Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, II

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Abstract

We study the Schrödinger equation on $\mathbb{R}$ with a potential behaving as x^{2l} at infinity, $l \in [1, +\infty)$ and with a small time quasiperiodic perturbation. We prove that, if the perturbation belongs to a class of unbounded symbols including smooth potentials and magnetic type terms with controlled growth at infinity, then the system is reducible.

1 Introduction

The present paper is a continuation of [Bam16] in which a reducibility result for the time dependent Schrödinger equation

$$\mathrm{i}\dot{\psi} = (H_0 + \epsilon W(\omega t))\psi \ , \ x \in \mathbb{R} \tag{1.1}$$

$$H_0 = -\partial_{xx} + V(x) \ , \tag{1.2}$$

with W a suitable unbounded perturbation was proved. The improvement we get here is that we deal with a more general class of perturbations. For example we prove here reducibility, if $V(x) \simeq |x|^{2l}$, $l \geq 1$, as $x \to \infty$, and

$$W(\omega t) = a_0(x, \omega t) - \mathrm{i}a_1(x, \omega t)\partial_x \ , \tag{1.3}$$

with a_i functions of class C^∞ fulfilling

$$\left|\partial_x^k a_0(x, \omega t)\right| \preceq \langle x \rangle^{\beta_2 - k} \ , \quad \beta_2 < l \ , \tag{1.4}$$

$$\left|\partial_x^k a_1(x, \omega t)\right| \preceq \langle x \rangle^{\beta_3 - k} \ , \quad \begin{cases} \beta_3 < l - 1 & \text{if} \quad 1 < l \leq 2 \\ \beta_3 < l/2 & \text{if} \quad 2 < l \end{cases} \ ; \tag{1.5}$$

in the case $l = 1$, a_1 must vanish identically. The theory developed in [Bam16] only allowed to deal with the case of polynomial a_0 and a_1, but a faster growth at infinity of both a_0 and a_1 was allowed.

As usual, boundedness of Sobolev norms and pure point nature of the Floquet spectrum follow.

We recall that previous results on the reducibility problem for perturbations of the Schrödinger equation have been obtained in quite a number of papers for the superquadratic case with bounded or unbounded perturbations (see in particular [DŠ96, DLŠV02, BG01, LY10, EK09]);

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in the quadratic case the only available results deal with bounded perturbations [Com87, Wan08, GT11, GP16]. The result of the present paper allows a growth of the perturbation at infinity faster then all the previous papers dealing with the one dimensional case (except [Bam16]). On the other hand, we assume here that W is a symbol with the property that its derivatives of suffitiently high order decay fast at infinity (essentially as in (1.4),(1.5)); this is not required in most papers on reducibility. Concerning the higher dimensional case, it is not clear if the present method can be extended in order to deal with it.

The idea of the proof (following [PT01, BBM14], see also [Mon14, FP15, BM16]) is to use pseudo-differential calculus in order to conjugate the original system to a system with a smoothing perturbation and then to apply KAM theory. In the present paper we just prove the smoothing result, since afterwards one can apply the KAM type theorem of [Bam16] in order to conclude the proof. From the technical point of view the result is obtained by introducing a new class of symbols. However, when working with such a class it becomes quite complicated to show that the function used to generate the smoothing transformation is actually a symbol. The proof of this property occupy the majority of the paper. We also would like to mention that the class of symbols we use is a variant of the class introduced by Hellfert and Robert in [HR82b].
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2 Statement of the Main Result

Fix a real number $l \geq 1$ and define the weights

$$\lambda(x,\xi) := \left(1+\xi^2+|x|^{2l}\right)^{1/2l} \quad , \quad \langle x\rangle := \sqrt{1+x^2} \tag{2.1}$$

Definition 2.1. *The space S^{m_1,m_2} is the space of the symbols $g \in C^\infty(\mathbb{R})$ such that $\forall k_1,k_2 \geq 0$ there exists C_{k_1,k_2} with the property that*

$$\left|\partial_\xi^{k_1}\partial_x^{k_2} g(x,\xi)\right| \leq C_{k_1,k_2}\left[\lambda(x,\xi)\right]^{m_1-k_1 l}\langle x\rangle^{m_2-k_2} \ . \tag{2.2}$$

The best constants C_{k_1,k_2} such that (2.2) hold form a family of seminorms for the space S^{m_1,m_2}.

To a symbol $g \in S^{m_1,m_2}$ we associate its Weyl quantization, namely the operator $g^w(x,D_x)$, $D_x := -\mathrm{i}\partial_x$, defined by

$$G\psi(x) \equiv g^w(x,D_x)\psi(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathrm{i}(x-y)\cdot\xi} g\left(\frac{x+y}{2};\xi\right)\psi(y)dyd\xi \ . \tag{2.3}$$

We will denote by a capital letter the Weyl quantized of a symbol denoted with the corresponding lower case letter. The only exception will be the perturbation W (we mainly think of it as a potential).

In the following we will denote by $\mathcal{S}^{m_1,m_2} := C^\infty(\mathbb{T}^n, S^{m_1,m_2})$ the space of C^∞ functions on $\mathbb{T}^n$ with values in S^{m_1,m_2}. The frequencies ω will be assumed to vary in the set

$$\Omega := [1,2]^n \ ,$$

or in suitable closed subsets $\widetilde{\Omega}$.

We denote by $S_N^{m_1,m_2}$ the space of the symbols which are only N times differentiable and fulfill the inequality (2.2) only for $k_1+k_2\leq N$. This is a Banach space with the norm

$$\|g\|_{S_N^{m_1,m_2}} := \sum_{k_1+k_2\leq N}\sup_{(x,\xi)\in\mathbb{R}^2}\frac{\left|\partial_x^{k_2}\partial_\xi^{k_1}g(x,\xi)\right|}{[\lambda(x,\xi)]^{m_1-lk_1}\langle x\rangle^{m_2-k_2}}\ . \tag{2.4}$$

We remark that for the space $\mathcal{S}^{m_1,m_2}$ a family of seminorms is given by the standard norms of $C^M(\mathbb{T}^n;S_N^{m_1,m_2})$ as M and N vary.

In the case $l>1$, the potential V defining H_0 is assumed to belong to $S^{0,2l}$ to be symmetric, namely

$$V(x)=V(-x)\ , \tag{2.5}$$

and furthermore to admit an asymptotic expansion of the form

$$V(x)\sim |x|^{2l}+\sum_{j\geq 1}V_{2l-2j}(x) \tag{2.6}$$

with V_k homogeneous of degree k, namely s.t., $V_k(\rho x)=\rho^k V(x)$, $\forall\rho>0$.
We also assume that

$$V'(x)\neq 0\ ,\quad \forall x\neq 0\ . \tag{2.7}$$

Remark 2.2. *The assumptions* (2.5), (2.6) *are used in order to simplify the proofs of Lemmas 3.13 and 3.14; they can probably be relaxed. Assumption* (2.7) *can also be weakened in order to deal with the case where the set of the critical points of V is bounded.*

An example of a non-polynomial potential fulfilling the assumptions is

$$V(x)=\langle x\rangle^{2l}\ .$$

In the case $l=1$ we assume that

$$V(x)=x^2\ .$$

The unperturbed Hamiltonian H_0 is the quantization of the classical Hamiltonian system with Hamiltonian function

$$h_0(x,\xi):=\xi^2+V(x)\ . \tag{2.8}$$

Remark 2.3. *As a consequence of the assumptions above all the solutions of the Hamiltonian system h_0 are periodic with a period $T(E)$ which depends only on $E=h_0(x,\xi)$.*

We will denote by $\Phi_{h_0}^t$ the flow of the Hamiltonian system (2.8).

We denote by λ_j^v the sequence of the eigenvalues of H_0. In what follows we will identify L^2 with ℓ^2 by introducing the basis of the eigenvector of H_0.

We use the symbol $\mathcal{A}(x,\xi):=(1+h_0(x,\xi))^{\frac{l+1}{2l}}$ to *define, for $s\geq 0$, the spaces $\mathcal{H}^s=D([\mathcal{A}^w(x,-\mathrm{i}\partial_x)]^s)$ (domain of the s- power of the operator operator $\mathcal{A}^w(x,-\mathrm{i}\partial_x)$) endowed by the graph norm.* For negative s, the space $\mathcal{H}^s$ is the dual of $\mathcal{H}^{-s}$.

We will denote by $B(\mathcal{H}^{s_1};\mathcal{H}^{s_2})$ the space of bounded linear operators from $\mathcal{H}^{s_1}$ to $\mathcal{H}^{s_2}$.

