REDUCIBILITY OF GENERALIZED PRINCIPAL SERIES REPRESENTATIONS

BY

BIRGIT SPEH and DAVID A. VOGAN, JR[1]

Mathematische Institut der Universität Bonn Bonn, West Germany

Massachusetts Institute of Technology, Cambridge Massachusetts, U.S.A.

1. Introduction

Let G be a connected semisimple matrix group, and $P \subseteq G$ a cuspidal parabolic subgroup. Fix a Langlands decomposition

$$P = MAN$$

of P, with N the unipotent radical and A a vector group. Let δ be a discrete series representation of M, and ν a (non-unitary) character of A. We call the induced representation

$$\pi(P, \delta \otimes \nu) = \operatorname{Ind}_P^G (\delta \otimes \nu \otimes 1)$$

(normalized induction) a *generalized principal series representation*. When ν is unitary, these are the representations occurring in Harish-Chandra's Plancherel formula for G; and for general ν they may be expected to play something of the same role in harmonic analysis on G as complex characters do in $\mathbf{R}^n$. Langlands has shown that any irreducible admissible representation of G can be realized canonically as a subquotient of a generalized principal series representation (Theorem 2.9 below). For these reasons and others (some of which will be discussed below) one would like to understand the reducibility of these representations, and it is this question which motivates the results of this paper. We prove

THEOREM 1.1. (Theorems 6.15 and 6.19). *Let $\pi(P, \delta \otimes \nu)$ be a generalized principal series representation. Fix a compact Cartan subgroup T^+ of M (which exists because M has a discrete series). Let*

$$\mathfrak{h} = \mathfrak{t}^+ + \mathfrak{a},\ \mathfrak{g}$$

(1) Support by an AMS Research Fellowship.

denote the complexified Lie algebras of T^+A and G, respectively. Let

$$\lambda \in (\mathfrak{t}^+)^*$$

denote the Harish-Chandra parameter of some constituent of the representation $\delta|_{M_0}$ (the identity component of M). Set

$$\gamma = (\lambda, \nu) \in \mathfrak{h}^*;$$

here we write $\nu \in \mathfrak{a}^*$ for the differential of ν. Let θ be the automorphism of $\mathfrak{h}$ which is 1 on $\mathfrak{t}^+$ and -1 on $\mathfrak{a}$; then θ preserves the root system Δ of $\mathfrak{h}$ in $\mathfrak{g}$. Then $\pi(P, \delta \otimes \nu)$ is reducible only if there is a root $\alpha \in \Delta$ such that

$$n = 2\langle \alpha, \gamma \rangle / \langle \alpha, \alpha \rangle \in \mathbf{Z};$$

and either

(a) *$\langle \alpha, \gamma \rangle > 0$, $\langle \theta\alpha, \gamma \rangle < 0$, and $\alpha \neq -\theta\alpha$, or*

(b) *$\alpha = -\theta\alpha$, and a parity condition (relating the parity of n and the action of δ on the disconnected part of M) is satisfied.*

Suppose further that $\langle \beta, \gamma \rangle \neq 0$ for any $\beta \in \Delta$. Then these conditions are also sufficient for $\pi(P, \delta \otimes \nu)$ to be reducible.

The parity condition is stated precisely in Proposition 6.1.

The simplest kind of direct application of this theorem is the analysis of so-called complementary series representations. Whenever ν is a unitary character of A, $\pi(P, \delta \otimes \nu)$ is a unitary representation. But $\pi(P, \delta \otimes \nu)$ can also be given a unitary structure for certain other values of ν; it is these representations which are called complementary series, and they have been studied by many people. The following theorem is well known, and we will not give a proof. It is included only to illustrate the applicability of Theorem 1.1 to the study of unitary representations.

THEOREM 1.2. *Suppose $P = MAN$ is a parabolic subgroup of G, $\delta \in \hat{M}$ is a unitary series representation, and* dim $A = 1$. *Assume that there is an element $x \in G$ normalizing M and A, fixing δ (in its action on $\hat{M}$) and acting by $a \to a^{-1}$ on A. Fix a non-trivial real-valued character $\nu \in \hat{A}$; and for $t \in \mathbf{R}$, write $t\nu$ for the character whose differential is t times that of ν. Let*

$$t_0 = \sup \{ t \in \mathbf{R} \mid \pi(P, \delta \otimes t_1 \nu) \text{ is irreducible for all } t_1 \text{ with } |t_1| < t \}.$$

Then whenever $|t| \leq t_0$, every irreducible composition factor of $\pi(P, \delta \otimes t\nu)$ is unitarizable.

The point is that Theorem 1.1 gives a lower bound on t_0 when δ is a discrete series. It is far from best possible, however, and the converse of Theorem 1.2 is not true; so this result (and its various simple generalizations) are very far from the final word on complementary series.

The techniques of this paper are actually directed at the more general problem of determining all of the irreducible composition factors of generalized principal series representations, and their multiplicities. (We will call this the composition series problem.) This is of interest for several reasons. First, it is equivalent to determining the distribution characters of all irreducible representations of G, a problem which is entertaining in its own right. Next, it would allow one to determine the reducibility of any representations induced from parabolic subgroups of G (and not merely those induced from discrete series). For technical reasons this general reducibility problem is not easy to approach directly; but results about it give more complementary series representations, because of results like Theorem 1.2. Unfortunately our results about the composition series problem are very weak. The first, described in Section 3, relies on the theory of integral intertwining operators. Corollary 3.15 provides a partial reduction of the composition series problem to the case when $\dim A = 1$; in particular, Theorem 1.1 is completely reduced to that case. Next, we study the Lie algebra cohomology of generalized principal series representations. When the parameter ν is not too large, this leads to a reduction of the composition series problem to a proper subgroup (Theorem 4.23).

Section 5 contains a series of technical results refining Zuckerman's "periodicity" ([21]). This leads easily to the "only if" part of Theorem 1.1. Section 6 is devoted to locating certain specific composition factors in generalized principal series representations, and thus to finding sufficient conditions for reducibility. All of the ideas described above appear as reduction techniques. We begin with two-well-known types of reducibility—the Schmid embeddings of discrete series into generalized principal series, and the embeddings of finite-dimensional representations into principal series—and do everything possible to complicate them. The main result is Theorem 6.9.

For the benefit of casual readers, here is a guide to understanding the theorems of this paper. We regard a generalized principal series representation as parametrized (roughly) by the Cartan subalgebra $\mathfrak{h}$ and weight γ defined in Theorem 1.1. This is made precise in 2.3–2.6. Accordingly, we write $\pi(\gamma)$ for such a representation. By a theorem of Langlands, $\pi(\gamma)$ has a canonical irreducible subquotient $\bar{\pi}(\gamma)$ (roughly), and in this way irreducible representations are also parametrized by weights of Cartan subalgebras. This is made precise in 2.8–2.9.

The main result of Section 3 is Theorem 3.14, which reduces the question of reducibility of $\pi(\gamma)$ to the case when $\dim A=1$. The notation is explained between 3.2 and 3.4, and between 3.12 and 3.13. For the reader already familiar with the factorization of intertwining operators, all of the results of Section 3 should be obvious consequences of Lemma 3.13.

The main result of Section 4 is Theorem 4.23, whose statement is self-contained. The proof consists of a series of tricks, of which the only serious one is Proposition 4.21. A more conceptual explanation of the results can be given in terms of recent unpublished work of Zuckerman; but this does not simplify the proofs significantly.

Section 5 consists of technical results on tensor products of finite dimensional representations and irreducible admissible representations. The major new results are Theorem 5.15 (for which notation is defined at 5.1) and Theorem 5.20 (notation after 5.5). We also include a complete account of Schmid's theory of coherent continuation (after 5.2–after 5.5), and a formulation of the Hecht-Schmid character identities for disconnected groups (Proposition 5.14; notation after 5.6, and 5.12–5.14). Proposition 5.22 (due to Schmid) describes one kind of reducibility for generalized principal series.

Section 6 begins by constructing more reducibility (Theorem 6.9). This leads to the precise forms of Theorem 1.1 (Theorems 6.15 and 6.19). Theorem 6.16 is a technical result about tensor products with finite dimensional representations; it can be interpreted as a calculation of the Borho-Jantzen-Duflo τ-invariant of a Harish-Chandra module, in terms of the Langlands classification. Theorem 6.18 states that any irreducible has a unique irreducible pre-image under Zuckerman's ψ-functor (Definition 5.1). In conjunction with Corollary 5.12, this reduces the composition series problem to the case of regular infinitesimal character.

Section 7 contains the proof of Theorem 6.9 for split groups of rank 2, which are not particularly amenable to our reduction techniques.

The questions considered in this paper have been studied by so many people that it is nearly impossible to assign credit accurately. We have indicated those results which we know are not original, but even then it has not always been possible to give a reference. Eearlier work may be found in [2], [7], [10], [11], and the references listed there.

2. Notation and the Langlands classification

It will be convenient for inductive purposes to have at our disposal a slightly more general class of groups than that considered in the introduction. Let G be a Lie group, with Lie algebra $\mathfrak{g}_0$ and identity component G_0; put $\mathfrak{g}=(\mathfrak{g}_0)_{\mathbf{C}}$. Notation such as H, H_0, $\mathfrak{h}_0$, and $\mathfrak{h}$ will be used analogously. Let $G_{\mathbf{C}}$ be the connected adjoint group of $\mathfrak{g}$, let $\mathfrak{g}_0^1=[\mathfrak{g}_0, \mathfrak{g}_0]$, and let G_0^1 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0^1$. (For the definitions and results of the next few paragraphs, see [3].)

Definition 2.1. G is *reductive* if

(1) $\mathfrak{g}_0$ is reductive, and $\operatorname{Ad}(G) \subseteq G_{\mathbf{C}}$,

(2) G_0^1 has finite center,

(3) G_0 has finite index in G.

Henceforth, G will denote a reductive linear group with abelian Cartan subgroups. (One reason for this last assumption will be indicated at the beginning of Section 4.)

Fix a Cartan involution θ_0 of G_0, with fixed point set a maximal compact subgroup K_0 of G_0. We can choose a compact subgroup K of G, meeting every component, so that $K \cap G_0 = K_0$; and θ_0 extends to an involution θ of G, with fixed point set K. Let $\mathfrak{p}_0$ denote the (-1) eigenspace of θ on $\mathfrak{g}_0$, and $P = \exp(\mathfrak{p}_0)$; then $G = KP$ as analytic manifolds. Fix on $\mathfrak{g}_0$ a G-invariant bilinear form $\langle\ ,\ \rangle$, positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$. We will frequently complexify, dualize and restrict $\langle\ ,\ \rangle$ without comment or change of notation.

PROPOSITION 2.2. *Let $\mathfrak{b}_0 \subseteq \mathfrak{g}_0$ be a θ-invariant reductive abelian subalgebra. Then the centralizer $G^{\mathfrak{b}_0}$ of $\mathfrak{b}_0$ in G is a closed linear reductive subgroup of G, with abelian Cartan subgroups. The subgroup $K \cap G^{\mathfrak{b}_0}$, the involution $\theta|_{G^{\mathfrak{b}_0}}$, and the bilinear form $\langle\ ,\ \rangle|_{\mathfrak{g}_0^{\mathfrak{b}_0}}$ satisfy the properties described in the preceding paragraph for the group G.*

The straightforward verification of this result is left to the reader. All of the reductive groups appearing in inductive arguments will be obtained in this way from a fixed reductive group G, and we will assume that they are endowed with Cartan involutions and so forth in accordance with Proposition 2.2.

Let H be a θ-invariant Cartan subgroup of G (i.e., the centralizer in G of a θ-invariant Cartan subalgebra). Then $H = T^+A$, a direct product; here $T^+ = H \cap K$ is compact, and $A = \exp(\mathfrak{h}_0 \cap \mathfrak{p}_0)$ is a vector group. The set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ is written $\Delta(\mathfrak{g}, \mathfrak{h})$. More generally, if $\mathfrak{h}^1 \subseteq \mathfrak{h}$ and $V \subseteq \mathfrak{g}$ is $\mathfrak{h}^1$-invariant, we write $\Delta(V, \mathfrak{h}^1)$ (or simply $\Delta(V)$ if the choice of $\mathfrak{h}^1$ is obvious from the context) for the set of roots of $\mathfrak{h}^1$ in V with multiplicities. We write $\varrho(V) = \varrho(\Delta(V)) = \frac{1}{2}\sum_{\alpha \in \Delta(V)} \alpha$. The roots are imaginary on $\mathfrak{t}_0^+$ and real on $\mathfrak{a}_0$; we write this as $\Delta(\mathfrak{g}, \mathfrak{h}) \subseteq i(\mathfrak{t}_0^+)' + \mathfrak{a}_0'$. In general, a prime will denote a real dual space, and an asterisk a complex dual. Any linear functional $\gamma \in \mathfrak{h}^*$ can be written as $(\operatorname{Re}\gamma) + i(\operatorname{Im}\gamma)$, with $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ in $i(\mathfrak{t}_0^+)' + \mathfrak{a}_0'$; unless the contrary is explicitly stated, Re and Im will be used in this way. Then $\langle\ ,\ \rangle$ is positive definite on real linear functionals (such as roots).

An element $\gamma \in \mathfrak{h}^*$ is called *regular* or *nonsingular* if $\langle \gamma, \alpha \rangle \neq 0$ whenever $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$. To each nonsingular element we attach a positive root system $\Delta_\gamma^+(\mathfrak{g}, \mathfrak{h}) = \Delta_\gamma^+$ as follows: $\alpha \in \Delta_\gamma^+$ iff $\operatorname{Re} \langle \gamma, \alpha \rangle > 0$, or $\operatorname{Re} \langle \gamma, \alpha \rangle = 0$ and $\operatorname{Im} \langle \gamma, \alpha \rangle > 0$. Conversely, to each positive root system Δ^+ we associate a Weyl chamber $C_{\Delta^+} = \mathfrak{h}^*$; $\gamma \in C_{\Delta^+}$ iff $\Delta_\gamma^+ = \Delta^+$. The closure $\bar{C}_{\Delta^+}$ of C_{Δ^+} is called a closed Weyl chamber; it is a fundamental domain for the action of the complex Weyl group $W(\mathfrak{g}/\mathfrak{h})$ on $\mathfrak{h}^*$. An element of $\bar{C}_{\Delta^+}$ is called *dominant*; an element of C_{Δ^+} is called *strictly dominant*. The Weyl group of H in G, $W(G/H)$, is defined as the normalizer of H in G, divided by H; it is in a natural way a subgroup of $W(\mathfrak{g}/\mathfrak{h})$. If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, we denote by $s_\alpha \in W(\mathfrak{g}/\mathfrak{h})$ the reflection about α.

To any element $\gamma \in \mathfrak{h}^*$ we can associate a parabolic subalgebra $\mathfrak{b} \supseteq \mathfrak{h}$, with Levi decomposition $\mathfrak{b} = \mathfrak{l} + \mathfrak{n}$, by the condition

$$\Delta(\mathfrak{n}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \operatorname{Re} \langle \alpha, \gamma \rangle > 0 \text{ or } \operatorname{Re} \langle \alpha, \gamma \rangle = 0 \text{ and } \operatorname{Im} \langle \alpha, \gamma \rangle > 0\}.$$

If $\gamma \in i(\mathfrak{t}_0^+)'$, then $\mathfrak{b}$ is θ-invariant; if $\gamma \in \mathfrak{a}_0'$, then $\mathfrak{b}$ is the complexification of a real parabolic subalgebra.

The set of infinitesimal equivalence classes of irreducible admissible representations of G is written $\hat{G}$; equivalence of representations will always mean infinitesimal equivalence. We consider Harish-Chandra modules (or compatible $(U(\mathfrak{g}), K)$-modules) as defined in [14]; essentially these are $\mathfrak{k}$-finite representations of the enveloping algebra of $\mathfrak{g}$. If X is such a module, we denote by X_{ss} (the semisimplification of X) the completely reducible Harish-Chandra module with the same composition series as X.

We turn now to the description of the standard representations of G, beginning with the discrete series. (The following construction is due to Harish-Chandra [3], and detailed proofs may be found there.) Choose a Cartan subgroup T of K; then G has a discrete series iff T is a Cartan subgroup of G, which we temporarily assume. Let Z be the center of G; then $T = ZT_0$.

Definition 2.3. A *regular pseudocharacter* (or simply *regular character*) λ of T is a pair $(\Lambda, \bar{\lambda})$, with $\Lambda \in \hat{T}$, and $\bar{\lambda} \in i\mathfrak{t}_0'$ regular, such that

$$d\Lambda = \bar{\lambda} + \varrho(\Delta_{\bar{\lambda}}^+(\mathfrak{g})) - 2\varrho(\Delta_{\bar{\lambda}}^+(\mathfrak{k})).$$

The set of regular pseudocharacters of T is written $\hat{T}'$. For definiteness, we may sometimes refer to a G-regular pseudocharacter. We will often write λ to mean $\bar{\lambda}$; thus if $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, $\langle \alpha, \lambda \rangle$ means $\langle \alpha, \bar{\lambda} \rangle$. We make $W(G/T)$ act on $\lambda \in \hat{T}'$ by acting on Λ and $\bar{\lambda}$ separately.

Fix $\lambda \in T'$, and let $\pi_{G_0}(\bar{\lambda}) = \pi(\bar{\lambda})$ denote the discrete series representation of G_0 with

Harish-Chandra parameter $\bar{\lambda}$. Then $\pi(\bar{\lambda})$ acts on $Z \cap G_0$ by the scalars $\Lambda|_{Z \cap G_0}$. Accordingly we can define a representation $\pi_{G_0}(\bar{\lambda}) \otimes \Lambda|_Z$ of ZG_0; and we define

$$\pi_G(\lambda) = \pi(\lambda) = \mathrm{Ind}_{ZG_0}^G(\pi_{G_0}(\bar{\lambda}) \otimes \Lambda|_Z).$$

PROPOSITION 2.4. *Suppose G has a compact Cartan subgroup T. Then for each $\lambda \in \hat{T}'$, $\pi_G(\lambda)$ is an irreducible square-integrable representation of G, and every such representation is obtained in this way. Furthermore, $\pi(\lambda) \cong \pi(\lambda')$ iff $\lambda = \sigma \cdot \lambda'$ for some $\sigma \in W(G/T)$.*

Now let $H = T^+A$ be an arbitrary θ-invariant Cartan subgroup of G. Let $MA = G^A = G^{\mathfrak{a}_0}$ be the Langlands decomposition of the centralizer of A in G; then M is a reductive linear group with abelian Cartan subgroups, and T^+ is a compact Cartan subgroup of M.

Definition 2.5. An *M-regular pseudocharacter* (or *M-regular character*) γ of H is a pair (λ, ν), with λ an M-regular pseudocharacter of T^+, and $\nu \in \hat{A}$.

The set of M-regular characters of H is written $\hat{H}'$. We use the same letter for ν and its differential; thus we may sometimes write $\gamma = (\lambda, \nu) \in \mathfrak{h}^*$.

Definition 2.6. Let $H = T^+A$ be a θ-invariant Cartan subgroup of G, and let $P = MAN$ be any parabolic subgroup of G with $MA = G^A$. If $\gamma \in \hat{H}'$, the *generalized principal series representation with parameter* (γ, P) is

$$\pi_G(P, \gamma) = \pi(P, \gamma) = \mathrm{Ind}_P^G(\pi_M(\lambda) \otimes \nu \otimes 1).$$

Here $\pi_M(\lambda) \otimes \nu \otimes 1$ is the obvious representation of $P = MAN$, and induction means normalized induction.

The distribution character of $\pi(P, \gamma)$ is written $\Theta_G(P, \gamma)$ or simply $\Theta(\gamma)$. For any θ-invariant Cartan subgroup H of G, one can find a finite set of parabolics associated to H as in Definition 2.6; they differ only in the choice of N. Our notational neglect of P is justified by

PROPOSITION 2.7. *With notation as above, if $P' = MAN'$ is another parabolic subgroup associated to H, then $\Theta(P, \gamma) = \Theta(P', \gamma)$.*

This standard result follows from the formulas for induced characters given in [16]. By a theorem of Harish-Chandra (cf. [5]) Proposition 2.7 has the following well-known corollary.

COROLLARY 2.8. *With notation as above, $\pi(P, \gamma)$ and $\pi(P', \gamma)$ have the same irreducible composition factors, occurring with the same multiplicities.*

So whenever a statement is independent of P, or if the choice of P is obvious, we simply write $\pi_G(\gamma)$ instead of $\pi_G(P, \gamma)$.

We now wish to consider a canonical family $\{\bar{\pi}^1(\gamma), \ldots, \bar{\pi}^r(\gamma)\}$ of irreducible subquotients of $\pi(P, \gamma)$, the *Langlands subquotients*. It turns out that they are precisely the subquotients which contain a lowest K-type of $\pi(P, \gamma)$ (Definition 4.1 below). This is proved in Section 4 (Corollary 4.6). Langlands' definition is more or less along the following lines: The parameter $\nu \in \hat{A}$ is said to be positive (respectively strictly positive) with respect to P if $\operatorname{Re}\langle \nu, \alpha\rangle$ is not negative (respectively positive) for all $\alpha \in \Delta(\mathfrak{a}_0, \mathfrak{n}_0)$. For any $\gamma \in \hat{H}'$, we can choose P so that ν is positive with respect to P, and $\bar{P}$, so that $-\nu$ is positive with respect to $\bar{P}$. In this situation there exists an intertwining operator

$$I(P, \bar{P}, \gamma): \pi(P, \gamma) \to \pi(\bar{P}, \gamma)$$

whose image is a direct sum of irreducible representations, namely, the Langlands subquotients of $\pi(\bar{P}, \gamma)$. D. Miličić observed that all irreducible subrepresentations of $\pi(\bar{P}, \gamma)$ are actually Langlands quotients. We write $\bar{\Theta}^i(\gamma)$ for the character of $\bar{\pi}^i(\gamma)$.

THEOREM 2.9. (Langlands [13], Knapp-Stein [11].) *Let G be a reductive linear group with all Cartan subgroups abelian. Every $\pi \in \hat{G}$ is infinitesimally equivalent to some $\bar{\pi}^i(\gamma)$, for an appropriate θ-invariant Cartan subgroup H, $\gamma \in \hat{H}'$, and index i. Furthermore, $\bar{\pi}^i(\gamma) \simeq \bar{\pi}^j(\gamma)$ iff $i = j$. If B is another θ-stable Cartan subgroup of G, and $\gamma' \in \hat{B}'$, then $\bar{\pi}^i(\gamma) \simeq \bar{\pi}^j(\gamma')$ only if H conjugate to B by an element of G taking γ to γ'.*

We conclude this section with some elementary but useful facts about the representations $\pi(\gamma)$. Let $\mathfrak{h}$ be any Cartan subalgebra of $\mathfrak{g}$, and let $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ be some system of positive roots; put $\varrho = \varrho(\Delta^+)$. Associated to Δ^+ there is an algebra isomorphism $\tilde{\xi}$ from $\mathfrak{Z}(\mathfrak{g})$, the center of $U(\mathfrak{g})$, onto $S(\mathfrak{h})^{\tilde{W}(\mathfrak{g}/\mathfrak{h})}$, the translated Weyl group invariants in the symmetric algebra of $\mathfrak{h}$. Composing $\tilde{\xi}$ with translation by ϱ gives an isomorphism $\mathfrak{Z}(\mathfrak{g}) \xrightarrow{\xi} S(\mathfrak{h})^{W(\mathfrak{g}/\mathfrak{h})}$, which we call the Harish-Chandra map; it is independent of Δ^+. In particular, the characters of $\mathfrak{Z}(\mathfrak{g})$ are identified with $W(\mathfrak{g}/\mathfrak{h})$ orbits in $\mathfrak{h}^*$; so if $\gamma \in \mathfrak{h}^*$, we may speak of "the infinitesimal character γ". (For all this see for example [8].) We say that a representation π has infinitesimal character γ if $\mathfrak{Z}(\mathfrak{g})$ acts in π by the character γ. Then $\pi_G(P, \gamma)$ has infinitesimal character γ; and hence so do all its composition factors.

PROPOSITION 2.10. (Cf. [1], [19]). *Suppose H_1 and H_2 are θ-invariant Cartan subgroups of G, and*

$$\gamma_1 = (\lambda_1, \nu_1) \in \hat{H}_1', \quad \gamma_2 = (\lambda_2, \nu_2) \in \hat{H}_2'.$$

If $\bar{\pi}^i(\gamma_2)$ occurs as a composition factor in $\pi(\gamma_1)$, then either $\pi(\gamma_1)$ and $\pi(\gamma_2)$ have equivalent composition series, or

$$\langle \lambda_1, \lambda_1 \rangle < \langle \lambda_2, \lambda_2 \rangle,$$

and

$$\langle \operatorname{Re} \nu_1, \operatorname{Re} \nu_1 \rangle > \langle \operatorname{Re} \nu_2, \operatorname{Re} \nu_2 \rangle.$$

Proof. Since γ_1 and γ_2 define the same infinitesimal character, the two inequalities are equivalent. The first is Lemma 8.8 of [19]; the second is a weak form of Theorem VII.3.2 in the Erratum Appendix to Chapter VII of [1].

3. Intertwining operators

Recall the theory of integral intertwining operators as developed in [4], [15].

Let H be a Cartan subgroup, $\gamma = (\lambda, \nu) \in \hat{H}'$, P a parabolic associated to H.

THEOREM 3.1. (Knapp-Stein, [11]). *Let $P' = MAN'$ be another parabolic associated to H. Then there exists an operator $I(P, P', \gamma)$ intertwining $\pi(P, \gamma)$ and $\pi(P', \gamma)$.*

Knapp and Stein use an embedding of the discrete series representations $\pi(\lambda)$ in a principal series representation to reduce the proof of Theorem 3.1 to the case of a minimal parabolic. Since their intertwining operator depends on the choice of the embedding of $\pi(\lambda)$ we will show instead that if γ satisfies certain positivity conditions with respect to P, P' we can choose an intertwining operator independent of an embedding of $\pi(\lambda)$.

Let $\bar{H}(\lambda)$ be the representation space of $\pi(\lambda)$ and $H(\lambda)$ the subspace of $M \cap K$-finite vectors in $\bar{H}(\lambda)$. Consider $H(\lambda)$ as the representation space for $\pi(\lambda) \otimes \nu \otimes 1$, $\varrho(\Delta(\mathfrak{a}, \mathfrak{n}_0))$ as a character of P, and

$$H(P, \gamma) = \{ f \in C^\infty(G, H(\lambda)),\ f(gp) = \varrho(p^{-1})(\pi(\lambda) \otimes \nu \otimes 1)(p^{-1}) f(g) \quad \text{for } p \in P,\ f\ K\text{-finite} \}$$

as the space of K-finite vectors of $\pi(P, \gamma)$. Here $U(\mathfrak{g})$ acts on $H(P, \gamma)$ by differentiation from the left.

Let P' be another parabolic associated to H. Then by [4], $N' = (N \cap N') U$, where U is a unipotent group. Define $I_0(P, P', \gamma)$ by

$$(I_0(P, P', \gamma) f)(g) = \int_U f(gu)\, du$$

for $f \in H(P, \gamma)$ and $g \in G$. As in [4] it follows that $I_0(P, P', \gamma) f \in H(P', \gamma)$, and

$$I_0(P, P', \gamma) \pi(P, \gamma) = \pi(P', \gamma) I_0(P, P', \gamma).$$

THEOREM 3.2. *The integral defining $I_0(P, P', \gamma)$ converges absolutely if γ is strictly negative with respect to all roots in $\Delta(\mathfrak{a}_0, \mathfrak{u}_0)$.*

We prove Theorem 3.2 by reducing it to the corresponding problem in the generalized rank one case, which was solved by Langlands [13]. Obviously, we may assume that G is connected and simple.

We call $\Sigma(\mathfrak{a}_0, \mathfrak{g}_0) \subset \Delta(\mathfrak{a}_0, \mathfrak{g}_0)$ a *set of positive roots* in $\Delta(\mathfrak{a}_0, \mathfrak{g}_0)$ if:

(a) $\Sigma(\mathfrak{a}_0, \mathfrak{g}_0) \cup -\Sigma(\mathfrak{a}_0, \mathfrak{g}_0) = \Delta(\mathfrak{a}_0, \mathfrak{g}_0)$;

(b) $\Sigma(\mathfrak{a}_0, \mathfrak{g}_0) \cap -\Sigma(\mathfrak{a}_0, \mathfrak{g}_0) = \varnothing$;

(c) if $\alpha, \beta \in \Sigma(\mathfrak{a}_0, \mathfrak{g}_0)$ and $\alpha + \beta \in \Delta(\mathfrak{a}_0, \mathfrak{g}_0)$, then $\alpha + \beta \in \Sigma(\mathfrak{a}_0, \mathfrak{g}_0)$.

LEMMA 3.3. *There is a one-to-one correspondence between the sets of positive roots and parabolics associated to H.*

Proof. Let $P = MAN$ be a parabolic associated to H; then $\Delta(\mathfrak{a}_0, \mathfrak{n}_0)$ is a set of positive roots. Define for $\alpha \in \Delta(\mathfrak{a}_0, \mathfrak{g}_0)$

$$\mathfrak{g}(\alpha) = \{X, X \in \mathfrak{g}_0, [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{a}_0\}.$$

For a set of positive roots Σ, put

$$\mathfrak{n}(\Sigma) = \bigoplus_{\alpha \in \Sigma} \mathfrak{g}(\alpha) \quad \text{and} \quad N(\Sigma) = \exp \mathfrak{n}(\Sigma).$$

The $P(\Sigma) = MAN(\Sigma)$ is a parabolic subgroup associated to H. Q.E.D.

Let Σ, Σ' be sets of positive roots. We call a sequence $\Sigma_1, \ldots, \Sigma_n$ of sets of positive roots a *chain* connecting Σ and Σ' if

(a) $\Sigma_1 = \Sigma$, $\Sigma_n = \Sigma'$;

(b) the span of $\Sigma_i \setminus \Sigma_i \cap \Sigma_{i+1}$ is one-dimensional.

The integer n is the *length* of the chain.

LEMMA 3.4. *Let Σ, Σ' be sets of positive roots. There exists a chain connecting Σ and Σ'.*

Proof. Choose an Iwasawa $\mathfrak{a}(I)_0$ such that $\mathfrak{a}_0 \subset \mathfrak{a}(I)_0$. Then there are sets $\tilde{\Sigma}$ and $\tilde{\Sigma}'$ of positive roots of $\Delta(\mathfrak{a}(I)_0, \mathfrak{g}_0)$, whose restrictions to $\mathfrak{a}_0$ are Σ and Σ' respectively.

Let $w \in W(\mathfrak{g}_0, \mathfrak{a}(I)_0)$ be such that $w\tilde{\Sigma} = \tilde{\Sigma}'$. Let $w = w_l \ldots w_1$ be a reduced decomposition of w. Then

$$\tilde{\Sigma}_1, w_1\tilde{\Sigma}_1, w_2w_1\tilde{\Sigma}_1, \ldots, w_{l-1} \ldots w_1\tilde{\Sigma}_1, w\tilde{\Sigma} = \tilde{\Sigma}'$$

is a chain connecting $\tilde{\Sigma}$ and $\tilde{\Sigma}'$.

Consider the restriction of this chain to $\mathfrak{a}_0$. This is a sequence of sets of positive roots. Two subsequent sets are either equal or have exactly one root α in common with the property $c\alpha \notin \Delta(\mathfrak{a}_0, \mathfrak{g}_0)$ for $0 < c < 1$. We can therefore select a subsequence with the required properties. Q.E.D.

Define the *distance* between Σ and Σ', dist (Σ, Σ'), to be the minimum of the length of chains connecting Σ and Σ'.

LEMMA 3.5. *Let Σ, Σ' be sets of positive roots. Then*

$$\mathfrak{n}(\Sigma, \Sigma')_0 = \bigoplus_{\alpha \in \Sigma \cap \Sigma'} \mathfrak{g}(\alpha)$$

and

$$\mathfrak{u}(\Sigma, \Sigma')_0 = \bigoplus_{\substack{\alpha \in \Sigma' \\ \alpha \notin \Sigma \cap \Sigma'}} \mathfrak{g}(\alpha)$$

are subalgebras, and $\mathfrak{n}(\Sigma') = \mathfrak{n}(\Sigma, \Sigma') \oplus \mathfrak{u}(\Sigma, \Sigma')$.

Proof. The last assertion follows immediately from the definitions. Since$[\mathfrak{g}(\alpha), \mathfrak{g}(\beta)] \subset \mathfrak{g}(\alpha+\beta)$ it suffices to show that if $\alpha, \beta \in \Sigma \cap \Sigma'$, then $\alpha+\beta \in \Sigma \cap \Sigma'$. This follows immediately from the definitions. Since $\{\alpha,\ \alpha \in \Sigma',\ \alpha \notin \Sigma \cap \Sigma'\} = \{\alpha,\ \alpha \in \Sigma' \cap -\Sigma\}$ the second claim follows as well. Q.E.D.

LEMMA 3.6. *Let Σ, Σ', Σ'' be sets of positive roots such that* dist $(\Sigma, \Sigma'') =$ dist $(\Sigma, \Sigma') +$ dist (Σ', Σ''). *Then the mapping*

$$\psi\colon U(\Sigma', \Sigma'') \times U(\Sigma, \Sigma') \to U(\Sigma, \Sigma''),$$
$$(u_1, u_2) \to u_1 u_2$$

is an isomorphism of analytic varieties.

