain interpoint distances. Each of these classes of geometric optimization problems is studied with three different distance metrics (Euclidean, Rectilinear, Infinity) as well as for feasible solution sets in the plane which are both discrete and infinite. All of these problems are shown to be polynomial-time reducible to each other and furthermore $D^p$ *complete*.

*Keywords*: Geometric Optimization, Polynomial-time Reducibility, Complexity Theory

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## 1. Introduction

A minimax location objective is one which minimizes the maximum cost resulting from a given location solution. Various applications have been raised in facility location theory [FW74],[KP79], where there exists sufficient justification in minimizing the effects of the worst situation. A common real world situation that might be formulated with a minimax objective is one of locating health clinics so that the maximum distance a patient must travel to a clinic is minimized. Another example concerns the placement of fire stations in a large metropolitan area such that the maximum distance between any location within the city and the nearest fire station is minimized.

Under this minimax location objective it is possible to distinguish two basic approaches that have been taken in the literature on sources location. The first suggests that a location site may be selected anywhere in the area of interest on the plane, giving an infinite number of possible location sites. The second approach considers only a finite number of known sites as feasible and models the constraints imposed on the possible location of sources, ensuring that undesirable and impractical locations need not be considered. The various distance metrics used, *Euclidean* ($l_2$), *Rectilinear* ($l_1$) and *Infinity* $l_\infty^{\dagger}$ , reflect the appropriate travel restrictions for the emergency vehicles of the problem, e.g. ambulances, helicopters, fire engines etc..

In this paper we analyze the complexity of certain geometric optimization problems which arise frequently in the above application areas, amongst others. We consider three different classes of multiple points location-allocation problems in the plane under a minimax optimization objective. We are given a set $T=\{(x_i,y_i),\ i=1..n\}$, the location of $n$ fixed destination points (destinations) in the plane and need to locate a number of points (sources) to service the destinations in a way which behooves the application on hand.

Three different optimizing objectives are considered as $P_1$, $P_2$ and $P_3$ below. In the case of locating multiple sources, the allocation of the destinations to the sources must also be ascertained. A common assumption for these problems [C63]

† Between two points $a=(a_x,a_y)$ and $b=(b_x,b_y)$ in the plane the $l_1$ distance is $|a_x-b_x|+|a_y-b_y|$ ; the $l_2$ distance is $\sqrt{[(a_x-b_x)^2+(a_y-b_y)^2]}$ ; and the $l_\infty$ distance is $\max(|a_x-b_x|,|a_y-b_y|)$

is that the sources are considered to be uncapacitated; that is there are no capacity limitations. As a consequence, each destination can be completely serviced by a single source though a source can itself service more than one destination. Furthermore in the optimal solution each destination is allocated to its closest located source. However this optimal allocation, which is just one of the exceedingly large number of possible allocations†, is not known a priori and needs to be determined.

Given the set $T$, as specified before, of $n$ destinations in the plane

($P_1$) Locate $k$ points (sources) so as to *minimize* the *maximum* of the weighted distances between the destinations and the sources closest to them.

($P_2$) Locate $k$ points (sources) so that for a *maximum* number of destinations the weighted distances of these destinations from their closest sources does not exceed a prescribed limit $R$.

($P_3$) Locate a *minimum* number of points (sources) so that the *maximum* of the weighted distances of the destinations and their closest sources does not exceed a prescribed limit $R$.

The weighted distances mentioned above come from a weight $w_j$ assigned to each destination $j$ and is some measure of the special cost of serving destination $j$ in traveling from its closest source . However in the following problems we assume that all weights are equal (similar to assuming that $w_j = 1$, for $j=1..n$) and show that even for this restricted case the above problems are quite difficult.

Problems $P_1$, $P_2$ and $P_3$ allow location of the sources to be anywhere in the plane.

Let problems $Q_1$, $Q_2$ and $Q_3$ correspond to problems $P_1$, $P_2$ and $P_3$ respectively, with the location of the sources being restricted to a finite discrete set $S$ of possible locations in the plane and of size polynomial in $n$.

The capacitated versions of these geometric location-allocation (with sources having finite capacities), turn out to be various cases of the more familiar transportation location problems and under discrete solution space constraints, to be the plant location and warehouse location problems [FW74].