In order to state the assumptions on the perturbation we define the average with respect to the flow of h_0:

$$\langle W\rangle(x,\xi,\omega t):=\frac{1}{T(E)}\int_0^{T(E)}W(\Phi_{h_0}^\tau(x,\xi),\omega t)d\tau\ ; \tag{2.9}$$

then, for $m \in \mathbb{R}$, we denote

$$[m] := \max\{0, m\} \ . \tag{2.10}$$

Concerning the perturbation, we assume that $W \in \mathcal{S}^{\beta_1,\beta_2}$ and we define

$$\tilde{\beta} := \begin{cases} 2\beta_1 + [\beta_2] + [\beta_2 - 1] - 2l + 1 & \text{if} \quad \langle W \rangle \equiv 0 \text{ and } l > 1 \\ \beta_1 + [\beta_2] & \text{otherwise} \end{cases} \ . \tag{2.11}$$

Theorem 2.4. *Assume*

$$\tilde{\beta} < l \quad \text{and} \quad \beta_1 + [\beta_2] < 2l - 1 \ ,$$

then there exists $\epsilon_* > 0$ and $\forall |\epsilon| < \epsilon_*$ a closed set $\Omega(\epsilon) \subset \Omega$ and, $\forall \omega \in \Omega(\epsilon)$ there exists a unitary (in L^2) time quasiperiodic map $U_\omega(\omega t)$ s.t. defining φ by $U_\omega(\omega t)\varphi = \psi$, it satisfies the equation

$$\mathrm{i}\dot{\varphi} = H_\infty \varphi \ , \tag{2.12}$$

with $H_\infty = \mathrm{diag}(\lambda_j^\infty)$, with $\lambda_j^\infty = \lambda_j^\infty(\omega, \epsilon)$ independent of time and

$$\left|\lambda_j^\infty - \lambda_j^v\right| \le C\epsilon j^{\frac{\tilde{\beta}}{l+1}} \ , \tag{2.13}$$

for some positive C. Furthermore one has

1. $\lim_{\epsilon \to 0} |\Omega - \Omega(\epsilon)| = 0$;
2. $\forall s, r \ge 0$, $\exists \epsilon_{s,r} > 0$ *and* s_r *s.t., if* $|\epsilon| < \epsilon_{s,r}$ *then the map* $\phi \mapsto U_\omega(\phi)$ *is of class* $C^r(\mathbb{T}^n; B(\mathcal{H}^{s+s_r}; \mathcal{H}^s))$; *in particular one has* $s_0 = 0$ *and* $s_1 = \beta_1 + [\beta_2]$.
3. $\exists b > 0$ *s.t.* $\forall |\epsilon| < \epsilon_{s,1}$, *one has* $\|U_\omega(\phi) - \mathbf{1}\|_{B(\mathcal{H}^{s+\beta_1+[\beta_2]}; \mathcal{H}^s)} \le C_s \epsilon^b$.

Remark 2.5. *If W is the sum of different addenda, then Theorem 2.4 applies also if its assumptions are fulfilled by each of the addenda separately. This is particularly relevant in the case where the average of some of the addenda vanishes. Thus in this case the value of $\tilde{\beta}$ can depend on the addendum one is considering.*

Corollary 2.6. *If W is given by* (1.3)*, then Theorem 2.4 applies under the conditions* (1.4) *and* (1.5)*.*

Proof. The condition on β_2 is obvious. Consider the addendum $-\mathrm{i}a_1(x, \omega t)\partial_x$, which has symbol

$$a_1(x, \omega t)\xi + S^{0,\beta_3 - 1} \ ,$$

and remark that, by Eq. (4.14) below, the average of the main term vanishes and therefore for this term $\tilde{\beta}$ is given by the first of (2.11) which is made explicit by (1.5). □

Remark 2.7. *In the case of the quartic oscillator ($l = 2$) and perturbation of the form* (1.3)*, we have the bounds $\beta_2 < 2$ and $\beta_3 < 1$. We recall that [LY10] had $\beta_2 \le 1$ and $\beta_3 \le 0$, but the perturbation was not assumed to be asymbol. In [Bam16] we were able to deal also with some cases with $\beta_2 = 4$ and $\beta_3 = 2$, but only when a_0, and a_1 are polynomial.*

We also remark that the assumption that the functions a_i are symbols rules out cases like $a_i(x, \omega t) = \cos(x - \omega t)$.

Remark 2.8. *In the case of the Harmonic oscillator we cover the perturbations of the class considered in [Wan08] (in which the decay at infinity of a_0 and its derivatives are exponential) and in the counterexample of [Del14].*

On the contrary the perturbations in [GT11] (which must decay at infinity) and in [GY00] can belong to a class of symbols in which the decay at infinity does not improve as one extracts derivatives.

3 Proof of Theorem 2.4

3.1 Some symbolic calculus

First we remark that $S^{m_1,m_2} \subset S^{m_1+[m_2],0}$.

In the proof we will also need the classes of symbols used in [Bam16], thus we recall the corresponding definitions

Definition 3.1. *The space S^m is the space of the symbols $g \in C^\infty(\mathbb{R})$ such that $\forall k_1, k_2 \geq 0$ there exists C_{k_1,k_2} with the property that*

$$\left|\partial_\xi^{k_1}\partial_x^{k_2} g(x,\xi)\right| \leq C_{k_1,k_2}\left[\lambda(x,\xi)\right]^{m-k_1 l-k_2} \ . \tag{3.1}$$

In order to deal with functions p such that there exist a $\tilde{p}$ with the property that

$$p(x,\xi) = \tilde{p}(h_0(x,\xi)) \ ,$$

we introduce the following class of symbols.

Definition 3.2. *A function $\tilde{p} \in \mathbb{C}^\infty$ will be said to be of class $\widetilde{S}^m$ if one has*

$$\left|\frac{\partial^k \tilde{p}}{\partial E^k}(E)\right| \preceq \langle E^{\frac{m}{2l}-k}\rangle \ . \tag{3.2}$$

By abuse of notation, we will say that $p \in \widetilde{S}^m$ if there exists $\tilde{p} \in \widetilde{S}^m$ s.t. $p(x,\xi) = \tilde{p}(h_0(x,\xi))$.

We will also need to use functions from $\mathbb{T}^n$ to $\widetilde{S}^m$. The corresponding class will be denoted by $\widetilde{\mathcal{S}}^m$.

We now give a reformulation of the results of sect. 4.1 of [Bam16] in the case of the symbols of the classes S^{m_1,m_2}.

The application of the Calderon Vaillencourt theorem yields the following Lemma.

Lemma 3.3. *Let $f \in S^{m_1,m_2}$, then one has*

$$f^w(x,D_x) \in B(\mathcal{H}^{s_1+s};\mathcal{H}^s) \ , \quad \forall s \ , \quad \forall s_1 \geq m_1 + [m_2] \ . \tag{3.3}$$

Given a symbol $g \in S^{m_1,m_2}$ we will write

$$g \sim \sum_{j\geq 0} g_j \ , \quad g_j \in S^{m_1^{(j)},m_2^{(j)}} \ , \quad m_1^{(j)} + [m_2^{(j)}] \leq m_1^{(j-1)} + [m_2^{(j-1)}] \ , \tag{3.4}$$

if $\forall \kappa$ there exist N and $r_N \in S^{-\kappa,0}$, s.t.

$$g = \sum_{j=0}^N g_j + r_N \ .$$

Lemma 3.4. *Given a couple of symbols $a \in S^{m_1,m_2}$ and $b \in S^{m_1',m_2'}$, denote by $a^w(x,D_x)$ and $b^w(x,D_x)$ the corresponding Weyl operators, then there exists a symbol c, denoted by $c = a\sharp b$ such that*

$$(a\sharp b)^w(x,D_x) = a^w(x,D_x)b^w(x,D_x) \ ,$$

furthermore one has

$$(a\sharp b) \sim \sum_{j\geq 0} c_j \tag{3.5}$$

with

$$c_j = \sum_{k_1+k_2=j} \frac{1}{k_1!k_2!} \left(\frac{1}{2}\right)^{k_1} \left(-\frac{1}{2}\right)^{k_2} (\partial_\xi^{k_1} D_x^{k_2} a)(\partial_\xi^{k_2} D_x^{k_1} b) \in S^{m_1+m_1'-lj, m_2+m_2'-j} \ .$$

In particular we have

$$\{a;b\}^q := -\mathrm{i}(a\sharp b - b\sharp a) = \{a;b\} + S^{m_1+m_1'-3l, m_2+m_2'-3} \ , \tag{3.6}$$

where

$$\{a;b\} := -\partial_\xi a \partial_x b + \partial_\xi b \partial_x a \in S^{m_1+m_1'-l, m_2+m_2'-1} \ ,$$

is the Poisson Bracket between a and b, while (3.6) *means that* $\{a;b\}^q = \{a;b\}$ *+some quantity belonging to* $S^{m_1+m_1'-3l, m_2+m_2'-3}$.

Definition 3.5. *An operator F will be said to be a pseudo-differential operator of class O^{m_1,m_2} if there exists a sequence $f_j \in S^{m_1^{(j)}, m_2^{(j)}}$ with $m_1^{(j)} + [m_2^{(j)}] \le m_1^{(j-1)} + [m_2^{(j-1)}]$ and, for any κ there exist N and an operator $R_N \in B(\mathcal{H}^{s-\kappa}; \mathcal{H}^s)$, $\forall s$ such that*

$$F = \sum_{j\ge 0}^{N} f_j^w + R_N \ . \tag{3.7}$$

In this case we will write $f \sim \sum_{j\ge0} f_j$ and f will be said to be the symbol of F; the function f_0 will be said to be the principal symbol of F.

Concerning maps we will use the following definition

Definition 3.6. *A map $\mathbb{T}^n \ni \phi \mapsto F(\phi) \in O^{m_1,m_2}$, will be said to be smooth of class $\mathcal{O}^{m_1,m_2}$ if the functions of the sequence f_j also depend smoothly on ϕ, namely $f_j \in \mathcal{S}^{m_1^{(j)}, m_2^{(j)}}$ and the operator valued map $\phi \mapsto R_N(\phi)$ has the property that for any $K \ge 1$ there exists $a_K \ge 0$ s.t. for any N one has*

$$R_N(.) \in C^K(\mathbb{T}^n; B(\mathcal{H}^{s-\kappa+a_K}; \mathcal{H}^s)) \ , \forall s \ . \tag{3.8}$$

Finally we need (Whitney) smooth functions of the frequencies. Following [Bam16] (and [Ste70]), we will denote by $Lip_\rho(\widetilde\Omega; \mathcal{B})$ the functions of $\omega \in \tilde\Omega$ with values in a Banach space $\mathcal{B}$ which have k derivatives of Hölder class $\rho - k$. Here k is the first integer strictly smaller then ρ and $\widetilde\Omega \subset \Omega$ is a closed set.