Proof. Note first that dist $(\tilde{\Sigma}, \tilde{\tilde{\Sigma}}') = |\tilde{\tilde{\Sigma}}' \setminus (\tilde{\Sigma} \cap \tilde{\tilde{\Sigma}}')|$ for any two sets $\tilde{\Sigma}$, $\tilde{\tilde{\Sigma}}'$ of positive roots. Choose a chain as in Lemma 3.4. By construction this chain has length $|\tilde{\tilde{\Sigma}}' \setminus (\tilde{\Sigma} \cap \tilde{\tilde{\Sigma}}')|$. On the other hand, dist $(\tilde{\tilde{\Sigma}}, \tilde{\Sigma}') \geqslant |(\tilde{\tilde{\Sigma}} \setminus (\tilde{\Sigma} \cap \tilde{\tilde{\Sigma}})|$ by the definition of a chain.

Observe next that $\mathfrak{u}(\Sigma, \Sigma'')_0 = \mathfrak{u}(\Sigma, \Sigma')_0 \oplus \mathfrak{u}(\Sigma', \Sigma'')_0$. By the previous remark it suffices to show that if $\alpha \in \Sigma \cap -\Sigma''$, then $\alpha \in \Sigma \cap -\Sigma'$ or $\alpha \in \Sigma' \cap -\Sigma''$. Let now $\alpha \in \Sigma \cap -\Sigma''$. If $\alpha \in \Sigma \cap -\Sigma'$, then we are done. Assume now $\alpha \notin \Sigma \cap -\Sigma'$; then $\alpha \in \Sigma'$ and hence $\alpha \in \Sigma \cap -\Sigma'' \cap \Sigma' \subset \Sigma' \cap -\Sigma''$. We proceed now by induction on dist (Σ, Σ'). Assume first dist $(\Sigma, \Sigma') = 1$. But then $[\mathfrak{u}(\Sigma, \Sigma')_0, \mathfrak{u}(\Sigma', \Sigma'')_0] \subset \mathfrak{u}(\Sigma', \Sigma'')_0$. Since $U(\Sigma, \Sigma'')$ is simply connected, we can apply [20, Lemma 1.1.41].

Assume now that the lemma is proved for all $\tilde{\Sigma}$ with dist $(\Sigma, \tilde{\Sigma}) <$ dist (Σ, Σ'). Choose a $\tilde{\tilde{\Sigma}}$ such that dist $(\Sigma, \Sigma') =$ dist $(\Sigma, \tilde{\tilde{\Sigma}}) +$ dist $(\tilde{\tilde{\Sigma}}, \Sigma')$ and dist $(\tilde{\tilde{\Sigma}}, \Sigma') = 1$. Then the commutative diagram

$$\begin{tikzcd} U(\Sigma, \tilde{\tilde{\Sigma}}) \times U(\tilde{\tilde{\Sigma}}, \Sigma') \times U(\Sigma', \Sigma'') \arrow[r] \arrow[d] & U(\Sigma, \Sigma') \times U(\Sigma', \Sigma'') \arrow[d] \\ U(\Sigma, \tilde{\tilde{\Sigma}}) \times U(\tilde{\tilde{\Sigma}}, \Sigma'') \arrow[r] & U(\Sigma, \Sigma'') \end{tikzcd}$$

completes the proof. Q.E.D.

THEOREM 3.7. *Let* $P = P(\Sigma)$, $P' = P(\Sigma')$, *and* $P'' = P(\Sigma'')$ *be such that* dist $(\Sigma, \Sigma') =$ dist $(\Sigma, \Sigma'') +$ dist (Σ'', Σ). *Then*

$$I_0(P, P', \gamma) = I_0(P'', P', \gamma)\, I_0(P, P'', \gamma).$$

Proof.

$$\begin{aligned}(I_0(P, P', \gamma) f)(g) &= \int_{U(\Sigma, \Sigma')} f(gu)\, du \\ &= \int_{U(\Sigma'', \Sigma')} \int_{U(\Sigma, \Sigma'')} f(gu_2 u_1)\, du_1\, du \\ &= (I_0(P'', P', \gamma)\, I_0(P, P'', \gamma) f)(g),\end{aligned}$$

where $u_1 \in U(\Sigma, \Sigma'')$, $u_2 \in U(\Sigma'', \Sigma')$. Q.E.D.

This theorem was proved by Knapp-Stein as well.

Now let $\Sigma_1, \ldots, \Sigma_r$ be a chain joining Σ and Σ' with $r =$ dist (Σ, Σ'). Then

$$I_0(P, P', \gamma) f(g) = (I_0(P_r(\Sigma_{r-1}), P', \gamma) \ldots I_0(P, P(\Sigma_2), \gamma) f)(g).$$

Thus to prove convergence of the integral it is enough to consider the case $P = P(\Sigma)$ and $P(\Sigma')$ with dist $(\Sigma, \Sigma') = 1$.

LEMMA 3.8. *Let* $P = P(\Sigma)$, $P' = P(\Sigma')$, *and* $P(\Sigma, \Sigma') = M(\Sigma, \Sigma') A(\Sigma, \Sigma') N_P(\Sigma, \Sigma')$ *be the smallest parabolic containing* P *and* P'. *Put* $P_M = M(\Sigma, \Sigma') \cap P$, $P'_M = M(\Sigma, \Sigma') \cap P'$. *Then* $I_0(P, P', \gamma) H(P, \gamma)$ *is equal to the set of* K*-finite vectors in* $\mathrm{Ind}^G_{P(\Sigma, \Sigma')} [I_0(P_M, P'_M, \gamma) H(P_M, \gamma)] \otimes 1$.

Proof. We identify $f \in H(P, \gamma)$, $\gamma = (\lambda, \nu)$ with a K-finite $\mathrm{Ind}_P^{P(\Sigma, \Sigma')} \pi(\lambda) \otimes \nu \otimes 1$-valued function $\bar{f}$ on G by the formula

$$f(gp) = (\bar{f}(g))(p), \quad g \in G,\ p \in P(\Sigma, \Sigma').$$

Then $\bar{f}(e) \in \mathrm{Ind}_P^{P(\Sigma, \Sigma')} \pi(\lambda) \otimes \nu \otimes 1$.

Define $\tilde{I}(P, P', \gamma)$ by the formula

$$[\tilde{I}(P, P', \gamma)\tilde{f}(g)](p) = I_0(P, P', \gamma) f(gp).$$

Since $I_0(P, P', \gamma)$ formally intertwines $\pi(P, \gamma)$ and $\pi(P', \gamma)$, so does $\tilde{I}(P, P', \gamma)$. Now

$$\begin{aligned}
[\tilde{I}(P, P', \gamma)\tilde{f}(g)](p) &= \pi(P', \gamma)(g^{-1})[\tilde{I}(P, P', \gamma)\tilde{f}(e)](p) \\
&= \pi(P', \gamma)(g^{-1}) I_0(P, P', \gamma) f(p) \\
&= \pi(P', \gamma)(g^{-1}) \int_{U(\Sigma, \Sigma')} f(gu)\, du \\
&= \pi(P', \gamma)(g^{-1}) \int_{U(\Sigma, \Sigma')} f(manu)\, du \\
&= \pi(P', \gamma)(g^{-1}) \int_{U(\Sigma, \Sigma')} f(mauu^{-1}nu)\, du \\
&= \pi(P', \gamma)(g^{-1}) \int_{U(\Sigma, \Sigma')} f(mau)\, du,
\end{aligned}$$

where $m \in M(\Sigma, \Sigma')$, $a \in A(\Sigma, \Sigma')$, $n \in N_P(\Sigma, \Sigma')$.

But f considered as a function of ma is in $\operatorname{Ind}_{P_M}^{M(\Sigma, \Sigma')} \pi(\lambda) \otimes \nu$, and

$$\int_{U(\Sigma, \Sigma')} f(mau)\, du = I_0(P_M, P'_M, \gamma) f(ma).$$

Hence

$$[\tilde{I}(P, P', \gamma)\tilde{f}](e) \in I_0(P_M, P'_M, \gamma) H(P_M, \gamma) \otimes 1,$$

and thus

$$\tilde{I}(P, P', \gamma)\tilde{f} \in \operatorname{Ind}_{P(\Sigma, \Sigma')}^{G} I_0(P_M, P'_M, \gamma) H(P_M, \gamma) \otimes 1. \qquad \text{Q.E.D.}$$

COROLLARY 3.9. *$I_0(P, P', \gamma)$ is injective iff $I_0(P_M, P'_M, \gamma)$ is injective.*

If $P = P(\Sigma)$, $P' = P(\Sigma')$ and dist $(\Sigma, \Sigma') = 1$, then P_M and P'_M have parabolic rank one. Hence if $P_M = P_M(\Sigma_M)$ for a set of positive roots Σ_M in $\Delta(\mathfrak{a}_0 \cap \mathfrak{m}(\Sigma, \Sigma')_0, \mathfrak{m}(\Sigma, \Sigma')_0)$, then $P'_M = P_M(-\Sigma_M)$.

THEOREM 3.10. (Langlands [13]). *Let $P = P(\Sigma)$, $P' = P(-\Sigma)$, and suppose γ is strictly positive with respect to Σ. Then $I_0(P, P', \gamma)$ converges absolutely.*

Since in the setting of Lemma 3.8, $I_0(P, P', \gamma)$ converges if and only if $I_0(P_M, P'_M, \gamma)$ converges, we see that if $P = P(\Sigma)$, $P' = P(\Sigma)$, dist $|\Sigma, \Sigma'| = 1$, then $I_0(P, P', \gamma)$ converges absolutely if γ is strictly positive with respect to all roots in $\Sigma' \setminus (\Sigma' \cap \Sigma)$. Using the product formula for $I_0(P, P', \gamma)$, Theorem 3.2 follows immediately.

THEOREM 3.11. (Langlands [13]). *Let $P=P(\Sigma)$, $P'=P(-\Sigma)$, and suppose $\gamma=(\lambda, \nu)$ is strictly positive with respect to P. Then*

$$\operatorname{Im} I_0(P, P', \gamma)$$

is irreducible.

Thus by Theorem 3.11, if γ is strictly positive, then $\pi(P, \gamma)$ is reducible iff $I_0(P, P', \gamma)$ has a nontrivial kernel. If $P'' \neq P$ is a parabolic subgroup associated to H, then by Proposition 2.7 $\pi(P'', \gamma)$ is reducible iff $\pi(P, \gamma)$ is reducible. Hence to give reducibility criteria for generalized principal series representations $\pi(P'', \gamma)$, with γ nonsingular with respect to $\Delta(\mathfrak{a}_0, \mathfrak{g}_0)$, is equivalent to finding necessary and sufficient conditions for the injectivity of $I(P, P', \gamma)$.

Now let $\Sigma_1, \ldots, \Sigma_r$ be a chain connecting Σ and $-\Sigma$. Then by Theorem 3.7

$$I(P, P', \gamma) = I(P(\Sigma_{r-1}, P(\Sigma_r), \gamma) \ldots I(P(\Sigma), P(\Sigma_2), \gamma).$$

Hence $I(P, P', \gamma)$ has a nontrivial kernel iff one of the factors does.

If $\alpha_i \in \Sigma_i$ and $\alpha_i \notin \Sigma_{i+1}$, put $G(\alpha_i) = M(\Sigma_i, \Sigma_{i+1}) A(\Sigma_i, \Sigma_{i+1})$. By Lemma 3.8 we have thus proved

THEOREM 3.12. *Under the assumptions above, $\pi(P, \gamma)$ is reducible iff one of the operators $I(P(\Sigma_i) \cap G(\alpha_i), P(\Sigma_{i+1}) \cap G(\alpha_i), \gamma)$ has a nontrivial kernel, or equivalently, iff one of the representations $\pi(P(\Sigma_i) \cap G(\alpha_i), \gamma)$ is reducible.*

Since $G(\alpha_i)$ is again a reductive linear group with abelian Cartan subgroups, we have thus reduced the reducibility problem for such parameters to the corresponding problem in the generalized rank one case.

Now consider the general case, so that ν may be singular with respect to $\Delta(\mathfrak{a}_0, \mathfrak{g}_0)$. Choose a parabolic $P=MAN$ so that P is positive with respect to $\gamma=(\lambda, \nu)$. Define a parabolic subgroup $P'=M'A'N'$ containing P by

$$\mathfrak{a}_0' = \bigcap_{\substack{\alpha \in \Delta(\mathfrak{a}_0, \mathfrak{n}_0) \\ \operatorname{Re}\langle \alpha, \nu\rangle = 0}} \ker(\alpha) \subseteq \mathfrak{a}$$

$$\Delta(\mathfrak{a}_0, \mathfrak{n}_0') = \{\alpha \in \Delta(\mathfrak{a}_0, \mathfrak{g}_0) \mid \operatorname{Re}\langle \alpha, \nu\rangle > 0\}$$

Then $\pi_{M'}(P \cap M', (\lambda, \nu))$ is unitarily induced, and so is a direct sum of irreducible tempered representations $\pi_1, \ldots, \pi_r$. Writing

$$\nu_2 = \nu|_{A'}, \quad \nu_1 = \nu|_{A \cap M'}$$

we get by step-by-step induction

$$\pi(P, \gamma) = \operatorname{Ind}_{P'}^{G} \pi_{M'}(P \cap M', (\lambda, \nu_1)) \otimes \nu_2 \otimes 1.$$

$$\pi(P, \gamma) = \bigoplus_{i=1}^{r} \operatorname{Ind}_{P'}^{G} \pi_i \otimes \nu_2 \otimes 1.$$

Notice that P' is strictly positive for ν_2.

Choose a chain $\Sigma_1^0, \ldots, \Sigma_t^0$ connecting

$$\Delta(\mathfrak{a}_0, \mathfrak{n}_0 \cap \mathfrak{m}'), \ -\Delta(\mathfrak{a}_0, \mathfrak{n}_0 \cap \mathfrak{m}'),$$

and set

$$\Sigma_i = \Sigma_i^0 \cup \Delta(\mathfrak{a}, \mathfrak{n}') \quad (1 \leqslant i \leqslant t).$$

Next, choose a chain $\Sigma_t, \ldots, \Sigma_n$ connecting Σ_t and $-\Delta(\mathfrak{a}_0, \mathfrak{n}_0)$. For $1 \leqslant i \leqslant n-1$, let α_i be a root in Σ_i not in Σ_{i+1}. If $1 \leqslant i \leqslant t-1$, then

$$\operatorname{Re} \langle \nu, \alpha_i \rangle = 0;$$

so by induction by stages,

$$\pi(P(\Sigma_i), \gamma), \pi(P(\Sigma_{i+1}), \gamma)$$

are induced from unitarily equivalent tempered representations; so we can choose an isomorphism $I(i, \gamma)$ between them. If $t \leqslant i \leqslant n-1$, then

$$\operatorname{Re} \langle \nu, \alpha_i \rangle > 0,$$

so the integral intertwining operator

$$I(i, \gamma) = I(P(\Sigma_i), P(\Sigma_{i+1}), \gamma)$$

is convergent. Set

$$\bar{I}(P, \gamma) = I(n-1, \gamma) \ldots I(1, \gamma).$$

By Theorem 3.7, $\bar{I}(P, \gamma)$ is (up to equivalences) exactly the integral used by Langlands in [13] to define the Langlands quotients of $\pi(P, \gamma)$. This proves

LEMMA 3.13. *Let $\bar{\pi}_i(\gamma)$ denote the Langlands quotient of $\operatorname{Ind}_{P'}^G (\pi_i \otimes \nu_2 \otimes 1)$. Then*

$$\bar{I}(P, \gamma) H(P, \gamma) = \bigoplus_{i=1}^{r} \bar{\pi}_i(\gamma),$$

a direct sum of r irreducible representations.

So one of the representations $\operatorname{Ind}_{P'}^G \pi_i \otimes \nu_2 \otimes 1$ is reducible if and only if $I(P, \gamma)$ has a nontrivial kernel; or, equivalently, if one of its factors does. This proves

THEOREM 3.14. *Let $\gamma = (\lambda, \nu) \in \hat{H}$; let $P = P(\Sigma)$ be a parabolic associated to H, positive with respect to γ, and $\Sigma_1, \ldots, \Sigma_n$ be a chain connecting Σ and $-\Sigma$. Then $\pi(P, \gamma)$ is reducible iff:*

(a) *one of the operators $I_0(P(\Sigma_i), P(\Sigma_{i+1}), \gamma)$ has a nontrivial kernel and $(\alpha_i, \nu) > 0$, or*

(b) *the tempered representation $\pi_{M'}((P \cap M'), (\lambda, \nu_1))$ is reducible.*

Again we reduced the problem to a problem for nonsingular $\mathfrak{a}$ parameter in the generalized rank one case and reducibility of unitarily induced tempered representations; and this latter problem is completely solved (cf. Theorem 4.5).

We call the operator $I(P, \gamma)$ the *long intertwining operator* for γ.

COROLLARY 3.15. *Let π be a composition factor of $\pi(P, \gamma)$. Then π is either a composition factor of the kernel of a factor of the long intertwining operator, or it contains a lowest K-type.*

Proof. This follows immediately from the product formula of the long intertwining operator, and Lemma 3.13. Q.E.D.

But by Lemma 3.8 the kernel of a factor is the representation induced from the kernel of the corresponding operator in the generalized rank one case. Hence, if in the generalized rank one case the kernel of the operator is again a generalized principal series representation or if we can find l generalized principal series representations such that each composition factor of the kernel is a composition factor of at least one of the l generalized principal series representations and vice versa, we can apply Corollary 3.15 again to compute the composition series of the kernel of the corresponding factor.

Example 3.16. Let G be a complex connected group. In this case there is only one conjugacy class of Cartan subgroups and thus the minimal parabolic is the only cuspidal parabolic. By Corollary 3.15 the computations of composition series for generalized principal series representations (up to multiplicities) are reduced to calculating the composition series for the kernels of the factors of the long intertwining operator. Using Lemma 3.8 and the fact that for $SL(2, \mathbf{C})$ the kernel of the corresponding intertwining operator is again a generalized principal series representation, we deduce that the kernel of each factor is either zero or a generalized principal series representation. Hence applying the above considerations again we can compute the composition series of the kernel of each factor of the long intertwining operator and thus compute the composition series of the generalized principal series representation we started with (up to multiplicities). This gives a partial answer to the composition series problem for complex groups.

4. Reducibility on the bottom layer of K-types

We are going to need the results of [19], including its unpublished second part. Unfortunately those results were proved only for connected groups. The extension to the present hypotheses on G poses various minor technical problems. Almost all of these involve questions of definitions. Since it is not practical to reproduce [19] here, we will give only a careful account of the definitions and main theorems, reformulated to allow for the disconnectedness of G. The theorems are certainly non-trivial; but following the arguments of [19] using the definitions here *is* trivial, and can safely be left to the skeptical reader. The main result is Theorem 4.5 below.

Choose a Cartan subgroup T of K, and a system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ of positive roots; put $2\varrho_c = \sum_{\alpha\in\Delta^+(\mathfrak{k})} \alpha$. Since $W(G/T) = W(K/T)$ may be larger than $W(\mathfrak{k}/\mathfrak{t})$, the closed Weyl chamber $\bar{C}_{\Delta^+(\mathfrak{k})} \subseteq \mathfrak{t}^*$ need not be a fundamental domain for the action of $W(K/T)$ on $\mathfrak{t}^*$; so we choose such a fundamental domain $C_1 \subseteq \bar{C}_{\Delta^+(\mathfrak{k})}$ arbitrarily. With this choice, every representation $\mu \in \hat{K}$ has a unique extremal weight $\bar{\mu} \in C_1$. Define $\|\mu\| = \langle \bar{\mu} + 2\varrho_c, \bar{\mu} + 2\varrho_c \rangle$; this is easily seen to be independent of all choices.

Definition 4.1. The Harish-Chandra module X is said to have μ as a *lowest K-type* if μ occurs in $X|_K$, and $\|\mu\|$ is minimal with respect to this property.

We want to describe the set of irreducible Harish-Chandra modules with lowest K-type μ. Just as in the connected case, we begin with a special situation; details may be found in Section 6 of [19].

G is said to be quasisplit if it has a parabolic subgroup $P = MAN$ which is a Borel subgroup; this is equivalent to having M compact and abelian. We assume for the next few paragraphs that G is quasisplit and fix such a Borel subgroup. Then MA is a maximally split Cartan subgroup of G. It should be pointed out that M is actually a subgroup of K, namely, the centralizer of A in K. To each root $\alpha \in \Delta(\mathfrak{n}, \mathfrak{a})$, we associate a connected semisimple subgroup $G^\alpha \subseteq G$: $P \cap G^\alpha$ is a Borel subgroup of G^α, $\mathfrak{n} \cap \mathfrak{g}^\alpha$ is the sum of the $r\alpha$ root spaces of $\mathfrak{a}$ in $\mathfrak{n}$ for $r > 0$, and $\mathfrak{a} \cap \mathfrak{g}^\alpha$ is the one-dimensional subspace of $\mathfrak{a}$ dual to α under $\langle\, , \rangle$. Up to isomorphism, there are only three possibilities for $\mathfrak{g}_0^\alpha$: $\mathfrak{sl}(2, \mathbf{R})$, $\mathfrak{su}(2, 1)$, and $\mathfrak{sl}(2, \mathbf{C})$. Put $\mathfrak{k}^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{k}$. If $\mathfrak{g}_0^\alpha \simeq \mathfrak{sl}(2, \mathbf{R})$, $\mathfrak{k}_0^\alpha$ has a natural element Z^α (defined up to sign) which corresponds to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbf{R}).$$

Set $\sigma_\alpha = \exp(\tfrac{1}{2}\pi Z_\alpha)$, $m_\alpha = \sigma_\alpha^2$. Then σ_α is defined only up to inverse; but $m_\alpha^2 = 1$, so m_α is well defined. Furthermore, σ_α normalizes MA; and $\bar{\sigma}_\alpha \in W(G/MA)$ is the reflection about α. Just as in the connected case, a representation $\delta \in \hat{M}$ is called *fine* if its restriction to the identity component of $M \cap G_0^1$ is trivial. A representation $\mu \in \hat{K}$ is called *fine* if whenever $\mathfrak{g}_0^\alpha \not\cong \mathfrak{sl}(2, \mathbf{R})$, $\mu|_{\mathfrak{k}^\alpha}$ is trivial; and whenever $\mathfrak{g}_0^\alpha \cong \mathfrak{sl}(2, \mathbf{R})$, $\mu(Z_\alpha)$ has only the eigenvalues 0 and $\pm i$. If $\delta \in \hat{M}$ is fine, we let $A(\delta) \subseteq \hat{K}$ denote the set of fine K-types whose restriction to M contains δ. Let M' be the normalizer of A in K, so that $W = M'/M = W(G/MA)$.

If $\delta \in \hat{M}$ is fine, we let $W_\delta \subseteq W$ be the stabilizer of δ with respect to the natural action of W on $\hat{M}$. The set of *good roots* of $\mathfrak{a}$ in $\mathfrak{g}$ with respect to δ is defined to be

$$\Delta_\delta = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid \mathfrak{g}_0^\alpha \ncong \mathfrak{sl}(2, \mathbf{R}), \text{ or } \mathfrak{g}_0^\alpha \cong \mathfrak{sl}(2, \mathbf{R}) \text{ and } \delta(m_\alpha) = 1\}.$$

Then Δ_δ is a root system just as in the connected case; and we define $W(\Delta_\delta) = W_\delta^0$. Just as in the connected case, one shows that $W_\delta^0 \subseteq W_\delta$, in fact as a normal subgroup. Let $\bar{\delta} = \delta|_{M \cap G_0}$. Then obviously $W_\delta^0 = W_{\bar{\delta}}^0$; and it is not hard to see that $W_\delta \subseteq W_{\bar{\delta}}$. Set $R_\delta = W_\delta / W_\delta^0$. It was shown in [19], Section 9, that if G is connected, then R_δ is a product of copies of $\mathbf{Z}/2\mathbf{Z}$. By the preceding remarks, $R_\delta \subseteq R^-$; so R_δ is a product of copies of $\mathbf{Z}/2\mathbf{Z}$ in general. In particular, $\hat{R}_\delta$ is a group. It is possible, just as in the connected case, to define a natural action of $\hat{R}_\delta$ on $A(\delta)$; this is rather complicated, and we will not repeat the definition here (cf. [19], Section 9). This action turns out to be simply transitive, so that in particular $A(\delta)$ is nonempty. If $\nu \in \hat{A}$, put

$$R_\delta(\nu) = \{\bar{\sigma} \in R_\delta \mid \sigma \in W_\delta \text{ and } \sigma \cdot \nu \in W_\delta^0 \cdot \nu\}.$$

Put $\gamma = (\delta, \nu) \in \hat{M} \times \hat{A}$. Then γ can obviously be identified with an M-regular pseudo-character of MA, so we get a principal series representation $\pi(P, \gamma)$. The set of lowest K-types of $\pi(P, \gamma)$ is $A(\delta)$—this is a trivial consequence of the corresponding result for connected groups ([19], Section 6). By an extension of the methods of [19], one can show that if $\mu \in A(\delta)$, then δ occurs only once in $\mu|_M$. Hence μ occurs exactly once in $\pi(P, \gamma)|_K$, so there is a unique subquotient $\bar{\pi}(\gamma, \mu)$ of $\pi(P, \gamma)$ containing the K-type μ. These results and several others are summarized in

THEOREM 4.2. *Let G be a quasisplit reductive linear group with all Cartan subgroups abelian, and let $P = MAN$ be a Borel subgroup of G. Suppose $\delta \in \hat{M}$ is fine. If $\mu \in A(\delta)$, then $\mu|_M$ is the direct sum of the M-types in the W orbit of δ in $\hat{M}$, each occurring with multiplicity one. The group $\hat{R}_\delta$ acts simply transitively on $A(\delta)$ in a natural way. Let $\gamma = (\delta, \nu) \in \hat{M} \times \hat{A}$. If $\mu' \in A(\delta)$, then μ' occurs in $\bar{\pi}(\gamma, \mu)$ iff μ' is in the orbit of μ under the action of $\hat{R}_\delta(\nu)$, the annihilator of $R_\delta(\nu)$ in $\hat{R}_\delta$.*

This can be proved by the methods of [19], where it is proved for connected reductive groups. Although the argument is not entirely trivial, we will not digress to give it here. It is worth remarking that the result can fail even for linear groups if the Cartan subgroup MA is nonabelian; the simplest example has $G_0 = \mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$, and $|G/G_0| = 4$. This gives an example of a tempered principal series representation of a reductive group, whose irreducible constituents have multiplicity two. The assumption that all Cartan subgroups are abelian is the simplest way to avoid such problems.

We return now to the general problem of determining all representations with lowest K-type $\mu \in \hat{K}$; recall the extremal weight $\bar{\mu} \in i\mathfrak{t}_0'$. Let $\bar{\lambda} = \bar{\lambda}(\bar{\mu}) \in i\mathfrak{t}_0'$ be the parameter associated to $\bar{\mu}$ by Proposition 4.1 of [19]. (If Δ^+ is a θ-invariant positive root system for the fundamental Cartan $\mathfrak{t} + \mathfrak{p}^t$ in $\mathfrak{g}$ such that $\mu + 2\varrho_c$ is dominant, then $\bar{\lambda}$ is close to $\bar{\mu} + 2\varrho_c - \varrho(\Delta^+)$. For details see [19].) Let $\mathfrak{b} = \mathfrak{l} + \mathfrak{n}$ be the parabolic subalgebra of $\mathfrak{g}$ defined by λ; then $\mathfrak{l}$ is quasisplit, and $\bar{\mu} - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ is the highest weight of a fine $L_0 \cap K$-type. Via $\langle\ ,\ \rangle$, we identify $\bar{\lambda}$ with an element $t_{\bar{\lambda}} \in \mathfrak{t}$; let L be the centralizer in G of $t_{\bar{\lambda}}$. Then the Lie algebra of L is in fact $\mathfrak{l}_0$, and L normalizes $\mathfrak{n}$. Suppose X is a Harish-Chandra module for G. If $\mu \in \hat{K}$, let X^μ denote the extremal weight vectors of the μ-primary subspace of X. Analogous notation is used for L. We know from [19] that $H^i(\mathfrak{n}, X)$ is a Harish-Chandra module for L_0. But $L \cap K$ acts on $H^i(\mathfrak{n}, X)$ in a natural way, so $H^i(\mathfrak{n}, X)$ becomes a Harish-Chandra module for L. Let μ_L denote the representation of $L \cap K$ generated by the $\bar{\mu}$ weight space in μ; then μ_L is irreducible. Recall that $R = \dim(\mathfrak{n} \cap \mathfrak{p})$; define

$$\mu_L - 2\varrho(\mathfrak{n} \cap \mathfrak{p}) = \mu_L \otimes [\Lambda^R(\mathfrak{n} \cap \mathfrak{p})]^*.$$

Just as in Section 3 of [19], one obtains a natural map

$$\pi_0^\mu\colon H^0(\mathfrak{n} \cap \mathfrak{k}, X)^{\mu_L} \otimes \Lambda^R(\mathfrak{n} \cap \mathfrak{p})^* \to H^R(\mathfrak{n}, X)^{\mu_L - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$$

which can be used to compute the action of $U(\mathfrak{g})^{K_0}$ on $H^0(\mathfrak{n} \cap \mathfrak{k}, X)^{\mu_L}$ whenever μ is a lowest K-type of X.

Now L is quasisplit; let $H = T^+A$ be a maximally split Cartan subalgebra. Furthermore, $\mu_L - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ is a fine $L \cap K$-type; so we can choose a fine T^+-type $\lambda_L = \lambda_L(\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})) \in \hat{T}^+$ such that $\mu_L - 2\varrho(\mathfrak{n} \cap \mathfrak{p}) \in A(\lambda_L)$.

LEMMA 4.3. *With notation as above, suppose that $\mu_L' - 2\varrho(\mathfrak{n} \cap \mathfrak{p}) \in A(\lambda_L)$. Then there is a unique K-type μ', containing some extremal weight $\bar{\mu}'$, such that μ_L' is the representation of $L \cap K$ generated by the $\bar{\mu}$, weight space in μ. Furthermore, $\bar{\lambda}(\mu')$ is conjugate to $\bar{\lambda}(\mu)$ under $W(G/T)$.*

The easy proof (using Proposition 7.15 of [19]) is left to the reader.

Let $P = MAN$ be a cuspidal parabolic subgroup of G associated to H. The pair

$$\lambda = \lambda_G(\mu) = (\lambda_L \otimes \Lambda^R(\mathfrak{n} \cap \mathfrak{p}), \bar{\lambda}(\mu)|_{\mathfrak{t}^+})$$

is an M-regular character of T^+ (cf. [19], Section 7). Suppose $\nu \in \hat{A}$; put $\gamma = (\lambda, \nu)$, and define $R_\lambda = R_{\lambda_L}$, $R_\lambda(\nu) = R_{\lambda_L}(\nu)$.

LEMMA 4.4. (cf. [19], Proposition 7.9). *The set of lowest K-types of $\pi(P,\gamma)$, which we write $A(\lambda)$, is precisely the set described by Lemma 4.3. These K-types occur with multiplicity one in $\pi(P,\gamma)$.*

For $\mu\in A(\lambda)$, we can now define $\bar{\pi}_G(\gamma,\mu)=\bar{\pi}(\gamma,\mu)$ to be the unique irreducible subquotient of $\pi(P,\gamma)$ containing the K-type μ. Since we have set up a natural bijection $A(\lambda)\cong A(\lambda_L)$, $\hat{R}_\lambda$ acts simply transitively on $A(\lambda)$.

THEOREM 4.5. (cf. [19], Theorem 10.1). *Let G be a reductive linear group with all Cartan subgroups abelian. Then the sets $A(\lambda)$ partition $\hat{K}$; and $A(\lambda)=A(\lambda')$ iff T^+ is conjugate to $(T^+)'$ by an element of G taking λ to λ'. If $\mu\in A(\lambda)$, then the set of irreducible representations of G with μ as a lowest K-type is $\{\bar{\pi}((\lambda,\nu),\mu)\}$. The set of lowest K-types of $\bar{\pi}((\lambda,\nu),\mu)$ is $\hat{R}_\lambda(\nu)\cdot\mu\subseteq A(\lambda)$. The $L\cap K$-type, $\mu_L-2\varrho(\mathfrak{n}\cap\mathfrak{p})$, occurs exactly once in $H^R(\mathfrak{n},\bar{\pi}_G((\lambda,\nu),\mu))$; this is accounted for by the occurrence of $\bar{\pi}_L((\lambda_L),\nu),(\mu_L-2\varrho(\mathfrak{n}\cap\mathfrak{p}))$ as a composition factor in the cohomology.*

If ν is unitary, then every component of $\pi((\lambda,\nu))$ contains a lowest K-type.

COROLLARY 4.6. (cf. [19], Corollary 10.15). *The set of Langlands subquotients of $\pi(\gamma)$ is $\{\bar{\pi}(\gamma,\mu)\,|\,\mu\in A(\lambda)\}$, which has order $|R_\lambda(\nu)|$; this is one unless ν annihilates some real root.*

Our next goal is to describe a family of representations which will be used to construct reducibility of generalized principal series. It has been known to several people (notably Schmid) for some time that the character identity of Proposition 5.14 describes the reducibility of certain generalized principal series representations, and that this reducibility is "analogous" to that of principal series representations for SL (2, **R**). The results below explain and generalize this analogy. To simplify the exposition, we assume for the remainder of this section that G is connected. Generalizations of the results to disconnected groups are needed in Section 6; the reader can easily supply the additional details. Accordingly we make notational simplifications as in [19]; for example, representations of K are now identified with their highest weights.