## 2. Problems with the Euclidean distance metric

Under the minimax criterion with *Euclidean* $l_2$ distance metrics each of the above location-allocation problems reduces in a direct fashion to the location of equal radius circular disks (circles) on the Euclidean plane, with the centers of the circles corresponding to the location of the sources. Further all the destinations covered by the same circle correspond also in a direct fashion to an allocation to the same source. Having equal weights for the above problems results in equal sized circles. Considering each of the above problems $P$'s and $Q$'s for equal

† The total number of possible assignments (allocations) of $n$ destinations to $k$ sources is $S(n,k)$, the Stirling number of the $2nd$ kind.

sized *circles* which we call problems $PC$'s and $QC$'s we show that each of these problems are complete for the complexity class $D^p$. This then gives us an idea of the inherent computational complexity of the above problems $P$'s and $Q$'s under the *Euclidean* distance metric.

The class of $D^p$ was defined in [PY82] as follows : $L$ is in $D^p$ *iff* $L$ is an intersection of $L_1$ and $L_2$ such that $L_1$ is in $NP$ and $L_2$ is in $Co\text{-}NP$. $D^p$ contains both $NP$ and $Co\text{-}NP$ and is contained in $\Delta_2^P = P^{NP}$. Alternatively $D^p$ can be defined as the class of all predicates $R(x)$ that can be expressed as $R(x) = [\exists y \; P(x,y)] \wedge [\forall z \; Q(x,z)]$ for some polynomially balanced and polynomial-time checkable $P$ and $Q$.

We denote the Euclidean plane by $E^2$ and a circle *locatable* anywhere in $E^2$ means the center of the circle can be any point in the Euclidean plane. Furthermore we define an *$R$-circle* to be a circle of radius $R$.

($PC_1$)Is $R$ the *minimum* radius of $k$ equal sized circles locatable anywhere in $E^2$ to cover the $n$ points ?

($PC_2$)Is $m$ the *maximum* number of points that $k$, *$R$-circles* locatable anywhere in $E^2$ can cover ?

($PC_3$)Is $k$ the *minimum* number of *$R$-circles* locatable anywhere in $E^2$ to cover $n$ points?

For the constrained location problems we assume that there exists a finite discrete set of points $S \subset E^2$ and the location of the circles is constrained to be from this set. A circle *locatable* anywhere in $S$ means the center of the circle is a point of this set.

($QC_1$)Is $R$ the *minimum* radius of $k$ equal sized circles locatable anywhere in $S$ to cover the $n$ points ?

($QC_2$)Is $m$ the *maximum* number of points that $k$, *$R$-circles* locatable anywhere in $S$ can cover ?

($QC_3$)Is $k$ the *minimum* number of *$R$-circles* locatable anywhere in $S$ to cover the $n$ points?

We first show that the problem $PC_3$ of locating the minimum number of *$R$-circles* in $E^2$ to cover all the $n$ demand points is *$D^p$ complete* by reducing (Sat,UnSat), a known *$D^p$ complete* problem [PY82], to it. We adapt certain constructions previously specified in [FPT81]. Next we show that all the above remaining problems are $D^p$ complete by a series of polynomial time reductions. To show a problem to be complete for this class $D^p$ we differ from [PY82] in that we use polynomial-time positive (disjunctive) reductions [LLS74],[S82] as opposed to polynomial-time many-one reductions. These positive reductions seem somewhat weaker than many-one reductions, however appear to be considerably stronger than Turing reductions.

In a simplistic fashion any form of polynomial-time oracle-reducibility (which includes both positive and Turing reductions), is a Boolean formula of a polynomial number of queries to the oracle. The essential restriction for positive reductions is that the Boolean formula is a positive formula. Sufficient to our purpose here, the Boolean formula is positive if it only contains disjunctive ($\vee$) and conjunctive ($\wedge$) Boolean connectives. Furthermore, in positive reductions, like in other truth-table reductions [LLS74], one is restricted to a prespecified list $f(x)$ based on input $x$, from which alone one can make queries. These restrictions are severe. First, these restrictions allow only a polynomial number of *feasible* queries for polynomial-time positive reductions, while an exponential number of *feasible* queries exist for polynomial-time Turing reductions (a tree of polynomial depth). Second, in positive reductions one is restricted from the use of the negation (*not*) Boolean connective which disallows using the 'No' answer of the oracle to say 'Yes' to the computation using the oracle.

It is also important to note that the special case of disjunctive, conjunctive positive reductions which we use here are by far the strongest of the various other positive and truth-table reducibilities known [LLS74],[S82]. In turn, any truth-table reduction is stronger than the Turing reduction.