Definition 3.7. *We will say that a function $f : \widetilde\Omega \to \mathcal{S}^{m_1,m_2}$ is of class $Lip_\rho^{m_1,m_2}(\widetilde\Omega)$ if forall N_1, N_2 it is of class $Lip_\rho(\widetilde\Omega; C^{N_1}(\mathbb{T}^n; S^{m_1,m_2}_{N_2}))$. Similarly we will say that $f \in \widetilde{Lip}^m_\rho(\widetilde\Omega)$ if forall N_1, N_2, one has $f \in Lip_\rho(\widetilde\Omega; C^{N_1}(\mathbb{T}^n; \widetilde S^m_{N_2}))$.*

3.2 Quantum Lie transform

Given a symbol χ, we consider the corresponding Weyl operator X. If X is selfadjoint, then we will consider the unitary operator $e^{-\mathrm{i}\epsilon X}$. The following Lemma gives a sufficient condition for selfadjointness.

Lemma 3.8. *Let $\chi \in S^{m,0}$ have the further property that $\partial_x \chi \in S^{m-1,0}$. Assume $m \le l+1$, then $X := \chi^w(x, D_x)$ is selfadjoint and $\mathrm{e}^{-\mathrm{i}\epsilon X}$ leaves invariant all the spaces $\mathcal{H}^s$.*

Proof. We use Proposition A.2 of [MR16]. To ensure the result it is enough to exhibit a positive selfadjoint operator K such that both the operators XK^{-1} and $[X,K]K^{-1}$ are bounded. To this end we take K to be the Weyl operator of the symbol $\mathcal{A} := (1+h_0)^{\frac{l+1}{2l}} \in S^{l+1}$. From symbolic calculus it follows that $XK^{-1} \in O^{0,0}$ which is thus bounded and, by the additional property on the x derivative of χ, one has $\{\chi;\mathcal{A}\} \in S^{2m-l-1,0}$ so that $[X,K]K^{-1} \in O^{m-l-1,0}$, which is bounded under the assumption of the Lemma. □

Next we use the operator $e^{-\mathrm{i}\epsilon X}$ to transform operators.

Definition 3.9. *Let X be a selfadjoint operator; we will say that*

$$(Lie_{\epsilon X}F) := e^{\mathrm{i}\epsilon X} F e^{-\mathrm{i}\epsilon X} \tag{3.9}$$

is the quantum Lie transform of F generated by ϵX.

It is easy to see that defining

$$F_0 = F \ ; \quad F_k := -\mathrm{i}[F_{k-1}; X] \ , \tag{3.10}$$

one has

$$\frac{d^k}{d\epsilon^k} Lie_{\epsilon X} F = e^{\mathrm{i}\epsilon X} F_k e^{-\mathrm{i}\epsilon X} \ . \tag{3.11}$$

and therefore (formally)

$$Lie_{\epsilon X} F = \sum_{k\geq 0} \frac{1}{k!} \epsilon^k F_k \ . \tag{3.12}$$

We will use these formulae in situations where the series are asymptotic.

We will use the same terminology also when X depends on time and/or on ω (which in this case play the role of parameters).

We are interested in the way Hamiltonian operators change their form in the case where X also depends on time. The following Lemma is Lemma 3.2 of [Bam16] to which we refer for the proof.

Lemma 3.10. *Let F be selfadjoint operator which can also depend on time, and let $X(t)$ be a family of selfadjoint operators smoothly dependent on time. Assume that $\psi(t)$ fulfills the equation*

$$\mathrm{i}\dot{\psi} = F\psi \ , \tag{3.13}$$

then φ defined by

$$\varphi = e^{\mathrm{i}\epsilon X(t)}\psi \ , \tag{3.14}$$

fulfills the equation

$$\mathrm{i}\dot{\varphi} = F_\epsilon(t)\varphi \tag{3.15}$$

with

$$F_\epsilon := Lie_{\epsilon X} F - Y_X \ , \tag{3.16}$$

$$Y_X := \int_0^\epsilon (Lie_{(\epsilon-\epsilon_1)X}\dot{X}) d\epsilon_1 \ . \tag{3.17}$$

In the case where both F and X are pseudo-differential operators one can reformulate everything in terms of symbols. Thus, if f and χ are symbols and χ fulfills the assumptions of Lemma 3.8 one can define

$$f_0^q := f \ , \quad f_k^q := \{f_{k-1}^q; \chi\}^q \ , \tag{3.18}$$

and one can expect the symbol of $Lie_{\epsilon X} F$ to be $\sum_{k\geq 0} \epsilon^k f_k^q/k!$. A sufficient condition is given by the following lemma:

Lemma 3.11. *Let $\chi \in S^{m,0}$ and let $f \in S^{m_1,m_2}$ be symbols, assume $m < l$, then $Lie_{\epsilon X} F \in O^{m_1,m_2}$, and furthermore its symbol, denoted by $lie_{\epsilon\chi} f$, fulfills*

$$lie_{\epsilon\chi} f \sim \sum_{k\geq 0} \frac{\epsilon^k f_k^q}{k!} \ . \tag{3.19}$$

Proof. First remark that $f_k^q \in S^{m_1+k(m-l),m_2-k}$. From (3.11) and the formula of the remainder of the Taylor expansion one has

$$Lie_{\epsilon X} F = \sum_{k=0}^{N} \frac{F_k}{k!}\epsilon^k + \frac{\epsilon^{N+1}}{N!}\int_0^1 (1+u)^J e^{-\mathrm{i}u\epsilon X} F_{N+1} e^{\mathrm{i}u\epsilon X} du \ ,$$

so that, by defining R_N to be the integral term of the previous formula, we have $R_N \in B(\mathcal{H}^{s-\kappa}, \mathcal{H}^s)$ with $\kappa = N(l-m) - m - [-N + m_2]$, which diverges as $N \to \infty$ and thus shows that the expansion (3.19) is asymptotic in the sense of definition 3.5. $\square$

Remark 3.12. *Let $\chi \in S^{m,0}$ be such that $\partial_x \chi \in S^{m-1,0}$, with $m < l$, then the operator Y_X defined by eq.* (3.17) *is a pseudo-differential operator of class $O^{m,0}$ with symbol*

$$y_x := \int_0^\epsilon (lie_{(\epsilon-\epsilon_1)\chi}\dot{\chi}) d\epsilon_1 = \dot{\chi} + \epsilon S^{2m-l-1,0} \ . \tag{3.20}$$

3.3 Main lemmas

The algorithm used in order to conjugate the original system to a system with a smoothing perturbation is the one described in Sect. 4.2 of [Bam16]. In order to make it effective in the present case we have to prove that the solutions of the homological equations are symbols. In this sub section we present the homological equations and give the Lemmas solving them; they will be used in the proof of the smoothing theorem which will be given in the next subsection. The proof of these lemmas is the main technical result of the paper and will be given in Sect. 4.

From now on we will use the notation

$$a \preceq b \tag{3.21}$$

to mean "there exists a constant C independent of all the relevant quantities, such that $a \leq Cb$".

As the example of the period $T(E)$ in the case $V(x) = x^{2l}$ (with l integer) shows, it is useful to deal with functions which have a singularity at zero. In order to avoid the problems it creates we will first regularize the functions at zero and solve the homological equations only outside a neighborhood of zero.

The first homological equation we have to solve is the following one

$$p + \{h_0; \chi\} = \langle p \rangle \ , \tag{3.22}$$

where $\langle p \rangle$ is defined by (2.9) with p in place of W. The problem is to determine χ s.t. (3.22) holds.

First we have the following Lemma.

Lemma 3.13. *Let $p \in S^{m_1,m_2}$ be a symbol supported outside a neighborhood of zero (in the phase space), then $\langle p \rangle$ is a symbol of class $\widetilde{S}^{m_1+[m_2]}$ and is supported outside a neighborhood of zero.*

Concerning the solution of the homological equation we have the following Lemma.

Lemma 3.14. *Let $p \in S^{m_1,m_2}$ be a symbol supported outside a neighborhood of zero, then the homological equation* (3.22) *has a solution χ which is a symbol of class $\chi \in S^{m_1+[m_2]-l+1,0}$ with the further property that $\partial_x \chi \in S^{m_1+[m_2]-l,0}$ and is supported outside a neighborhood of zero.*

Remark 3.15. *In the above lemmas p can also depend on the angles ϕ and on the frequencies ω, but they only play the role of parameters, so in that case the result is still valid substituting the classes $\mathcal{S}$ or Lip_ρ with the same indexes to the classes S.*

In order to iterate the procedure, when $l > 1$, we will have to solve an equation of the form of (3.22) with h_0 replaced by

$$h_1 := h_0 + \epsilon f(h_0) \ , \tag{3.23}$$

with $f \in \tilde{S}^m$ and $m < l$, namely equation

$$p + \{h_1; \chi\} = \langle p \rangle \ , \tag{3.24}$$

Lemma 3.16. *Let $l > 1$ and $p \in S^{m_1,m_2}$ be a symbol supported outside a neighborhood of zero, then the homological equation* (3.24) *has a solution χ which is a symbol of class $\chi \in S^{m_1+[m_2]-l+1,0}$ and $\partial_x \chi \in S^{m_1+[m_2]-l,0}$.*

The third homological equation we have to solve is

$$-\,\omega \cdot \frac{\partial \chi}{\partial \phi} = p - \bar{p} \ , \tag{3.25}$$

where p is a symbol and $\bar{p}$ is defined by

$$\bar{p}(x,\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} p(x,\xi,\phi) d\phi \ . \tag{3.26}$$

Such an equation was already studied in [Bam16] and the solution was obtained in Lemma 4.20 of that paper which is already in the form we need in the present paper. We now give its statement (for the proof we refer to [Bam16]).