Let $\mathfrak{b}=\mathfrak{l}+\mathfrak{n}$ be a θ-invariant parabolic subalgebra of $\mathfrak{g}$, compatible with our fixed choice of $\Delta^+(\mathfrak{k})$. Let L be the normalizer of $\mathfrak{b}$ in G; the Lie algebra of L is $\mathfrak{l}_0$. Let V be a Harish-Chandra module for L. We want to construct a representation of G which is "holomorphically induced" from V, in analogy with the Borel-Weil theorem for compact groups. Formally there is an obvious way to do this, using the cohomology groups of a certain sheaf defined by V on the complex manifold G/L. Unfortunately, this approach presents formidable analytic problems: The sheaf in question is not coherent unless V is finite-dimensional. So we use some infinitesimal properties which this holomorphically induced object ought to have as its definition. Fix an $L \cap K$-type $\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ occurring in V, and assume that μ is dominant for $\Delta^+(\mathfrak{k})$. (Such μ need not exist, of course.) Recall that $V^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ is the space of extremal weight vectors in the $L \cap K$-primary subspace of V corresponding to the $L \cap K$-type $\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$. Then $V^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ is a finite-dimensional module for $U(\mathfrak{l})^{L \cap K}$. Recall from [19], (3.2), the map ξ: $U(\mathfrak{g})^K \to U(\mathfrak{l})^{L \cap K}$. Via ξ, we can consider $V^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ as a module for $U(\mathfrak{g})^K$. Since it is finite-dimensional, it has a finite composition series; write W^μ for the direct sum of the composition factors. By a theorem of Harish-Chandra, the action of $U(\mathfrak{g})^K$ on a single K-type of an irreducible Harish-Chandra module determines the module completely. Therefore, there is at most one completely reducible Harish-Chandra module $X = X_G(\mathfrak{b}, V, \mu) = X(\mathfrak{b}, V, \mu)$ with the following properties:

(a) Every irreducible constituent of X contains the K-type μ.

(b) X^μ is isomorphic to the $U(\mathfrak{g})^K$ module W_μ defined above.

Since not every module for $U(\mathfrak{g})^K$ can occur as Y_μ for a Harish-Chandra module Y, the existence of $X(\mathfrak{b}, V, \mu)$ is not obviouus. We want to prove this existence, and derive some simple properties of $X(\mathfrak{b}, V, \mu)$. For this purpose it clearly suffices to consider irreducible V, as we do from now on. We begin with a holomorphic "induction by stages" result.

LEMMA 4.7. *Suppose $\mathfrak{b}' \supseteq \mathfrak{b}$; say $\mathfrak{b}' = \mathfrak{l}' + \mathfrak{n}'$, with $\mathfrak{l}' \supseteq \mathfrak{l}$, $\mathfrak{n}' \subseteq \mathfrak{n}$. Then*

$$X_G(\mathfrak{b}', X_{L'}(\mathfrak{b} \cap \mathfrak{l}', V, \mu - 2\varrho(\mathfrak{n}' \cap \mathfrak{p})), \mu) = X(\mathfrak{b}, V, \mu);$$

i.e., if the left side exists, then so does the right, and they are equal.

Proof. Write ξ': $U(\mathfrak{g})^K \to U(\mathfrak{l}')^{L' \cap K}$, $\xi_{\mathfrak{l}'}$: $U(\mathfrak{l}')^{L' \cap K} \to U(\mathfrak{l})^{L \cap K}$ for the maps of [19], (3.2). Then $\xi = \xi_{\mathfrak{l}'} \circ \xi'$, as follows trivially from the definition. The assertion of the lemma is now a formal consequence of the definition of $X(\mathfrak{b}, V, \mu)$. Q.E.D.

LEMMA 4.8. *Let Y be a Harish-Chandra module for G such that V is a composition factor of $H^R(\mathfrak{n}, Y)$ (with $R = \dim(\mathfrak{n} \cap \mathfrak{p})$). Suppose that $V^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ lies in the image of*

$$\pi_0^\mu: H^0(\mathfrak{n} \cap \mathfrak{k}, Y)^\mu \otimes [\Lambda^R(\mathfrak{n} \cap \mathfrak{p})]^* \to H^R(\mathfrak{n}, Y)^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}.$$

Then $X(\mathfrak{b}, V, \mu) \subseteq Y_{ss}$.

Proof. By Theorem 3.5 of [19], the pullback of $V^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ to a $U(\mathfrak{g})^K$ module via ξ is a subquotient of Y^μ. The conclusion is immediate. Q.E.D.

LEMMA 4.9. *Suppose $\mathfrak{b}^0$ is the θ-invariant parabolic subalgebra associated to μ^0, with $\mu^0 - 2\varrho(\mathfrak{n} \cap \mathfrak{p}) \in A(\lambda^0_{L^0})$. Define $\lambda^0_G = \lambda_G(\mu^0)$ using $\lambda^0_{L^0}$ as in the first part of this section. Then*

$$X(\mathfrak{b}^0, \bar{\pi}((\lambda^0_{L^0}, \nu), \mu^0 - 2\varrho(\mathfrak{n}^0 \cap \mathfrak{p})), \mu^0) = \bar{\pi}((\lambda^0_G, \nu), \mu^0).$$

Proof. This is simply an assertion about the action of $U(\mathfrak{g})^K$ on $\bar{\pi}((\lambda_G^0, \nu), \mu^0)^{\mu^0}$. It follows from Theorem 4.5 and Lemma 4.8. Q.E.D.

Suppose X is a Harish-Chandra module containing the K-type μ. Recall from [19], Definition 3.13, that μ is said to be *strongly* $\mathfrak{n}$-*minimal* in X if whenever F is an irreducible representation of K such that $(H^J(\mathfrak{n}\cap\mathfrak{k}, F)\otimes[\Lambda^{J'}(\mathfrak{n}\cap\mathfrak{p})]^*)^{\mu-2\varrho(\mathfrak{n}\cap\mathfrak{p})}\neq 0$, then either $(J, J') = (0, R)$, or F does not occur in X. By [19], Theorem 3.14,

$$\pi_0^\mu\colon X^\mu\otimes[\Lambda^R(\mathfrak{n}\cap\mathfrak{p})]^*\to H^R(\mathfrak{n}, X)^{\mu-2\varrho(\mathfrak{n}\cap\mathfrak{p})}$$

is bijective if μ is strongly $\mathfrak{n}$-minimal in X. Because of Lemma 4.8, therefore, we are very interested in this condition. One can sometimes reformulate the condition much more simply.

LEMMA 4.10. *There is a constant $N = N(G)$ with the following properties: Let $\mathfrak{b} = \mathfrak{l}+\mathfrak{n}\subseteq\mathfrak{g}$ be a θ-invariant parabolic subalgebra compatible with $\Delta^+(\mathfrak{k})$, and suppose $\mu\in\hat{K}$ satisfies $\langle\mu, \alpha\rangle > N$ for every root α of $\mathfrak{t}$ in $\mathfrak{n}$.*

(a) *If X is a Harish-Chandra module not containing the K-type of highest weight*

$$\mu-(\beta_1+\dots+\beta_l)$$

for any non-empty subset $\{\beta_1, \dots, \beta_l\}$ of the roots of $\mathfrak{t}$ in $\mathfrak{n}\cap\mathfrak{p}$, then μ is strongly $\mathfrak{n}$-minimal in X.

(b) *Let $\tilde{\mathfrak{b}} = \tilde{\mathfrak{l}}+\tilde{\mathfrak{n}}\subseteq\mathfrak{l}$ be the θ-invariant parabolic associated to $\mu-2\varrho(\mathfrak{n}\cap\mathfrak{p})$. Then $\bar{\mathfrak{b}} = \tilde{\mathfrak{l}}+(\tilde{\mathfrak{n}}+\mathfrak{n}) = \bar{\mathfrak{l}}+\bar{\mathfrak{n}}\subseteq\mathfrak{b}$ is the θ-invariant parabolic in $\mathfrak{g}$ associated to μ.*

Proof. Let F be a K-type of highest weight γ and suppose

$$(H^J(\mathfrak{n}\cap\mathfrak{k}, F)\otimes[\Lambda^{J'}(\mathfrak{n}\cap\mathfrak{p})]^*)^{\mu-2\varrho(\mathfrak{n}\cap\mathfrak{p})}\neq 0,$$

with $(J, J')\neq(0, R)$. Let $l = R-J'$ (with $R = \dim\mathfrak{n}\cap\mathfrak{p}$). By the computation at the beginning of Section 5 of [19], there is an element $\sigma\in W(K/T)$ such that

$$\sigma(\gamma+\varrho_c) = \mu-\beta_1 \dots -\beta_l+\varrho_c$$

with the β_i distinct roots of $\mathfrak{t}$ in $\mathfrak{n}\cap\mathfrak{p}$. Furthermore σ satisfies the following condition: let

$$\Delta_\sigma^+ = \{\alpha\in\Delta^+(\mathfrak{k}, \mathfrak{t})\,|\,\sigma^{-1}\alpha\notin\Delta^+(\mathfrak{k}, \mathfrak{t})\}.$$

Then Δ_σ^+ consists of exactly J roots of $\mathfrak{t}$ in $\mathfrak{n}\cap\mathfrak{k}$, and the length of σ is J. We claim that the hypothesis on μ forces $J = 0$. So suppose $\alpha\in\Delta_\sigma^+$. Then

$$\langle\alpha, \sigma(\gamma+\varrho_c)\rangle = \langle\sigma^{-1}\alpha, \gamma+\varrho_c\rangle < 0$$

since $\gamma+\varrho_c$ is dominant, and $\sigma^{-1}\alpha \notin \Delta^+(\mathfrak{k}, \mathfrak{t})$. On the other hand

$$\langle \alpha, \sigma(\gamma+\varrho_c)\rangle = \langle \alpha, \mu+\varrho_c-\beta_1 \dots -\beta_l\rangle > N - \langle \alpha, \sum \beta_i\rangle.$$

If N is large enough, this is positive, a contradiction. So Δ_σ^+ is empty, $J=0$, and σ has length 0. So $\sigma=1$. Since $(J, J')\neq(0, R)$, $J'\neq R$; so $l\neq 0$. The equation for γ reduces to

$$\gamma = \mu - (\beta_1 + \dots + \beta_l).$$

So if X satisfies the hypothesis (a), γ does not occur in X, and μ is strongly $\mathfrak{n}$-minimal.

For (b), we have to recall how the parabolic is associated to a K-type ([19], Proposition 4.1). Notice first that

$$2\varrho(\mathfrak{n}) = 2\varrho(\mathfrak{n}\cap\mathfrak{p}) + 2\varrho(\mathfrak{n}\cap\mathfrak{k})$$

is the weight of the one-dimensional representation $\Lambda^{R+S}(\mathfrak{n})$ of $\mathfrak{l}$. So $\tilde{\mathfrak{b}}$ is also the parabolic associated to

$$(\mu - 2\varrho(\mathfrak{n}\cap\mathfrak{p})) + 2\varrho(\mathfrak{n}) = \mu + 2\varrho(\mathfrak{n}\cap\mathfrak{k}).$$

Now $\tilde{\mathfrak{b}}$ is defined as follows. We choose a positive root system $\Delta^+(\mathfrak{l}, \mathfrak{t})$ for $\mathfrak{t}$ in $\mathfrak{l}$, so that

$$[\mu + 2\varrho(\mathfrak{n}\cap\mathfrak{k})] + 2\varrho(\Delta^+(\mathfrak{l}\cap\mathfrak{k}, \mathfrak{t})) = \mu + 2\varrho_c$$

is dominant for $\Delta^+(\mathfrak{l}, \mathfrak{t})$. Next choose $\{\beta_i, c_i\}$ as in Proposition 4.1 of [19]; this means in particular that

$$\tilde{\lambda}(\mathfrak{l}) = \mu + 2\varrho_c - \varrho(\Delta^+(\mathfrak{l})) + \tfrac{1}{2}\sum c_i\beta_i$$

is dominant for $\Delta^+(\mathfrak{l})$, and that

$$\Delta(\tilde{\mathfrak{b}}, \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{l}, \mathfrak{t}) \,|\, \langle \alpha, \tilde{\lambda}(\mathfrak{l})\rangle \geq 0\}.$$

Now we consider the parabolic defined by μ. We must begin by choosing a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ so that $\mu+2\varrho_c$ is dominant. By the hypotheses on μ,

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \Delta(\mathfrak{n}, \mathfrak{t}) \cup \Delta^+(\mathfrak{l}, \mathfrak{t})$$

has this property. Next we must look at

$$\lambda(\mathfrak{g}) = \mu + 2\varrho_c - \varrho = \mu + 2\varrho_c - \varrho(\Delta^+(\mathfrak{l})) - \varrho(\Delta(\mathfrak{n}, \mathfrak{t})).$$

Since $\varrho(\Delta(\mathfrak{n}, \mathfrak{t}))$ is orthogonal to all the roots of $\mathfrak{t}$ in $\mathfrak{l}$, and $\lambda(\mathfrak{g})$ has a large inner product with the roots of $\mathfrak{t}$ in $\mathfrak{n}$, the set $\{\beta_i, c_i\}$ satisfies the hypotheses of Proposition 4.1 of [19] for μ. Thus

$$\tilde{\lambda}(\mathfrak{g}) = \mu + 2\varrho_c - \varrho + \tfrac{1}{2}\sum c_i\beta_i = \tilde{\lambda}(\mathfrak{l}) - \varrho(\Delta(\mathfrak{n}, \mathfrak{t}))$$

defines the parabolic $\bar{\mathfrak{b}}$ associated to μ. So

$$\begin{aligned}\Delta(\bar{\mathfrak{b}}, \mathfrak{t}) &= \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \alpha, \lambda(\mathfrak{g})\rangle \geq 0\} \\ &= \Delta(\mathfrak{n}, \mathfrak{t}) \cup \{\alpha \in \Delta(\mathfrak{l}, \mathfrak{t}) \mid \langle \alpha, \tilde{\lambda}(\mathfrak{g})\rangle \geq 0\} \\ &= \Delta(\mathfrak{n}, \mathfrak{t}) \cup \{\alpha \in \Delta(\mathfrak{l}, \mathfrak{t}) \mid \langle \alpha, \tilde{\lambda}(\mathfrak{l})\rangle \geq 0\} \\ &= \Delta(\mathfrak{n}, \mathfrak{t}) \cup \Delta(\tilde{\mathfrak{b}}, \mathfrak{t}).\end{aligned}$$

Here the second equality follows from the hypotheses on μ, and the third from the fact that $\varrho(\Delta(\mathfrak{n}, \mathfrak{t}))$ is orthogonal to the roots of $\mathfrak{l}$. This proves (b). Q.E.D.

Because of the condition in Lemma 4.10 (a), the following technical result is useful.

LEMMA 4.11. *In the situation of Lemma* 4.7, *every K-type μ^1 occurring in $\pi(\lambda^0, \nu)$ is of the form $\mu^0 + Q$, with Q a sum of roots in $\mathfrak{b}^0$.*

This is established in the proof of Lemma 8.8 of [19].

Definition 4.12. The K-type μ of the Harish-Chandra module X is said to lie on the *$\mathfrak{b}$-bottom layer* if every K-type occurring in X is of the form $\mu + Q$, with Q a sum of roots in $\mathfrak{b}$. The K-types of X of the form $\mu + Q_0$, with Q_0 a sum of roots in $\mathfrak{l}$, constitute the *$\mathfrak{b}$-bottom layer of K-types*.

PROPOSITION 4.13. *With notation as above, $X(\mathfrak{b}, V, \mu)$ exists; and μ lies on the $\mathfrak{b}$-bottom layer of K-types of X.*

This requires a lemma, which is borrowed from the proof of Lemma 7.3 of [19], and inspired by results of Jantzen and Zuckerman.

LEMMA 4.14. *Suppose $\mathfrak{b} = \mathfrak{l} + \mathfrak{n}$ is a θ-invariant parabolic compatible with $\Delta^+(\mathfrak{k})$, and $\mu \in \hat{K}$. Suppose that μ is on the $\mathfrak{b}$-bottom layer of K-types of a Harish-Chandra module X, and that* (*with N as in Lemma* 4.10)

$$\langle \mu, \alpha \rangle > N$$

for all $\alpha \in \Delta(\mathfrak{n})$. Let γ be a positive integral multiple of $2\varrho(\mathfrak{n})$; then we can regard $\mathbf{C}_{-\gamma}$ as a one-dimensional representation of $\mathfrak{l}$ (namely a tensor product of copies of $(\Lambda^t \mathfrak{n})^*$, with $t = \dim \mathfrak{n}$). Suppose finally that $\mu - \gamma$ is dominant for $\Delta^+(\mathfrak{k})$. Then there is a Harish-Chandra module Y (*depending on X, μ, and γ*) *such that*

(a) *Every K-type of Y is of the form*

$$\mu - \gamma + Q,$$

with Q a sum of roots of $\mathfrak{t}$ in $\mathfrak{b}$.

(b) *If $\mu'-\gamma$ is a K-type such that $\mu-\mu'$ is a sum of roots of $\mathfrak{t}$ in $\mathfrak{l}$, then the multiplicity of $\mu'-\gamma$ in Y is the multiplicity of μ' in X.*

(c) *The natural map (defined before Lemma* **4.3**)

$$\pi_0^{\mu-\gamma}: H^0(\mathfrak{n}\cap\mathfrak{k}, Y)^{\mu-\gamma}\otimes[(\Lambda^R(\mathfrak{n}\cap\mathfrak{p}))]^*\to H^R(\mathfrak{n}, Y)^{\mu-\gamma-2\varrho(\mathfrak{n}\cap\mathfrak{p})}$$

is injective, and its image is isomorphic to

$$H^R(\mathfrak{n}, X)^{\mu-2\varrho(\mathfrak{n}\cap\mathfrak{p})}\otimes\mathbf{C}_{-\gamma}$$

as a module for $U(\mathfrak{l})^{L\cap K}$.

(d) *If δ is another K-type of X satisfying the hypotheses of the lemma, then the Harish-Chandra module Z associated to X, δ, and γ is isomorphic to Y.*

We refer to the process of going from X to Y as "shifting".

Proof. Since γ is integral and dominant for any positive root system Δ^+ containing $\Delta(\mathfrak{n})$, there is an irreducible finite dimensional representation $F_{-\gamma}$ of lowest weight $-\gamma$. We define

$$Y = X\otimes F_{-\gamma};$$

then (d) is obvious. Let φ be a K-type of Y. Then there is a K-type φ_1 of X and a weight φ_2 of $\mathfrak{t}$ in $F_{-\gamma}$ such that

$$\varphi=\varphi_1+\varphi_2.$$

Since μ lies on the $\mathfrak{h}$-bottom layer of X, and $-\gamma$ is the lowest weight of $F_{-\gamma}$, we can find sums of roots of $\mathfrak{t}$ in $\mathfrak{h}$, Q_1 and Q_2, such that

$$\varphi_1=\mu+Q_1$$

$$\varphi_2=-\gamma+Q_2.$$

Setting $Q=Q_1+Q_2$, this gives

$$\varphi=\mu-\gamma+Q$$

proving (a).

For (b), let $E^{\mu'-\gamma}$ be a copy of the representation of K of highest weight $\mu'-\gamma$. We want to compute

$$\dim \operatorname{Hom}_K(E^{\mu'-\gamma}, Y).$$

Now

$$\operatorname{Hom}_K(E^{\mu'-\gamma}, Y)=\operatorname{Hom}_K(E^{\mu'-\gamma}, X\otimes F_{-\gamma})\cong\operatorname{Hom}_K(E^{\mu'-\gamma}\otimes F^*_{-\gamma}, X).$$

The representation contragredient to $F_{-\gamma}$ is the one F^γ of highest weight γ; so we want

$$\operatorname{Hom}_K(E^{\mu'-\gamma}\otimes F^\gamma, X).$$

Now one knows that $E^{\mu'-\gamma}\otimes F^{\gamma}$ contains the K-type $E^{\mu'}$ of highest weight μ' (the "Cartan product" of $E^{\mu'-\gamma}$ and E^{γ}) exactly once, and that every other constituent is of the form $E^{\mu'-\gamma+\delta}$, with $\delta \neq \gamma$ a weight of $\mathfrak{t}$ in F^{γ}. Such a δ is of the form

$$\delta = \gamma - \varepsilon - Q,$$

with ε a weight of $\mathfrak{t}$ in $\mathfrak{n}$ and Q a sum of roots in $\mathfrak{b}$. So these other constituents are

$$E^{\mu'-\varepsilon-Q},$$

which cannot occur in X since μ' is on the $\mathfrak{b}$-bottom layer. So the multiplicity we want is

$$\dim \operatorname{Hom}_K (E^{\mu'}, X),$$

which proves (b).

To prove (c), write

$$F^1_{-\gamma} = \mathfrak{n}\cdot F_{-\gamma};$$

then

$$F_{-\gamma}/F^1_{-\gamma} \cong \mathbf{C}_{-\gamma}$$

as an $\mathfrak{l}$-module. To study Y, we use the long exact sequences in cohomology attached to the short exact sequence

$$0 \to X\otimes F^1_{-\gamma} \to X\otimes F_{-\gamma} \to X \otimes \mathbf{C}_{-\gamma} \to 0$$

of $\mathfrak{g}$ modules. Write

$$A = \Lambda^R(\mathfrak{n}\cap\mathfrak{p})^*;$$

then we have a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H^R(\mathfrak{n}, X\otimes F^1_{-\gamma})^{\mu-\gamma-2\varrho(\mathfrak{n}\cap\mathfrak{p})} \arrow[r] & H^R(\mathfrak{n}, Y)^{\mu-\gamma-2\varrho(\mathfrak{n}\cap\mathfrak{p})} \arrow[r, "\beta"] & H^R(\mathfrak{n}, X)^{\mu-2\varrho(\mathfrak{n}\cap\mathfrak{p})}\otimes\mathbf{C}_{-\gamma} \arrow[r] & \cdots \\
0 \arrow[r] & H^0(\mathfrak{n}\cap\mathfrak{k}, X\otimes F^1_{-\gamma})^{\mu-\gamma}\otimes A \arrow[r] & H^0(\mathfrak{n}\cap\mathfrak{k}, Y)^{\mu-\gamma}\otimes A \arrow[r, "\alpha"] \arrow[u, "\pi_0^{\mu-\gamma}"] & {[H^0(\mathfrak{n}\cap\mathfrak{k}, X)^{\mu}\otimes A]\otimes\mathbf{C}_{-\gamma}} \arrow[r] \arrow[u, "\pi_0^{\mu}\otimes 1"] & \cdots
\end{tikzcd}$$

Now (c) says that $\pi_0^{\mu-\gamma}$ is injective, and β is an isomorphism of the image of $\pi_0^{\mu-\gamma}$ onto that of $\pi_0^{\mu}\otimes 1$. We will show that

$$H^0(\mathfrak{n}\cap\mathfrak{k}, X\otimes F^1_{-\gamma})^{\mu-\gamma} = 0,$$

and

$$\pi_0^{\mu} \text{ is bijective.} \tag{*}$$

Assume this for a moment. The first assertion shows that α is injective; by (b), the domain and range have the same dimension, so α is bijective. Thus $(\pi_0^\mu \otimes 1)\circ\alpha$ is also bijective. But

$$(\pi_0^\mu \otimes 1)\circ\alpha = \beta\circ(\pi_0^{\mu-\gamma}),$$

so the assertions of (c) are immediate.

To prove the first claim of (*), choose a filtration of $F^1_{-\gamma}$ as a $\mathfrak{g}\cap\mathfrak{k}$ module, so that the successive subquotients are one-dimensional. Using the spectral sequence of the filtration, we are reduced to showing that

$$H^0(\mathfrak{n}\cap\mathfrak{k}, X)^{\mu-\gamma-\delta} = 0$$

whenever δ is a weight of $F^1_{-\gamma}$. Such a weight is of the form $-\gamma+\varepsilon+Q$, with $\varepsilon\in\Delta(\mathfrak{n})$ and Q a sum of roots in $\mathfrak{b}$; so we must show that

$$H^0(\mathfrak{n}\cap\mathfrak{k}, X)^{\mu-\varepsilon-Q} = 0.$$

This follows from the fact that μ is on the $\mathfrak{b}$-bottom layer of X.

Finally, we must show that π_0^μ is bijective. As was remarked before Lemma 4.10, it is enough to show that μ is strongly $\mathfrak{n}$-minimal in X. This follows from the hypotheses and Lemma 4.10. Q.E.D.

COROLLARY 4.15. *In the setting of Lemma* 4.14, *suppose* $\tilde{V}$ *is a Harish-Chandra module for* $\mathfrak{l}$; *set* $V = \tilde{V}\otimes\mathbb{C}_{-\gamma}$. *Suppose that*

$$X = X(\mathfrak{b}, \tilde{V}, \mu).$$

Then $X(\mathfrak{b}, V, \mu-\gamma)$ *exists, and is contained in* Y_{ss}.

Proof. This follows from Lemmas 4.8 and 4.14 (c). Q.E.D.

This corollary allows us to reduce some problems about $X(\mathfrak{b}, V, \mu)$ to the case when μ is very nonsingular (in the sense of Lemma 4.10) by "shifting".

Proof of Proposition 4.13. Choose N as in Lemma 4.10. Let $\mu^0 - 2\varrho(\mathfrak{n}\cap\mathfrak{p})$ be a lowest $L\cap K$-type of V, and $\bar{\mathfrak{b}}^0 = \mathfrak{l}^0 + \bar{\mathfrak{n}}^0 \subseteq \mathfrak{l}$ the associated parabolic. Then

$$\mathfrak{b}^0 = \mathfrak{l}^0 + (\bar{\mathfrak{n}}^0 + \mathfrak{n}) = \mathfrak{l}^0 + \mathfrak{n}^0 \subseteq \mathfrak{b} \subseteq \mathfrak{g}$$

is a θ-invariant parabolic, compatible with $\Delta^+(\mathfrak{k})$.

Assume first that $\langle\mu, \alpha\rangle > N$ and $\langle\mu_0, \alpha\rangle > N$ for all $\alpha\in\Delta(\mathfrak{n})$. Then μ^0 is dominant for $\Delta^+(\mathfrak{k})$, and $\mathfrak{b}^0$ is the associated θ-invariant parabolic by Lemma 4.10 (b). Let $H = T^+A \subseteq L^0$

be a maximally split Cartan subgroup; say $\mu^0 - 2\varrho(\mathfrak{n}^0 \cap \mathfrak{p}) \in A(\lambda^0_{L^0})$. Construct $\lambda^0_L = \lambda_L(\mu^0 - 2\varrho(\mathfrak{n} \cap \mathfrak{p}))$ and $\lambda^0_G = \lambda^0 = \lambda_G(\mu^0)$ using $\lambda^0_{L^0}$ as before. By Theorem 4.5, $V = \bar{\pi}_L((\lambda^0_L, \nu), \mu^0 - 2\varrho(\mathfrak{n} \cap \mathfrak{p}))$ for some $\nu \in \hat{A}$. Define

$$X = \bar{\pi}_G((\lambda^0_G, \nu), \mu^0).$$

Then X is our candidate for $X(\mathfrak{b}, V, \mu)$. Since $\mathfrak{b} \subseteq \mathfrak{b}^0$, Lemma 4.11 implies that μ and μ^0 lie in the $\mathfrak{b}$-bottom layer of K-types of X. In particular, if $\{\beta_i\} \subseteq \Delta(\mathfrak{n} \cap \mathfrak{p})$, $\mu - \sum \beta_i$ and $\mu^0 - \sum \beta_i$ do not occur in X. By Lemma 4.10 (a), μ and μ^0 are strongly $\mathfrak{n}$-minimal in X. In particular, $\mu^0 - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ occurs exactly once in $H^R(\mathfrak{n}, X)$. Let V_1 denote the corresponding irreducible composition factor. We claim that $V = V_1$. By Lemma 4.8 $X = X(\mathfrak{b}, V_1, \mu^0)$. By Theorem 4.5 there is some $\nu_1 \in \hat{A}$ such that $V_1 = \bar{\pi}_L((\lambda^0_L, \nu_1), \mu^0 - 2\varrho(\mathfrak{n} \cap \mathfrak{p}))$; then by Lemma 4.9, $V_1 = X(\mathfrak{b}^0, \bar{\pi}_{L^0}((\lambda^0_{L^0}, \nu_1), \mu^0 - 2\varrho(\mathfrak{n} \cap \mathfrak{p})), \mu^0 - 2\varrho(\mathfrak{n} \cap \mathfrak{p}))$. By Lemma 4.7, this implies that $X = X(\mathfrak{b}^0, \bar{\pi}_{L^0}((\lambda^0_{L^0}, \nu_1), \mu^0 - 2\varrho(\mathfrak{n}^0 \cap \mathfrak{p})), \mu)$. By Lemma 4.8 again, this is $\bar{\pi}_G((\lambda^0, \nu_1), \mu^0)$. Now we can use the uniqueness statement in Theorem 2.9 to deduce that ν is conjugate to ν_1 under the stabilizer of λ in $W(G/H)$. It follows easily that $V = V_1$.

We have now established that V occurs in $H^R(\mathfrak{n}, X)$. But we also know that π^μ_0 is bijective. By Lemma 4.8, it follows that $X(\mathfrak{b}, V, \mu) \subseteq X_{ss}$; since X is irreducible, in fact, $X = X(\mathfrak{b}, V, \mu)$. That μ is on the $\mathfrak{b}$-bottom layer of X follows from Lemma 4.11.

We now drop the hypotheses $\langle \mu, \alpha \rangle > N$ and $\langle \mu^0, \alpha \rangle > N$ for $\alpha \in \Delta(\mathfrak{n})$. Let γ be a multiple of $2\varrho(\mathfrak{n})$ so large that $\langle \gamma + \mu, \alpha \rangle > N$ and $\langle \gamma + \mu^0, \alpha \rangle > N$ for all $\alpha \in \Delta(\mathfrak{n})$. Set $\tilde{V} = V \otimes \mathbf{C}_\gamma$. By the first part of the proof,

$$X = X(\mathfrak{b}, \tilde{V}, \mu + \gamma)$$

exists, and $\mu + \gamma$ lies on the $\mathfrak{b}$-bottom layer of X. By Corollary 4.15, $X(\mathfrak{b}, V, \mu)$ exists, and is contained in the semisimplification of the module Y associated to X and γ. By Lemma 4.14 (a), μ lies on the $\mathfrak{b}$-bottom layer of Y, and so also on the $\mathfrak{b}$-bottom layer of $X(\mathfrak{b}, V, \mu)$.

PROPOSITION 4.16. *In the setting of Proposition* 4.13, *suppose μ^1 lies on the $\mathfrak{b}$-bottom layer of $X(\mathfrak{b}, V, \mu)$. Then the multiplicity of μ^1 in $X(\mathfrak{b}, V, \mu)$ is less than or equal to the multiplicity of $\mu^1 - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ in V. Equality holds if $\langle \mu^1, \alpha \rangle > N$ and $\langle \mu, \alpha \rangle > N$ for every $\alpha \in \Delta(\mathfrak{n})$).*

Proof. We first establish the inequality. This is clearly consistent with Lemma 4.14 and Corollary 4.15, so we may assume that (with μ^0 defined as in the proof of Proposition 4.13) $\langle \mu^1, \alpha \rangle > N$, $\langle \mu, \alpha \rangle > N$, and $\langle \mu^0, \alpha \rangle > N$ for all $\alpha \in \Delta(\mathfrak{n})$. Then μ^1 is strongly $\mathfrak{n}$-minimal in $X(\mathfrak{b}, V, \mu)$; so $H^R(\mathfrak{n}, X)$ has a unique irreducible subquotient V^1 containing the $L \cap K$-type $\mu^1 - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$; and $X(\mathfrak{b}, V, \mu) = X(\mathfrak{b}, V^1, \mu^1)$. Let $\mu^2 - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ be a lowest K-type of V^1.

By a further shift we may assume that $\langle\mu^2, \alpha\rangle > N$ for all $\alpha \in \Delta(\mathfrak{n})$. By the proof of Proposition 4.13, $X(\mathfrak{b}, V^1, \mu^1) = X(\mathfrak{b}, V^1, \mu^2)$, and $X(\mathfrak{b}, V, \mu) = X(\mathfrak{b}, V, \mu^0)$; furthermore, μ^2 and μ^0 are lowest K-types of the corresponding modules. Since $X(\mathfrak{b}, V, \mu) = X(\mathfrak{b}, V^1, \mu^1)$, it follows that μ^2 is a lowest K-type of $X(\mathfrak{b}, V, \mu^0)$. Using Theorem 4.5, it is easy to deduce that $V = V^1$. This completes the proof of the inequality.

For the last statement, notice that we can still define V^1 as before. Since μ does occur in $X(\mathfrak{b}, V, \mu)$, the proposition implies that $\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ occurs in V^1. Since $H^R(\mathfrak{n}, X)^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ pulls back to an irreducible module for $U(\mathfrak{g})^K$, it is irreducible for $U(\mathfrak{l})^{L \cap K}$. It follows that $V = V^1$, proving the desired equality. Q.E.D.

For completeness we state now a result which will be proved in Section 5. Notation is as above.

THEOREM 4.17. *Let* $\mathfrak{b} \subseteq \mathfrak{g}$ *be a* θ*-invariant parabolic subalgebra,* $V = \bar{\pi}((\lambda_L^0, \nu), \mu^0 - 2\varrho(\mathfrak{n} \cap \mathfrak{p}))$ *an irreducible Harish-Chandra module, and* $\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ *an* $L \cap K$*-type of* V*, such that* μ *is dominant for* $\Delta^+(\mathfrak{k})$. *Suppose further that if* $\gamma = (\lambda^0, \nu)$, *then* $\operatorname{Re}\langle\gamma, \alpha\rangle > 0$ *or* $\operatorname{Re}\langle\gamma, \alpha\rangle = 0$ *and* $\operatorname{Im}\langle\gamma, \alpha\rangle \geqslant 0$ *for every root* α *of* $\mathfrak{h}$ *in* $\mathfrak{n}$. *Then* $X(\mathfrak{b}, V, \mu)$ *is irreducible; and a* K*-type* μ^1 *on the* $\mathfrak{b}$*-bottom layer occurs exactly as often as* $\mu^1 - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ *occurs in* V.

It seems likely that the hypotheses on γ are completely unnecessary; at least they are far too strong.