In [S82], it is proved that, similar to polynomial-time many-one reductions, polynomial-time positive reductions preserve the class of $NP$ . That is if a language $L_1$ polynomial-time positive reduces (or polynomial-time many-one reduces) to a language $L_2$ then $L_2 \epsilon NP \Rightarrow L_1 \epsilon NP$. A similar fact is true for the class $Co-NP$. We feel therefore that these positive reductions are adequate to separate the class of $D^p$ *complete* languages from the classes of $NP$ and $Co-NP$, (assuming $NP \neq co-NP$). A similar argument is given when using polynomial-time Turing reductions as opposed to polynomial-time many-one reductions, in separating the class of $NP$*complete* languages from the class of $P$, since both Turing reductions as well as many-one reductions preserve the class of $P$. It is also important to note that polynomial-time Turing reductions which do not preserve the class of $NP$, are not adequate in separating $D^p$ languages from $NP$ and $Co-NP$. Thus, for instance, it is possible to polynomial-time Turing reduce (Sat,Unsat), a known $D^p$ *complete* problem to (Sat) a known $NP$*complete* problem.

On the above classes of problems, $P$'s and $Q$'s, the question of irrational quantities which could result due to square roots of the Euclidean distance metric [GGJ76] can be eliminated by squaring, since any comparison in these problems involves at most two Euclidean distances.

**Theorem 1:** $PC_3$ is $D^p$ *complete*.

*Proof :* The problem is in $D^p$ since it can be rephrased as the conjunction of a predicate in $NP$ and a predicate in $Co-NP$: $(\exists(b_{i_1},..,b_{i_k}) in E^2)[R-circles$ with centers at $b_{i_1},..,b_{i_k}$ cover $n$ points$] \wedge (\forall(b_{j_1},..,b_{j_{k-1}}) in E^2)[R-circles$ with centers at $b_{j_1},..,b_{j_{k-1}}$ cover $<n$ points$]$.

To prove the completeness we reduce (Sat, Unsat) to $PC_3$, using polynomial-time positive (disjunctive) reductions.

Starting from $(F_1, F_2)$ and adapting a polynomial time construction in [FPT81] we construct two separate sets of points $S_1$ and $S_2$ in the plane such that for $i=1,2$, exactly $k_i$, $R-circles$ are required to cover all the $n_i$ points in $S_i$

if $F_i$ is satisfiable. Further if $F_i$ is not satisfiable, at least $k_i+1$ and at most $k_i+c_i$, *R−circles* are needed to cover all the $n_i$ points of $S_i$, where $c_i$ is the number of clauses in the *CNF* formula $F_i$.

Now construct $c_2$ additional copies of the set of points $S_1$. We now have $(c_2+1)$ copies of sets of points $S_1$ and a single set of points $S_2$. It is important to note why $(c_2+1)$ copies of $S_1$ are required. Let $n=(c_2+1)n_1+n_2$. It is not hard to see that $k$, the minimum number of circles of radius $R$ needed to cover all the $n$ points, satisfies $(c_2+1)k_1+k_2+1 \le k \le (c_2+1)k_1+k_2+c_2$ *iff* $F_1$ is satisfiable and $F_2$ is not satisfiable. Since this is a disjunction of at most $c_2$ calls of $PC_3$, problem $PC_3$ is $D^p$ *complete* under a polynomial-time positive (disjunctive) reduction from (Sat, Unsat). □

**Theorem 2:** $PC_2$ is $D^p$ *complete*.

*Proof :* The problem is in $D^p$ since it can be rephrased as before, as the conjunction of a predicate in *NP* and a predicate in *Co−NP*: $(\exists(b_{i_1},..,b_{i_k}) in E^2)$[*R−circles* with centers at $b_{i_1},..,b_{i_k}$ cover $m$ points] $\wedge$ $(\forall(b_{j_1},..,b_{j_k}) in E^2)$[*R−circles* with centers at $b_{j_1},..,b_{j_k}$ cover $\le m$ points].

To prove the completeness we again reduce (Sat, Unsat) to $PC_3$, using polynomial-time positive (disjunctive) reductions in a way very similar to above.

Starting from $(F_1, F_2)$ we construct two separate sets of points $S_1$ and $S_2$ in the plane such that for $i=1,2$, $k_i$, *R−circles* are required to cover all the $n_i$ points in $S_i$ if $F_i$ is satisfiable. Further, if $F_i$ is not satisfiable, $k_i$, *R−circles* can cover at least $n_i-c_i$ points and at most $n_i-1$ points of $S_i$, where $c_i$ is the number of clauses in the *CNF* formula $F_i$.