Fix $\tau > n-1$ and denote

$$\Omega_{0\gamma} := \left\{ \omega \in \Omega \ : \ |k \cdot \omega| \geq \gamma |k|^{-\tau} \right\} \ , \tag{3.27}$$

then it is well known that

$$|\Omega - \Omega_{0\gamma}| \preceq \gamma \ . \tag{3.28}$$

Lemma 3.17. *Let $p \in \widetilde{Lip}^m_\rho(\Omega_{0\gamma})$, be a symbol, then there exists a solution $\chi \in \widetilde{Lip}^m_\rho(\Omega_{0\gamma})$ of Eq.* (3.25)*. Furthermore $\bar{p} \in \widetilde{Lip}^m_\rho(\Omega_{0\gamma})$.*

Finally, in the case of the Harmonic oscillator $l=1$, we will meet the following homological equation

$$\{h_0,\chi\}-\dot\chi+p=\overline{\langle p\rangle}\ . \tag{3.29}$$

In order to solve it, define the set

$$\Omega_{1\gamma}:=\left\{\omega\in\Omega\ :\ \left|\omega\cdot k+k_0\right|\geq\frac{\gamma}{1+|k|^\tau}\right|\ ,\ (k_0,k)\in\mathbb{Z}^{n+1}-\{0\}\right\}\ . \tag{3.30}$$

Lemma 3.18. *Let $p\in Lip_\rho^{m_1,m_2}(\Omega_{1\gamma})$, then there exists a solution $\chi\in Lip_\rho^{m_1+[m_2],0}(\Omega_{1\gamma})$ of* (3.29)*. Furthermore $\overline{\langle p\rangle}\in\widetilde{Lip}_\rho^{m_1+[m_2]}(\Omega_{1\gamma})$.*

3.4 The smoothing theorem and end of the proof of Theorem 2.4

Theorem 3.19. *Fix $\gamma>0$ small, $\rho>2$ and an arbitrary $\kappa>0$. Assume*

$$\beta_1+[\beta_2]<2l-1\quad\text{and}\quad\tilde\beta<l \tag{3.31}$$

then there exists a (finite) sequence of symbols $\chi_1,...,\chi_N$ with $\chi_j\in Lip_\rho^{m_1^{(j)},m_2^{(j)}}(\Omega_{0\gamma})$, $m_1^{(j)}+[m_2^{(j)}]\leq\beta_1+[\beta_2]$ $\forall j$, s.t., defining

$$X_j:=\chi_j^w(x,D_x,\omega t)\ ,\quad\omega\in\Omega_{0\gamma}\ , \tag{3.32}$$

such operators are selfadjoint and the transformation

$$\psi=e^{-\mathrm{i}\epsilon X_1(\omega t)}....e^{-\mathrm{i}\epsilon X_N(\omega t)}\varphi\ , \tag{3.33}$$

transforms $H_\epsilon(\omega t)$ (c.f. (1.2)*) into a pseudo-differential operator $H^{(reg)}$ with symbol $h^{(reg)}$ given by*

$$h^{(reg)}=h_0+\epsilon z+\epsilon\widetilde{z}+\epsilon r \tag{3.34}$$

where $z\in\tilde S^{\tilde\beta}$ is a function of h_0 independent of time and of ω; $\tilde z\in\widetilde{Lip}_\rho^{2\tilde\beta-2l+1}(\Omega_{0\gamma})$ is an ω dependent function of h_0 independent of time, and r depends on (x,ξ,ϕ,ω). Furthermore one has

$$r\in Lip_\rho^{-\kappa,0}(\Omega_{0\gamma})\ . \tag{3.35}$$

In the case $l=1$ the set $\Omega_{0\gamma}$ must be substituted by the set $\Omega_{1\gamma}$.

Proof of Theorem 3.19 in the case $l>1$. Denote

$$\beta:=\beta_1+[\beta_2]\ ,\quad m:=\beta-l+1\ .$$

Let η be a C^∞ function such that

$$\eta(E)=\begin{cases}1 & \text{if}\quad |E|>2\\ 0 & \text{if}\quad |E|<1\end{cases} \tag{3.36}$$

and split

$$W=W_0+W_\infty\ ,\quad W_\infty(x,\xi)=W(x,\xi)(1-\eta(h_0(x,\xi)))\ ,\quad W_0(x,\xi)=W(x,\xi)\eta(h_0(x,\xi))\ , \tag{3.37}$$

then $W_\infty \in \mathcal{S}^{-\kappa_1,-\kappa_2}$ for any κ_1, κ_2, and $W_0 \in \mathcal{S}^{\beta_1,\beta_2}$ is the actual perturbation that has to be transformed into a regularizing operator.

The proof of the smoothing theorem is based only on the solution of the homological equation and the computation of symbols of commutators, which (up to operators which are smoothing of all orders) are operations preserving the property of symbols of being zero in the region $E < 1$.

So, we forget W_∞ and transform $h_0 + \epsilon W_0$ using the operator X_1 with symbol χ_1 obtained by solving the homological equation (3.22) with $p = W_0$, so that $\chi_1 \in \mathcal{S}^{m,0}$, with $\partial_x \chi_1 \in \mathcal{S}^{m-1,0}$ so that by Lemma 3.8 the corresponding Weyl operator is selfadjoint provided $m \leq l+1$ and Lemma 3.11 applies provided $m < l$ (implied by (3.31)).

Then the symbol of the transformed Hamiltonian is given by

$$
\begin{aligned}
h^{(1)} :=& \ h_0 + \epsilon(\langle W_0\rangle - W_0) + \epsilon \mathcal{S}^{m-l,-3} + \epsilon^2 S^{\beta+m-l-1,0} + \epsilon^2 S^{\beta_1+m-l,\beta_2-1} && (3.38)\\
&+ \epsilon W_0 + \epsilon^2 S^{\beta_1+m-l,\beta_2-1} && (3.39)\\
&- \epsilon\dot\chi_1 + \epsilon^2 S^{2m-(l+1),0} && (3.40)\\
=& \ h_0 + \epsilon\langle W_0\rangle - \epsilon\dot\chi_1 + \epsilon p_1 \ , && (3.41)
\end{aligned}
$$

with $p_1 \in S^{\beta+m-l-1,0} + S^{\beta_1+m-l,\beta_2-1}$.

Consider first the case where $\langle W_0 \rangle \equiv 0$. In this case we determine χ_2 by solving the homological equation (3.22) with p_1 in place of p, A simple analysis shows that

$$
\langle p_1\rangle \in \tilde S^{2\beta_1+[\beta_2]+[\beta_2-1]-2l+1} \equiv \tilde S^{\tilde\beta} \ , \quad \chi_2 \in S^{\tilde\beta-l+1,0} \ .
$$

Since $\tilde\beta < l$, $lie_{\epsilon\chi_2}$ has the property that, if $f \in S^{m_1,m_2}$, then

$$
lie_{\epsilon\chi_2} f - f \in \sum_j S^{m_1^{(j)},m_2^{(j)}} \ , \quad m_1^{(j)} < m_1 \text{ and } \quad m_2^{(j)} < m_2 \ . \tag{3.42}
$$

Thus, the transformed Hamiltonian has the form

$$
h^{(1_2)} = h_0 + \epsilon\langle p_1\rangle - \epsilon\dot\chi_1 + l.o.t \tag{3.43}
$$

where l.o.t. means terms with the property analogue to (3.42). Next we eliminate $-\dot\chi_1$. To this end we determine χ_3 by solving (3.22) with $-\dot\chi_1$ in place of p_1. Remark that $\langle\dot\chi_1\rangle \equiv 0$ so that $\chi_3 \in S^{\beta_1+[\beta_2]-2l+2,0}$ transforms $h^{(1_2)}$ into

$$
h^{(1_3)} := h_0 + \epsilon\langle p_1\rangle - \epsilon\dot\chi_3 + l.o.t \ .
$$

Then (if needed) we iterate again until we get

$$
\tilde h^{(1)} = h_0 + \epsilon\langle p_1\rangle + \epsilon \sum_j S^{\beta_1^{(j)},\beta_2^{(j)}} \ ,
$$

with $\beta_1^{(j)} + [\beta_2^{(j)}] < \tilde\beta$, $\forall j$.

Thus, both in the case $\langle W_0\rangle = 0$ and in the case $\langle W_0\rangle \neq 0$, we are reduced to a Hamiltonian of the form

$$
h^{(1')} := h_0 + \epsilon f(h_0, \omega t) + \epsilon p_2 \ , \tag{3.44}
$$

with $f(h_0, .) \in \mathcal{S}^{\tilde\beta}$ and p_2 a lower order correction in the above sense.

We now continue, following [Bam16], by eliminating the time dependence from f. Thus take χ_4 to be the solution of Eq. (3.25) with $p = f(h_0)$, so that $\chi_4 \in \widetilde{Lip}_\rho^{\tilde\beta}(\Omega_{0\gamma})$. Provided

$$
\tilde\beta < l \ ,
$$

one gets that the corresponding Weyl operator is selfadjoint and the quantum lie transform it generates, transforms symbols into symbols and has the property (3.42). Then the symbol of the transformed Hamiltonian takes the form

$$h^{(2)} = h_0 + \epsilon \overline{f(h_0)} + \epsilon p_2 + l.o.t.$$

where all the functions are defined on $\Omega_{0\gamma}$ and

$$p_2 \in \sum_j S^{\beta_1^{(j)}, \beta_2^{(j)}} \ , \quad \beta_1^{(j)} + [\beta_2^{(j)}] < \tilde{\beta} - l \ .$$

In particular the l.o.t. is the lowest order term with a nontrivial dependence on ω.