To see how these modules fit into the generalized principal series, we need to compute the action of $U(\mathfrak{g})^K$ on the bottom layer of such representations. We begin with

LEMMA 4.18. *Let* μ^1 *be a* K*-type on the* $\mathfrak{b}$*-bottom layer of* $X(\mathfrak{b}, V, \mu)$. *Then* $X(\mathfrak{b}, V, \mu)$ *and* $X(\mathfrak{b}, V, \mu^1)$ *have a composition factor in common; more precisely,* $X(\mathfrak{b}, V, \mu)^{\mu^1}$ *is a composition factor of the pull-back of* $V^{\mu^1 - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ *to* $U(\mathfrak{g})^K$.

Proof. This is consistent with shifting, so we may assume that $\langle\mu, \alpha\rangle > N$ and $\langle\mu^1, \alpha\rangle > N$ for all $\alpha \in \Delta(\mathfrak{n})$. In that case we have seen that $X(\mathfrak{b}, V, \mu) = X(\mathfrak{b}, V, \mu^1)$. Q.E.D.

Fix now a K-type μ^0, with $\mathfrak{b}^0 = \mathfrak{l}^0 + \mathfrak{n}^0$ the associated parabolic; say $\mu^0 \in A(\lambda^0)$. A parabolic $\mathfrak{b} \supseteq \mathfrak{b}^0$ is called *permissible* for the principal series $\pi(\lambda^0, \nu)$. Define λ_L^0 and $\lambda_{L^0}^0$ as usual. Suppose μ occurs on the $\mathfrak{b}$-bottom layer of X; we want to compute the action of $U(\mathfrak{g})^K$ on $\pi(\lambda^0, \nu)^\mu$.

PROPOSITION 4.19. *The* $U(\mathfrak{g})^K$ *module* $\pi_G(\lambda^0, \nu)^\mu$ *has the same composition series as the pullback of* $\pi_L(\lambda_L^0, \nu)^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ *via* ξ.

Proof. It suffices to establish this for an algebraically dense set of $\nu \in \hat{A}$. So we consider only unitary ν which are annihilated by no real root. In this case $\pi(\lambda^0, \nu)$ and $\pi_L(\lambda_L^0, \nu)$ are irreducible by Theorem 4.5, and so

$$\pi(\lambda^0, \nu) = X(\mathfrak{b}, \pi_L(\lambda_L^0, \nu), \mu^0),$$

as we saw in the proof of Proposition 4.13. Lemma 4.18 now implies that $\pi(\lambda^0, \nu)^\mu$ is a composition factor of the pullback of $\pi(\lambda_L^0, \nu)^{\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})}$ via ξ. But a straightforward (and tedious) computation using Frobenius reciprocity and the Blattner multiplicity formula shows that these two spaces have the same dimension. Q.E.D.

(The last multiplicity computation can be avoided using the proof of Theorem 4.17; the argument is left to the reader.)

COROLLARY 4.20. *In the setting of Proposition* 4.19, *suppose that* $\pi(\lambda_L^0, \nu)$ *has composition factors* $V_1 \dots V_s$ *(listed with multiplicity), and that* V_i $(1 \leqslant i \leqslant r, r \leqslant s)$ *contains an* $L \cap K$*-type* $\mu_i - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ *such that* μ_i *is dominant for* $\Delta^+(\mathfrak{k})$. *Then* $\pi(\lambda^0, \nu)$ *has (perhaps among others) the composition factors* $X(\mathfrak{b}, V_1, \mu_1) \dots X(\mathfrak{b}, V_r, \mu_r)$ *(occurring with at least the multiplicities listed).*

This result gives a great deal of information about reducibility on the $\mathfrak{b}$-bottom layer of K-types. We conclude with a sufficient condition for this to be all the reducibility. Let $\mathfrak{b}$ be a permissible parabolic for the representation $\pi(\lambda^0, \nu)$, and $\mathfrak{h} = \mathfrak{t}^+ + \mathfrak{a}$ the Cartan subalgebra associated to the representation. We may assume that if $\mathfrak{t}^- = (\mathfrak{t}^+)^\perp \cap \mathfrak{t}$, then $\mathfrak{a}$ is spanned by $\mathfrak{p}^t$ and $\mathfrak{a}^-$, with $\mathfrak{a}^-$ obtained from $\mathfrak{t}^-$ by successive Cayley transforms through imaginary roots orthogonal to λ^0. This gives an isomorphism from $\mathfrak{h}^0 = \mathfrak{t} + \mathfrak{p}^t$ to $\mathfrak{h}$. (For more details see [19].) In this way we identify (λ, ν) with a weight $(\lambda, \tilde{\nu})$ of $\mathfrak{h}^0$. The parameter (λ, ν) is called *positive for* $\mathfrak{b}$ if $\operatorname{Re}\langle \alpha, (\lambda, \nu)\rangle \geqslant 0$ for all $\alpha \in \Delta(\mathfrak{n}, \mathfrak{h})$. In this case $\mathfrak{b}$ is called a *positive permissible parabolic*. Since $\langle \alpha, \lambda\rangle > 0$ for $\alpha \in \Delta(\mathfrak{n})$, the condition means that $\operatorname{Re} \nu$ is not too large.

PROPOSITION 4.21. *If* $\mathfrak{b}$ *is a positive permissible parabolic for* $\pi(\lambda, \nu)$, *then every composition factor of* $\pi(\lambda, \nu)$ *contains a* K*-type in the* $\mathfrak{b}$*-bottom layer*.

Proof. Choose $\gamma \in i(\mathfrak{t}^+)'$ so that $\langle \gamma, \alpha\rangle > 0$ if $\alpha \in \Delta(\mathfrak{n})$, and $\langle \gamma, \alpha\rangle = 0$ if $\alpha \in \Delta(\mathfrak{l})$. Let $V^1 \subseteq \pi(\lambda, \nu)$ be a composition factor with lowest K-type μ^1. We want to show that $\langle \gamma, \mu^1\rangle = \langle \gamma, \mu\rangle$. Following [19], Proposition 4.1, we can write $\lambda^1 = \bar{\lambda}(\mu^1) = \mu^1 + 2\varrho_c - \varrho + \sum c_i \beta_i$; here $\varrho = \varrho(\Delta^+)$, with Δ^+ a θ-invariant positive root system such that $\mu + 2\varrho_c$ is dominant, and the β_i are orthogonal imaginary roots spanning a subspace $(\mathfrak{t}^1)^-$ of $\mathfrak{t}$. Then $(\mathfrak{t}^1)^+ = ((\mathfrak{t}^1)^-)^\perp$

is the compact part of the Cartan subalgebra associated to μ^1, and $V^1 = \pi(\lambda^1, \nu^1)$ for some ν^1. As in the preceding paragraph we may identify ν^1 with a weight $\tilde{\nu}^1$ of $(\mathfrak{t}^1)^- + \mathfrak{p}^t$. We are free to modify $\tilde{\nu}^1$ by reflections about any of the β_i, so we may assume that $\operatorname{Re}\langle\gamma, \tilde{\nu}^1\rangle > 0$. We know that $\mu^1 = \mu + Q$, with Q a sum of roots in $\mathfrak{b}$. As in the proof of Lemma 8.8 of [19], one sees that $\lambda^1 = \lambda + Q + Q_1$, with Q_1 a sum of positive (nonintegral) multiples of roots in $\mathfrak{b}$. So

$$\operatorname{Re}\langle\gamma, (\lambda^1, \tilde{\nu}^1)\rangle \geqslant \langle\gamma, \lambda^1\rangle \geqslant \langle\gamma, \lambda\rangle; \tag{4.22}$$

equality holds only if Q involves only roots in $\mathfrak{l}$, which is what we wish to prove.

On the other hand, $(\lambda, \tilde{\nu})$ and $(\lambda^1, \tilde{\nu}^1)$ must define the same infinitesimal character, so they differ by some $\sigma \in W(\mathfrak{g}, \mathfrak{h}^0)$. Choose a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h}^0) \supseteq \Delta(\mathfrak{n}, \mathfrak{h}^0)$ so that $\operatorname{Re}\langle\alpha, (\lambda, \tilde{\nu})\rangle \geqslant 0$ for all $\alpha \in \Delta^+$; this is possible by the positivity of $\mathfrak{b}$. Then $\sigma\cdot(\lambda, \tilde{\nu}) = (\lambda, \tilde{\nu}) - \sum_{\alpha_i \in \Delta^+} n_i \alpha_i$, with $\operatorname{Re} n_i \geqslant 0$. Hence

$$\begin{aligned}\operatorname{Re}\langle\gamma, (\lambda^1, \tilde{\nu}^1)\rangle &= \operatorname{Re}\langle\gamma, \sigma(\lambda, \tilde{\nu})\rangle \\ &= \operatorname{Re}\langle\gamma, (\lambda, \tilde{\nu})\rangle - \sum \operatorname{Re} n_i\langle\gamma, \alpha_i\rangle \leqslant \langle\gamma, \lambda\rangle.\end{aligned}$$

So equality must hold in (4.22). Q.E.D.

For the convenience of the reader, we summarize the definitions of this section and Theorem 4.17, Proposition 5.18, Corollary 4.20, and Proposition 4.21 in one theorem; although the first two results will not be proved until the next section, we will not use this theorem until after they are proved. We will formulate this for disconnected groups, leaving to the reader the necessary extensions of the intermediate results.

THEOREM 4.23. *Suppose $H = T^+A$ is a θ-stable Cartan subgroup of G, and $\gamma = (\lambda, \nu) \in \hat{H}'$. Let $\mathfrak{b} = \mathfrak{l} + \mathfrak{n}$ be a θ-invariant parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{l}$, and let L be the normalizer of $\mathfrak{b}$ in G. Assume that*

(a) *γ is nonsingular, i.e.*

$$\langle\alpha, \gamma\rangle \neq 0$$

whenever $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$

(b) *$\mathfrak{b}$ is a positive permissible parabolic for γ, i.e.*

$$\operatorname{Re}\langle\alpha, \gamma\rangle \geqslant 0$$

for all $\alpha \in \Delta(\mathfrak{n}, \mathfrak{h})$.

Then the composition series of $\pi_G(\gamma)$ can be computed in terms of the subgroup L of G. More precisely, define

$$\gamma_L = (\lambda_L, \nu) \in \hat{H}'_L$$

by

$$\lambda_L = (\Lambda \otimes [\Lambda^R(\mathfrak{n} \cap \mathfrak{p})]^*, \bar{\lambda} - \varrho(\mathfrak{n})),$$

and list the composition factors of $\pi_L(\gamma_L)$ as $V_L^1 \dots V_L^r$ (with multiplicity); say

$$V_L^i \cong \bar{\pi}_L(\gamma_L^i), \quad \gamma_L^i \in (\hat{H}^i)'_L, \quad \gamma_L^i = (\lambda_L^i, \nu^i).$$

Define

$$\gamma_i = (\lambda^i, \nu^i) \in (\hat{H}^i)'$$

by

$$\lambda^i = (\Lambda_L^i \otimes \Lambda^R(\mathfrak{n} \cap \mathfrak{p}), \bar{\lambda}_L^i + \varrho(\mathfrak{n})).$$

Then the composition factors of $\pi_G(\gamma)$ (with multiplicity) are $\{\bar{\pi}_G(\gamma^i)\}$.

5. Coherent continuation of characters

We begin by recalling the basic facts of character theory for G (cf. [5]). Let π be an admissible representation of G on a Hilbert space, with a finite composition series, such that $\pi|_K$ is unitary. If $f \in C_c^\infty(G)$, we define $\pi(f) = \int_G f(g)\pi(g)\,dg$. Then $\pi(f)$ is an operator of trace class, and $f \mapsto \operatorname{tr} \pi(f)$ defines a distribution on G. This distribution is called the *character* of π, and is written $\Theta(\pi)$. $\Theta(\pi)$ depends only on the infinitesimal equivalence class of π; so if X is the Harish-Chandra module of K-finite vectors for π, we may define $\Theta(X) = \Theta(\pi)$. Every irreducible Harish-Chandra module can be realized in this way ([14]) and therefore has a well-defined character. In the above situation, suppose X has the irreducible composition factors $X_1, \dots, X_r$ (listed with multiplicity). Then $\Theta(X) = \sum_{i=1}^r \Theta(X_i)$. We take this as the definition of $\Theta(X)$ whenever X has a finite composition series. If $X_1, \dots, X_r$ are inequivalent irreducible Harish-Chandra modules, then $\Theta(X_1), \dots, \Theta(X_r)$ are linearly independent.

By a *virtual representation* we will mean a formal finite combination of irreducible representations with integer coefficients. By the preceding remarks, such an object can be assigned a distribution character, which vanishes only if the virtual representation does. By a *character* we will always mean the character of a virtual representation.

Let X be a Harish-Chandra module, and $\delta: \mathfrak{Z}(\mathfrak{g}) \to \mathbf{C}$ an infinitesimal character. Define

$$P_\delta(X) = \{x \in X \mid \forall z \in \mathfrak{Z}(\mathfrak{g})\ \exists n > 0 \quad \text{such that } (z - \delta(z))^n \cdot x = 0\}.$$

Then $P_\delta(X)$ is a submodule of X, and every composition factor of $P_\delta(X)$ has infinitesimal character δ. Furthermore,

$$X = \sum_\delta P_\delta(X).$$

The sum is direct; if X has finite composition series, it is finite, Clearly, P_δ gives rise to a unique homomorphism P_δ from the group of characters to itself, satisfying $P_\delta(\Theta(X)) =$

$\Theta(P_\delta(X))$. Suppose next that F is a finite-dimensional representation of G. Then $X\otimes F$ is a Harish-Chandra module; it has finite composition series if X does. In that case $\Theta(X\otimes F)=\Theta(X)\cdot\Theta(F)$; it is enough to verify this when X is irreducible, in which case it follows easily from the definitions. (Notice that $\Theta(F)$ is a smooth function on G, so that $\Theta(X)\cdot\Theta(F)$ is well defined.) It follows that multiplication by $\Theta(F)$ defines a homomorphism from the group of characters to itself.

Formally, we need no more results from character theory. However, the consistency and completeness of the definitions to be made below rely heavily on a deep theorem of Harish-Chandra: that every character is integration against a function in $L^1_{\mathrm{loc}}(G)$.

We fix for the remainder of this section a maximally split θ-invariant Cartan subalgebra $H^0=(T^+)^0A^0$ of G, and a positive root system $\Delta^+_{H^0}\subseteq\Delta(\mathfrak{g},\mathfrak{h}^0)$ compatible with an Iwasawa decomposition of G. Every irreducible finite-dimensional representation F of G has a unique highest weight $\mu\in\hat{H}^0$, which occurs with multiplicity one and characterizes F; we write $F=F(\mu)$. (We will also write $F(-\mu)$ for the dual of $F(\mu)$, which has lowest weight $-\mu$.) Every dominant weight of F is also the highest weight of some finite-dimensional irreducible representation. The set of weights of F is written $\Delta(F)\subseteq\hat{H}^0$. Suppose $\mu\in\Delta(F)$; say $0\neq v\in F$, $h\cdot v=\mu(h)v$ for $h\in H^0$. If H is another Cartan subgroup of G, and $\gamma\in\mathfrak{h}^*$ is nonsingular, we can consider μ as a character of H in the following way. Let $\tilde{G}_{\mathbf{C}}$ be the simply connected cover of $G_{\mathbf{C}}$. Then the complexified differential of F exponentiates to a representation of $G_{\mathbf{C}}$ on F. Choose $c\in G_{\mathbf{C}}$ so that $c\cdot\mathfrak{h}^0=\mathfrak{h}$, and $c\cdot\Delta^+_{H^0}=\Delta^+_\gamma$. Then $c^{-1}\cdot v$ is a weight vector for H; we call the corresponding weight μ_γ or $\mu_{\Delta^+_\gamma}$. It is independent of c and F.

We summarize now the main results of Zuckerman on coherent continuation.

Definition 5.1. Suppose $\mu\in\hat{H}^0$ is a dominant weight of a finite-dimensional representation, and $\lambda\in(\mathfrak{h}^0)^*$ is dominant. If X is a Harish-Chandra module, put

$$\varphi^\lambda_{\lambda+\mu}X=P_{\lambda+\mu}(P_\lambda(X)\otimes F(\mu))$$
$$\psi^{\lambda+\mu}_\lambda X=P_\lambda(P_{\lambda+\mu}(X)\otimes F(-\mu)).$$

Analogous definitions are made for characters. Finally, we define $\mathcal{A}(\lambda)$ to be the category of Harish-Chandra modules with infinitesimal character λ. Recall that a module all of whose composition factors are isomorphic is called *primary*.

THEOREM 5.2. (Zuckerman [21]). *$\psi^{\lambda+\mu}_\lambda$ maps primary modules to primary modules. If λ and $\lambda+\mu$ have the same stabilizer in $W(\mathfrak{g}/\mathfrak{h})$, then $\varphi^\lambda_{\lambda+\mu}$ restricts to an isomorphism of $\mathcal{A}(\lambda)$ with $\mathcal{A}(\lambda+\mu)$, with natural inverse $\psi^{\lambda+\mu}_\lambda$.*

Our goal is to reformulate this theorem in character-theorctic terms, and then to consider natural generalizations of it. Suppose then that $\gamma \in (\mathfrak{h}^0)^*$ is dominant and nonsingular, and that the character Θ has infinitesimal character γ. Let $\mu \in \hat{H}^0$ be a weight of a finite-dimensional representation. We want to define a new character $S_\mu \cdot \Theta$, which is to have infinitesimal character $\gamma + \mu$. (The following construction is due to Hecht and Schmid [6], and Zuckerman [21], among others; but apparently no complete account of it has been published.) We begin by defining $S_\mu \cdot \Theta$ simply as a function on G'. Now Θ is invariant under conjugation, and we want $S_\mu \cdot \Theta$ to be also; so it suffices to define $S_\mu \cdot \Theta$ on each connected component of $H \cap G'$, with H an arbitrary Cartan subgroup of G. Fix such a component H_i'. Then $\Theta|_{H_i'}$ can be written uniquely as a sum of terms of the form $a \cdot \exp(\lambda(\log hh_0))/\Delta(h)$; here Δ is a "Weyl denominator", $\lambda \in \mathfrak{h}^*$ is a weight defining the infinitesimal character of Θ, and one must choose h_0 and the definition of log appropriately. (For all this see [21].) We define $S_\mu \cdot \Theta$ to be a similar sum, but with the above term multiplied by the weight $\mu_\lambda \in \hat{H}$.

LEMMA 5.3. *If F is a finite-dimensional representation of G, then*

$$\Theta \cdot \Theta(F) = \sum_{\mu \in \Delta(F)} S_\mu \cdot \Theta$$

(*an identity of functions on G'*).

Proof. If $\gamma \in \mathfrak{h}^*$ is nonsingular, then obviously

$$\Theta(F)(h) = \sum_{\mu \in \Delta(F)} \mu_\gamma(h)$$

for all $h \in H$. The result follows immediately from the definitions. Q.E.D.

LEMMA 5.4. *In the setting of Lemma 5.3, suppose $\delta \in (\mathfrak{h}^0)^*$, and that Θ has infinitesimal character $\gamma^0 \in (\mathfrak{h}^0)^*$. Then*

$$P_\delta(\Theta \cdot \Theta(F)) = \sum_{\substack{\mu \in \Delta(F) \\ \mu + \gamma^0 \in W(\mathfrak{g}/\mathfrak{h}^0) \cdot \delta}} S_\mu \cdot \Theta.$$

Proof. This follows from the earlier remarks on the form of a character with a given infinitesimal character. Q.E.D.

LEMMA 5.5. *Suppose Θ has a nonsingular infinitesimal character represented by the dominant weight $\gamma^0 \in (\mathfrak{h}^0)^*$, and that $\mu \in \hat{H}^0$ is a dominant weight of a finite-dimensional representation. Then*

$$S_\mu \cdot \Theta = \varphi^{\gamma^0}_{\gamma^0 + \mu}(\Theta).$$

If also $\gamma^0-\mu$ is dominant, then

$$S_{-\mu}\cdot\Theta=\psi^{\gamma^0}_{\gamma^0-\mu}(\Theta).$$

In particular, $S_\mu\cdot\Theta$ and $S_{-\mu}\cdot\Theta$ are characters in this case.

Proof. Consider the first statement; it asserts that

$$S_\mu\cdot\Theta=P_{\gamma^0+\mu}(\Theta\cdot\Theta(F(\mu))).$$

By Lemma 5.4, the right side is just

$$\sum_{\substack{\tilde\mu\in\Delta(F)\\ \tilde\mu+\gamma^0\in W(\mathfrak{g}/\mathfrak{h}^0)\cdot(\mu+\gamma^0)}} S_{\tilde\mu}\cdot\Theta.$$

Suppose $\tilde\mu\in\Delta(F)$, $w\in W(\mathfrak{g}/\mathfrak{h}^0)$, and $\tilde\mu+\gamma^0=w^{-1}\cdot(\mu+\gamma^0)$. We want to show that $\tilde\mu=\mu$. We have $\mu-w\cdot\tilde\mu=w\gamma^0-\gamma^0$. Since $w\cdot\tilde\mu\in\Delta(F)$, $\mu-w\cdot\tilde\mu$ is a sum of positive roots. On the other hand, we can write $w\gamma^0-\gamma^0=\sum n_i\alpha_i$, with $\alpha_i\in\Delta^+$, and Re $n_i\leqslant 0$. This is possible only if $\mu=w\tilde\mu$ and $\gamma^0=w\gamma^0$. Since γ^0 is nonsingular, it follows that $w=1$. This proves the first statement. The second is similar. Q.E.D.

PROPOSITION 5.5. *$S_\mu\cdot\Theta$ is a character.*

Proof. The proof of Lemma 5.2 of [6] carries over without change to the present situation. Q.E.D.

Passage from Θ to $S_\mu\Theta$ is called *coherent continuation*. For general considerations, the following notation will be useful. Suppose Θ has infinitesimal character represented by a nonsingular dominant weight $\gamma\in(\mathfrak{h}^0)^*$. Then we write $\Theta=\Theta(\gamma)$, and $S_\mu\cdot\Theta=\Theta(\gamma+\mu)$. Lemma 5.3, for example, can be written as

$$\Theta(\gamma)\cdot\Theta(F)=\sum_{\mu\in\Delta(F)}\Theta(\gamma+\mu).$$

Other notation for more specific representations will be modelled on this.

Suppose for a moment that $\Theta(\gamma)$ is irreducible, and that $\gamma+\mu$ is dominant and nonsingular. It follows from Zuckerman's theorem that $\Theta(\gamma+\mu)$ is also irreducible. (For we choose ν dominant so that $\nu-\mu$ is also dominant. Then $\Theta(\gamma+\nu)$ is irreducible by one application of Theorem 5.2, and so $\Theta((\gamma+\nu)-(\nu-\mu))=\Theta(\gamma+\mu)$ is irreducible by a second application. Such arguments are henceforth left to the reader.) If $\gamma+\mu$ is dominant but possibly singular, then $\Theta(\gamma+\mu)$ is at least primary. We will first prove that in this last situation $\Theta(\gamma+\mu)$ is in fact irreducible or zero. This result will then be used to get information about $\Theta(\gamma+\mu)$ when $\gamma+\mu$ is not even dominant.

We need to understand coherent continuation of generalized principal series representations. We begin with the discrete series; so suppose for a moment that G has a compact Cartan subgroup T. Fix a positive root system $\Psi \subseteq \Delta(\mathfrak{g}, \mathfrak{t})$.

Definition 5.6. A Ψ*-pseudocharacter* (or Ψ*-character*) of T is a pair $(\Lambda, \bar{\lambda})$, with $\Lambda \in \hat{T}$, $\bar{\lambda} \in i\mathfrak{t}_0'$, and $d\Lambda = \bar{\lambda} + \varrho(\Psi) - 2\varrho(\Psi \cap \Delta(\mathfrak{k}))$.

The set of Ψ-pseudocharacters of T is written $\hat{T}_\Psi$. If μ is a weight of a finite-dimensional representation, we write $\lambda + \mu_\Psi = (\Lambda \otimes \mu_\Psi, \bar{\lambda} + \mu_\Psi)$. Notice that if $\bar{\lambda}$ is strictly dominant with respect to Ψ, then λ is a regular pseudocharacter of T. To each $\lambda \in \hat{T}_\Psi$ we associate a character $\Theta(\Psi, \lambda)$ as follows: $\Theta(\Psi, \bar{\lambda})$ as defined by Hecht and Schmid [6] is a character of G_0. Extend this to ZG_0 so that $\Theta(gz) = \Theta(g)\Lambda(z)$, and then to G by making it zero off ZG_0. Clearly $\Theta(\Psi, \lambda)$ has infinitesimal character λ; and if $\bar{\lambda}$ is strictly dominant for Ψ, then $\Theta(\Psi, \lambda) = \Theta(\pi(\lambda))$.

LEMMA 5.7. *Suppose $\lambda \in \hat{T}'$, with $\Psi = \Delta_{\bar{\lambda}}^+$. Then*

$$S_\mu \cdot \Theta(\pi(\lambda)) = \Theta(\Psi, \lambda + \mu).$$

Proof. For connected G this is the definition of $\Theta(\Psi, \lambda + \mu)$ ([6], p. 133). The extension to the present case is trivial. Q.E.D.

Now let $H = T^+ A$ be an arbitrary θ-invariant Cartan subgroup of G, $P = MAN$ an associated parabolic subgroup, and Ψ a system of positive roots for $\mathfrak{t}^+$ in $\mathfrak{m}$. We define the set $\hat{H}_\Psi$ of Ψ-pseudocharacters of H in the obvious way. If $x \in \mathfrak{h}^*$ is regular, $\gamma \in \hat{H}_\Psi$, and μ is a weight of a finite-dimensional representation, we define $\gamma + \mu_x$ in analogy with the case when H is compact. Set

$$\Theta(\Psi, \gamma) = \operatorname{Ind}_P^G \Theta(\Psi, \lambda) \otimes \nu \otimes 1.$$

(The representation induced by a formal difference is the formal difference of the induced representations, so this makes sense. As the notation indicates, $\Theta(\Psi, \gamma)$ is independent of P, cf. Proposition 2.7.) If $\bar{\lambda}$ is strictly dominant for Ψ, then $\Theta(\Psi, \gamma) = \Theta(\pi(\gamma))$.

LEMMA 5.8. *Suppose π_{MA} is a representation of MA with finite composition series; put $\pi_G = \operatorname{Ind}_P^G \pi_{MA} \otimes 1$. If F is a finite-dimensional representation of G, choose a family $0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = F$ of P-invariant subspaces of F, such that N acts trivially in $V_i = F_i/F_{i-1}$. Then $\pi_G \otimes F$ has a family $0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n = \pi_G \otimes F$ of G-invariant subspaces, such that*

$$H_i/H_{i-1} \cong \operatorname{Ind}_P^G [(\pi_{MA} \otimes V_i) \otimes 1].$$

Proof. For formal reasons, $\pi_G \otimes F \cong \mathrm{Ind}_P^G [(\pi_{MA} \otimes 1) \otimes_P F|_P]$. The result now follows from the exactness of Ind. Q.E.D.

Possibly replacing H by a conjugate, we may assume that $A^0 \supseteq A$; then $H^0 \subseteq MA$.

COROLLARY 5.9. *In the setting of Theorem* 5.2 *and Lemma* 5.8, *suppose π_{MA} has infinitesimal character $\lambda + \mu$. Then so does π_G; and*

$$\psi_\lambda^{\lambda+\mu}(\pi_G) = \mathrm{Ind}_P^G (\psi_\lambda^{\lambda+\mu} \pi_{MA}).$$

Proof. That π_G has infinitesimal character $\lambda + \mu$ is obvious. In Lemma 5.8, take $F = F(-\mu)$; we may as well choose the V_i to be irreducible. An argument like that given for Lemma 5.5 shows that $\mathrm{Ind}_P^G [(\pi_{MA} \otimes V_i) \otimes 1]$ has a composition factor of infinitesimal character λ only if V_i contains the $-\mu$. In that case $V_i = F_{MA}(-\mu)$, since $-\mu$ is extremal in F. Furthermore, only $P_\lambda(\pi_{MA} \otimes F_{MA}(-\mu))$ contributes to $P_\lambda(\pi_G \otimes F(-\mu))$. This shows that $\mathrm{Ind}_P^G (\psi_\lambda^{\lambda+\mu} \pi_{MA})$ is a subquotient of $\pi_G \otimes F(-\mu)$, containing all the composition factors of infinitesimal character λ. The corollary is immediate. Q.E.D.

The preceding result is hinted at in the closing remarks of [21].

COROLLARY 5.10. *In the setting of Lemma* 5.8, *$\pi_G \otimes F$ has the same composition series as* $\mathrm{Ind}_P^G [(\pi_{MA} \otimes F|_{MA}) \otimes 1]$.

COROLLARY 5.11. *Suppose $\lambda \in (\mathfrak{h}^0)^*$ is G-regular. If Θ_{MA} is a character for MA with infinitesimal character λ, and μ is a weight of a finite-dimensional representation of G, then*

$$S_\mu \cdot [\mathrm{Ind}_P^G (\Theta \otimes 1)] = \mathrm{Ind}_P^G [(S_\mu \cdot \Theta) \otimes 1].$$

COROLLARY 5.12. *Suppose $\gamma \in \hat{H}'$, with $\Psi = \Delta_\lambda^+(\mathfrak{m})$. Then*

$$S_\mu \cdot \Theta(\pi(\gamma)) = \Theta(\Psi, \gamma + \mu_\gamma).$$

These are obvious.

The Langlands classification theorem provides a natural basis for the space of characters with a fixed nonsingular infinitesimal character γ, namely, the characters of generalized principal series representations. We want to express the various $\Theta(\Psi, \gamma)$ in terms of this basis. Evidently it suffices to do this in case $H = T$ is compact. For this purpose we use the character identities of Hecht and Schmid [17]. Their extension to the present situation is straightforward, but requires a brief discussion. The first identity says that if $\alpha \in \Psi$ is a simple compact root, then $\Theta(\Psi, \lambda) + \Theta(s_\alpha \Psi, \lambda) = 0$; this is a trivial consequence of the result for connected groups. The second begins with a noncompact simple root $\beta\in\Psi$, and involves a Cartan subgroup $H^\beta = T^\beta A^\beta$. Here $\mathfrak{a}_0^\beta$ is a one-dimensional subalgebra of $\mathfrak{p}_0$ contained in the sum of the β and $-\beta$ root spaces, and $\mathfrak{t}^\beta$ is the orthogonal complement of β in $\mathfrak{t}$. Let $P^\beta = M^\beta A^\beta N^\beta$ be an associated parabolic subgroup. The roots of T^β in M^β are identified with the roots of $\mathfrak{t}$ in $\mathfrak{g}$ orthogonal to β; so $\Psi\cap(\mathfrak{t}^\beta)^* = \Psi^\beta$ is a positive system. Put $H_1^\beta = (T^\beta\cap T)\cdot A^\beta$, $T_1^\beta = T^\beta \cap T$. If $\lambda = (\Lambda, \bar\lambda)$, we define $\lambda_1^\beta \in (\hat{T}_1^\beta)_{\Psi^\beta}$ as $\lambda_1^\beta = (\Lambda|_{T_1^\beta}, \bar\lambda|_{\mathfrak{t}^\beta})$; that $d\Lambda|_{\mathfrak{t}_1^\beta} = \bar\lambda + \varrho(\Psi^\beta) - 2\varrho(\Psi^\beta\cap\Delta(\mathfrak{m}\cap\mathfrak{k}))$ follows from (7.21) of [16]. Finally, we define $\nu^\beta \in \hat{A}^\beta$ so that if $\tilde\beta$ is the unique real root of $\mathfrak{h}^\beta$ in $\mathfrak{g}$, then $\langle \nu^\beta, \tilde\beta\rangle = \langle\lambda,\beta\rangle$. Set $M_1^\beta = T_1^\beta M_0^\beta$.

LEMMA 5.13. *With notation as above,*

$$\Theta(\Psi,\lambda) + \Theta(s_\beta\Psi,\lambda) = \operatorname{Ind}_{M_1^\beta A^\beta N^\beta}^G \Theta(\Psi^\beta, \lambda_1^\beta)\otimes\nu^\beta\otimes 1.$$

Proof. This is in essence the Hecht-Schmid identity ([16], Theorem 9.4), combined with the definitions of the $\Theta(\Psi,\lambda)$ for disconnected groups. The definition given in [16] for the inducing distribution $\Theta(\Psi^\beta, \lambda_1^\beta)$ is formulated in a slightly different way, but it is easy to check that the two definitions agree. Q.E.D.

Put $M_2^\beta = Z(M^\beta)\cdot M_0^\beta \supseteq M_1^\beta$. Then $M_2^\beta/M_1^\beta \cong T^\beta/T_1^\beta$. This group is nontrivial exactly when the reflection $s_\beta\in W(\mathfrak{g}/\mathfrak{t})$ about the root β lies in $W(G/T)$. In that case it has order 2; so λ_1^β has exactly two extensions λ_+^β and λ_-^β to T^β; these are the constituents of $\operatorname{Ind}_{T_1^\beta}^{T^\beta}\lambda_1^\beta$. Set $\gamma_\pm^\beta = (\lambda_\pm^\beta, \nu^\beta)$. If $T_1^\beta = T^\beta$, set $\lambda^\beta = \lambda_1^\beta$, $\gamma^\beta = (\lambda^\beta, \nu^\beta)$.