Now construct $c_2$ additional copies of the set of points $S_1$. We now have $(c_2+1)$ copies of sets of points $S_1$ and a single set of points $S_2$. It is important to note why $(c_2+1)$ copies of $S_1$ are required. Let $k=(c_2+1)k_1+k_2$. It is not hard to see that $m$, the maximum number of points that can be covered by $k$ circles of radius $R$, satisfies $(c_2+1)n_1+n_2-c_2 \le m \le (c_2+1)n_1+n_2-1$ *iff* $F_1$ is satisfiable and $F_2$ is not satisfiable. Since this is a disjunction of at most $c_2$ calls of $PC_2$, problem $PC_2$ is $D^p$ *complete* under a polynomial-time positive (disjunctive) reduction from (Sat, Unsat). □

**Theorem 3:** $PC_1$ is $D^p$ *complete*.

*Proof :* The problem is in $D^p$, when $R$ is restricted to integers[†], since it can be rephrased as before, as the conjunction of a predicate in *NP* and a predicate in *Co−NP*: $(\exists(b_{i_1},..,b_{i_k}) in E^2)$[*R−circles* with centers at $b_{i_1},..,b_{i_k}$ cover $n$ points] $\wedge$ $(\forall(b_{j_1},..,b_{j_k}) in E^2)$[ $(R-1)$*−circles* with centers at $b_{j_1},..,b_{j_k}$ cover $<n$ points].

To prove it complete we show that $PC_3$ polynomial-time positive reduces to $PC_1$. We construct a set S of the radii of all possible circles which minimally cover $n$ points in the plane. Since the minimum enclosing circle for a set of points is defined by exactly two or three of the points, the total size of S is at

† Otherwise the problem appears to be $D^p$ *hard* when $R$ is in general, a real number.

most $\binom{n}{2} + \binom{n}{3}$ which is $O(n^3)$. We claim that $k$ is the minimum number of $R$*-circles* that cover all $n$ points ***iff*** for some $s \epsilon S$, $s \leq R$, $s$ is the minimum radius of $k$ circles to cover all $n$ points and for some $s \epsilon S$, $s > R$, $s$ is the minimum radius of $k-1$ circles to cover all $n$ points. The proof is straightforward and follows from the definitions of the two problems $PC_1$ and $PC_3$. Again since we have a conjunction of two sets of disjunctive calls of $PC_1$, {at most $O(n^3)$ calls}, we have a polynomial-time positive reduction from $PC_3$ to $PC_1$. □

**Theorem 4:** $QC_3$ is $D^p$ *complete*.

*Proof :* The problem is in $D^p$ since it can be rephrased as the conjunction of a predicate in *NP* and a predicate in *Co–NP*: $(\exists(b_{i_1},..,b_{i_k})\epsilon S)$[$R$*-circles* with centers at $b_{i_1},..,b_{i_k}$ cover $n$ points] $\wedge$ $(\forall(b_{j_1},..,b_{j_{k-1}})\epsilon S)$[$R$*-circles* with centers at $b_{j_1},..,b_{j_{k-1}}$ cover $<n$ points].

To prove it complete we prove that $PC_3$ polynomial-time many-one reduces to $QC_3$. It suffices to show that for any set $T$ of $n$ destination points in the plane there exists a finite set $S \subset E^2$, such that if a minimum of $k$, $R$*-circles* can cover $T$ then these $R$*-circles* can be chosen to have their centers in $S$. Furthermore, $S$ must be constructible in time polynomial in $n$.

We claim one can choose such an $S = T \bigcup$ {intersection points of $R$*-circles* centered at the points of $T$}. For a proof of this claim let $F$ be a (minimal) set of circles of radius $R$ covering $T$ and let circle $C \epsilon F$. If $C$ contains only a single point $p \epsilon T$, replace $C$ by an $R$*-circle* centered at $p \epsilon T \subset S$. Otherwise, if $C$ contains more than one point, move $C$ without uncovering any point of $T$, until two points $p,q \epsilon T$, lie on the boundary of the moved circle $C'$. Clearly the center $c$ of $C'$ lies at an intersection of the $R$*-circles* centered at $p$ and $q$. Thus $c \epsilon S$. {A similar duality was proved and exploited in [BL83]}.