Denote now

$$h_1 := h_0 + \epsilon f(h_0)$$

and iterate the construction with h_1 in place of h_0. At each step of the iteration one gains l, in the sense that one passes from a perturbation (of a time independent Hamiltonian) which belongs to some classes $S^{\tilde{\beta}_1, \tilde{\beta}_2}$ to perturbations belonging to classes $S^{\tilde{\beta}'_1, \tilde{\beta}'_2}$ with

$$\tilde{\beta}'_1 + [\tilde{\beta}'_2] \leq \tilde{\beta}_1 + [\tilde{\beta}_2] - l \ .$$

Thus the result follows. □

Proof of Theorem 3.19 in the case $l = 1$. First remark that $\beta < 1$ implies $\beta_1 < 1$ and $\beta_2 < 1$. We make a first step by taking $\chi_1 \in Lip^{\beta}_{\rho}$ to be the solution of Eq. (3.29) with $p = W$. Remarking that in this case, for any symbol f, one has

$$\{h_0, f\}^q = \{h_0, f\} \ ,$$

it follows that the transformed Hamiltonian is

$$h^{(1)} = h_0 + \epsilon \overline{\langle W \rangle} + \epsilon^2 r_1 \ ,$$

with

$$r_1 \in Lip^{2\beta-2,0}_{\rho} + Lip^{\beta+\beta_1-1,0}_{\rho} \subset Lip^{\beta^{(1)},0}_{\rho} \ , \quad \beta^{(1)} := \beta + \beta_1 - 1 \ .$$

Then we iterate getting

$$h^{(2)} = h_0 + \epsilon \overline{\langle W \rangle} + \epsilon^2 \overline{\langle r_1 \rangle} + \epsilon^3 r_2 \ ,$$

with $r_2 \in Lip^{\beta+\beta^{(1)}-2,0} + Lip^{\beta^{(1)}+\beta^{(1)}-1,0}$. If $\beta - 2 > \beta^{(1)} - 1$ the dominant term is the first one and we put $\beta^{(2)} := \beta^{(1)} - 2 + \beta$, otherwise we define $\beta^{(2)} := 2\beta^{(1)} - 1$. Thus in particular we have $\beta^{(2)} < \beta^{(1)}$. Then we iterate and at each step we get a remainder $r_N \in Lip^{\beta^{(N)},0}$, with a sequence $\beta^{(N)}$ diverging at $-\infty$. We remark that, after some steps, one will get $\beta - 2 > \beta^{(N)} - 1$, and therefore, from such a step one will have simply $\beta^{(N+1)} = \beta^{(N)} - 2 + \beta$.

Finally we remark that the average of r_1 is the first term in the time independent part which depends on ω. □

After the smoothing Theorem 3.19, the Hamiltonian of the system is reduced to the form (3.34) to which we apply the methods (and the results) of [Bam16]. Precisely using, Lemmas 5.1 and 5.2 and Corollary 5.4 of [Bam16] one has the following Lemma

Lemma 3.20. *For any $\gamma > 0$ and $\rho \geq 2$ there exists a positive ϵ_* s.t., if $|\epsilon| < \epsilon_*$ then there exists a set $\Omega^{(0)}_{\gamma}$, and a unitary (in L^2) operator U_1 Whithney smooth in $\omega \in \Omega^{(0)}_{\gamma}$, fulfilling*

$$\left| \Omega - \Omega^{(0)}_{\gamma} \right| \preceq \gamma^a \tag{3.45}$$

$$U_1^* H^{(reg)} U_1 = A^{(0)} + \epsilon R_0 \ , \tag{3.46}$$

where a is a positive constant (independent of γ, ϵ). The operator $A^{(0)}$ is given by

$$A^{(0)} := \mathrm{diag}(\lambda_j^{(0)}) \ , \tag{3.47}$$

with $\lambda_j^{(0)} = \lambda_j^{(0)}(\omega)$ Whitney smooth in ω fulfilling the following inequalities

$$\left|\lambda_j^{(0)} - \lambda_j^v\right| \preceq j^{\frac{\tilde\beta}{l+1}} \ , \tag{3.48}$$

$$\left|\lambda_i^{(0)} - \lambda_j^{(0)}\right| \succeq \left|i^d - j^d\right| \ , \tag{3.49}$$

$$\left|\frac{\Delta(\lambda_i^{(0)} - \lambda_j^{(0)})}{\Delta\omega}\right| \preceq \epsilon|i^d - j^d| \ . \tag{3.50}$$

$$\left|\lambda_i^{(0)} - \lambda_j^{(0)} + \omega\cdot k\right| \geq \frac{\gamma(1+|i^d - j^d|)}{1+|k|^\tau} \ , \quad |i-j| + |k| \neq 0 \ , \tag{3.51}$$

where, as usual, for any Lipschitz function f of ω, we denoted $\Delta f = f(\omega) - f(\omega')$.

Furthermore, $\forall s$ $\exists \epsilon_s$, s.t., if $|\epsilon| < \epsilon_s$ then

$$\|U_1 - \mathbf{1}\|_{Lip_\rho(\Omega_\gamma^{(0)};B(\mathcal{H}^{s-\delta};\mathcal{H}^s))} \preceq \epsilon \ , \quad \delta := \tilde\beta - (l+1) \ , \tag{3.52}$$

$$R_0 := U_1^{-1} R U_1 \in Lip_\rho(\Omega_\gamma^{(0)}; C^\ell(\mathbb{T}^n; B(\mathcal{H}^{s-\kappa};\mathcal{H}^s))) \ , \quad \forall \ell \ . \tag{3.53}$$

End of the proof of Theorem 2.4. Now Theorem 2.4 is obtained immediately by applying Theorem 7.3 of [Bam16] to the system (3.46). □

4 Proof of the main lemmas

In this section we prove Lemmas 3.13, 3.14, 3.16 and 3.18.

To prove that $\langle p\rangle$ and χ are symbols we use some explicit formulae for the solution of second order equations in order to write in a quite explicit form the integrals over the orbits of h_0.

Consider the Hamilton equations of h_0, namely

$$\dot\xi = -\frac{\partial V}{\partial x} \ , \quad \dot x = \xi \ . \tag{4.1}$$

It is well known that one can exploit the conservation of energy in order to reduce the system to quadrature, namely to compute the time as a function of the position:

$$t(x, x_0) = \int_{x_0}^{x} \frac{dq}{\sqrt{E - V(q)}} \ . \tag{4.2}$$

One also has that the period $T(E)$ is given by

$$T(E) = 4\int_0^{q_M(E)} \frac{dq}{\sqrt{E - V(q)}} \ , \tag{4.3}$$

where $q_M = q_M(E)$ is the positive solution of the equation

$$E = V(q_M) \ . \tag{4.4}$$

Before giving the proof of the main Lemmas, we need some preliminary results. First, in order to compute and estimate integrals of the form (4.2), (4.3), we will often use the change of variables

$$q(y) = q_M y \ . \tag{4.5}$$

Furthermore it is useful to define the function

$$\tilde{v}(E,y) := \sqrt{\frac{1-|y|^{2l}}{1-\frac{V(q(y))}{E}}} \ , \tag{4.6}$$

so that one has

$$\frac{1}{\sqrt{1-\frac{V(q(y))}{E}}} = \frac{\tilde{v}(E,y)}{\sqrt{1-|y|^{2l}}} \ . \tag{4.7}$$

Lemma 4.1. *The quantity q_M has the form*

$$q_M(E) \sim E^{1/2l}\bar{q}(E) \ , \tag{4.8}$$

where the function $\bar{q}$ admits an asymptotic expansion in powers of $\mu^2 := E^{-1/l}$ and its first term is 1.

Proof. Consider equation (4.4), divide by $E = \mu^{-2l}$; using the asymptotic expansion (2.6) it takes the form

$$1 \sim \sum_{j\geq 0} \mu^{2l} V_{2l-2j}(q_M) = \sum_{j\geq 0} \mu^{2l-2j}\mu^{2j} V_{2l-2j}(q_M) = \sum_{j\geq 0} \mu^{2j} V_{2l-2j}(\mu q_M) = \bar{q}^{2l} + \sum_{j\geq 1} \mu^{2j} V_{2l-2j}(\bar{q}) \ .$$

Thus one sees that $\bar{q}$ admits an asymptotic expansion in powers of μ^2. □

Lemma 4.2. *For all $E_0 > 0$, the function $\tilde{v}(E,y)$ is a $C^\infty([E_0,\infty))$ function of E and one has*

$$\left| \frac{\partial^k \tilde{v}}{\partial E^k}(E,y) \right| \preceq \frac{1}{E^k} \ , \quad \forall y \in [-1,1] \ , \quad \forall E \geq E_0 \ . \tag{4.9}$$

Proof. Denote $\tilde{V}_E(y) := \frac{V(q(y))}{E}$ and remark that, due to the definition of $q(y)$, one has $\tilde{V}_E(\pm 1) \equiv 1$, so that $\tilde{v}$ is regular at $y = \pm 1$. Furthermore, by Lemma 4.1 (and its proof), one has

$$\tilde{V}_E(y) \sim \bar{q}^{2l}|y|^{2l} + \sum_{j\geq 1} \mu^{2j} V_{2l-2j}(\bar{q}y) \ , \tag{4.10}$$

(with $\mu = E^{-1/2l}$) which shows that $\tilde{V}_E(y)$ admits an asymptotic expansion in μ. First we remark that, by eq. (4.10) and Lemma A.1, the thesis of the Lemma holds true for y outside a neighborhood of ± 1. We discuss now the result for y near 1.