PROPOSITION 5.14. *Suppose $T\subseteq G$ is a compact Cartan subgroup, $\Psi\subseteq\Delta(\mathfrak{g},\mathfrak{t})$ is a positive root system, $\lambda\in\hat{T}_\Psi$, and $\beta\in\Psi$ is a noncompact simple root. If $s_\beta\notin W(G/T)$,*

$$\Theta(\Psi,\lambda) + \Theta(s_\beta\Psi,\lambda) = \Theta(\Psi^\beta, \gamma^\beta).$$

If $s_\beta\in W(G/T)$,

$$\Theta(\Psi,\lambda) + \Theta(s_\beta\Psi,\lambda) = \Theta(\Psi^\beta,\gamma_+^\beta) + \Theta(\Psi^\beta,\gamma_-^\beta).$$

This is an immediate consequence of Lemma 5.13. It should be emphasized that the result is only a reformulation of the Hecht-Schmid character identity. Most of the technicalities involved have appeared already in [12], formula (7b). It is an easy exercise to see that these identities allow us to write any $\Theta(\Psi,\lambda)$ as an integral combination of characters of generalized principal series, at least if λ is nonsingular. In particular, the various $S_\mu\cdot\Theta(\pi(\gamma))$ are computable.

For future reference we record the condition for a character $\Theta(\Psi,\gamma)$ to occur on the right side of one of the identities of Proposition 5.14. Suppose γ is a Ψ-pseudocharacter of

$H=T^+A$, with $\dim A=1$. Then there must be a real root $\tilde\beta$ of $\mathfrak{h}$ in $\mathfrak{g}$. By a Cayley transform, $\tilde\beta$ gives rise to the compact Cartan subgroup T of G. Choose a positive root system Ψ_1 for $\mathfrak{t}$ in $\mathfrak{g}$, such that if β is the Cayley transform of $\tilde\beta$, then β is a simple positive root, and (identifying $\mathfrak{t}^\beta=\langle\beta\rangle^\perp$ with $\mathfrak{t}^+$) $\Psi_1\cap\mathfrak{t}^\beta=\Psi$; this is certainly possible. Let $\bar\lambda\in\mathfrak{t}^*$ be the Cayley transform of γ. Fix a map φ_β: SL $(2, \mathbf{R})\to G$, a three-dimensional subgroup through the real root $\tilde\beta$; we may assume that $\varphi_\beta({}^tx^{-1})=\theta(\varphi_\beta(x))$.

Then

$$m_\beta=\varphi_\beta\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\in H\cap T$$

is independent of the choice of φ_β, and $m_\beta^2=1$. Suppose there is a Ψ_1-pseudocharacter $\lambda=(\Lambda,\bar\lambda)$ of T so that $\Theta(\Psi,\gamma)$ occurs on the right in the corresponding character identity. Then

$$\gamma(m_\beta)=\Lambda(m_\beta)=(-1)^n,$$

where $n=2\langle\beta,\lambda+\varrho(\Psi_1)-2\varrho(\Psi_1\cap\Delta(\mathfrak{k})\rangle/\langle\beta,\beta\rangle$ is an integer. Write $n_\beta=2\langle\beta,\varrho(\Psi_1)-2\varrho(\Psi_1\cap\Delta(\mathfrak{k})\rangle/\langle\beta,\beta\rangle$, and $\varepsilon_\beta=(-1)^{n_\beta}$. Since $\langle\beta,\lambda\rangle=\langle\tilde\beta,\gamma\rangle$, the condition can now be written as

$$\gamma(m_\beta)=\varepsilon_\beta\cdot(-1)^{2\langle\gamma,\beta\rangle/\langle\beta,\beta\rangle}. \tag{*}$$

We leave to the reader the verification that ε_β is independent of the choice of Ψ_1, and that the condition (*) is in fact sufficient for the existence of a character identity.

One might expect that the characters $\Theta(\Psi,\lambda)$ with λ dominant (but possibly singular) have special properties. This is the case—they are called *limits of discrete series*, and are tempered and irreducible (or zero). (See [6], Lemma 3.1, [21], Theorem 5.7, and [19], Lemma 7.3.) The corresponding representations are written $\pi(\Psi,\lambda)$. More generally, $\Theta(\Psi,\gamma)$ is called a *limit of generalized principal series* if $\gamma=(\lambda,\nu)$ and λ is dominant for Ψ, and we define $\pi(P,\Psi,\gamma)=\operatorname{Ind}_P^G\pi(\Psi,\lambda)\otimes\nu\otimes 1$. Most of the theory of generalized principal series holds for these representations as well, since Langlands discusses induction from any tempered representation in [13]. If γ is unitary (and λ is dominant for Ψ) then $\pi(P,\Psi,\gamma)$ is tempered; it is not necessarily irreducible, but every tempered irreducible representation arises in this way (cf. [12]).

THEOREM 5.15. *Suppose $\tilde\gamma\in(\mathfrak{h}^0)^*$ is dominant and nonsingular, $\mu\in\hat H^0$ is a dominant weight of a finite-dimensional representation, and $\tilde\gamma-\mu$ is dominant. Suppose $\pi\in\hat G$ has infinitesimal character $\tilde\gamma$. Then $\psi^{\tilde\gamma}_{\tilde\gamma-\mu}(\pi)$ is irreducible or zero.*

Proof. Zuckerman's Theorem 5.2 asserts that $\psi^{\tilde{\gamma}}_{\gamma-\mu}(\pi)$ is primary. By Theorem 2.9, there is a θ-invariant Cartan subgroup $H=T^+A$ of G, and a regular character $\gamma=(\lambda,\nu)\in\hat{H}'$ such that $\pi\cong\bar{\pi}(\gamma)$. Choose $P=MAN$ so that ν is negative with respect to P. Then there is an exact sequence

$$0\to\bar{\pi}(\gamma)\to\pi(P,\gamma)\to Q\to 0.$$

Set $X_1=\psi^{\tilde{\gamma}}_{\gamma-\mu}(\bar{\pi}(\gamma))$, $X_2=\psi^{\tilde{\gamma}}_{\gamma-\mu}(Q)$. Suppose λ defines the positive root system $\Psi\subseteq\Delta(\mathfrak{t}^+,\mathfrak{m})$. Put $\gamma^1=(\lambda^1,\nu^1)=\gamma-\mu_\gamma$. Then λ^1 is dominant for Ψ. By Corollary 5.9 and Lemma 5.7,

$$\psi^{\tilde{\gamma}}_{\gamma-\mu}(\pi(P,\gamma))=\pi(P,\Psi,\gamma^1),$$

so we have an exact sequence

$$0\to X_1\to\pi(P,\Psi,\gamma^1)\to X_2\to 0.$$

We know that X_1 is primary; we want to show that it is irreducible. Define $P^1=MAN^1$ so that

$$\Delta(\mathfrak{n}^1,\mathfrak{a})=\{\alpha\in\Delta(\mathfrak{g},\mathfrak{a})\,|\,\mathrm{Re}\,\langle\nu^1,\alpha\rangle<0\quad\text{or}\quad\mathrm{Re}\,\langle\nu^1,\alpha\rangle=0\quad\text{and}\quad\alpha\in\Delta(\mathfrak{n},\mathfrak{a})\}.$$

Now $\pi_M(\Psi,\lambda^1)$ is tempered and irreducible, so by a theorem of Harish-Chandra we can find a θ-invariant Cartan subgroup H^2 of M and a unitary pseudocharacter $\gamma^2\in(\hat{H}^2)'$ such that $\pi_M(\Psi,\lambda^1)$ is a constituent of $\pi_M(\gamma^2)$. Put $H^3=H^2A$, and let $\gamma^3\in(\hat{H}^3)'$ be defined using γ^2 and ν^1 in the obvious way. Choose a parabolic subgroup $P^3\subseteq P^1$ associated to H^3. Then clearly $\pi(P^1,\Psi,\gamma^1)$ is a direct summand of $\pi(P^3,\gamma^3)$. Furthermore, ν^3 is negative with respect to P^3. By Theorem 2.9, the irreducible subrepresentations of $\pi(P^3,\gamma^3)$ are inequivalent and occur exactly once in the composition series, so the same statement holds for $\pi(P^1,\Psi,\gamma^1)$. In particular, a primary subrepresentation of $\pi(P^1,\Psi,\gamma^1)$ is irreducible or zero. To complete the proof, we need only show that $\pi(P^1,\Psi,\gamma^1)\cong\pi(P,\Psi,\gamma^1)$. Using $\psi^{\tilde{\gamma}}_{\gamma-\mu}$, it is enough to show that $\pi(P^1,\gamma)\cong\pi(P,\gamma)$.

LEMMA 5.16. *Suppose $H=T^+A$ is a θ-invariant Cartan subgroup of G, with* $\dim A=1$. *Suppose $\gamma\in\hat{H}'$ is nonsingular, and that $\Delta^+_\gamma(\mathfrak{g},\mathfrak{h})$ is θ-invariant. Then $\pi(\gamma)$ is irreducible.*

Assuming this lemma for a moment, we show that $\pi(P^1,\gamma)\cong\pi(P,\gamma)$. Recall from Section 3 the intertwining operator

$$I(P^1,P)\colon\pi(P^1,\gamma)\to\pi(P,\gamma).$$

We want to show that $I(P^1,P)$ is an isomorphism. Using Theorem 3.7 to factor $I(P^1,P)$, we may assume that $\dim A=1$, and that $P=\theta P^1$; in this situation we want to show that

$\pi(P, \gamma)$ is irreducible. If γ is unitary, this follows from Theorem 2.9, so we may assume $\operatorname{Re} \nu \neq 0$. Suppose $\alpha \in \Delta(\mathfrak{n}^1, \mathfrak{a})$. Since $P = \theta P^1$, $\alpha \notin \Delta(\mathfrak{n}, \mathfrak{a})$, so $\operatorname{Re}\langle \nu^1, \alpha\rangle < 0$, and $\operatorname{Re}\langle \nu, \alpha\rangle > 0$. We claim that $\Delta_\gamma^+(\mathfrak{g}, \mathfrak{h})$ is θ-invariant; for let $(\beta, \alpha) \in \Delta_\gamma^+(\mathfrak{g}, \mathfrak{h})$. (Here $\beta \in (\mathfrak{t}^+)^*$, $\alpha \in \mathfrak{a}^*$.) Of course $\theta(\beta, \alpha) = (\beta, -\alpha)$, so we may assume $\alpha \neq 0$. Suppose $(\beta, -\alpha) \notin \Delta_\gamma^+$. Then clearly $\operatorname{Re}\langle \alpha, \nu\rangle > 0$, so $\alpha \in \Delta(\mathfrak{n}^1, \mathfrak{a})$, and $(-\beta, \alpha) \in \Delta_\gamma^+$. Since γ^1 is dominant for Δ_γ^+,

$$2 \operatorname{Re}\langle \alpha, \nu^1\rangle = \operatorname{Re}\langle (\beta, \alpha), (\lambda^1, \nu^1)\rangle + \operatorname{Re}\langle (-\beta, \alpha), (\lambda^1, \nu^1)\rangle \geqslant 0,$$

a contradiction. Q.E.D.

Proof of Lemma 5.16. By Corollary 4.6, we may assume that G s connected. Since Δ_γ^+ is fixed by θ, so is $\varrho(\Delta_\gamma^+)$; so $\varrho(\Delta_\gamma^+) \in \mathfrak{t}^+$. By an application of Theorem 5.2, it suffices to prove the lemma with γ replaced by $\gamma + 2\varrho(\Delta_\gamma^+)$. In this case it is easy to check that if $\alpha \in \Delta_\gamma^+$, then $\langle \alpha, \lambda\rangle > 0$. Thus the θ-invariant parabolic subalgebra $\mathfrak{b}$ associated to λ is just the Borel subalgebra corresponding to Δ_γ^+. Hence the $\mathfrak{b}$-bottom layer of K-types of $\pi(\gamma)$ consists of the lowest K-type alone. By Proposition 4.21 $\pi(\gamma)$ is irreducible. Q.E.D.

As a corollary of the proof, we have

COROLLARY 5.17. *If $\gamma \in \hat{H}'$ is nonsingular, μ is a weight of a finite-dimensional representation, and $\gamma - \mu_\gamma$ is dominant for Δ_γ^+, then $S_{-\mu}(\bar{\pi}(\Psi, \gamma))$ is a Langlands subquotient of $\pi(\Psi, \gamma - \mu_\gamma)$. In particular, if $\gamma - \mu_\gamma$ is strictly dominant,*

$$S_{-\mu}(\bar{\pi}(\gamma)) = \bar{\pi}(\gamma - \mu_\gamma).$$

We will see much later (Theorem 6.18) that if $S_{-\mu}(\bar{\pi}(\Psi, \gamma)) \neq 0$, then $\pi(\Psi, \gamma - \mu_\gamma)$ has a unique Langlands subquotient.

We can now prove Theorem 4.17. With notation as in its statement, choose a large multiple γ^1 of $2\varrho(\mathfrak{n})$, so that $\mu + \gamma^1$ and $\mu^1 + \gamma^1$ both satisfy the strong nonsingularity condition of Proposition 4.16. As in Lemma 4.14, we can regard γ^1 and $-\gamma^1$ as the weights of one dimensional representations $\mathbf{C}_{\gamma^1}$ and $\mathbf{C}_{-\gamma^1}$ of L; set $\tilde{V} = V \otimes \mathbf{C}_{\gamma^1}$. If we also write γ^1 for the restriction of γ^1 to the Cartan subgroup H of L to which V is attached in the Langlands classification, then

$$\tilde{V} \cong \bar{\pi}_L((\lambda_L^0, \nu) + \gamma^1, \mu^0 + \gamma^1 - 2\varrho(\mathfrak{n} \cap \mathfrak{p})).$$

By the proof of Proposition 4.15,

$$X(\mathfrak{b}, \tilde{V}, \mu + \gamma^1) \cong \bar{\pi}_G(\gamma + \gamma^1, \mu^0 + \gamma^1).$$

Let F denote the finite dimensional irreducible representation of G of lowest weight $-\gamma^1$. The representation Y attached to $X(\mathfrak{b}, \tilde{V}, \mu + \gamma^1)$ by Lemma 4.14 is

$$Y \cong X(\mathfrak{b}, \tilde{V}, \mu + \gamma^1) \otimes F$$

by definition; and $X(\mathfrak{b}, V, \mu) \subseteq Y_{ss}$ by Corollary 4.15. Set $Y_0 = P_\gamma(Y)$ (the submodule of infinitesimal character γ); then

$$\begin{aligned} Y_0 &= \Psi_\gamma^{\gamma+\gamma^1}(X(\mathfrak{b}, \tilde{V}, \mu+\gamma^1)) \\ &\cong \Psi_\gamma^{\gamma+\gamma^1}(\bar{\pi}_G(\gamma+\gamma^1, \mu^0+\gamma^1)) \\ &\cong \bar{\pi}_G(\gamma, \mu^0) \end{aligned}$$

by Corollary 5.17. On the other hand, Lemma 4.14 calculates the action of $U(\mathfrak{g})^K$, and hence, of $\mathfrak{Z}(\mathfrak{g})$, on the K-primary subspaces $Y(\mu)$ and $Y(\mu^1)$. This calculation shows that

$$Y_0 \supseteq Y(\mu), \quad Y_0 \supseteq Y(\mu^1);$$

so the multiplicity of μ and μ^1 in Y_0 is the same as in Y. In particular

$$\bar{\pi}_G(\gamma, \mu^0) \cong Y_0 \supseteq X(\mathfrak{b}, V, \mu);$$

since the first of these is irreducible, equality holds. Lemma 4.14 (b) and Proposition 4.16 complete the proof. Q.E.D.

PROPOSITION 5.18. *Suppose G is connected. Let $\mathfrak{b} = \mathfrak{l} + \mathfrak{n}$ be a θ-invariant parabolic subalgebra of G. Let H be a θ-invariant Cartan subgroup of G contained in L, and $\gamma_L \in \hat{H}'$ a regular pseudocharacter with respect to L. Suppose that γ_L is nonsingular for $\mathfrak{l}$ and that for all $\alpha \in \Delta(\mathfrak{n}, \mathfrak{h})$, $\mathrm{Re}\,\langle\alpha, \gamma_L\rangle > 0$ or $\mathrm{Re}\,\langle\alpha, \gamma_L\rangle = 0$ and $\mathrm{Im}\,\langle\alpha, \gamma_L\rangle > 0$. Associate to $\gamma_L = (\lambda_L, \nu)$ a regular pseudocharacter $\gamma_G = (\lambda_G, \nu)$ of H with respect to G as in Section 4 (proof of Proposition 4.13). Then whenever μ is a K-type such that $\mu - 2\varrho(\mathfrak{n} \cap \mathfrak{p})$ occurs in $\bar{\pi}_L(\gamma_L)$, we have*

$$X(\mathfrak{b}, \bar{\pi}_L(\gamma_L), \mu) = \bar{\pi}_G(\gamma_G).$$

Proof. This follows from the preceding proof, together with Corollary 5.17. Q.E.D.

Like the results of Section 4, Proposition 5.18 generalizes readily to disconnected G.

To study the problem of coherent continuation across walls, we will make heavy use of Theorem 5.15. This means that we want to be able to stop on a wall, which in turn requires that we have lots of weights of finite-dimensional representation available. So we need

LEMMA 5.19. *Let G be a linear reductive group with abelian Cartan subgroups. Then there is a linear reductive group $\tilde{G}$, with abelian Cartan subgroups, and a surjective map $\tilde{G} \to G$ with finite kernel, with the following property: Whenever $\lambda \in \mathfrak{h}^*$ is an integral weight of a Cartan subalgebra of $\mathfrak{g}$, there is a character Λ of the corresponding Cartan subgroup $\tilde{H}$ of $\tilde{G}$, occurring in a finite-dimensional representation of $\tilde{G}$, such that $d\Lambda - \lambda$ annihilates every root of $\mathfrak{h}$ in $\mathfrak{g}$.*

Proof. Let $H = T^+A$ be a maximally split Cartan subgroup of G; recall that Z is the center of G. Put $R_1 = \{t \in T^+ \mid t^2 \in Z\}$. It is easy to show that $G = R_1(ZG_0)$. Set $R_1^0 = R_1 \cap ZG_0$; then R_1/R_1^0 is a product of s copies of $\mathbf{Z}/2\mathbf{Z}$. Choose elements $g_1, \ldots, g_s \in R_1$ of finite order so that the $\bar{g}_i$ generate R_1/R_1^0. Let R be the group generated by the g_i, and $R^0 = R \cap ZG_0$. Then R is finite and abelian, and acts by automorphisms on $\mathfrak{g}_0$. Choose a linear covering $\tilde{G}_0$ of G_0 such that the automorphisms of R lift to $\tilde{G}_0$, and such that integral weights lift to characters as described in the theorem. Let

$$\tilde{G} = R \times \tilde{G}_0 \times Z,$$

a semidirect product with Z central and $\tilde{G}_0$ normal. This group clearly satisfies the conditions of the lemma. Q.E.D.

THEOREM 5.20. *Let $\Theta(\gamma)$ be an irreducible character, with γ strictly dominant, and let α be a simple positive root. Suppose r is a positive integer such that $(\gamma - r\alpha)$ lies in the same Weyl chamber as $s_\alpha \cdot \gamma$.*

(a) *If $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle$ is not an integer, then $\Theta(\gamma - r\alpha)$ is an irreducible character.*

(b) *If $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle = n$ is an integer, then either $\Theta(\gamma - n\alpha) = -\Theta(\gamma)$, or $\Theta(\gamma - n\alpha)$ is the character of a representation.*

Proof. Consider first (a). Choose a dominant weight of a finite-dimensional representation so large that $\gamma - r\alpha + \mu$ is strictly dominant. Then $\Theta(\gamma - r\alpha + \mu)$ is an irreducible character by Theorem 5.2. Put $\Theta_0 = P_{\gamma - r\alpha}(\Theta(\gamma - r\alpha + \mu) \otimes F(-\mu))$, which is the character of a representation. We claim $\Theta_0 = \Theta(\gamma - r\alpha)$. By Lemma 5.4, it suffices to show that if $\tilde{\mu} \in \Delta(F(-\mu))$, $w \in W(\mathfrak{g}/\mathfrak{h}^0)$, and $\gamma - r\alpha + \mu + \tilde{\mu} = w(\gamma - r\alpha)$, then $w = 1$ and $\tilde{\mu} = \mu$. Write $\tilde{\mu} = -\mu + Q$, with Q a sum of positive roots. We can write $w(\gamma - r\alpha) = \gamma - r\alpha - Q_1 + s\alpha$. Here

$$Q_1 = \sum_{\substack{\alpha_i \in \Delta^+ \\ \alpha_i \neq \alpha}} n_i \alpha_i,$$

where Re $n_i \geqslant 0$, and Re $s \geqslant 0$. Thus

$$\gamma - r\alpha + Q = \gamma - r\alpha - Q_1 + s\alpha.$$

Such an equation can hold only if $Q = s\alpha$, and $Q_1 = 0$. In particular, s is an integer. It follows easily that $w = s_\alpha$ or 1 and that $s = 2\langle\alpha, \gamma - r\alpha\rangle/\langle\alpha, \alpha\rangle$ or 0 accordingly. Since $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle$ is not an integer, the first case is impossible. So $\Theta(\gamma - r\alpha)$ is the character of a representation. It follows immediately from the definitions that $S_{-r\alpha}(\Theta(\gamma - r\alpha)) = \Theta(\gamma)$. In particular, $\Theta(\gamma - r\alpha) \neq 0$. Suppose it is not irreducible; say $\Theta(\gamma - r\alpha) = \Theta_1(\gamma - r\alpha) + \Theta_2(\gamma - r\alpha)$, with Θ_1 and Θ_2 characters of representations. By the preceding results, $\Theta(\gamma) = \Theta_1(\gamma) + \Theta_2(\gamma)$ (here

$\Theta_i(\gamma) = S_{-r\alpha}\Theta_i(\gamma - r\alpha))$ and $\Theta_i(\gamma)$ is a nonzero character of a representation. This contradicts the irreducibility of $\Theta(\gamma)$ and proves a).

For (b), after passing to a covering group of G in accordance with Lemma 5.19, we choose a weight μ of a finite-dimensional representation such that $\gamma - n\alpha + \mu$ is dominant, and $2\langle\mu, \alpha\rangle/\langle\alpha, \alpha\rangle = n$. Suppose $\Theta(\gamma)$ is the character of $\pi(\gamma)$. We can choose a dominant weight ν of a finite-dimensional representation, so that $\nu - \mu + n\alpha$ is dominant. By Theorem 5.2, it clearly suffices to establish the analogue of b) with $\gamma + \nu$ replacing γ. Set $\pi(\gamma - n\alpha + \mu) = \psi^{\gamma+\nu}_{\gamma-n\alpha+\mu}(\pi(\gamma+\nu))$. By Theorem 5.15, $\pi(\gamma - n\alpha + \mu)$ is irreducible or zero. Define

$$\pi_0 = \varphi^{\gamma-n\alpha+\mu}_{\gamma+\nu}(\pi(\gamma - n\alpha + \mu)).$$

Say $2\langle\alpha, \gamma+\nu\rangle/\langle\alpha, \alpha\rangle = m$. Arguing as for (a), one sees that

$$\Theta(\pi_0) = \Theta(\gamma+\nu) + \Theta(\gamma+\nu-m\alpha).$$

If $\pi(\gamma - n\alpha + \mu) = 0$, then $\Theta(\gamma+\nu) + \Theta(\gamma+\nu-m\alpha) = 0$, and we are done. Otherwise we have

$$\begin{aligned}\mathrm{Hom}\,(\pi(\gamma+\nu), \pi_0) &= \mathrm{Hom}\,(\pi(\gamma+\nu), \varphi^{\gamma-n\alpha+\mu}_{\gamma+\nu}\psi^{\gamma+\nu}_{\gamma-n\alpha+\mu}\,\pi(\gamma+\nu))\\ &= \mathrm{Hom}\,(\psi^{\gamma+\nu}_{\gamma-n\alpha+\mu}\,\pi(\gamma+\nu), \psi^{\gamma+\nu}_{\gamma-n\alpha+\mu}\,\pi(\gamma+\nu))\\ &= \mathrm{Hom}\,(\pi(\gamma-n\alpha+\mu), \pi(\gamma-n\alpha+\mu)) = \mathbf{C}\end{aligned}$$

since ψ is left adjoint to φ ([21], Lemma 4.1). So $\pi(\gamma+\nu)$ is a composition factor of π_0; so $\Theta(\gamma+\nu-m\alpha) = \Theta(\pi_0) - \Theta(\gamma+\nu)$ is the character of a representation. Q.E.D.

Definition 5.21. In the setting of 5.20 (b), $\Theta(\gamma)$ is called *α-singular* or *α-nonsingular* according as $\Theta(\gamma - n\alpha) = -\Theta(\gamma)$ or not.

By the proof of Theorem 5.20, $\Theta(\gamma)$ is α-singular iff its coherent continuation to the α wall of the Weyl chamber is zero. Suppose $\Theta(\gamma)$ is α-nonsingular, and suppose $\gamma+\mu$ lies on the α wall. Clearly $S_\mu\cdot\Theta(\gamma-n\alpha) = \Theta(\gamma - n\alpha + s_\alpha\mu) = \Theta(\gamma+\mu)$, which is irreducible. But S_μ takes each irreducible constituent of $\Theta(\gamma - n\alpha)$ to an irreducible character or zero; so we can write

$$\Theta(\gamma - n\alpha) = \Theta_0(\gamma - n\alpha) + \sum_{i=1}^{r} \Theta_i(\gamma - n\alpha);$$

here Θ_i is an irreducible character, Θ_i is α-singular for $i \geq 1$, and $S_\mu\cdot\Theta_0(\gamma - n\alpha) = S_\gamma\cdot\Theta(\gamma)$ whenever $\gamma+\mu$ lies on the α wall. Corollary 6.17 says that $\Theta_0(\gamma - n\alpha) = \Theta(\gamma)$, a fact which has many consequences in representation theory. Even Theorem 5.19 can be useful; however, we conclude this section with a simple application of it.

PROPOSITION 5.22. (Schmid). *In the setting of Proposition* 5.14, *suppose* $\lambda \in \hat{T}'$ *is dominant for* Ψ'. *Then* $\pi(\lambda)$ *is a composition factor of both* $\pi(\gamma_+^\beta)$ *and* $\pi(\gamma_-^\beta)$ *(if* $s_\beta \in W(G/T)$*) or of* $\pi(\gamma^\beta)$ *(if* $s_\beta \notin W(G/T)$*).*

Proof. Since G is linear, $2\langle\lambda, \beta\rangle/\langle\beta, \beta\rangle = n$ is an integer. Define $s_\beta\lambda = \lambda - n\beta$. For definiteness we assume $s_\beta \notin W(G/T)$; the other case follows by a fairly easy argument. By Proposition 5.14,

$$\begin{aligned}\Theta(\gamma^\beta) - \Theta(\lambda) &= \Theta(s_\beta\Psi', \lambda)\\ &= S_{-n\beta}[\Theta(s_\beta\Psi', s_\beta\lambda)]\\ &= S_{-n\beta}[\Theta(s_\beta\lambda)].\end{aligned}$$

By 5.20(b), the only irreducible character which can occur on the right with negative multiplicity is $\Theta(s_\beta\lambda)$. Since $\Theta(s_\beta\lambda) \neq \Theta(\lambda)$, $\Theta(\lambda)$ occurs with non-negative multiplicity in $\Theta(\gamma^\beta) - \Theta(\lambda)$; so $\Theta(\lambda)$ occurs with positive multiplicity in $\Theta(\gamma^\beta)$. Q.E.D.

Schmid actually computed the composition series of $\pi(\gamma^\beta)$. His results follow from Theorem 4.23, applied to the parabolic $\mathfrak{b}$ defined by $\bar{\lambda}^\beta = \bar{\lambda}|_{\mathfrak{t}^\beta}$. That theorem reduces us to the case $\mathfrak{g}_0 \cong \mathfrak{sl}(2, \mathbf{R})$, where the composition series of principal series are well known ([20], 457–458). The conclusion is that if $s_\beta \notin W(G/T)$, $\pi(\gamma^\beta)$ has exactly three composition factors, namely $\bar{\pi}(\gamma^\beta)$, $\pi(\lambda)$, and $\pi(s_\beta\lambda)$. If $s_\beta \in W(G/T)$, then $\pi(\gamma_+^\beta)$ has two composition factors, namely $\bar{\pi}(\gamma_+^\beta)$ and $\pi(\lambda)$. These facts will be used in Section 7.

6. Conditions for reducibility

PROPOSITION 6.1. *Let* G *be a reductive linear group with abelian Cartan subgroups, and let* $H = T^+A$ *be a* θ*-invariant Cartan subgroup. Fix* $\gamma = (\lambda, \nu) \in \hat{H}'$ *such that the corresponding weight* $\gamma \in \mathfrak{h}^*$ *is nonsingular; write* $\Delta^+ = \Delta_\gamma^+$. *Then the generalized principal series representation* $\pi(\gamma)$ *is reducible only if*

(a) *there is a complex root* $\alpha \in \Delta_\gamma^+$ *such that* $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle$ *is an integer, and* $\theta\alpha \notin \Delta_\gamma^+$; *or*

(b) *there is a real root* $\alpha \in \Delta_\gamma^+$ *with the following property. Let* $\varphi_\alpha: \mathrm{SL}(2, \mathbf{R}) \to G$ *be the three-dimensional subgroup corresponding to* α, *with* φ_α *chosen so that*

$$\varphi_\alpha\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix} \in H.$$

Set

$$m_\alpha = \varphi_\alpha\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix} \in T^+.$$

Then $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle$ *is an integer, even or odd according as* $\lambda(m_\alpha)$ *is* ε_α *or* $-\varepsilon_\alpha$. *(Recall that* ε_α *was defined after the proof of Proposition* 5.14.)

Proof. This result is consistent with the reduction technique of Theorem 3.14, so we may assume that $\dim A=1$. We proceed by induction on the number of complex roots $\beta\in\Delta_\gamma^+$ with $\theta\beta\notin\Delta_\gamma^+$. Suppose first that there are no such roots. Then if $\beta\in\Delta_\gamma^+$ and β is not real, $\theta\beta\in\Delta_\gamma^+$. If there are no real roots, $\pi(\gamma)$ is irreducible by Lemma 5.16. So suppose that there is a real root α. If α is not simple, we write $\alpha=\varepsilon_1+\varepsilon_2$, with $\varepsilon_i\in\Delta_\gamma^+$. Then $-\alpha=\theta\alpha=\theta\varepsilon_1+\theta\varepsilon_2\in\Delta_\gamma^+$, a contradiction; so α is simple. Let $\mathfrak{b}=\mathfrak{l}+\mathfrak{n}$ be the corresponding parabolic (i.e., $L_0=H_0\cdot\varphi_\alpha(\mathrm{SL}(2,\mathbf{R}))$. Clearly $\mathfrak{b}$ is θ-invariant. Possibly shifting by $2\varrho(\mathfrak{n})$ in accordance with Theorem 5.2, we see that $\mathfrak{b}$ is the parabolic defined by λ. Suppose $\pi(\gamma)$ is reducible. By Proposition 4.19, every constituent of $\pi(\gamma)$ contains a K-type on the $\mathfrak{b}$-bottom layer. By Theorem 4.15 and the other results of Section 4, we deduce that the principal series representation $\pi_L(\gamma_L)$ is reducible. But the semisimple part of $\mathfrak{l}_0$ is $\mathfrak{sl}(2,\mathbf{R})$; so it follows from known results about $\mathrm{SL}(2,\mathbf{R})$ that condition 6.1 (b) holds. (Notice that this argument also establishes the converse of Proposition 6.1 in this case.)

Now suppose that Proposition 6.1 has been established whenever there are $n-1$ complex roots $\beta\in\Delta_{\gamma'}^+$ with $\theta\beta\notin\Delta_{\gamma'}^+$, and that there are n such roots in Δ_γ^+, with $n>0$. It follows that there is a simple root $\alpha\in\Delta_\gamma^+$ with $\theta\alpha\notin\Delta_\gamma^+$. If α is real, suppose $\beta\in\Delta_\gamma^+$ is complex and $\theta\beta\notin\Delta_\gamma^+$. Clearly $\theta\beta=s_\alpha\beta$; since α is simple, $\theta\beta\in\Delta_\gamma^+$, a contradiction. So α is a complex root. Suppose $\pi(\gamma)$ is reducible. If $2\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle$ is an integer, there is nothing to prove; so suppose it is not. Possibly shifting γ by $2\varrho(\Delta_\gamma^+)$ in accordance with Theorem 5.2, we may assume that $2\operatorname{Re}\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle\geqslant 1$. In this case we can find an integer $r>0$ such that $\gamma-r\alpha$ is dominant and nonsingular for $s_\alpha(\Delta_\gamma^+)$. By Corollary 5.12 and Theorem 5.20, the generalized principal series representation $\pi(\gamma-r\alpha)$ is reducible. Clearly the set of complex roots $\beta\in\Delta_{\gamma-r\alpha}^+$ such that $\theta\beta\notin\Delta_{\gamma-r\alpha}^+$ consists of the corresponding set for Δ_γ^+, with α removed; so it has order $n-1$. By induction, 6.1 (a) or 6.1 (b) holds with $\gamma-r\alpha$ replacing γ. It follows easily that 6.1 (a) or 6.1 (b) holds for γ. Q.E.D.

Our goal is to establish the sufficiency of the reducibility criterion of Proposition 6.1. We begin with a simple but very useful computation, and continue with a series of technical lemmas.

LEMMA 6.2. *Let $\mathfrak{h}=\mathfrak{t}^++\mathfrak{a}$ be θ-invariant Cartan subalgebra of $\mathfrak{g}$. Suppose $\gamma=(\lambda,\nu)\in\mathfrak{h}^*$, and $\alpha\in\Delta(\mathfrak{g},\mathfrak{h})$. Put $n=2\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle$, $\gamma_\alpha=\gamma-n\alpha=s_\alpha\gamma=(\lambda_\alpha,\nu_\alpha)$. Then*

$$\langle\lambda_\alpha,\lambda_\alpha\rangle-\langle\lambda,\lambda\rangle=n\langle\gamma+s_\alpha\gamma,\ -\theta\alpha\rangle.$$

The proof is left to the reader.