Finally note that $S$ contains at most $O(n^2)$ points and can be constructed in $O(n^2)$ time. □

**Theorem 5:** $QC_2$ is $D^p$ *complete*.

*Proof :* The problem is in $D^p$ since it can be rephrased as before, as the conjunction of a predicate in *NP* and a predicate in *Co–NP*: $(\exists(b_{i_1},..,b_{i_k})\epsilon S)$[$R$*-circles* with centers at $b_{i_1},..,b_{i_k}$ cover $m$ points] $\wedge$ $(\forall(b_{j_1},..,b_{j_k})\epsilon S)$[$R$*-circles* with centers at $b_{j_1},..,b_{j_k}$ cover $\leq m$ points]

To prove it complete we exhibit a polynomial-time reduction from $PC_2$ to $QC_2$ similar to the proof of Theorem 4. Again construct the set of points $S = T \bigcup$ {intersection points of $R$*-circles* centered at the points of $T$}. As before there is no loss of coverage and it suffices to consider only the set of points $S$ as possible location centers; $n$ remains the maximum number of points covered by $k$, $R$*-circles* if $n$ is the maximum number of points covered by locating the $k$, $R$*-circles* anywhere in $E^2$. □

**Theorem 6:** $QC_1$ is $D^p$ *complete*.

*Proof :* The problem is in $D^p$, when $R$ is restricted to integers, since it can be rephrased as before, as the conjunction of a predicate in *NP* and a predicate in *Co-NP*: $(\exists(b_{i_1},..,b_{i_k})\epsilon S)$[*R-circles* with centers at $b_{i_1},..,b_{i_k}$ cover $n$ points] $\wedge$ $(\forall(b_{j_1},..,b_{j_k})\epsilon S)$[$(R-1)$-*circles* with centers at $b_{j_1},..,b_{j_k}$ cover $<n$ points].

To prove it complete we exhibit a polynomial time reduction from $PC_1$ to $QC_1$ similar to the proofs of Theorems 4 and 5. Again construct the set of points $S=T \bigcup$ {intersection points of *R-circles* centered at the points of $T$}. It suffices to see that there is no loss of coverage in considering only the set $S$ as possible location centers and $R$ remains to be the minimum radius of the $k$ circles covering the $n$ points if $R$ is the minimum radius of the $k$ circles locatable anywhere in $E^2$ to cover the $n$ points. □

## 3. Problems with the rectilinear and infinity distance metrics

With the *rectilinear* $l_1$ distance metrics each of the above location-allocation problems reduces to the location-allocation of equal sized diamonds (squares rotated by 45°) with the intersection points of their diagonals corresponding to the location of the sources. Again assuming equal weights for the destinations results in our considering each of the previous problems $P$'s and $Q$'s for equal sized *diamonds* which we call the problems $PD$'s and $QD$'s.

These problems $PD$'s and $QD$'s are exactly similar to the problems $PC$'s and $QC$'s listed in section 2, with the geometric objects to be located now, being equal sized diamonds of half-diagonal length $R$ instead of *R-circles*. Also a diamond *locatable* anywhere in $S$ or $E^2$ means the intersection point of the diagonals of the diamond can be any point in the finite discrete set $S$ or the Euclidean plane respectively.

For these problems both membership and completeness for the class of $D^p$ carry over in a fashion quite similar to the proofs of Theorems 1 to 6. Each of the constructions used here as well as the adapted constructions of [FPT81], can be modified in a direct fashion for diamonds (as well as squares). Thus we have the following result.

**Theorem 7:** $PD_1$,$PD_2$,$PD_3$ and $QD_1$,$QD_2$,$QD_3$ are all $D^p$ *complete*

An identical set of arguments as above, apply to the *infinity* $l_\infty$ distance metric where now each of the above location problems reduces to the location of equal sized squares of half-edge length $R$, having sides parallel to the respective coordinate axes. Again the intersection points of the square's diagonals corresponds to the location of the sources. The problems $P$'s and $Q$'s for equal sized *squares* are called the problems $PS$'s and $QS$'s and again by adapting the proofs of Theorems 1 to 6 we have the following result.

**Theorem 8:** $PS_1$,$PS_2$,$PS_3$ and $QS_1$,$QS_2$,$QS_3$ are all $D^p$ *complete*

## 4. Conclusion

We have shown that a number of different geometric optimization problems arising from multiple points location-allocation in the plane, under a discrete optimization criterion which minimizes the maximum cost based on certain interpoint distances, are all computationally equivalent with respect to polynomial-time positive reductions. Furthermore, all these above problems having an infinite feasible solution region (a subset of the Euclidean plane) are polynomial-time many-one reducible to similar problems where the feasible solution sets are constrained to be discrete and of size polynomial in the number of given points. Furthermore each of these problems lies in the complexity class $D^p$ and is also $D^p$ *complete* .

We can also claim that all of the above problems are ***strongly*** $D^p$ ***complete*** analogous to the similar concept for *NPcomplete* languages, since all the above constructions hold even when the largest number occurring in any instance of the problems, that is parameter $R$ and the coordinate points in set $T$, are restricted to be of size bounded by a polynomial in $n$.

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