We use the Faa di Bruno formula in order to compute the derivatives of

$$\tilde{v} \equiv \frac{\sqrt{1-|y|^{2l}}}{\sqrt{1-\tilde{V}_E(y)}}$$

with respect to E. Denote $f(x) := (1-x)^{-1/2}$. Remark that

$$f^{(j)}(x) = C_j \frac{(1-x)^{-j}}{\sqrt{1-x}} \ ,$$

and compute

$$\frac{\partial^k}{\partial E^k} f(\tilde{V}_E) \asymp \sum_{j=1}^{k} f^{(j)}(\tilde{V}_E) \sum_{h_1+...+h_j=k} \partial_E^{h_1} \tilde{V}_E ... \partial_E^{h_j} \tilde{V}_E$$
$$\asymp \frac{1}{\sqrt{1-x}} \sum_{j=1}^{k} \sum_{h_1+...+h_j=k} \frac{\partial_E^{h_1} \tilde{V}_E}{1-\tilde{V}_E} ... \frac{\partial_E^{h_j} \tilde{V}_E}{1-\tilde{V}_E} \ . \tag{4.11}$$

We study the single fraction at r.h.s.. Compute the Taylor expansion of $\tilde{V}_E(y)$ at $y = 1$, it is given by

$$\tilde{V}_E(y) \simeq 1 + \sum_{k \geq 1} \frac{1}{E} V^{(k)}(E^{1/2l}\bar{q})(E^{1/2l}\bar{q})^k \frac{(y-1)^k}{k!} \ , \tag{4.12}$$

from which we get

$$\frac{\partial_E^h \tilde{V}_E}{1-\tilde{V}_E} \simeq \frac{\sum_{k\geq 1} \partial_E^h \left[\frac{1}{E} V^{(k)}(E^{1/2l}\bar{q})(E^{1/2l}\bar{q})^k \frac{(y-1)^{k-1}}{k!} \right]}{\sum_{k\geq 1} \frac{1}{E} V^{(k)}(E^{1/2l}\bar{q})(E^{1/2l}\bar{q})^k \frac{(y-1)^{k-1}}{k!}} \ ,$$

which is regular at $y = 1$. To get a more usable expression and an estimate of this fraction we remark that the single term of the sum in the numerator is a multiple of

$$\partial_E^h [\partial_y^k \tilde{V}_E]_{y=1} = [\partial_y^k \partial_E^h \tilde{V}_E]_{y=1} \ ,$$

and one can compute the r.h.s. exploiting the asymptotic expansion (4.10) of $\tilde{V}_E$. So one gets that $\partial_y \tilde{V}_E$ admits an asymptotic expansion in μ^2. Thus one can apply Lemma A.1 which shows that the single term in the sum in the numerator of the fraction is estimated by $E^{-(h+1/l)}$. Inserting in (4.11) one gets the thesis. □

Lemma 4.3. *The period $T = T(E)$ is s.t. $T\eta \in S^{1-l}$, where η is the cutoff function defined in* (3.36).

Proof. Due to the presence of the cutoff function it is enough to study the behavior of $T(E)$ at infinity. Making the change of variables (4.5) in the integral (4.3), we get

$$T = \frac{4q_M}{E^{1/2}} \int_0^1 \frac{dy}{\sqrt{1-\tilde{V}_E(y)}} = \frac{4\bar{q}}{E^{\frac{1}{2}-\frac{1}{2l}}} \int_0^1 \frac{\tilde{v}(E,y)}{\sqrt{1-y^{2l}}} \ ; \tag{4.13}$$

exploiting the property (4.9) of the function $\tilde{v}$ one immediately gets the thesis. □

We are now ready for proving that the average of a symbol is a symbol.

Proof of Lemma 3.13 Remark that $\langle p \rangle$ is a function of E only. To compute it we first make a change of variables in the phase space, namely we will use the variables (E, x) instead of (x, ξ). Such a change of variables is well defined in the region $\xi > 0$ (or $\xi < 0$) and for $-q_M < x < q_M$. In these variables the flow Φ_{h_0} is given by $E(t) = E$ and $x(t)$ given by the inverse of the formula (4.2). Thus, using the definition of the average and making the change of variables $t(q)$ in the integrals, we have

$$\langle p \rangle(E) = \frac{1}{T(E)} \int_{-q_M}^{q_E} \frac{p(q, \sqrt{E-V(q)})}{\sqrt{E-V(q)}} dq + \frac{1}{T(E)} \int_{-q_M}^{q_E} \frac{p(q, -\sqrt{E-V(q)})}{\sqrt{E-V(q)}} dq \ . \tag{4.14}$$

Consider the first term (the second one can be treated in the same way); making the change of variables (4.5) it takes the form

$$\frac{q_M}{T(E)E^{1/2}} \int_{-1}^{1} \frac{p\left(q(y), E^{1/2}\sqrt{1-\tilde{V}_E(y)}\right)\tilde{v}(E,y)}{\sqrt{1-|y|^{2l}}} dy \ . \tag{4.15}$$

This quantity and its derivatives with respect to E can be easily estimate using Lemma A.3 and Lemma A.4. □

We recall a first representation formula for χ. The next lemma is Lemma 5.3 of [BG93] to which we refer for the proof (see also Lemma 4.21 of [Bam16]).

Lemma 4.4. *The solution of the homological equation* (3.22) *is given by*

$$\chi = \frac{1}{T(E)} \int_0^{T(E)} t(p - \langle p \rangle) \circ \Phi_{h_0}^t dt \ . \tag{4.16}$$

To estimate the function χ we need some more preliminary work.

Lemma 4.5. *Let* p *be a function, denote* $\check{p} := p - \langle p \rangle$ *and*

$$t_S(x) := \int_{-q_M}^{x} \frac{dq}{\sqrt{E - V(q)}} \ , \quad t_S^-(x) := \int_x^{q_M} \frac{dq}{\sqrt{E-V(q)}} \equiv t_S(-x) \ , \tag{4.17}$$

$$d\mu^+(q) := \frac{\check{p}(q, \sqrt{E-V(q)})}{\sqrt{E-V(q)}} dq \ , \quad d\mu^-(q) := \frac{\check{p}(q, -\sqrt{E-V(q)})}{\sqrt{E-V(q)}} dq \tag{4.18}$$

(t_S is the time taken to go from $-q_M$ to x) then, in the coordinates (E,x) for the upper half plane, the function χ defined by (4.16) *is given by*

$$\chi(E,x) = \frac{1}{T(E)} \int_{-q_M}^{q_M} (t_S(q) d\mu^+(q) + t_S^-(q) d\mu^-(q)) + \frac{1}{2} \int_{-q_M}^{q_M} d\mu^-(q) \tag{4.19}$$

$$+ \int_{-q_M}^{x} d\mu^+(q) \ . \tag{4.20}$$

Proof. We use again the formula (4.2). In all the integrals E will play the role of a parameter, so we do not write it in the argument of the functions. We split the interval of integration in (4.16) into three subintervals. For this purpose we define $t_M(x) := \frac{T}{2} - t_S(x)$, and remark that this is the time at which a solution starting at (x, ξ) reaches $(q_M, 0)$. We write

$$[0, T] = [0, t_M(x)] \cup [t_M, t_M + \frac{T}{2}] \cup [t_M + \frac{T}{2}, T] \ ,$$

and we study separately the integrals over the intervals.

The first integral is given by

$$\int_0^{t_M} t\check{p}(\Phi_{h_0}^t(x,\xi)) dt = \int_x^{q_M} \frac{t(q,x)\check{p}(q, \sqrt{E-V(q)})}{\sqrt{E-V(q)}} dq \tag{4.21}$$

$$= \int_x^{q_M} t_S(q) d\mu^+(q) - t_S(x) \int_x^{q_M} d\mu^+(q) \ , \tag{4.22}$$

where of course $t(q, x)$ is defined by (4.2). The integral over the second interval is given by

$$-\int_{q_M}^{-q_M}(\frac{T}{2} - t_S(x) + t_S^-(q))d\mu^-(q) = \tag{4.23}$$

$$= \frac{T}{2}\int_{-q_M}^{q_M} d\mu^-(q) - t_S(x)\int_{-q_M}^{q_M} d\mu^-(q) + \int_{-q_M}^{q_M} t_S^-(q)d\mu^-(q)\ . \tag{4.24}$$

Finally the third integral is given by

$$\int_{-q_M}^{x}\left[\frac{T}{2} + \left(\frac{T}{2} - t_S(x)\right) + t_S(q)\right]d\mu^+(q) = \tag{4.25}$$

$$= T\int_{-q_M}^{x} d\mu^+(q) - t_S(x)\int_{-q_M}^{x} d\mu^+(q) + \int_{-q_M}^{x} t_S(q)d\mu^+(q)\ . \tag{4.26}$$

Summing up we get

$$\int_{-q_M}^{q_M}(t_S(q)d\mu^+(q) + t_S^-(q)d\mu^-(q)) \tag{4.27}$$

$$-t_S(x)\int_{-q_M}^{q_M}(d\mu^+(q) + d\mu^-(q)) \tag{4.28}$$

$$+\frac{T}{2}\int_{-q_M}^{q_M} d\mu^-(q) + T\int_{-q_M}^{x} d\mu^+(q)\ , \tag{4.29}$$

but the integral in (4.28) is exactly the integral of $\check{p}$ along an orbit of h_0 and thus it vanishes, thus we get (4.19) and (4.20). □

Lemma 4.6. *Let $g \in S^{m_1,m_2}$ be a symbol, consider the function*

$$G(E, x) := \int_{-q_M}^{x} \frac{g(q, \sqrt{E - V(q)})}{\sqrt{E - V(q)}}dq\ , \tag{4.30}$$

and the function

$$\widehat{G}(x, \xi) := G(\xi^2 + V(x), x)\ .$$

Then $\eta(h_0)\widehat{G} \in S^{m_1+[m_2]-l+1,0}$ and $\eta(h_0)\partial_x\widehat{G} \in S^{m_1+[m_2]-l,0}$.