COROLLARY 6.3. *In the setting of Lemma 6.2, suppose α is complex and positive simple for Δ_γ^+, and that n is an integer. Then $\langle \lambda_\alpha, \lambda_\alpha\rangle - \langle \lambda, \lambda\rangle$ is positive iff $-\theta\alpha \in \Delta_\gamma^+$.*

Proof. Since $-\theta\alpha \neq \pm\alpha$, $-\theta\alpha \in \Delta_\gamma^+$ iff $-\theta\alpha \in \Delta_{s_\alpha\gamma}^+$. Q.E.D.

LEMMA 6.4. *Let $H = T^+A$ be a θ-invariant Cartan subgroup of G, Ψ a system of positive roots for $\mathfrak{t}^+$ in $\mathfrak{m}$, and $\gamma = (\lambda, \nu) \in \hat{H}'$ with λ dominant for Ψ. Suppose γ is nonsingular, and that $\alpha \in \Psi$ is a compact simple root which is also simple for Δ_γ^+. Suppose π_1 is an irreducible constituent of $\pi(\gamma)$, with character Θ_1. If $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle = n$, then*

$$S_{-n\alpha}(\Theta_1) = -\Theta_1.$$

Proof. Write $\Theta(\gamma) = \Theta_1 + \ldots + \Theta_r$, a sum of irreducible characters. By Corollary 5.12, $S_{-n\alpha}(\Theta(\gamma)) = \Theta(\Psi, \gamma - n\alpha) = \Theta(\Psi, (\lambda - n\alpha, \nu)) = \Theta(s_\alpha\Psi, (\lambda, \nu)) = -\Theta(\Psi, (\lambda, \nu)) = -\Theta(\gamma)$. Here we have used the fact that discrete series characters depend only on the $W(G/T)$ orbit of the parameter, and the first Hecht-Schmid character identity. Define the *rank* of a character to be the sum of the multiplicities of its irreducible constituents; we write rk (Θ) for the rank of Θ. Then rk $(\Theta(\gamma)) = r$, and rk $(S_{-n\alpha}(\Theta(\gamma))) = -r$. By Theorem 5.20, rk $(S_{-n\alpha}(\Theta_i)) \geq -1$; equality holds iff $S_{-n\alpha}(\Theta_i) = -\Theta_i$. So we must have $S_{-n\alpha}(\Theta_i) = -\Theta_i$ for all i. Q.E.D.

We will write $\overline{\Theta}(\gamma)$ for the character of $\bar{\pi}(\gamma)$.

LEMMA 6.5. *Let $H = T^+A$ be a θ-invariant Cartan subgroup of G, and $\gamma = (\lambda, \nu) \in \hat{H}'$. Suppose γ is nonsingular, and that $\alpha \in \Delta_\gamma^+$ is a complex simple root such that $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle = n$ and $\theta\alpha \in \Delta_\gamma^+$. Put $\gamma_\alpha = \gamma - n\alpha$. Then $S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma_\alpha) + \Theta_0$, with Θ_0 the character of a representation.*

Proof. For a fixed infinitesimal character, we proceed by downward induction on $|\lambda|$. Write $\Theta(\gamma) = \overline{\Theta}(\gamma) + \Theta_1 + \ldots + \Theta_r$, with Θ_i an irreducible character, and $\Theta(\gamma_\alpha) = \overline{\Theta}(\gamma_\alpha) + \Theta'$, with Θ' the character of a representation. By Corollary 5.12, $S_{-n\alpha}(\Theta(\gamma)) = \Theta(\gamma - n\alpha)$; so

$$S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma_\alpha) + \Theta' - \sum_{i=1}^{r} S_{-n\alpha}(\Theta_i). \tag{6.6}$$

By Theorem 5.20, it is enough to show that $\overline{\Theta}(\gamma_\alpha)$ cannot be a constituent of any $S_{-n\alpha}(\Theta_i)$. Suppose then that $\overline{\Theta}(\gamma_\alpha)$ is a constituent of $S_{-n\alpha}(\Theta_1)$, say; put $\Theta_1 = \overline{\Theta}(\gamma_1)$, with $\gamma_1 \in \hat{H}_1'$, and $\alpha_1 \in \Delta_{\gamma_1}^+$ the simple root corresponding to α. If α_1 is imaginary and compact, Lemma 6.4 implies that $S_{-n\alpha}(\Theta_1) = -\Theta_1$, a contradiction. Suppose α_1 is imaginary and noncompact; for definiteness, say $s_{\alpha_1} \in W(M_1/T_1^+)$ (the other case being easier). Construct $(\gamma_1^{\alpha_1})_\pm$ as in Proposition 5.14, and let Ψ be the positive root system in M_1 determined by λ_1. Then

$$\begin{aligned} S_{-n\alpha}(\Theta(\gamma_1)) &= \Theta(\Psi, (\gamma_1)_{\alpha_1}) \\ &= \Theta((\gamma_1^{\alpha_1})_+) + \Theta((\gamma_1^{\alpha_1})_-) - \Theta(\gamma_1) \end{aligned}$$

by Proposition 5.14 and Corollary 5.12. So if $\Theta(\gamma_1)=\bar\Theta(\gamma_1)+\Theta''$,

$$S_{-n\alpha}(\bar\Theta(\gamma_1))=\Theta((\gamma_{\mathrm{I}}^{\alpha_1})_+)+\Theta((\gamma_{\mathrm{I}}^{\alpha_1})_-)-\Theta(\gamma_1)-S_{-n\alpha}(\Theta'').$$

Since $\bar\Theta(\gamma_\alpha)$ occurs on the left with positive multiplicity, Theorem 5.20 implies that either $\bar\Theta(\gamma_\alpha)$ occurs in $\Theta((\gamma_{\mathrm{I}}^{\alpha_1})_\pm)$, or $\bar\Theta(\gamma_\alpha)$ occurs in $\Theta''\subseteq\Theta(\gamma_1)\subseteq\Theta((\gamma_{\alpha_1}^{1})_\pm)$ by Proposition 5.22. By Proposition 2.10, $\langle\lambda_{\mathrm{I}}^{\alpha_1},\lambda_{\mathrm{I}}^{\alpha_1}\rangle\leqslant\langle\lambda_\alpha,\lambda_\alpha\rangle$. But $\lambda_{\mathrm{I}}^{\alpha_1}$ is just the projection of λ_1 orthogonal to α_1; so

$$\langle\lambda_{\mathrm{I}}^{\alpha_1},\lambda_{\mathrm{I}}^{\alpha_1}\rangle=\langle\lambda_1,\lambda_1\rangle-\frac{n^2}{4}\langle\alpha_1,\alpha_1\rangle$$
$$=\langle\lambda_1,\lambda_1\rangle-\frac{n^2}{4}\langle\alpha,\alpha\rangle.$$

Combining this with Lemma 6.2, we get

$$\langle\lambda_\alpha,\lambda_\alpha\rangle-\langle\lambda_{\mathrm{I}}^{\alpha_1},\lambda_{\mathrm{I}}^{\alpha_1}\rangle=\langle\lambda_\alpha,\lambda_\alpha\rangle-\langle\lambda,\lambda\rangle+\langle\lambda,\lambda\rangle-\langle\lambda_1,\lambda_1\rangle+\langle\lambda_1,\lambda_1\rangle-\langle\lambda_{\mathrm{I}}^{\alpha_1},\lambda_{\mathrm{I}}^{\alpha_1}\rangle$$
$$=n\langle\gamma+s_\alpha\gamma,\ -\theta\alpha\rangle+\frac{n^2}{4}\langle\alpha,\alpha\rangle-(\langle\lambda_1,\lambda_1\rangle-\langle\lambda,\lambda\rangle).$$

We now shift γ by a dominant weight of a finite-dimensional representation in accordance with Theorem 5.2, so that after shifting, $\langle\gamma,\alpha\rangle$ is still small, but $\operatorname{Re}\langle\gamma,\varepsilon\rangle$ is large for every other simple root ε. Since $\theta\alpha$ must involve such other roots, the first term $n\langle\gamma+s_\alpha\gamma,\ -\theta\alpha\rangle$ above becomes large and negative, while the second remains small, and the third is always negative by Proposition 2.10. So we get $\langle\lambda_\alpha,\lambda_\alpha\rangle<\langle\lambda_{\mathrm{I}}^{\alpha_1},\lambda_{\mathrm{I}}^{\alpha_1}\rangle$, a contradiction. So α_1 is not imaginary, and therefore $S_{-n\alpha}(\Theta(\gamma_1))=\Theta(\gamma_1-n\alpha_1)$, a generalized principal series character. An argument similar to several already given shows that $\bar\Theta(\gamma_\alpha)$ is a constituent of either $\Theta(\gamma_1)$ or $\Theta(\gamma_1-n\alpha_1)$. Since $|\lambda_\alpha|<|\lambda|<|\lambda_1|$, the first is impossible. If $\theta\alpha_1$ is negative, then $|(\lambda_1)_{\alpha_1}|>|\lambda_1|$, and we would still have a contradiction. So $\theta\alpha_1$ is positive—in particular α_1 is complex—and $\bar\Theta(\gamma_\alpha)$ occurs in $\Theta((\gamma_1)_{\alpha_1})$. By inductive hypothesis, $S_{-n\alpha}(\bar\Theta(\gamma_1))=\bar\Theta((\gamma_1)_{\alpha_1})+\Theta''$, with Θ'' the character of a representation. Consider the occurrence of $\bar\Theta((\gamma_1)_{\alpha_1})$ in (6.6). We have $|\lambda|>|\lambda_\alpha|\geqslant|(\lambda_1)_{\alpha_1}|$, so $\bar\Theta((\gamma_1)_{\alpha_1})\neq\bar\Theta(\gamma)$. By Theorem 5.20, $\bar\Theta((\gamma_1)_{\alpha_1})$ occurs with nonnegative multiplicity on the right side of (6.6). So either $\Theta_i=\bar\Theta((\gamma_1)_{\alpha_1})$ for some i, or $\bar\Theta((\gamma_1)_{\alpha_1})$ is a constituent of Θ', or $\bar\Theta(\gamma_\alpha)=\bar\Theta((\gamma_1)_{\alpha_1})$. Since $|\lambda|>|\lambda_\alpha|\geqslant|(\lambda_1)_{\alpha_1}|$, Proposition 2.10 implies that the first two are impossible; so $\bar\Theta(\gamma_\alpha)=\bar\Theta((\gamma_1)_{\alpha_1})$. From the uniqueness statement in Theorem 2.9, one deduces easily that $\gamma=\gamma_1$, a contradiction. Q.E.D.

LEMMA 6.7. *Let $H=T^+A$ be a θ-invariant Cartan subgroup of G, with* $\dim A=1$. *Suppose $\gamma\in\hat H'$ is real and nonsingular, and $\beta\in\Delta_\gamma^+$ is a complex simple root such that*

$2\langle\beta, \gamma\rangle/\langle\beta, \beta\rangle$ *is* **not** *an integer. If r is a positive integer such that $\gamma - r\beta$ is nonsingular and dominant for $s_\beta(\Delta_\gamma^+)$, then $S_{-r\beta}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma - r\beta)$.*

Proof. After shifting γ by a weight of a finite-dimensional representation, we may assume that r and $2\langle\beta, \gamma\rangle/\langle\gamma, \gamma\rangle$ are small, but that $\langle\gamma, \varepsilon\rangle$ is large for any other simple root $\varepsilon \in \Delta_\gamma^+$. Since the expression of $\pm\theta\beta$ as a sum of simple roots must involve such other simple roots, $|\langle\gamma, \theta\beta\rangle|$ is large; so $|\langle\gamma, \beta - \theta\beta\rangle| = |\langle\nu, \beta - \theta\beta\rangle|$ is large. Let $P = MAN$ be the parabolic defined by $-\nu$. Choose a small weight μ of a finite-dimensional representation, such that $\mu - r\beta$ is dominant. Set

$$\pi_1 = P_{\gamma - r\beta}(\bar{\pi}(\gamma + \mu) \otimes F(-\mu + r\beta));$$

then, as we saw in the proof of Theorem 5.20, $\Theta(\pi_1) = S_{-r\beta}(\overline{\Theta}(\gamma))$. Now $\bar{\pi}(\gamma + \mu)$ is a subrepresentation of $\pi(P, \gamma + \mu)$; so π_1 is a subrepresentation of $P_{\gamma - r\beta}(\pi(P, \gamma + \mu) \otimes F(-\mu + r\beta))$. Using Lemma 5.8, one sees that this last representation is just $\pi(P, \gamma - r\beta)$. Since $|\langle\nu, \beta - \theta\beta\rangle|$ is large and r is small, P is the parabolic defined by $-\nu + r\beta$. It follows that $\pi_1 = \bar{\pi}(P, \gamma - r\beta)$. Q.E.D.

LEMMA 6.8. *In the setting of Lemma 6.5, suppose* $\dim A = 1$.

(a) *If $\overline{\Theta}(\gamma)$ is a constituent of $\Theta(\gamma_\alpha)$, then $S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma) + \overline{\Theta}(\gamma_\alpha) + \Theta_0$, with Θ_0 the character of a representation.*

(b) *If $\overline{\Theta}(\gamma)$ is not a constituent of $\overline{\Theta}(\gamma_\alpha)$, then $S_{-n\alpha}(\overline{\Theta}(\gamma_\alpha)) = \overline{\Theta}(\gamma) + \Theta_0$, with Θ_0 the character of a representation (so $\overline{\Theta}(\gamma_\alpha)$ is not α-singular).*

Proof. We begin by shifting γ as in Lemma 6.7, so that $\langle\gamma, \alpha\rangle$ is small, but $\langle\gamma, \varepsilon\rangle$ is large for $\alpha \neq \varepsilon \in \Delta_\gamma^+$; then let P be the parabolic defined by $-\nu$. Possibly passing to a cover of G, we choose a weight μ of a finite-dimensional representation, such that $2\langle\mu, \alpha\rangle/\langle\alpha, \alpha\rangle = n$, and $\langle\mu, \varepsilon\rangle = 0$ for $\varepsilon \neq \alpha$, ε a simple root of Δ_γ^+. Put $\gamma_0 = \gamma - \mu$; then ν_α, ν_0, and ν are all negative for P. By Lemma 6.5, $\bar{\pi}(\gamma)$ is *not* α-singular; so by Corollary 5.17, $\psi_{\gamma_0}^{\gamma}(\bar{\pi}(\gamma)) = \bar{\pi}(\gamma_0)$. By Corollary 5.9, $\psi_{\gamma_0}^{\gamma}(\pi(P, \gamma)) = \psi_{\gamma_0}^{\gamma}(\pi(P, \gamma_\alpha)) = \pi(P, \gamma_0)$. A short computation with Lemma 5. 8 shows that there is an exact sequence

$$0 \to \pi(P, \gamma) \to \varphi_{\gamma}^{\gamma_0}(\pi(P, \gamma_0)) \to \pi(P, \gamma_\alpha) \to 0.$$

By the remarks following Definition 5.21, $S_{-n\alpha}(\overline{\Theta}(\gamma))$ has a unique constituent Θ_1 such that $S_{-\mu}(\Theta_1) = \overline{\Theta}(\gamma_0)$. Let π_1 be an irreducible representation such that $\psi_{\gamma_0}^{\gamma}\pi_1 = \bar{\pi}(\gamma_0)$. Then $\operatorname{Hom}_{\mathfrak{g}}(\pi_1, \varphi_{\gamma}^{\gamma_0}(\pi(P, \gamma_0)) = \operatorname{Hom}_{\mathfrak{g}}(\psi_{\gamma_0}^{\gamma}\pi_1, \pi(P, \gamma_0))$ by [21], Lemma 4.1. But $\psi_{\gamma_0}^{\gamma}\pi_1 = \bar{\pi}(\gamma_0)$; so the right side has dimension one. It follows from the exact sequence above that π_1 is a subrepresentation of $\pi(P,\gamma)$ or of $\pi(P,\gamma_\alpha)$. Since ν and ν_α are negative with respect to P, $\pi_1 \simeq \bar{\pi}(\gamma)$ or $\bar{\pi}(\gamma_\alpha)$. Finally, $\psi^{\gamma}_{\gamma_0}\varphi^{\gamma_0}_{\gamma}(\overline{\Theta}(\gamma_0)) = 2\overline{\Theta}(\gamma_0)$ by [21], Lemma 3.1; so $\varphi^{\gamma_0}_{\gamma}(\overline{\Theta}(\gamma_0)) = \overline{\Theta}(\gamma) + S_{-n\alpha}(\overline{\Theta}(\gamma))$ has exactly two constituents Θ_1 and Θ_2 such that $S_{-\mu}(\Theta_i) = \overline{\Theta}(\gamma_0)$. Consider now 6.8 (b). By the exact sequence above, $\overline{\Theta}(\gamma)$ occurs only once in $\varphi^{\gamma_0}_{\gamma}(\bar{\pi}(P,\gamma_0))$; so $\overline{\Theta}(\gamma_\alpha)$ must satisfy $S_{-\mu}(\overline{\Theta}(\gamma_\alpha)) = \overline{\Theta}(\gamma_0) \neq 0$. In particular, $\overline{\Theta}(\gamma_\alpha)$ is α-nonsingular; so by Lemma 6.5, $S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma_\alpha) + \Theta_0$, with Θ_0 α-singular. Applying $S_{-n\alpha}$ to both sides, we obtain 6.8 (b). For 6.8 (a), suppose $S_{-\mu}(\overline{\Theta}(\gamma_\alpha))$ is nonzero. By Corollary 5.17, it must be $\overline{\Theta}(\gamma_0)$; so $S_{-\mu}(\Theta(\gamma_\alpha)) = S_{-\mu}(\overline{\Theta}(\gamma_\alpha)) + S_{-\mu}(\overline{\Theta}(\gamma)) + S_{-\mu}(\Theta_1)$ would contain $\overline{\Theta}(\gamma_0)$ twice. But $S_{-\mu}(\Theta(\gamma_\alpha)) = \Theta(\gamma_0)$, a contradiction. So the only character Θ satisfying $S_{-\mu}(\Theta) = \overline{\Theta}(\gamma_0)$ is $\overline{\Theta}(\gamma)$. So $\overline{\Theta}(\gamma)$ must occur in $S_{-n\alpha}(\overline{\Theta}(\gamma))$. Q.E.D.

THEOREM 6.9. *Let G be a reductive linear group with abelian Cartan subgroups, and let $H = T^+A$ be a θ-invariant Cartan subgroup. Fix $\gamma = (\lambda, \nu) \in \hat{H}'$ such that the corresponding weight $\gamma \in \mathfrak{h}^*$ is a nonsingular; write $\Delta^+ = \Delta^+_\gamma$. Suppose there is a complex simple root $\alpha \in \Delta^+$ with the following properties:*

(a) $2\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle = n$ *a positive integer,*

(b) $\theta\alpha \notin \Delta^+$.

Then $\bar{\pi}(\gamma - n\alpha)$ is a composition factor of $\pi(\gamma)$.

Proof. Write $\gamma_\alpha = \gamma - n\alpha$. We proceed by induction: We may assume that the result has been established for all groups of lower dimension than G. By step-by-step induction, we may assume $\dim A = 1$. Using a covering group argument like that given for Lemma 5.19 to write G as a direct product, one sees that we may assume G is simple. Define $r(\gamma) = |\{\alpha \in \Delta^+ \mid \alpha \text{ is complex, and } \theta\alpha \notin \Delta^+_\gamma\}|$. We may assume the result is known for representations $\pi_G(\gamma')$ with $r(\gamma') < r(\gamma)$. By induction, the result also holds for generalized principal series attached to Cartan subgroups H' of G with $\dim A' > 1$. Write $\alpha = (\alpha^+, \alpha^-)$ in accordance with $\mathfrak{h} = \mathfrak{t}^+ + \mathfrak{a}$. Then $\theta\alpha = (\alpha^+, -\alpha^-)$. Since $\lambda \in i(\mathfrak{t}_0^+)'$, $\langle\alpha^+, \lambda\rangle$ is real; it follows that ν is real, and that in fact $\langle\nu, \alpha^-\rangle > 0$. In particular, γ is real.

The strategy of the proof is now simple. We will give three more reduction techniques (Lemmas 6.10, 6.11, and 6.12). An examination of cases will determine when all of these techniques fail—certain cases involving rank one groups, and the split real forms of rank two. The rank one cases are dealt with as they arise, using essentially only the existence of the trivial representation. The split real forms of rank two are discussed in Section 7.

LEMMA 6.10. *Suppose there is a proper θ-invariant parabolic $\mathfrak{b} = \mathfrak{l} + \mathfrak{n} \subseteq \mathfrak{g}$, with $\mathfrak{l} \supseteq \mathfrak{h}$, such that $\Delta(\mathfrak{n}) \subseteq \Delta^+$. Then Theorem 6.9 holds.*

Proof. If $\alpha \in \Delta(\mathfrak{n})$, then also $\theta\alpha \in \Delta(\mathfrak{n})$; but this contradicts $\theta\alpha \notin \Delta^+$. So $\alpha \in \Delta(\mathfrak{l})$. Since $\mathfrak{n}$ is θ-invariant, $2\varrho(\mathfrak{n}) \in \mathfrak{t}^+$; by Theorem 5.2, we can replace λ by $\lambda + 2\varrho(\mathfrak{n})$ without changing anything, so we may assume that the parabolic $\mathfrak{b}^0 = \mathfrak{l}^0 + \mathfrak{n}^0$ defined by λ is contained in $\mathfrak{b}$. Let L be the normalizer of $\mathfrak{b}$ in G; the Lie algebra of L is $\mathfrak{l}_0$. Then λ_L can be defined as in Section 4, and $\gamma_L = (\lambda_L, \nu)$. Since $\dim L < \dim G$, the inductive hypothesis implies that $\bar{\pi}_L(\gamma_L - n\alpha) \subseteq \bar{\pi}_L(\gamma_L)$. The generalizations of Lemma 4.18 and Proposition 5.18 to disconnected G now imply Theorem 6.9. Q.E.D.

LEMMA 6.11. *Suppose there is a complex simple root $\beta \neq -\theta\alpha$ orthogonal to α, with $\theta\beta \notin \Delta^+$. Then Theorem 6.9 holds.*

Proof. If $2\langle\gamma, \beta\rangle / \langle\beta, \beta\rangle$ is not an integer, this follows easily from Lemma 6.7 and Theorem 5.20; details are left to the reader. So suppose $2\langle\gamma, \beta\rangle / \langle\beta, \beta\rangle = m$, a positive integer. Define $\gamma_\beta = \gamma - m\beta$, $\gamma_{\alpha\beta} = \gamma - m\beta - n\alpha$. Furthermore, $r(\gamma_\beta) = r(\gamma_\alpha) = r(\gamma) - 1$; so Theorem 6.1 holds for $\pi(\gamma_\beta)$ and $\pi(\gamma_\alpha)$; i.e., $\bar{\pi}(\gamma_{\alpha\beta})$ is a composition factor of $\pi(\gamma_\beta)$ and $\pi(\gamma_\alpha)$. By Lemma 6.8, $S_{-m\beta}(\overline{\Theta}(\gamma_{\alpha\beta})) = \overline{\Theta}(\gamma_\alpha) + \overline{\Theta}(\gamma_{\alpha\beta}) + \Theta_0$, with Θ_0 the character of a representation. Write

$$\Theta(\gamma_\beta) = \overline{\Theta}(\gamma_\beta) + \overline{\Theta}(\gamma_{\alpha\beta}) + \Theta_1 + \ldots + \Theta_r.$$

Then

$$S_{-n\alpha}(\overline{\Theta}(\gamma_\beta)) = \Theta(\gamma) - (\overline{\Theta}(\gamma_\alpha) + \Theta(\gamma_{\alpha\beta}) + \Theta_0) - \sum_{i=1}^{r} S_{-m\beta}(\Theta_i).$$

By Lemma 6.5, the left side is the character of a representation; so the β-singular character $\overline{\Theta}(\gamma_\alpha)$ must occur with nonnegative multiplicity. By Theorem 5.20, this is possible only if $\overline{\Theta}(\gamma_\alpha)$ occurs in $\Theta(\gamma)$, or if $\Theta_i = \overline{\Theta}(\gamma_\alpha)$ for some i. In the second case, $\Theta(\gamma_\beta) = \overline{\Theta}(\gamma_\alpha) + \Theta'$, with Θ' the character of a representation. Applying $S_{-n\alpha}$ to both sides, we get $\Theta(\gamma_{\alpha\beta}) = \overline{\Theta}(\gamma) + \Theta'' + S_{-n\alpha}(\Theta')$, with Θ'' the character of a representation. By Lemma 6.2, $|\lambda_{\alpha\beta}| > |\lambda_\beta| > |\lambda|$; so by Proposition 2.10, $\overline{\Theta}(\gamma)$ does not occur in $\Theta(\gamma_{\alpha\beta})$. By Theorem 5.20, we deduce that $\overline{\Theta}(\gamma)$ occurs in $\Theta' \subseteq \Theta(\gamma_\beta)$, which is impossible since $|\lambda_\beta| > |\lambda|$. This contradiction proves that $\overline{\Theta}(\gamma_\alpha) \subseteq \Theta(\gamma)$. Q.E.D.

The last reduction technique is by far the most subtle. Through it, the structure of the discrete series enters. It is quite complicated in its most general form; to convey the idea we give here only a simplified version, which suffices to prove Theorem 6.9 for the classical groups. Generalizations are discussed as they are needed below.

LEMMA 6.12. *Suppose there is a simple imaginary noncompact root $\beta \in \Delta_\gamma^+$, such that α^+ is not a multiple of β. Then Theorem 6.9 holds.*

Proof. We begin with a simple observation, which will also be the basis of generalizations of the lemma. Suppose Θ_0 is a character of a representation, $\overline{\Theta}(\gamma_\alpha)$ occurs in Θ_0, and $\overline{\Theta}(\gamma)$ does not occur in $S_{-n\alpha}(\Theta_0)$. Then Theorem 6.9 holds. For suppose not; write $\Theta_0 = \overline{\Theta}(\gamma_\alpha) + \sum_{i=1}^r \Theta_i$ as a sum of irreducible characters. By Lemma 6.5, $S_{-n\alpha}(\Theta_0) = \overline{\Theta}(\gamma) + \Theta' + \sum_{i=1}^r S_{-n\alpha}(\Theta_i)$, with Θ' the character of a representation. By Lemma 6.8 (b), $\overline{\Theta}(\gamma)$ is α-nonsingular. (It should be pointed out that the roles of γ and γ_α are reversed here with respect to the notation of Lemmas 6.5 and 6.8.) Theorem 5.20 and the remarks after it now imply that $\overline{\Theta}(\gamma)$ has nonnegative multiplicity in each $S_{-n\alpha}(\Theta_i)$. Hence $\overline{\Theta}(\gamma)$ occurs in $S_{-n\alpha}(\Theta_0)$, a contradiction. So our goal is simply to construct Θ_0; this will be the character $\Theta(\gamma^\beta)$ defined below.

We use obvious notation based on that introduced before Lemma 5.13. Thus $H^\beta = (T^+)^\beta A^\beta$ will be the Cartan subgroup obtained from H by a Cayley transform through β. Fix a character $\gamma^\beta \in (\hat{H}^\beta)'$ as described there— there are two choices $\gamma^\beta_\pm$, simply take one of them. By Proposition 5.22, $\pi(\gamma)$ is a subquotient of $\pi(\gamma^\beta)$. Similarly, we can define γ^β_α from γ_α, and obtain $\pi(\gamma_\alpha)$ as a subquotient of $\pi(\gamma^\beta_\alpha)$. Write $\bar{\varepsilon}$ for the root of $\mathfrak{h}^\beta$ in $\mathfrak{g}$ corresponding to $\varepsilon \in \Delta(\mathfrak{g}, \mathfrak{h})$ under the Cayley transform. We may choose $\gamma^\beta_\alpha = \gamma^\beta - n\alpha$. We claim that $\bar{\alpha}$ is a complex root; since α^+ is not a multiple of β, this is clear. Furthermore, $\theta\bar{\alpha} = s_{\bar{\beta}}(\overline{\theta\alpha}) = \overline{s_\beta(\theta\alpha)}$ is a negative root, since $\theta\alpha \neq -\beta$ is negative, and β is simple. Since $\dim A^\beta = 2$, Theorem 6.9 is available by inductive hypothesis; we deduce that $\bar{\pi}(\gamma^\beta_\alpha)$ occurs in $\pi(\gamma^\beta)$. We claim that $\bar{\pi}(\gamma)$ does not occur in $\pi(\gamma^\beta_\alpha)$. To see this, we may assume that $\langle\gamma, \alpha\rangle$ and $\langle\gamma, \beta\rangle$ are small, but that $\langle\gamma, \varepsilon\rangle$ is large for every other simple root ε. Since $\alpha^+ = \frac{1}{2}(\alpha + \theta\alpha) \neq c\beta$, it is easy to see that $-\theta\alpha$ must involve simple roots other than α and β; so $\langle\gamma, -\theta\alpha\rangle$ is large. By Lemma 6.2, $\langle\lambda_\alpha, \lambda_\alpha\rangle - \langle\lambda, \lambda\rangle$ is large. On the other hand, $\langle\lambda_\alpha, \lambda_\alpha\rangle - \langle\lambda^\beta_\alpha, \lambda^\beta_\alpha\rangle = \langle\lambda_\alpha, \beta\rangle^2/\langle\beta, \beta\rangle = \langle\gamma - n\alpha, \beta\rangle^2/\langle\beta, \beta\rangle$ is small (since $n = 2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle$); so $\langle\lambda^\beta_\alpha, \lambda^\gamma_\alpha\rangle - \langle\lambda, \lambda\rangle > 0$. By Proposition 2.10, $\bar{\pi}(\gamma)$ does not occur in $\pi(\gamma^\beta_\alpha)$.

Now $S_{-n\alpha}(\Theta(\gamma^\beta)) = \Theta(\gamma^\beta_\alpha))$; so to complete the argument sketched at the beginning of the proof, we need only show that $\bar{\pi}(\gamma_\alpha)$ occurs in $\pi(\gamma^\beta)$. Let $0 \neq X \in (\mathfrak{a}^\beta)^*$ be orthogonal to $\bar{\alpha}$. Theorem 6.9 implies that $\bar{\pi}(\gamma^\beta_\alpha + cX)$ occurs in $\pi(\gamma^\beta + cX)$ for all sufficiently small $c \in \mathbf{C}$ (i.e., whenever $\gamma^\beta + cX$ is strictly dominant for $\Delta^+_{\gamma^\beta}$). Let μ_α be a lowest K-type of $\pi(\gamma_\alpha)$, and let m be the multiplicity of μ_α in $\pi(\gamma^\beta_\alpha)$. Possibly after an appropriate shift of γ, we claim that μ_α has multiplicity m in $\bar{\pi}(\gamma^\beta_\alpha + cX)$ for an algebraically dense set of c. Assume this result for a moment. Then the $U(\mathfrak{g})^{K_0}$ module $\pi(\gamma^\beta_\alpha + cX)^{\mu_\alpha}$ is a subquotient of $\pi(\gamma^\beta + cX)^{\mu_\alpha}$ for an algebraically dense set of c. By a simple analytic continuation argument, every composition factor of $\pi(\gamma^\beta_\alpha)^{\mu_\alpha}$—in particular $\pi(\gamma_\alpha)^{\mu_\alpha}$—is a composition factor of $\pi(\gamma^\beta)^{\mu_\alpha}$. Thus $\bar{\pi}(\gamma_\alpha)$ occurs in $\pi(\gamma^\beta)$.

It remains to establish the multiplicity assertion. We consider those (small) c with the property that if $\gamma_\alpha^\beta + cX$ is integral with respect to $\bar{\varepsilon} \in \Delta(\mathfrak{g}, \mathfrak{h}^\beta)$, then $\bar{\varepsilon}^-$ is proportional to $\bar{\alpha}^-$. This is clearly an algebraically dense set. For such c, the only factor of the long intertwining operator for $\pi(\gamma_\alpha^\beta + cX)$ which can fail to be an isomorphism is the one corresponding to the restricted root $\bar{\alpha}^-$. The corresponding parabolic subgroup has Levi factor $\bar{M}^\alpha \bar{A}^\alpha = \bar{G}^\alpha$; $\mathfrak{a}^\alpha$ is spanned by X, and

$$\Delta(\bar{\mathfrak{g}}^\alpha, \mathfrak{h}^\beta) = \{\bar{\varepsilon} \,|\, \bar{\varepsilon}^- \text{ is proportional to } \bar{\alpha}^-\}.$$

List the composition factors of $\pi_{\bar{G}^\alpha}(\gamma_\alpha^\beta + cX)$ as $\bar{\pi}_{\bar{G}^\alpha}(\gamma_\alpha^\beta + cX), \bar{\pi}_{\bar{G}^\alpha}(\tilde{\gamma}_1 + cX), \ldots, \bar{\pi}_{\bar{G}^\alpha}(\tilde{\gamma}_r + cX)$. (Since $\bar{A}^\alpha$ is central in $\bar{G}^\alpha$, γ_i may be chosen to be independent of c.) By Corollary 3.15, every composition factor of $\pi_G(\gamma_\alpha^\beta + cx)$ other than the Langlands subquotient occurs in some $\pi_G(\tilde{\gamma}_i + cX)$. All of this data transforms coherently after shifting in accordance with Theorem 5.2. To prove the multiplicity assertion, it is therefore enough to show that, after shifting γ appropriately, the K-type μ_α does not occur in $\pi_G(\tilde{\gamma}_i + cX)$ for any i. Because of Lemma 8.8 of [19], it suffices to prove that $|\lambda_\alpha| < |\tilde{\lambda}_i|$ for all i (and appropriately shifted γ). Suppose this is *not* the case, i.e. that for all shifted γ there exists an i with $|\lambda_\alpha| \geqslant |\tilde{\lambda}_i|$.