Proof. Due to the presence of the cutoff function, it is enough to study the behavior of $\widehat{G}$ as $E \to \infty$. First we estimate the modulus of G (and of $\widehat{G}$). To this end it is better to represent the integral in terms of integral over the flow of h_0. Preliminarly remark that

$$|g(x, \xi)| \preceq \lambda^{m_1}(x, \xi)\langle x\rangle^{m_2} \preceq \lambda^{m_1+[m_2]}(x, \xi) \preceq \langle h_0(x, \xi)\rangle^{m_1+[m_2]}\ . \tag{4.31}$$

Using the notation (4.2) one has

$$|G(E, x)| = \left|\int_0^{t_S(x)} g(\Phi_{h_0}^t(-q_M, 0))dt\right| \preceq \int_0^{T/2}\langle h_0(\Phi_{h_0}^t(-q_M, 0))\rangle^{m_1+[m_2]}dt$$

$$= \frac{T}{2}\langle E\rangle^{\frac{m_1+[m_2]}{2l}} \preceq \lambda^{m_1+[m_2]-l+1}\ .$$

To compute the derivatives of G and of $\widehat{G}$ it is better to use the formula (4.30), to make the change of variables (4.5) and to use the function $\tilde{v}$ defined in (4.6), so that one gets

$$G(E,x) = \frac{\bar{q}}{E^{\frac{1}{2}-\frac{1}{2l}}} \int_{-1}^{\frac{\mu x}{\bar{q}}} \frac{\tilde{v}(E,y) g(q(y), \sqrt{E-V(q(y))})}{\sqrt{1-|y|^{2l}}} dy \tag{4.32}$$

with $\mu = E^{-1/2l}$. From this formula one can easily compute

$$\partial_E G = \partial_E \left(\frac{\bar{q}}{E^{\frac{1}{2}-\frac{1}{2l}}} \right) \int_{-1}^{\frac{\mu x}{\bar{q}}} \frac{\tilde{v}(E,y) g(q(y), \sqrt{E-V(q(y))})}{\sqrt{1-|y|^{2l}}} dy \tag{4.33}$$

$$+ E^{\frac{1}{2l}} \bar{q} \frac{g(x, \sqrt{E-V(x)})}{\sqrt{E-V(x)}} \partial_E \left(\frac{\mu x}{\bar{q}} \right) \tag{4.34}$$

$$+ \frac{\bar{q}}{E^{\frac{1}{2}-\frac{1}{2l}}} \int_{-1}^{\frac{\mu x}{\bar{q}}} \frac{\partial_E \tilde{v}(E,y)\ g(q(y), \sqrt{E-V(q(y))})}{\sqrt{1-|y|^{2l}}} dy \tag{4.35}$$

$$+ \frac{\bar{q}}{E^{\frac{1}{2}-\frac{1}{2l}}} \int_{-1}^{\frac{\mu x}{\bar{q}}} \frac{\tilde{v}(E,y) \partial_E g(q(y), \sqrt{E-V(q(y))})}{\sqrt{1-|y|^{2l}}} dy\ , \tag{4.36}$$

where, in order to simplify (4.34) we used the definition of $\tilde{v}$.

Remark now that one has

$$\frac{\partial \widehat{G}}{\partial x} = \frac{\partial G}{\partial E} V' + \frac{\partial G}{\partial x}\ . \tag{4.37}$$

We study the contribution of (4.34) to $\partial \widehat{G}/\partial x$, which is the most singular one. To this end we compute

$$\frac{\partial G}{\partial x} + V'(x)\,(4.34) = \frac{g(x,\xi)}{\xi} \left[1 + q_M V'(x) \partial_E \left(\frac{x}{q_M} \right) \right]\ , \tag{4.38}$$

where, when explicitly possible we introduced the variables (x, ξ). We study now the square bracket in (4.38) in order to show that (4.38) is regular on the line $\xi = 0$; we denote by

$$\mathcal{T}(E,x) := q_M V'(x) \partial_E \left(\frac{x}{q_M} \right) \tag{4.39}$$

the second term in the bracket and we simplify it. First remark that the line $(x, \xi) = (x, 0)$, in terms of the variables (E, x), becomes the curve $(V(x), x)$, which can also be parametrized by E and in such a parametrization has the form $(E, q_M(E))$. Expanding at $\xi = 0$, one has

$$\widehat{\mathcal{T}}(x,\xi) := \mathcal{T}(\xi^2 + V(x), x) = \mathcal{T}(V(x), x) + \partial_E \mathcal{T}(V(x), x) 2\xi + O(\xi^2) = \mathcal{T}(E, q_M) + 2\partial_E \mathcal{T}(E, q_M)\xi + O(\xi^2)\ . \tag{4.40}$$

Now, using (4.39) and the definition of q_M, one gets

$$\mathcal{T}(E, q_M) = -V'(q_M) \partial_E(q_M) = -V'(q_M) \frac{1}{V'(q_M)} = -1\ .$$

Inserting in (4.40) and substituting in (4.38) one sees that (4.38) is regular at $\xi = 0$.

In conclusion we have

$$\partial_x \widehat{G} = V'(x)\frac{E^{\frac{1}{2}}}{q_M}\partial_E\left(\frac{q_M}{E^{\frac{1}{2}}}\right)\widehat{G}(x,\xi) \tag{4.41}$$

$$+ g(x,\xi)\left[\frac{1+\widehat{\mathcal{T}}(x,\xi)}{\xi}\right] \tag{4.42}$$

$$+ V'(x)q_M\int_{-q_M}^{x}\frac{(\partial_E\tilde{v})(E,y(q))\ g(q,\sqrt{E-V(q)})}{\tilde{v}(E,y(q))\sqrt{E-V(q)}}dq \tag{4.43}$$

$$+ V'(x)q_M\int_{-q_M}^{x}\frac{\partial_E g(q(y),\sqrt{E-V(q(y))})}{\sqrt{E-V(q(y))}}dy\ . \tag{4.44}$$

Remark that (4.41) and (4.44) clearly have the same structure as $\widehat{G}$, so these terms are suitable to start an iteration which shows that the original quantity is a symbol. One has still to deal with the other two terms. We start by (4.42).

The analysis of the square bracket in (4.42) (the only nontrivial part) has to be done by analyzing separately a neighborhood of $\xi = 0$. Such a region can be analyzed by exploiting the expansion (4.40), which allows to show that it is a symbol in such a neighborhood. The other region is trivial since the function is smooth in that region. Doing the explicit computations one easily shows that it is a symbol.

We come to (4.43). We wrote it in that form, since exploiting it one can compute its derivative with respect to x. An explicit computation shows that, mutatis mutandis, such a derivative is given again by (4.41)-(4.44). The main difference is that (4.42) has to be substituted by

$$\frac{g(x,\xi)\partial_E\tilde{v}(E,x/q_M)}{\tilde{v}(E,x/q_M)}\left[\frac{1+\widehat{\mathcal{T}}(x,\xi)}{\xi}\right]\ ,$$

which is again a symbol.

To conclude the proof we estimate the different terms of (4.41)-(4.44). The estimate of all the terms, but (4.42) is obtained by the same argument used to estimate G which gives that all such terms are bounded by $\langle x\rangle^{2l-1}\lambda^{m_1+[m_2]-3l+1}$.

In order to estimate (4.42) we consider its main term in the expansion in inverse powers of E:

$$\mathcal{T}(E,x) = V'(x)\left[-x\frac{\partial_E E^{1/2l}}{E^{1/2l}}\right] = -\frac{V'(x)x}{2lE} \simeq -\frac{|x|^{2l}}{E}\ ,$$

so that

$$\left|\frac{1+\mathcal{T}(E,x)}{\xi}\right| \simeq \left|\frac{E-|x|^{2l}}{\xi E}\right| = \left|\frac{\xi}{E}\right| \preceq \lambda^{-l}\ .$$

It follows that

$$|(4.42)| \preceq \lambda^{m-(l-1)-1}\ .$$

□

Proof of Lemma 3.14. First remark that, from Lemma 4.6, $\eta t_S \in S^{-l+1,0}$ and $\eta\partial_x t_S \in S^{-l,0}$. It follows that $(4.19)\eta \in \tilde{S}^{m_1+[m_2]-l+1}$ and $\eta(4.20) \in S^{m_1+[m_2]-l+1,0}$ with $\eta\partial_x(4.20) \in S^{m_1+[m_2]-l,0}$, which gives the thesis. □

Proof of Lemma 3.16. The proof is based on the fact that the flow of h_1 is essentially a rescaling of the flow of h_0. Precisely, $\Phi^t_{h_1}$ leaves invariant the level surfaces of h_0 and on a level surface $h_0 = E$ one has

$$\Phi^t_{h_1} \equiv \Phi^{(1+\epsilon f'(E))t}_{h_0}\ . \tag{4.45}$$

So, we apply the formulae for the average and for χ getting the result. We give the explicit proof of the fact that the solution χ is a symbol. From (4.16) with $\Phi^t_{h_1}$ in place of $\Phi^t_{h_0}$ we have

$$\begin{aligned}\chi &= \frac{1}{T_{h_1}}\int_0^{T_{h_1}} t\check{p}\circ\Phi^t_{h_1}dt = \frac{1}{(1+\epsilon f')^2 T_{h_1}}\int_0^{T_{h_1}} t(1+\epsilon f')\check{p}\circ\Phi^t_{h_1}(1+\epsilon f')dt \\ &= \frac{1}{1+\epsilon f'}\frac{1}{T_{h_0}}\int_0^{T_{h_0}} \tau\check{p}\circ\Phi^\tau_{h_0}d\tau\ ,\end{aligned}$$

Now this is just $(1+\epsilon f')^{-1}$ times the solution of the homological equation with the original unperturbed Hamiltonian h_0. Since, by the assumption $(1+\epsilon f')^{-1}$ is a symbol, which is a lower order correction of the identity, the thesis follows. $\square$

4.1 Solution (3.29)

The homological equation (4.1) will be relevant only when $l = 1$, where we assume that $V(x) = x^2$ is a Harmonic potential.

Lemma 4.7. *(Lemma 6.4 of [Bam97]) The solution of the homological equation* (4.1) *is given by*

$$\chi(x,\xi,\phi) := \sum_{k\in\mathbb{Z}^n}\chi_k(x,\xi)e^{\mathrm{i}k\cdot\phi}\ ,$$

where

$$\chi_0 = \frac{1}{T(E)}\int_0^{T(E)} t(\overline{p} - \overline{\langle p\rangle})\circ\Phi^t_{h_0}dt \tag{4.46}$$

$$\chi_k(x,\xi) = \frac{1}{e^{\mathrm{i}\omega\cdot kT(E)}-1}\int_0^{T(E)} e^{\mathrm{i}\omega\cdot kt}p_k(\Phi^t_{h_0}(x,\xi))dt\ , \tag{4.47}$$

and p_k is defined by

$$p_k(x,\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} p(x,\xi,\phi)e^{-\mathrm{i}k\phi}d\phi\ .$$

Lemma 4.8. *Let $p\in S^{m_1,m_2}$, fix $\alpha\in\mathbb{R}$ and consider*

$$I(x,\xi) := \int_0^{2\pi} e^{\mathrm{i}\alpha t}p\left(\Phi^t(x,\xi)\right)dt\ . \tag{4.48}$$

One has $I\in S^{m_1+[m_2],0}$ with $\partial_x p\in S^{m_1+[m_2]-1,0}$.

Proof. First we write the integral using the action angle variables (A,θ) for the Harmonic oscillator. Thus we make the change of variables

$$x = \sqrt{A}\sin\theta\ ,\quad \xi = \sqrt{A}\cos\theta\ ;$$

In these variables the flow is simply $\theta\to\theta+t$, so we have

$$\begin{aligned}I(A,\theta) &= \int_0^{2\pi} e^{\mathrm{i}\alpha t}p_a(A,\theta+t)dt = e^{-\mathrm{i}\alpha\theta}\int_0^{2\pi} e^{\mathrm{i}\alpha t}p_a(A,t)dt \\ &= e^{-\mathrm{i}\alpha\theta}\int_0^{2\pi} e^{\mathrm{i}\alpha t}p(\sqrt{\xi^2+x^2}\cos t, -\sqrt{\xi^2+x^2}\sin t)dt\ ,\end{aligned}$$

where $p_a(A,\theta) = p(\sqrt{A}\sin\theta, \sqrt{A}\cos\theta)$.

Now using a technique similar to that used in the proof of Lemmas A.3 and A.4, one can see that the integral is of class $S^{m_1+[m_2]}$.

In order to conclude the proof we have to check the prefactor. The prefactor can be written as

$$\left(\frac{\xi - \mathrm{i}x}{A^{1/2}}\right)^{\alpha} ,$$

which is easily seen to be a symbol which is bounded and has the property that its x derivative is bounded by $A^{-1/2}$, from which the thesis immediately follows. □

Proof of Lemma 3.18. The result follows using the previous Lemmas once one has a lower bound of the small denominators. This is easily obtained by remarking that, in $\Omega_{1\gamma}$ one has

$$\left|e^{\mathrm{i}\omega\cdot kT} - 1\right| = \left|2\sin\left(\frac{\omega\cdot kT}{2}\right)\right| \geq 2\left|\frac{\omega\cdot kT}{2} - k_0\pi\right|$$
$$= |\omega\cdot k - k_0| \geq \frac{\gamma}{1+|k|^{\tau}} .$$

□

A Some technical lemmas

Lemma A.1. *Let f be a function of class C^k, and consider $f(1/E^{1/l})$. For $E \to \infty$ one has:*

$$\frac{\partial^k}{\partial E^k}\left[f\left(\frac{1}{E^{1/l}}\right)\right] \asymp \frac{1}{E^{k+\frac{1}{l}}}\sum_{j=1}^{k}\frac{1}{E^{\frac{j-1}{l}}}f^{(j)}\left(\frac{1}{E^{\frac{1}{l}}}\right) . \tag{A.1}$$

By $a \asymp b$ we mean $|a| \preceq |b|$ and $|b| \preceq |a|$, at least for sufficiently large values of E.

Proof. We use the Faa di Bruno formula which gives

$$\frac{\partial^k f}{\partial E^k} \asymp \sum_{j=1}^{k} f^{(j)}(\mu)\sum_{h_1+\ldots+h_j=k}\frac{\partial^{h_1}\mu}{\partial E^{h_1}}\cdots\frac{\partial^{h_j}\mu}{\partial E^{h_j}} ,$$

where we denoted $\mu = E^{-1/l}$. The indexes h_i always fulfill $h_i \geq 1$. On the other hand one has

$$\frac{\partial^h\mu}{\partial E^h} \asymp \frac{1}{E^{h+1/l}} ;$$

substituting in the previous formula one gets the result. □

Lemma A.2. *Let $W(y,x)$ be a C^∞ function fulfilling*

$$\left|\partial_x^k W(y,x)\right| \preceq \langle x\rangle^{m-k} , \tag{A.2}$$

denote

$$I(M) := \int_{-1}^{1}\frac{W(y,My)}{\sqrt{1-|y|^{2l}}}dy \tag{A.3}$$

then one has

$$\left|\frac{\partial^k I}{\partial M^k}(M)\right| \preceq \langle M\rangle^{[m]-k} . \tag{A.4}$$

Proof. The difficulty in estimating the integral is that when $y = 0$ the quantity My does not diverge. One has

$$\frac{\partial^k}{\partial M^k}\int_{-1}^{1}\frac{W(y,My)}{\sqrt{1-|y|^{2l}}}dy = \int_{-1}^{1}\frac{\partial_x^k W(y,My)y^k}{\sqrt{1-|y|^{2l}}}dy \tag{A.5}$$

We fix a small a and split the interval of integration: $[-1,1] = [-1,-a]\cup(-a,a)\cup[a,1]$. The integral over the first and the last intervals are estimated in the same way. Consider the one over $[a,1]$. One has

$$\left|\int_{a}^{1}\frac{\partial_x^k W(y,My)y^k}{\sqrt{1-|y|^{2l}}}dy\right| \le \left|\int_{a}^{1}\frac{\langle My\rangle^{m-k}y^k}{\sqrt{1-|y|^{2l}}}dy\right| \preceq \langle Ma\rangle^{m-k} \ .$$

Over the interval $(-a,a)$ one has $\sqrt{1-|y|^{2l}} > 1/2$ provided a is small enough. Thus one has

$$\left|\int_{-a}^{a}\frac{\partial_x^k W(y,My)y^k}{\sqrt{1-|y|^{2l}}}dy\right| \preceq \int_{-a}^{a}\langle My\rangle^{m-k}|y|^k dy = 2\int_0^{Ma}\langle q\rangle^{m-k}\left(\frac{q}{M}\right)^k\frac{dq}{M}$$
$$= \frac{2}{M^{k+1}}\int_0^{Ma}\langle q\rangle^{m-k}q^k dq \preceq M^{[m]-k} \ ,$$

which immediately gives the thesis. □

Lemma A.3. *Under the same assumption of Lemma A.2, one has* $I(E\bar{q})\in S^{[m]}$.

Proof. First remark that, denoting $M = E^{\frac{1}{2l}}\bar{q}$, by Lemma A.1, one has

$$\partial_E^k M \asymp \sum_{j=0}^{k}\partial_E^{k-j}E^{\frac{1}{2l}}\partial_E^j\bar{q} \asymp \frac{E^{\frac{1}{2l}}}{E^k}\bar{q} + \sum_{j=1}^{k}\frac{E^{\frac{1}{2l}}}{E^{k-j}}\frac{1}{E^{j+\frac{1}{l}}}\sum_{i=1}^{j}\frac{\partial^i\bar{q}}{\partial\mu^i}\frac{1}{E^{\frac{i-1}{l}}} \asymp \frac{E^{1/2l}}{E^k} \ .$$

Now, from the Faa di Bruno formula one has

$$\partial_E^k I(M) \asymp \sum_{j=1}^{k} I^{(j)}(M)\sum_{h_1+...+h_j=k}\partial_M^{h_1}M...\partial_M^{h_j}M \asymp \sum_{j=1}^{k}\langle M\rangle^{[m]-j}\sum_{h_1+...+h_j=k}\frac{M}{E^{h_1}}...\frac{M}{E^{h_j}} = \frac{M^{[m]}}{E^k} \ ,$$

from which the thesis follows. □

By working as in the proof of the above lemmas one gets also the following useful result.

Lemma A.4. *Let* $g(y,\xi)$ *be such that*

$$\left|\partial_\xi^k g(x,\xi)\right| \preceq \lambda^{m-kl} \ ,$$

consider

$$I(E) := \int_{-1}^{1}\frac{g(y,\sqrt{E-V(q(y))})}{\sqrt{1-|y|^{2l}}}dy \ ,$$

then one has $I\in S^m$.

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