Define $P_i(\gamma_\alpha^\beta) = \langle \tilde{\lambda}_i, \tilde{\lambda}_i \rangle - \langle \lambda_\alpha^\beta, \lambda_\alpha^\beta \rangle$; this can be regarded as a homogeneous quadratic polynomial on $(\mathfrak{h}^\beta)^*$ by coherent continuation. Since $\tilde{\gamma}_i$ and γ_α^β define the same infinitesimal character for the group $\bar{G}^\alpha$, $P_i(\gamma_\alpha^\beta)$ is a function of the various $\langle \varepsilon, \gamma_\alpha^\beta \rangle$, with ε a root of $\mathfrak{h}^\beta$ in $\bar{\mathfrak{g}}^\alpha$. Denote by B the projection of $(\mathfrak{h}^\beta)^*$ on the span of $\Delta(\bar{\mathfrak{g}}^\alpha, \mathfrak{h}^\beta)$; then $P_i(\gamma_\alpha^\beta) = P_i(B\gamma_\alpha^\beta)$. Now consider the set $C(c_1, c_2)$ of $\gamma_\alpha^\beta + \mu_{\gamma_\alpha^\beta}$, with μ a dominant weight of a finite-dimensional representation, and $\langle \gamma + \mu_\gamma, \alpha \rangle = c_1$, $\langle \gamma + \mu_\gamma, \beta \rangle = c_2$. If $(\gamma')_\alpha^\beta \in C(c_1, c_2)$, it is easy to compute that $\langle \lambda'_\alpha, \lambda'_\alpha \rangle - \langle (\lambda')_\alpha^\beta, (\lambda')_\alpha^\beta \rangle = f(c_1, c_2)$. If $\langle \tilde{\lambda}'_i, \tilde{\lambda}'_i \rangle \leqslant \langle \lambda'_\alpha, \lambda'_\alpha \rangle$, it follows that $0 \leqslant P_i((\gamma')_\alpha^\beta) \leqslant f(c_1, c_2)$. Define a *semilattice* in a Euclidean space to be the intersection of a cone with non-empty interior and a lattice. Let T be the real subspace (i.e., the real span of the roots) in $B(\mathfrak{h}^\beta)^*$, and

$$T_{\bar{\alpha}} = \{x \in T \,|\, \langle \bar{\alpha}, x \rangle = 0\}.$$

Because $\alpha \in T$ and $\beta \notin T$, it is easy to see that the projection $B(c_1, c_2)$ of $C(c_1, c_2)$ on T is translate of a semilattice in $T_{\bar{\alpha}}$. Our hypothesis says that for each $x \in B(c_1, c_2)$ there is an i such that $P_i(x) \leqslant f(c_1, c_2)$. An elementary argument (which is left to the reader) now implies that for some i, $P_i(x) = c(\langle \bar{\alpha}, x \rangle)^2$.

Suppose $\tilde{\gamma}_i$ is associated to the Cartan subalgebra $\mathfrak{h}_i$ of $\mathfrak{g}$. Choose an automorphism σ of $\bar{\mathfrak{g}}_\alpha$, inner for $(\bar{G}_\alpha)_{\mathbf{C}}$, such that σ maps $\mathfrak{h}_i$ to $\mathfrak{h}^\beta$ and $\tilde{\gamma}_i$ to γ_α^β. Let θ' be the involution of $\mathfrak{h}^\beta$ induced by $\theta|_{\mathfrak{h}_i}$ and σ. Then

$$\begin{aligned} c\langle \bar{\alpha}, \gamma_\alpha^\beta\rangle^2 = P_i(\gamma_\alpha^\beta) &= \langle \tilde{\lambda}_i, \tilde{\lambda}_i\rangle - \langle \lambda_\alpha^\beta, \lambda_\alpha^\beta\rangle \\ &= \tfrac{1}{4}\langle \tilde{\gamma}_i + \theta\tilde{\gamma}_i, \tilde{\gamma}_i + \theta\tilde{\gamma}_i\rangle - \tfrac{1}{4}\langle \gamma_\alpha^\beta + \theta\gamma_\alpha^\beta, \gamma_\alpha^\beta + \theta\gamma_\alpha^\beta\rangle \\ &= \tfrac{1}{2}\langle \tilde{\gamma}_i, \theta\tilde{\gamma}_i\rangle - \tfrac{1}{2}\langle \gamma_\alpha^\beta, \theta\gamma_\alpha^\beta\rangle \\ &= \tfrac{1}{2}[\langle \gamma_\alpha^\beta, \theta'\gamma_\alpha^\beta\rangle - \langle \gamma_\alpha^\beta, \theta\gamma_\alpha^\beta\rangle]. \end{aligned}$$

Now we make use of a simple geometric result. (We would like to thank Jorge Vargas for a helpful discussion.)

LEMMA 6.13. *Let V be a finite-dimensional real vector space with positive definite inner product $\langle\ ,\ \rangle$. Suppose θ and θ' are self-adjoint involutive automorphisms of V, and that*

$$\langle (\theta - \theta')v, v\rangle = c\langle \alpha, v\rangle^2$$

for some $0 \neq \alpha \in V$ and some constant c. Then θ and θ' commute; and either $\theta = \theta'$, and $c = 0$, or $\theta\alpha = \pm\alpha$, and $\theta' = s_\alpha\theta$. (Here s_α is the reflection about α.)

Proof. Recall that V is the orthogonal direct sum of the $+1$ and -1 eigenspaces of either θ or θ'. By polarization,

$$\langle (\theta - \theta')v, w\rangle = c\langle \alpha, v\rangle\langle \alpha, w\rangle.$$

If $c = 0$, obviously $\theta = \theta'$ and we are done. So suppose $c \neq 0$. It follows that $\theta - \theta'$ annihilates $\alpha^\perp$. If $v \in V^\theta$ and $w \in V^{\theta'}$, then $\langle \theta v, w\rangle = \langle \theta' v, w\rangle = \langle v, w\rangle$; so for such v and w,

$$0 = c\langle \alpha, v\rangle\langle \alpha, w\rangle.$$

It follows that either $V^\theta \subseteq \alpha^\perp$, or $V^{\theta'} \subseteq \alpha^\perp$; assume the first. The -1 eigenspace of θ is $(V^\theta)^\perp$, so $\theta\alpha = -\alpha$. In particular $\alpha^\perp$ is θ-invariant. Since $\theta - \theta'$ annihilates $\alpha^\perp$, $\theta|_{\alpha^\perp} = \theta'|_{\alpha^\perp}$. In particular $\alpha^\perp$ is θ' invariant, so $\theta'\alpha = \pm\alpha$. Since $c \neq 0$, we see that $\theta'\alpha = \alpha$. Q.E.D.

Applying this lemma to the present situation, we deduce that $\theta\bar{\alpha} = \pm\bar{\alpha}$, contradicting the fact that $\bar{\alpha}$ is complex. Q.E.D.

We now begin a case-by-case analysis, determining when these reduction techniques fail and analyzing the remaining cases. Recall that $\mathfrak{g}_0$ is assumed to be simple, that $P = MAN$ is cuspidal, and that $\dim A = 1$. If $G = ZG_0$, we may also assume that G is connected. Suppose that $\{\varepsilon_i\}$ are the simple roots of Δ_γ^+, and that $0 \neq \sum n_i\varepsilon_i \in \mathfrak{a}$. Then the parabolic subalgebra corresponding to the simple roots $\{\varepsilon_i \mid n_i \neq 0\}$ is θ-invariant. By Lemma 6.10, we may therefore assume that $n_i \neq 0$ for all i. If G is complex, these conditions force $G \cong$ SL (2, C) (or its adjoint group—we will often be somewhat vague about such distinctions). In this case Theorem 6.9 is well known. Alternatively, the argument given below for $G = \mathrm{Spin}\,(2n+1, 1)$ applies to SL (2, C) $\cong$ Spin (3, 1). So we may assume $\mathfrak{g}$ is simple. Suppose first that rk $G =$ rk K, or equivalently, that there is a real root δ. Then the expression of δ in terms of simple roots must involve all of them. If ε is simple, then $\theta\varepsilon < 0$ iff $\langle \varepsilon, \delta \rangle > 0$.

If $\mathfrak{g}$ is of type A_n, list the simple roots as $\varepsilon_1, \ldots, \varepsilon_n$, with ε_i adjacent to ε_{i+1}. The only root involving all the simple roots, and hence the only possibility for δ, is $\varepsilon_1 + \ldots + \varepsilon_n$. The only complex simple roots are ε_1 and ε_n; $\theta\varepsilon_1$ and $\theta\varepsilon_n$ are both negative, and $\varepsilon_1 \perp \varepsilon_n$ if $n \geqslant 3$. So Lemma 6.11 applies if $n \geqslant 3$. If $n = 1$, $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbf{R})$, and there are no complex roots. If $n = 2$, necessarily $\mathfrak{g}_0 \cong \mathfrak{su}(2, 1)$. Theorem 6.9 is known in this case (cf. [1]), but for completeness we sketch a proof. One can argue as for SO $(2n, 1)$ below; but for variety we give another argument, which also applies whenever the SO$(2n, 1)$ argument is used. Since G is linear, λ is the restriction to $\mathfrak{t}^+$ of some integral weight x of $\mathfrak{h}$; so $\gamma - x = c\delta = c(\varepsilon_1 + \varepsilon_2)$. But $\gamma - x$ is integral with respect to either ε_1 or ε_2, so we deduce that $c \in \mathbf{Z}$. Hence γ is integral. After a shift we may assume $\gamma = \varrho$. Since Δ_γ^+ is clearly invariant under $-\theta$, we must have $\varrho \in \mathfrak{a}^*$, i.e., $\lambda = 0$. Now G has no outer automorphisms which are inner for $G_{\mathbf{C}}$; so $G = ZG_0$, and thus $G = G_0$ under our current assumptions. So $M = M_0$, and $\pi_M(\lambda)$ is the trivial representation. In particular $\pi(\gamma)$ contains the trivial K-type. Now the generalized principal series representations with infinitesimal character ϱ are $\pi(\gamma), \pi(\gamma_{\varepsilon_1}), \pi(\gamma_{\varepsilon_2})$, and three discrete series. Of these, only $\pi(\gamma)$ contains the trivial K-type. Since the trivial representation of G has infinitesimal character ϱ, this forces $\bar{\pi}(\gamma)$ to be the trivial representation. Let μ be the lowest K-type of $\pi(\gamma_\alpha)$. By computation, μ has an M-invariant subspace, and hence, occurs in $\pi(\gamma)$; but of course it cannot occur in $\bar{\pi}(\gamma)$. A computation shows that μ does not occur in $\pi(\gamma_\beta)$ for $\beta \neq \alpha$, or in the discrete series representations with infinitesimal character ϱ. Therefore $\pi(\gamma)$ must contain $\bar{\pi}(\gamma_\alpha)$ as a subquotient.

Next suppose $\mathfrak{g}$ is of type B_n $(n \geqslant 2)$. List the simple roots as $\varepsilon_1, \ldots, \varepsilon_n$, with ε_i adjacent to ε_{i+1}, and ε_n short. Then $\delta = \varepsilon_1 + \ldots + \varepsilon_n$, or $\delta = \varepsilon_1 + \ldots + \varepsilon_{i-1} + 2\varepsilon_i + \ldots + 2\varepsilon_n$, with $2 \leqslant i \leqslant n$. Consider first the second possibility (so that δ is long). The complex simple roots ε_i with $\theta\varepsilon_i < 0$ are ε_1 and ε_i (if $i > 2$) or ε_2. If $i > 2$, $\varepsilon_1 \perp \varepsilon_i$; so by Lemma 6.11 we may assume $i = 2$. In that case δ is dominant, so every root $\beta \in \Delta^+(\mathfrak{t}^+, \mathfrak{m})$ which is simple for $\mathfrak{m}$ is also simple for $\mathfrak{g}$. Now the real root of the rank one form $\mathfrak{so}(2n, 1)$ of $\mathfrak{g}$ is short; so $\mathfrak{m}$ is noncompact. Hence we can find a simple noncompact imaginary root β. Now $2\alpha^+ = \varepsilon_2 + \theta\varepsilon_2 = \varepsilon_2 + s_\delta \varepsilon_2$ involves all the simple roots except perhaps ε_2. So if $n > 2$, α^+ cannot be proportional to β. By Lemma 6.12, we may assume $n = 2$. By the classification of real forms, $\mathfrak{g}_0 \cong \mathfrak{so}(3, 2)$, which is split. This case is treated in Section 7. We are left with the case $\delta = \varepsilon_1 + \ldots + \varepsilon_n$, which is dominant. By the argument just given, either $n=2$ and $\mathfrak{g}_0 \cong \mathfrak{so}(3,2)$, or $\mathfrak{g}_0 \cong \mathfrak{so}(2n,1)$. The first case is treated in Section 7. The second is known (cf. [1]), but again we sketch a proof for completeness. The only simple root α with $\theta\alpha<0$ is ε_1. The real dual of $\mathfrak{h}$ may be identified with $\mathbf{R}^n$; if $\{e_i\}$ is the standard basis of $\mathbf{R}^n$, then we can arrange $\varepsilon_i=e_i-e_{i+1}$ $(i<n)$, $\varepsilon_n=e_n$. Then $\delta=e_1$. Write

$$\gamma=(\nu,\lambda_2,\ldots,\lambda_n).$$

Now γ is integral with respect to the imaginary compact roots $\varepsilon_2,\ldots,\varepsilon_n$; and by hypothesis γ is integral with respect to ε_1. So γ is integral, and after a shift we may assume that $\gamma=\varrho=(n-\frac{1}{2}, n-\frac{3}{2},\ldots,\frac{1}{2})$. Since $M\cap G_0$ is connected, it is easy to deduce that $\bar{\pi}(\gamma)$ is a one-dimensional representation. By Lemma 6.8, Theorem 6.9 amounts to showing that $\bar{\pi}(\gamma)$ is α-singular. But the coherent continuation of a finite-dimensional representation to a wall is a finite-dimensional representation with singular infinitesimal character, and therefore it vanishes. So $\bar{\pi}(\gamma)$ is α-singular.

Suppose next that $\mathfrak{g}$ is of type D_n $(n\geqslant 4)$, with simple roots $\varepsilon_1,\ldots,\varepsilon_n$, with ε_{i-1} adjacent to ε_i for $i<n$, and ε_n adjacent to ε_{n-2}. Then necessarily $\delta=\varepsilon_1+\ldots+\varepsilon_{i-1}+2\varepsilon_i+\ldots+2\varepsilon_{n-2}+\varepsilon_{n-1}+\varepsilon_n$ $(2\leqslant i\leqslant n-1)$. The complex simple roots ε_i with $\theta\varepsilon_i<0$ are ε_1 and ε_i (if $2\leqslant i\leqslant n-2$), or ε_1, ε_n, and ε_{n-1} (if $i=n-1$), or ε_2 (if $i=2$). In the first two cases these sets are mutually orthogonal; so by Lemma 6.11 we may assume $i=2$. Then δ is dominant. Arguing as for type B_n, we deduce that G must have real rank one. But then by the classification of rank one real forms, $\mathfrak{g}_0\cong\mathfrak{so}(2n-1,1)$, contradicting $\operatorname{rk} G=\operatorname{rk} K$.

Next take $\mathfrak{g}$ of type C_n, $n\geqslant 3$, with simple roots $\varepsilon_1,\ldots,\varepsilon_n$, ε_i adjacent to ε_{i+1}, and ε_n long. The possibilities for δ are $\delta=2\varepsilon_1+2\varepsilon_2+\ldots+2\varepsilon_{n-1}+\varepsilon_n$, or $\delta=\varepsilon_1+\ldots+\varepsilon_{i-1}+2\varepsilon_i+\ldots+2\varepsilon_{n-1}+\varepsilon_n$, with $2\leqslant i\leqslant n$. Consider the first possibility. In this case δ is dominant; as usual we may assume by Lemma 6.12 that $\mathfrak{g}_0$ has real rank 1. But the real root of the rank one form of $\mathfrak{g}$ is short, and δ is long, a contradiction. In the second possibility, the complex roots ε_i with $\theta\varepsilon_i<0$ are ε_1 and ε_i (if $i>2$) or ε_2 (if $i=2$). In the first case $\varepsilon_1\perp\varepsilon_i$; so by Lemma 6.11 we may assume $\delta=\varepsilon_1+2\varepsilon_2+\ldots+2\varepsilon_{n-1}+\varepsilon_n$. Then δ is dominant; so as before Lemma 6.12 allows us to assume $\mathfrak{g}_0\cong\mathfrak{sp}(n-1,1)$. This real form has no outer automorphisms in $G_{\mathbf{C}}$, so $G=ZG_0$, and we may assume G is connected. The simple roots ε_1 and ε_3 through ε_n are imaginary, so γ is integral on those roots. Also γ is integral on ε_2 by hypothesis, so after a shift we may assume $\gamma=\varrho$. Just as in the case of type B_n, it follows that $\bar{\pi}(\gamma)$ is one-dimensional, and hence α-singular.

Before considering the exceptional groups, we dispose of the possibility that there is no real root. In this case H is a fundamental Cartan subalgebra of $\mathfrak{g}$, so that we must have $\operatorname{rk}\mathfrak{g}=\operatorname{rk} k+1$. There are very few such algebras: By the classification of real forms (cf. [20])

they are $\mathfrak{sl}(3, \mathbf{R})$, $\mathfrak{sl}(4, \mathbf{R}) \cong \mathfrak{so}(3, 3)$, $\mathfrak{su}^*(4) \cong \mathfrak{so}(5, 1)$, and $\mathfrak{so}(p, q)$, with p and q odd. SL (3, R) is dealt with in Section 7. Consider then $\mathfrak{so}(p, q)$; say $p+q=2n$. We may identify h with $\mathbf{R}^n$, and the simple roots with $\varepsilon_i = e_i - e_{i+1}$ $(i<n)$ and $\varepsilon_n = e_{n-1}+e_n$. The involution θ is just reflection about some e_i. Writing e_i in terms of the simple roots (recall that it must involve all of them), we see that $i=1$; so ε_2 through ε_n are imaginary, and we must have $\alpha=\varepsilon_1$. As usual it follows that γ is integral. If M is noncompact, then some ε_i $(i \geqslant 2)$ is noncompact, and we can apply Lemma 6.12. So we may assume M is compact, i.e., $\mathfrak{g}_0 = \mathfrak{so}(2n-1, 1)$. This real form has no outer automorphisms in $G_{\mathbf{C}}$, so we may assume $G=G_0$. After a shift, we have $\gamma=\varrho$. The argument now proceeds exactly as for SO$(2n, 1)$.

Finally, we turn to the exceptional groups. The split form of G_2 (which is the only noncompact, noncomplex form) is treated in Section 7. Recall that there is a real root δ, which involves all the simple roots in its expression. For each type of root system, one begins by listing the roots involving all simple roots. Given an explicit realization of the root system, this is not difficult. One simply computes the fundamental weights corresponding to the two or three "extremal" simple roots. The roots δ under consideration are those having a positive inner product with these fundamental weights. (Even for E_8 there are only 44 such roots.) It is then a simple matter to determine which simple roots α satisfy $\theta\alpha<0$; they are the simple roots having positive inner product with δ. If there are two such roots orthogonal to each other, Lemma 6.11 applies. (It is an amusing exercise to verify that for $\mathfrak{g}$ not of type A_2, two simple roots having positive inner product with a root involving all simple roots are necessarily orthogonal. We will not need this, however.) This much of the computation will be left to the reader. For each root system, we will simply present a list of the remaining possibilities for δ. Next we list the simple roots of $\mathfrak{t}^+$ in $\mathfrak{m}$; the roots of $\mathfrak{m}$ are just those orthogonal to δ, so this is a straightforward computation. If $\mathfrak{m}$ is compact, then G has real rank one; so $\mathfrak{g}$ is of type F_4, and δ is short. This case will be treated last. Otherwise, there is a noncompact root β, simple for $\Delta^+(\mathfrak{t}^+, \mathfrak{m})$. If β is actually simple in Δ^+_γ, we apply Lemma 6.12. Otherwise we can write $\beta = \sum n_i \varepsilon_i$, with $\varepsilon_i \in \Delta^+_\gamma$ simple; say, $n_i \neq 0$. Now $\langle \beta, \delta \rangle = 0$; but if $\langle \varepsilon_i, \delta \rangle = 0$ for all i, then β is not simple in $\Delta(\mathfrak{t}^+, \mathfrak{m})$. So $\langle \varepsilon_i, \delta \rangle > 0$ for some i. It follows from a remark made above that $\varepsilon_i = \alpha$; or one can simply observe that in each case computed below, β involves α. If β involves only one other simple root, the proof of Lemma 6.12 goes through with almost no change. (Notice that if α^+ is proportional to β, then β must involve all the simple roots except perhaps α by the argument given for type B_n. This never happens, as follows from the computations below; we make no further mention of the point.) So serious problems arise only when β involves at least three simple roots; this will happen only for types E_7 and E_8. The main conclusion of our case-by-case computations is

Observation 6.14. Suppose β involves $n \geq 3$ simple roots $\varepsilon_1, \ldots, \varepsilon_n$. Then $n=3$ or 4, and the ε_i span a root system of type A_n. We may assume $2\langle\varepsilon_1, \delta\rangle/\langle\varepsilon_1, \varepsilon_1\rangle = -2\langle\varepsilon_n, \delta\rangle/\langle\varepsilon_n, \varepsilon_n\rangle = 1$, that $\varepsilon_1 = \alpha$, and that ε_i is adjacent to ε_{i+1}. In this case $\varepsilon_2 \ldots \varepsilon_{n-1}$ are orthogonal to δ and to $\beta = \varepsilon_1 + \ldots + \varepsilon_n$. Before verifying this observation, we show how to extend Lemma 6.12 to cover this case. Just as in that situation, we can use β to construct a Cartan subgroup H^β, and representations $\pi(\gamma^\beta)$ and $\pi(\gamma^\beta_\alpha)$. The simplicity of β first entered in the verification that $\bar{\pi}(\gamma)$ does not occur in $\pi(\gamma^\beta_\alpha)$. To see this, we now shift γ so that $\langle\gamma, \varepsilon_i\rangle$ is small for $i=1, \ldots, n$, but $\langle\gamma, \varepsilon\rangle$ is large for every other simple root ε. Since $\theta\alpha = s_\delta(\alpha) = \alpha - r\delta$, $-\theta\alpha$ must involve all the simple roots except perhaps α. Since $\mathfrak{g}$ has rank 7 or 8, and $n \leq 4$, it follows that $-\theta\alpha$ involves some simple root $\varepsilon \notin \{\varepsilon_i\}$. So $\langle -\theta\alpha, \gamma\rangle$ is large. The argument now proceeds as in Lemma 6.12. The next use of the simplicity of β was to verify the following fact, with notation as in Lemma 6.12: For some strictly dominant shifted γ, $|\lambda_\alpha| < |\tilde{\lambda}_i|$ for all i. Suppose not. We consider the set

$$C(c_1 \ldots c_{n-1}) \quad \text{of } \gamma^\beta_\alpha + \mu_{\gamma^\beta_\alpha},$$

with μ a dominant weight of a finite-dimensional representation, and $\langle\gamma + \mu_\gamma, \varepsilon_i\rangle = c_i$ for $i = 1, \ldots, n-1$. Now $\langle\lambda^\beta_\alpha, \lambda^\beta_\alpha\rangle - \langle\lambda_\alpha, \lambda_\alpha\rangle$ depends only on $\langle\beta, \gamma_\alpha\rangle = \langle s_\alpha\beta, \gamma\rangle$. Since $s_\alpha\beta = \varepsilon_1 + \ldots + \varepsilon_{n-1}$ by Observation 6.14, $\langle\lambda^\beta_\alpha, \lambda^\beta_\alpha\rangle - \langle\lambda_\alpha, \lambda_\alpha\rangle = f(c_1 \ldots c_{n-1})$. Just as in Lemma 6.12 it follows that $0 \leq P_i((\gamma')^\beta_\alpha) \leq f(c_1 \ldots c_{n-1})$ whenever $(\gamma')^\beta_\alpha \in C(c_1 \ldots c_{n-1})$. We want to describe the projection of $C(c_1 \ldots c_{n-1})$ on T. Recall that the roots of $\mathfrak{h}^\beta$ in $\bar{\mathfrak{g}}^\alpha$ are $\{\bar{\varepsilon} \mid \bar{\varepsilon}|_{\mathfrak{a}^\beta} = c\alpha|_{\mathfrak{a}^\beta}\}$. This set is just

$$\{\bar{\varepsilon} \mid \langle\bar{\varepsilon}, \bar{\delta}\rangle = c\langle\bar{\alpha}, \bar{\delta}\rangle, \langle\bar{\varepsilon}, \bar{\beta}\rangle = c\langle\bar{\alpha}, \bar{\beta}\rangle\}.$$

Since $\bar{\varepsilon}_2$ through $\bar{\varepsilon}_{n-1}$ are orthogonal to $\bar{\delta}$ and $\bar{\beta}$, they lie in $\Delta(\bar{\mathfrak{g}}^\alpha, \mathfrak{h}^\beta)$, and are in fact imaginary roots. Furthermore, $\bar{\alpha} = \bar{\varepsilon}_1$ obviously lies in $\Delta(\bar{\mathfrak{g}}^\alpha, \mathfrak{h}^\beta)$. Put $T_0 = \{x \in T \mid \langle\bar{\varepsilon}_i, x\rangle = 0, i = 1, \ldots, n-1\}$. Then $B(c_1 \ldots c_{n-1})$ is a translate of a semilattice in T_0. An elementary argument like that omitted in Lemma 6.12 now shows that for some i, $P_i(x)$ is a function only of the various $\langle\bar{\varepsilon}_j, x\rangle$ for $j = 1, \ldots, n-1$. On the other hand, this polynomial was rewritten as $\frac{1}{2}[\langle x, \theta' x\rangle - \langle x, \theta x\rangle]$. Let W denote the span of ε_1 through ε_{n-1}. Exactly as in the proof of Lemma 6.13, we deduce that $\theta' - \theta$ annihilates $W^\perp$. Since $\theta' - \theta$ is self-adjoint, $\theta' - \theta$ preserves W. For $2 \leq i \leq n-1$, $\theta\varepsilon_i = \varepsilon_i$; so we can write $\theta'\varepsilon_i = c_{i1}\varepsilon_1 + \sum_{j=2}^{n-1} c_{ij}\varepsilon_i$. We claim $c_{i1} = 0$ for all i. Suppose not. Then

$$\theta'\varepsilon_1 = \frac{1}{c_{i1}}\left((\theta')^2\varepsilon_i - \sum_{2 \leq j \leq n-1} c_{ij}\theta'\varepsilon_i\right) = \frac{1}{c_{i1}}\left(\varepsilon_i - \sum_{\substack{2 \leq j \leq n-1 \\ 1 \leq k \leq n-1}} c_{ij}c_{jk}\varepsilon_k\right) \in W.$$

Therefore, $\theta\varepsilon_1=\theta\alpha$ also lies in W; but we have seen already that $\theta\alpha$ cannot be expressed in terms of the ε_i. So $c_{i1}=0$. Let W_1 denote the span of $\varepsilon_2, \ldots, \varepsilon_{n-1}$; then we have shown that θ and θ' preserve W_1. Since θ is the identity on W_1, θ and θ' commute on W_1. Let θ'' denote the involution of V which is $+1$ on W_1 and θ' on $W_1^\perp$. Then

$$P''(x)=\tfrac{1}{2}[\langle x, \theta''x\rangle-\langle x, \theta x\rangle]$$

is a function of $\langle\varepsilon_i, x\rangle$; or if we choose $0\neq\delta_1\in W_1^\perp\cap W$, and let $\delta_2, \ldots, \delta_{n-1}$ be a basis of W_1, we can write

$$P''(x)=\sum_{1\leqslant i,j\leqslant n-1} c_{ij}\langle\delta_i, x\rangle\langle\delta_j, x\rangle.$$

Just as in the proof of Lemma 6.13, we consider

$$\begin{aligned}P''(x, y)&=\tfrac{1}{2}[\langle x, \theta''y\rangle-\langle x, \theta y\rangle]\\&=\sum c_{ij}\langle\delta_i, x\rangle\langle\delta_j, y\rangle.\end{aligned}$$

If $y\in W_1$, $\theta''y=\theta y=y$, so $P''(x, y)=0$. It follows immediately that $c_{ij}=0$ unless $(i, j)=(1, 1)$, i.e., that

$$P''(x)=c_{11}\langle\delta_1, x\rangle^2.$$

If $c_{11}\neq 0$, Lemma 6.13 implies that $\theta\delta_1=\pm\delta_1$, so that $W=\operatorname{span}\{\delta_i\}$ is in fact θ-invariant. Again this contradicts $\theta\alpha\notin W$; so we conclude that $P''(x)=0$, and hence that $\theta=\theta''$. So $\theta=\theta'$ on $W_1^\perp$. Let π_{W_1} denote projection on W_1. Since θ is the identity on W_1, we have

$$\begin{aligned}P(x)&=\tfrac{1}{2}[\langle x, \theta'x\rangle-\langle x, \theta x\rangle]\\&=\tfrac{1}{2}[\langle\pi_W x, \theta'\pi_W x\rangle-\langle\pi_W x, \pi_W x\rangle].\end{aligned}$$

But this is obviously nonpositive for all x, contradicting $P(\gamma_\alpha^\beta)>0$. So the desired shift of γ exists, completing the extension of Lemma 6.12.

We now verify Observation 6.14. Suppose first that $\mathfrak{g}$ is of type E_8. We can identify the real dual of $\mathfrak{h}$ with $\mathbf{R}^8$, which is given the standard basis $e_1, \ldots, e_8$. The roots are $\pm e_i\pm e_j$ $(i\neq j)$, and $\tfrac{1}{2}(\pm e_1\pm\ldots\pm e_8)$, with an even number of plus signs. As a system of simple roots, we can take $\varepsilon_1=-\tfrac{1}{2}\sum e_i$, $\varepsilon_2=e_7+e_8$, $\varepsilon_3=e_6-e_7$, $\varepsilon_4=e_5-e_6$, $\varepsilon_5=e_4-e_5$, $\varepsilon_6=e_3-e_4$, $\varepsilon_7=e_2-e_3$, and $\varepsilon_8=e_7-e_8$. Then ε_i is adjacent to ε_{i-1} for $i\leqslant 7$, and ε_8 is adjacent to ε_3. In accordance with earlier remarks, we now list the possible δ to which Lemma 6.11 does not apply, together with the simple roots of $\mathfrak{m}$. Verification of Observation 6.14 is left to the reader; in all cases it is obvious by inspection. (This choice of simple roots makes the fundamental weights for ε_1 and ε_7 quite simple, so the computation of possible δ is not difficult.)

δ	Simple Roots of $\mathfrak{m}$.
$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_8$
$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$\varepsilon_2, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_1+\varepsilon_2+\varepsilon_3, \varepsilon_3+\varepsilon_8$
$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$\varepsilon_1, \varepsilon_3, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_2+\varepsilon_3+\varepsilon_4$
$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_6, \varepsilon_7, \varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_8$
$-e_1+e_2$	$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_8$
$-e_1+e_3$	$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_8, \varepsilon_6+\varepsilon_7$
$-e_1+e_4$	$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_7, \varepsilon_8, \varepsilon_5+\varepsilon_6$
$-e_1+e_5$	$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_4+\varepsilon_5$
$-e_1+e_6$	$\varepsilon_1, \varepsilon_2, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_3+\varepsilon_4$
$-e_1+e_8$	$\varepsilon_1, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_2+\varepsilon_3+\varepsilon_8$.

The root systems of type E_7 and E_6 are entirely similar and less complicated; verification of 6.14 in these cases is left to the reader.

We are left with the possibility that $\mathfrak{g}$ is of type F_4. In this case the real dual of $\mathfrak{h}$ may be identified with $\mathbf{R}^4$, with roots $\pm e_i$, $\pm e_i \pm e_j$, and $\frac{1}{2}(\pm e_1 \pm \ldots \pm e_4)$. As simple roots we may take $\varepsilon_1 = \frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_2 - \frac{1}{2}e_4$, $\varepsilon_2 = e_4$, $\varepsilon_3 = e_3 - e_4$, and $\varepsilon_4 = e_2 - e_3$. Then ε_i is adjacent to ε_{i+1}. The possible δ to which Lemma 6.11 does not apply, together with the simple roots of $\mathfrak{m}$, are

δ	Simple Roots of $\mathfrak{m}$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$\varepsilon_1, \varepsilon_4, 2\varepsilon_2+\varepsilon_3$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$\varepsilon_3, \varepsilon_4, e_1+e_2$
$(1, 0, 0, 0)$	$\varepsilon_2, \varepsilon_3, \varepsilon_4$
$(1, 0, 1, 0)$	$\varepsilon_1, \varepsilon_2, \varepsilon_3+\varepsilon_4$
$(1, 0, 0, 1)$	$\varepsilon_1, \varepsilon_4, \varepsilon_2+\varepsilon_3$
$(1, 1, 0, 0)$	$\varepsilon_1, \varepsilon_2, \varepsilon_3$.

By the remarks already made, a slight modification of Lemma 6.12 always applies when $\mathfrak{m}$ is noncompact. So we may assume $\mathfrak{m}$ is compact, and therefore that $\mathfrak{g}_0$ is the rank one form of F_4, with $\mathfrak{k}_0 \cong \mathfrak{so}(9)$. Since $\mathfrak{k}_0$ has no outer automorphisms, neither does $\mathfrak{g}_0$; so we may assume G is connected. The real root of $\mathfrak{h}$ is known to be short, so we are in one of the first three cases listed above. Furthermore, M is connected, and hence T^+ is also; so we may identify pseudocharacters with their differentials. Since γ is integral with respect to the imaginary roots and the complex root α, it is easy to check that in each of the three cases γ must be integral. (We note that α is either ε_3, ε_2, or ε_1, respectively, according to which case is under consideration.) After a shift, we may assume $\gamma = \varrho$; one calculates easily that $\varrho=(11/2, 5/2, 3/2, 1/2)$. By Lemma 6.8, what must be shown is that $\bar{\pi}(\gamma)$ is α-singular in each case. We write γ^{I}, γ^{II}, and γ^{III} to distinguish the cases. It is easy to see that $\bar{\pi}(\gamma^{\mathrm{III}})$ is one-dimensional, so as usual $\bar{\pi}(\gamma^{\mathrm{III}})$ is ε_1-singular. Now $\gamma_{\varepsilon_1}^{\mathrm{III}}=s_{\varepsilon_1}(\gamma^{\mathrm{III}})$. If we change our identification of $\mathfrak{h}^*$ with $\mathbf{R}^4$ so that $s_{\varepsilon_1}(\gamma^{\mathrm{III}})$ is dominant, δ becomes $s_{\varepsilon_1}(1, 0, 0, 0)=(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Therefore, $\gamma_{\varepsilon_1}^{\mathrm{III}}$ is conjugate under G to γ^{II}, i.e., $\bar{\pi}(\gamma^{\mathrm{II}})$ occurs in $\pi(\gamma^{\mathrm{III}})$. Now in case II, $\alpha=\varepsilon_2$. But ε_2 is a compact imaginary root for γ^{III}, so by Lemma 6.4, every constituent of $\pi(\gamma^{\mathrm{III}})$—in particular $\bar{\pi}(\gamma^{\mathrm{II}})$—is ε_2-singular. Similarly, $\gamma_{\varepsilon_2}^{\mathrm{II}}$ is conjugate to γ^{I}, so $\bar{\pi}(\gamma^{\mathrm{I}})$ occurs in $\pi(\gamma^{\mathrm{II}})$ and is therefore ε_3-singular.

Except for the split groups of real rank two (to be treated in Section 7), this completes the proof of Theorem 6.9. Q.E.D.

THEOREM 6.15. *Let G be a reductive linear group with abelian Cartan subgroups, and let $H=T^+A$ be a θ-invariant Cartan subgroup. Fix $\gamma=(\lambda, \nu)\in\hat{H}'$ such that the corresponding weight $\gamma\in\mathfrak{h}^*$ is nonsingular; write $\Delta=\Delta_\gamma^+$. Then $\pi(\gamma)$ is reducible if and only if there is a root $\alpha\in\Delta_\gamma^+$ such that $\theta\alpha\notin\Delta_\gamma^+$, with $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle=n\in\mathbf{Z}$, and either*

(a) *α is complex; or*

(b) *α is real, and (with notation as in Proposition 6.1)*

$$(-1)^n=\varepsilon_\alpha\cdot\lambda(m_\alpha).$$

Proof. By Proposition 6.1, the condition is necessary for reducibility. So suppose that it holds. The condition is consistent with the reduction technique given by Theorem 3.14, so we may assume $\dim A=1$. Just as for Theorem 6.1, we proceed by induction on the number $r(\gamma)$ of complex roots $\beta\in\Delta_\gamma^+$ with $\theta\beta\notin\Delta_\gamma^+$. If $r(\gamma)=0$, then we saw in the first part of the proof of Theorem 6.1 that $\pi(\gamma)$ is reducible. (In this case its composition factors are $\bar{\pi}(\gamma)$ and the discrete series described by Proposition 5.22. This result, which follows from our argument, is due to Schmid.) So suppose $r(\gamma)=n>0$, and the result is known when $r(\gamma)\leqslant n-1$. Clearly, there must be a simple root α with $\theta\alpha\notin\Delta_\gamma^+$. If α is real, then $\Delta_\gamma^+-\{\alpha\}$ is θ-invariant, contradicting $r(\gamma)>0$. So α is complex. If $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle=n\in\mathbf{Z}$, then $\pi(\gamma)$ is reducible by Theorem 6.9. Otherwise, after shifting γ, we can choose a positive integer r so that $\gamma-r\alpha$ is strictly dominant for $s_\alpha(\Delta_\gamma^+)$. Then $r(\gamma-r\alpha)=n-1$. By Theorem 5.20, $\pi(\gamma)$ and $\pi(\gamma-r\alpha)$ have the same number of composition factors. But any root $\beta\in\Delta_\gamma^+$ satisfying the condition of the theorem does so for $\gamma-r\alpha$ as well; so $\pi(\gamma-r\alpha)$ is reducible by induction. So $\pi(\gamma)$ is reducible. Q.E.D.

It is clear from this argument that a more complete understanding of composition series is virtually equivalent to a more complete understanding of coherent continuation across walls. Therefore, we summarize our results on that subject.

THEOREM 6.16. *Let $\pi(\gamma)$ be a generalized principal series representation of G, with $\gamma \in \mathfrak{h}^*$ nonsingular. Suppose $\alpha \in \Delta_\gamma^+$ is simple, with $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle = n \in \mathbf{Z}$.*

(a) *If α is real and $\gamma(m_\alpha) = -(-1)^n \cdot \varepsilon_\alpha$ (with notation as in Proposition 6.1), then $S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma) + \Theta_0$.*

(b) *If α is real and $\gamma(m_\alpha) = (-1)^n \cdot \varepsilon_\alpha$, then $\overline{\Theta}(\gamma)$ is α-singular.*

(c) *If α is complex and $\theta\alpha \in \Delta_\gamma^+$, then $S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma) + \overline{\Theta}(\gamma_\alpha) + \Theta_0$.*

(d) *If α is complex and $\theta\alpha \notin \Delta_\gamma^+$, then $\overline{\Theta}_\gamma$ is α-singular.*

(e) *If α is compact imaginary, then $\overline{\Theta}(\gamma)$ is α-singular.*

(f) *If α is noncompact imaginary, and $s_\alpha \in W(M/T^+)$, then (with notation as in Proposition 5.14)*

$$S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma) + \overline{\Theta}(\gamma_+^\alpha) + \overline{\Theta}(\gamma_-^\alpha) + \Theta_0.$$

(g) *If α is noncompact imaginary, and $s_\alpha \notin W(M/T^+)$, then (with notation as in Proposition 5.14)*

$$S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma) + \overline{\Theta}(\gamma^\alpha) + \Theta_0.$$

In each case, Θ_0 denotes the character of some representation (possibly zero).

Proof. Assertions (c), (d), and (e) have already been proved (cf. Lemmas 6.4, 6.8 and Theorem 6.9). (It should be pointed out that (c) and (d) are essentially equivalent by Lemma 6.8.) Consider (g). Put $\gamma_\alpha = \gamma - n\alpha$. By Propositions 5.14 and 5.22,

$$S_{-n\alpha}(\Theta(\gamma)) = \Theta(\gamma^\alpha) - \Theta(\gamma_\alpha),$$

and the right side is a character of a representation, containing $\overline{\Theta}(\gamma)$ and $\overline{\Theta}(\gamma^\alpha)$. Write $\Theta(\gamma^\alpha) - \Theta(\gamma_\alpha) = \overline{\Theta}(\gamma) + \overline{\Theta}(\gamma^\alpha) + \Theta'$, and $\Theta(\gamma) = \overline{\Theta}(\gamma) + \Theta_1 + \ldots + \Theta_r$; here Θ_i is an irreducible character, and Θ' is a character of a representation. Then

$$S_{-n\alpha}(\overline{\Theta}(\gamma)) = \overline{\Theta}(\gamma) + \overline{\Theta}(\gamma_\alpha) + \Theta' = \sum_{i=1}^{r} S_{-n\alpha}(\Theta_i).$$

We claim first that $\overline{\Theta}(\gamma^\alpha)$ appears with positive multiplicity. Suppose not; then $\overline{\Theta}(\gamma^\alpha)$ must occur in some $S_{-n\alpha}(\Theta_i)$ with positive multiplicity. Say $\Theta_i = \Theta(\gamma')$; let $\alpha' \in \Delta_{\gamma'}^+$ correspond to α. Arguing as in the proof of Lemma 6.5, we see that α' is complex, $\theta\alpha'$ is positive, and $\overline{\Theta}(\gamma^\alpha)$ occurs in $\overline{\Theta}(\gamma'_{\alpha'})$. By Proposition 2.10, $|\lambda'_{\alpha'}| < |\lambda^\alpha|$. On the other hand, $\overline{\Theta}(\gamma'_{\alpha'})$ occurs in $S_{-n\alpha}(\overline{\Theta}(\gamma'))$ by Lemma 6.5. Since $|\lambda'_{\alpha'}| < |\lambda^\alpha| < |\lambda|$, $\overline{\Theta}(\gamma'_{\alpha'}) \neq \Theta_j$ for any j, and $\overline{\Theta}(\gamma'_{\alpha'})$ does not occur in Θ'. Theorem 5.20 now implies that $\overline{\Theta}(\gamma'_{\alpha'})$ has multiplicity $\leqslant -1$ in $S_{-n\alpha}(\overline{\Theta}(\gamma))$, which contradicts Theorem 5.20. This shows that $\overline{\Theta}(\gamma^\alpha)$ does in fact occur in $S_{-n\alpha}(\overline{\Theta}(\gamma))$; and (g) can be deduced just as Lemma 6.8 (a) is deduced from Lemma 6.5. Assertion (f) is proved in precisely the same way. In case (b), an easy argument shows that γ is of the form $(\tilde{\gamma})^{\alpha}_{\pm}$ for some $\tilde{\gamma}$ as in cases (f) or (g). By the remarks after Theorem 5.20, $S_{-n\alpha}(\overline{\Theta}(\tilde{\gamma}))$ has only one α-nonsingular constituent, which is of course $\overline{\Theta}(\gamma)$; so $\overline{\Theta}((\tilde{\gamma})^{\alpha}_{\pm}) = \overline{\Theta}(\gamma)$ is α-singular. This proves (b).

For (a), we claim that $S_{-n\alpha}(\Theta(\gamma)) = \Theta(\gamma)$, i.e., that $\gamma - n\alpha$ is conjugate to γ under $W(G/H)$. Let φ_α: $\mathrm{SL}(2, \mathbf{R}) \to G$ be the three-dimensional subgroup through the real root α. Define

$$\sigma_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then σ_α normalizes H, and $\bar{\sigma}_\alpha = s_\alpha \in W(G/H)$. We want to show that $s_\alpha \cdot \gamma = \gamma - n\alpha$. This is obvious on the Lie algebra level; the only problem is the value of γ on other connected components of H. Now

$$\gamma|_Z = s_\alpha \gamma|_Z = \gamma - n\alpha|_Z,$$

so we need to consider only H/ZH_0. Each component of this factor group has a representative $m \in T^+ \subseteq K$, with $m^2 = 1$. For such m, we must show that

$$\gamma(\sigma_\alpha m \sigma_\alpha^{-1}) = \gamma(m) \cdot \alpha(m)^{-n}. \tag{*}$$

Let $X_\alpha \in \mathfrak{g}_0$ be a root vector for α; then $\sigma_\alpha = \exp(cX_\alpha + dX_{-\alpha})$. Since $m^2 = 1$, $\alpha(m) = \pm 1$. If $\alpha(m) = 1$, then $\mathrm{Ad}(m) \cdot X_\alpha = X_\alpha$, so m and σ_α commute, and both sides of (*) are equal to $\gamma(m)$. If $\alpha(m) = -1$, $\mathrm{Ad}(m) \cdot X_\alpha = -X_\alpha$, so $m^{-1}\sigma_\alpha m = \sigma_\alpha^{-1}$. It follows that $\sigma_\alpha m \sigma_\alpha^{-1} = m\sigma_\alpha^{-2} = mm_\alpha$, so the left side of (*) is $\gamma(m)\gamma(m_\alpha)$. So we must show that $\gamma(m_\alpha) = (-1)^n$; by hypothesis this amounts to $\varepsilon_\alpha = -1$. Recall the definition of ε_α after Proposition 5.14; if H^α is the Cartan subgroup obtained from H by a Cayley transform through α, and $G^{A^\alpha} = M^\alpha A^\alpha$, we choose a certain positive root system Ψ_1 for $(\mathfrak{t}^+)^\alpha$ in $\mathfrak{m}^\alpha$; and set

$$n_\alpha = 2\langle \tilde{\alpha}, \varrho(\Psi_1) - 2\varrho(\Psi_1 \cap \Delta(\mathfrak{m}^\alpha \cap \mathfrak{k}))\rangle / \langle \alpha, \alpha \rangle,$$

$\varepsilon_\alpha = (-1)^{n_\alpha}$. Now clearly the element m defined above normalizes H^α; and $\bar{m} = s_{\tilde{\alpha}} \in W(G/H^\alpha)$. Now $\tilde{\alpha}$ is a noncompact simple root in Ψ_1. Any element of $W(G/H^\alpha)$ preserves $\Delta(\mathfrak{m}^\alpha \cap \mathfrak{k})$; so $s_{\tilde{\alpha}}$ preserves $\Delta(\mathfrak{m}^\alpha \cap \mathfrak{k}) \cap \Psi_1$. Thus $\langle \tilde{\alpha}, 2\varrho(\Psi_1 \cap \Delta(\mathfrak{m}^\alpha \cap \mathfrak{k}))\rangle = 0$; so $n_\alpha = 2\langle \tilde{\alpha}, \varrho(\Psi_1)\rangle / \langle \alpha, \alpha \rangle = 1$, and $\varepsilon_\alpha = -1$. This proves that $S_{-n\alpha}(\Theta(\gamma)) = \Theta(\gamma)$. To prove (a), we now apply the usual argument; we need only show that if $\overline{\Theta}(\gamma') \neq \overline{\Theta}(\gamma)$ occurs in $\Theta(\gamma)$, then $S_{-n\alpha}(\overline{\Theta}(\gamma'))$ does not contain $\overline{\Theta}(\gamma)$. Let $\alpha' \in \Delta^+_{\gamma'}$ correspond to α. Using arguments which have been given several times, one sees that this can only happen if α' is imaginary and noncompact. in this case we would have to have $\overline{\Theta}(\gamma)$ occurring in $\Theta((\gamma')^{\alpha}_{\pm})$; and, investigating the occurrence of

$\overline{\Theta}((\gamma')^{\alpha}_{\pm})$ in $S_{-n\alpha}(\overline{\Theta}(\gamma))$, we would find that $\overline{\Theta}((\gamma')^{\alpha}_{\pm})$ occurs in $\Theta(\gamma)$. By Proposition 2.10, this forces $\gamma=(\gamma')^{\alpha}_{\pm}$; but by the construction of $(\gamma')^{\alpha}_{\pm}$, this contradicts $\gamma(m_\alpha)=-(-1)^n\varepsilon_\alpha$.
Q.E.D.

COROLLARY 6.17. *Let Θ be an irreducible character with nonsingular infinitesimal character γ. Let $\alpha\in\Delta^+_\gamma$, and suppose $2\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle=n\in\mathbf{Z}$. Then either*

(a) $S_{-n\alpha}(\Theta)=-\Theta$, *or*

(b) $S_{-n\alpha}(\Theta)=\Theta+\Theta_0$, *with Θ_0 the character of a representation.*

With this corollary, it is a simple matter to discuss singular infinitesimal characters. Thus let $\pi(\Psi,\gamma_0)$ be a representation in the limits of the generalized principal series. Choose a positive root system Δ^+ for $\mathfrak{h}$ in $\mathfrak{g}$ so that γ_0 is dominant and $\Psi\subseteq\Delta^+$, and a dominant weight μ of a finite-dimensional representation. Suppose $\gamma_0+\mu=\gamma$ is strictly dominant; this can always be arranged by proper choice of μ. Then by Corollary 5.12, $\psi^{\gamma}_{\gamma_0}(\pi(\Psi,\gamma))=\pi(\Psi,\gamma_0)$. By Theorem 5.15, the functor $\psi^{\gamma}_{\gamma_0}$ maps irreducible representations to irreducible representations or zero. Once we know how to compute $\psi^{\gamma}_{\gamma_0}$, information about the composition series of $\pi(\Psi,\gamma)$ immediately provides information about the composition series of $\pi(\Psi,\gamma_0)$. The computation of $\psi^{\gamma}_{\gamma_0}$ is given by

THEOREM 6.18. *With notation as above, every irreducible representation with infinitesimal character γ_0 has a unique irreducible preimage under $\psi^{\gamma}_{\gamma_0}$. $\psi^{\gamma}_{\gamma_0}(\bar{\pi}(\Psi,\gamma))=0$ iff there is a simple root $\alpha\in\Delta^+_\gamma$ such that $\langle\alpha,\gamma_0\rangle=0$ and either*

(a) *α is compact imaginary,*

(b) *α is complex and $\theta\alpha\notin\Delta^+_\gamma$, or*

(c) *α is real and (with notation as in Proposition 6.1)*

$$\lambda(m_\alpha)=\varepsilon_\alpha(-1)^{2\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle}.$$

If $\psi^{\gamma}_{\gamma_0}(\bar{\pi}(\Psi,\gamma))\neq 0$, then it is the unique Langlands subquotient of $\pi(\Psi,\gamma_0)$.

Proof. Suppose γ satisfies (a), (b), or (c) with respect to some root α. Choose γ_1 so that $\gamma-\gamma_1$ and $\gamma_1-\gamma_0$ are dominant weights of finite-dimensional representations, and γ_1 is singular with respect to only α. By Theorem 6.16, $\psi^{\gamma}_{\gamma_1}(\bar{\pi}(\Psi,\gamma))=0$. Hence $\psi^{\gamma}_{\gamma_0}(\bar{\pi}(\Psi,\gamma))=\psi^{\gamma_1}_{\gamma_0}\psi^{\gamma}_{\gamma_1}(\bar{\pi}(\Psi,\gamma))=0$. (The composition law for Zuckerman's ψ-functor is an easy exercise.)

Conversely, suppose no such root α exists. Set $\Delta_0=\{\alpha\in\Delta\mid\langle\alpha,\gamma_0\rangle=0\}$, $\Delta_0^+=\Delta_0\cap\Delta^+$, $W_0=W(\Delta_0)\subseteq W(\mathfrak{g}/\mathfrak{h})$. For μ a weight of a finite-dimensional representation, define $\Theta(\gamma+\mu)=S_\mu(\Theta(\bar{\pi}(\Psi,\gamma)))$. If $w\in W_0$, $\gamma-w\gamma$ is a sum of roots, which is a weight of some tensor product of copies of the adjoint representation; accordingly we can write $\Theta(w\gamma)$ for $\Theta(\gamma+(w\gamma-\gamma))$. We claim that for every $w\in W_0$, $\Theta(w\gamma)=\Theta(\gamma)+\Theta_w$, and every irreducible constituent Θ_w^i of Θ_w is α-singular for some simple root $\alpha \in \Delta_0^+$. This is clear when $w=1$. Suppose then that it is true for some w, and that $\alpha \in \Delta_0^+$ is simple with $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle = n$. It is clear from the definitions that $S_{-n\alpha}(\Theta(w\gamma)) = \Theta(ws_\alpha\gamma)$; so by Corollary 6.17,

$$\Theta(ws_\alpha\gamma) = \Theta(\gamma) + S_{-n\alpha}(\Theta_w) + \Theta_0,$$

with Θ_0 α-singular. So $\Theta_{ws_\alpha} = \Theta_0 + S_{-n\alpha}(\Theta_w)$. If Θ' is a constituent of Θ_0, then Θ' is α-singular. If Θ' is a constituent of $S_{-n\alpha}(\Theta_w)$, then by Corollary 6.17 again, either Θ' is a constituent of Θ_w, or Θ' is α-singular. Since the simple reflections generate W_0, this proves the claim. Using Lemma 5.4, one finds that

$$\varphi_\gamma^{\gamma_0}(\psi_{\gamma_0}^{\gamma}\Theta(\gamma)) = \sum_{w \in W_0} \Theta(w\gamma);$$

by what we have just proved, this is $|W_0| \cdot \Theta(\gamma) + \Theta_0$, with Θ_0 a combination of characters of representations which are singular with respect to some simple root $\alpha \in \Delta_0^+$. Theorem 6.16 implies that $\Theta(\gamma)$ is not a constituent of Θ_0, so $\varphi_\gamma^{\gamma_0}(\psi_{\gamma_0}^{\gamma}(\Theta(\gamma))) \neq 0$, and in particular $\psi_{\gamma_0}^{\gamma}(\Theta(\gamma)) \neq 0$. This proves the vanishing criterion for $\psi_{\gamma_0}^{\gamma}(\bar{\pi}(\Psi, \gamma))$. For the unique preimage statement, Zuckerman has shown that every irreducible preimage of $\psi_{\gamma_0}^{\gamma}(\bar{\pi}(\Psi, \gamma))$ under $\psi_{\gamma_0}^{\gamma}$ is a constituent of $\varphi_\gamma^{\gamma_0}(\psi_{\gamma_0}^{\gamma}(\bar{\pi}(\Psi, \gamma)))$ ([21], Theorem 1.3). But by our computation of the character of this last representation, the only constituent satisfying $\psi_{\gamma_0}^{\gamma}(\pi) \neq 0$ is $\bar{\pi}(\Psi, \gamma)$ itself.

Finally, we must show that if $\psi_{\gamma_0}^{\gamma}(\bar{\pi}(\Psi, \gamma)) \neq 0$, then it is the unique Langlands subquotient of $\pi(\Psi, \gamma_0)$. By Corollary 5.17, it is a Langlands subquotient. By the proof of Theorem 5.15, we can choose a parabolic $P = MAN$ associated to $H = T^+A$ in such a way that the Langlands subquotients of $\pi(P, \Psi, \gamma)$ and $\pi(P, \Psi, \gamma_0)$ are precisely the irreducible subrepresentations. Let ϱ_0 be such a subrepresentation of $\pi(P, \Psi, \gamma_0)$, and choose an irreducible representation ϱ so that $\psi_{\gamma_0}^{\gamma}(\varrho) = \varrho_0$. Then by Lemma 4.1 of [21],

$$\begin{aligned} \mathbf{C} &\cong \mathrm{Hom}_{\mathfrak{g}}(\varrho_0, \pi(P, \Psi, \gamma_0)) = \mathrm{Hom}_{\mathfrak{g}}(\psi_{\gamma_0}^{\gamma}(\varrho), \psi_{\gamma_0}^{\gamma}(\pi(P, \Psi, \gamma)) \\ &\cong \mathrm{Hom}_{\mathfrak{g}}(\varphi_\gamma^{\gamma_0}\psi_{\gamma_0}^{\gamma}(\varrho), \pi(P, \Psi, \gamma)). \end{aligned}$$

Since $\pi(P, \Psi, \gamma)$ has $\bar{\pi}(\Psi, \gamma)$ as its unique subrepresentation, $\bar{\pi}(\Psi, \gamma)$ is a constituent of $\varphi_\gamma^{\gamma_0}\psi_{\gamma_0}^{\gamma}(\varrho)$; and of course $\psi_{\gamma_0}^{\gamma}(\bar{\pi}(\Psi, \gamma)) \neq 0$ by assumption. Applying the theory just developed to ϱ instead of $\bar{\pi}(\Psi, \gamma)$, we deduce that $\varrho = \bar{\pi}(\Psi, \gamma)$, and hence that $\varrho_0 = \psi_{\gamma_0}^{\gamma}(\bar{\pi}(\Psi, \gamma))$. Hence $\pi(P, \Psi, \gamma_0)$ has a unique Langlands subquotient. Q.E.D.

Thus, as promised, the computation of composition series is completely reduced to the case of nonsingular infinitesimal character. Our reducibility criterion does not extend so easily: A reducible representation frequently becomes irreducible after continuation to a wall. Nevertheless, an irreducible representation remains irreducible; so we have the following necessary condition for reducibility.

THEOREM 6.19. *Let G be a reductive linear group with abelian Cartan subgroups, and let $H=T^+A$ be a θ-stable Cartan subgroup. Fix a positive root system $\Psi\subseteq\Delta(\mathfrak{m}, \mathfrak{t}^+)$ (with MA the centralizer of A in G) and a dominant Ψ-pseudocharacter $\gamma=(\lambda, \nu)\in\hat{H}_\Psi$. Then the limit of generalized principal series $\pi(\Psi, \gamma)$ is reducible only if*

(a) *there is a complex integral root α such that $\langle\alpha, \gamma\rangle$ is positive and $\langle\theta\alpha, \gamma\rangle$ is negative;*

or

(b) *there is a real integral root α such that if $n=2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle$, then in the notation of Proposition* 6.1,

$$(-1)^n=\varepsilon_\alpha\cdot\lambda(m_\alpha).$$

(*Here $n=0$ is allowed.*)

(The conditions given are not sufficient for reducibility in general.)

Proof. Define a positive root system Δ^+ for $\mathfrak{h}$ in $\mathfrak{g}$ as follows. First, set

$$\Delta_0=\{\alpha\in\Delta(\mathfrak{g}, \mathfrak{h})\,|\,\langle\alpha, \gamma\rangle=\langle\theta\alpha, \gamma\rangle=0\}.$$

Choose a positive system Δ_0^+ containing $\Delta_0\cap\Psi$, so that if $\alpha\in\Delta_0^+$ and $\theta\alpha\notin\Delta_0^+$, then α is real; this is possible. Define Δ^+ to consist of those roots α of $\mathfrak{h}$ in $\mathfrak{g}$ such that either

(a) $\operatorname{Re}\langle\alpha, \gamma\rangle>0$; or
(b) $\operatorname{Re}\langle\alpha, \gamma\rangle=0$, and $\operatorname{Im}\langle\alpha, \gamma\rangle>0$; or
(c) $\langle\alpha, \gamma\rangle=0$, but $\theta\alpha$ satisfies (a) or (b); or
(d) $\alpha\in\Delta_0^+$.

Then $\Delta^+\supseteq\Psi$.

Now let μ be a regular dominant weight of a finite dimensional representation of G, and set $\gamma^1=\gamma+\mu_{\Delta^+}\in\hat{H}'$. By Corollary 5.12,

$$\pi(\Psi, \gamma)=\Psi_\gamma^{\gamma^1}(\pi(\gamma^1)).$$

Suppose now that $\pi(\Psi, \gamma)$ is reducible. By Theorem 5.15, $\pi(\gamma^1)$ is as well; so there is a root $\alpha\in\Delta^+_{\gamma^1}$ satisfying the conditions in Theorem 6.15. By the choice of μ, $\Delta^+_{\gamma^1}\supseteq\Delta^+$; so $\alpha\in\Delta^+$. Since γ and γ^1 differ by a weight of a finite dimensional representation, α is integral for γ; and if it satisfies Theorem 6.15 (b) for γ^1, then it satisfies Theorem 6.19 (b) for γ. So suppose α satisfies Theorem 6.15 (a) for γ^1; thus α is complex, and

$$\langle\gamma^1, \alpha\rangle>0,\ \langle\gamma^1, \theta\alpha\rangle<0.$$

In particular, $\alpha \in \Delta^+$ and $\theta\alpha \notin \Delta^+$. So α satisfies one of (a)–(d) above. We want to show that

$$\langle \gamma, \alpha \rangle > 0, \langle \gamma, \theta\alpha \rangle < 0.$$

Suppose (d) holds. Then, in particular, α and $\theta\alpha$ are orthogonal to γ, so they both lie in Δ_0. By definition of Δ^+,

$$\alpha \in \Delta_0^+, \theta\alpha \notin \Delta_0^+.$$

But by the choice of Δ_0^+, this forces α to be real, a contradiction. So suppose (c) holds. Then by definition of Δ^+, $\theta\alpha$ is positive; a contradiction. Since $\langle \alpha, \gamma^1 \rangle$ is real, $\langle \alpha, \gamma \rangle$ is also; so (b) is impossible. So α must satisfy (a), proving that $\langle \gamma, \alpha \rangle > 0$. Exactly the same argument shows that $\langle \gamma, \theta\alpha \rangle < 0$. Q.E.D.

The proof of Theorem 6.15 provides some explicitly computable composition factors of $\pi(\gamma)$. Theorem 6.18 shows how to translate this to singular infinitesimal characters; so we could formulate a (rather complicated) sufficient condition for reducibility in the singular case. This condition is unfortunately not necessary, as can be seen in the group Sp(3, 1); so it does not seem worthwhile to state it carefully.

The following conjectures are true in groups of real rank one, Sp(3, $\mathbf{R}$), SL(4, $\mathbf{R}$), and the complex groups of rank less than or equal to three.

Conjecture 6.20. If $\gamma \in \hat{H}'$, and $H = T^+A$ with $\dim A = 1$, then the irreducible composition factors of $\pi(\gamma)$ occur with multiplicity one.

Conjecture 6.21. Let Θ be an irreducible character with nonsingular infinitesimal character $\gamma \in \mathfrak{h}^*$, and suppose $\alpha \in \Delta_\gamma^+$ is simple. If $2\langle \alpha, \gamma \rangle / \langle \alpha, \alpha \rangle = n \in \mathbf{Z}$, then the irreducible constituents of $S_{-n\alpha}(\Theta)$ occur with multiplicity one.

The second conjecture is closely connected with applications of coherent continuation to computing extensions of Harish-Chandra modules, a problem which we hope to pursue in a later paper.

We believe that the techniques described in this paper are sufficient to construct an algorithm for computing composition series. The idea (which is illustrated in the proof of Theorem 6.9) is this: Using Theorem 2.9, one lists all the generalized principal series with a fixed infinitesimal character (which we may as well assume to be nonsingular). Then one writes down a list of composition factors for each generalized principal series, with the multiplicities as unknowns. Proposition 2.10 says immediately that many of these are zero, and our various reduction techniques show how to compute some of these unknowns (in terms of composition series for smaller groups), or at least show that some must be positive, or equal to others, and so forth. Given these multiplicities (as unknowns), one can express the characters of the Langlands quotients in terms of the characters of generalized principal series and the unknown multiplicities. Thus whenever Θ is an irreducible character, we get a formula for the various $S_{-n\alpha}(\Theta)$ as a combination of irreducible characters, with coefficients involving the original unknown multiplicities. Corollary 6.17 now gives a new family of conditions on the multiplicities, since it says that some constituents of $S_{-n\alpha}(\Theta)$ occur with nonnegative multiplicities. Roughly speaking, this should provide enough conditions to solve for all the unknown multiplicities. Actually, one has to do a little more thinking than this, mainly by using the ideas of Section 3 more carefully; but these ideas have been extremely effective in examples which have not previously been treated.

7. The split groups of rank two

In this section G denotes a connected linear split simple Lie group of rank two.

Let $H_0 = MA$ be a maximally split Cartan subgroup, $P_0 = MAN$ a parabolic associated to H_0, α_1 and α_2 the simple roots of $\Delta(\mathfrak{a}_0, \mathfrak{n}_0)$, $H_i = T_i A_i$ $(i=1, 2)$ Cartan subgroups so that $(\mathfrak{a}_i)_0 = \ker \alpha_i$, and $P_i = M_i A_i N_i$ a parabolic associated to H_i containing P_0. Choose a set Δ_i of positive roots in $\mathfrak{h}_i^*$ compatible with the choice of P_i. Then $A = H_i^{\beta_i}$, the Cayley transform of H_i for a simple imaginary root β_i.

For each α_i we can choose an injection φ_i: $\mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}_0$, so that

$$\varphi_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}_0$$

$$\varphi_i(-{}^tX) = \theta\varphi_i(X), \quad X \in \mathfrak{sl}(2, \mathbf{R}),$$

and

$$\varphi_i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

lies in the α_i root space of $\mathfrak{a}_0$ in $\mathfrak{g}_0$. Write

$$Z_i = \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}$$

$$m_{\alpha_i} = m_i = \exp(\pi \cdot Z_i)$$

Then $m_i^2 = 1$ and M is generated by m_1 and m_2. If $H_i = T_i A_i$ is connected, then $|W(M_i/T_i)| = 2$; otherwise H_i has 2 connected components, $|W(M_i/T_i)| = 1$, and $H_i \cong$

$\mathbf{T}_1 \times \mathbf{R}^+ \times \{m_j, 1\}$ $(j \neq i)$. Consider $\varrho(\mathfrak{n}_0)$, $\varrho(\Delta_i)$ as pseudocharacters of H_0 and H_i respectively, extended to be trivial on the $\mathbf{Z}_2$ factors given above.

If H_i is connected the representation $\pi(\varrho(\Delta_i))$ is independent of the choice of Δ_i. Otherwise write Δ_i^1, Δ_i^2 for the two choices of Δ_i and $\pi(\varrho(\Delta_i^1))$ and $\pi(\varrho(\Delta_i^2))$ for the corresponding non-equivalent representations.

By induction by stages and Proposition 5.22 $\pi(P_i, \varrho(\Delta_i))$ is a subrepresentation of $\pi(P_0, \varrho(\mathfrak{n}_i))$ if H_i is connected; and otherwise $\pi(P_i, \varrho(\Delta_i^1)) \oplus \pi(P_i, \varrho(\Delta_i^2))$ is a subrepresentation of $\pi(\varrho(\mathfrak{n}_0))$. Define $\delta_i \in \mathfrak{a}_0'$ so that $\langle \delta_i, \alpha_j \rangle = \delta_{ij}$. Passing to a suitable covering group we may assume that δ_i is the highest weight of a finite dimensional representation of G.

LEMMA 7.1. *Let π be an irreducible representation of G, and Θ its character. Assume there exists a parabolic P_i associated to a connected Cartan subgroup H_i, and that π is a composition factor of $\pi(\varrho(\mathfrak{n}_0))/\pi(P_i, \varrho(\Delta_i))$; or there exists a parabolic P_i associated to a disconnected Cartan subgroup H_i, and that π is a composition factor of $\pi(\varrho(\mathfrak{n}_0))/\pi(P_i, \varrho(\Delta_i^1)) \oplus \pi(P_i, \varrho(\Delta_i^2))$. Then $S_{-\delta_i}\Theta = 0$.*

Proof. Assume for definiteness that H_i is connected and $i = 1$; put $\Psi = \Delta_1 \cap \Delta(\mathfrak{m}_1)$. Then

$$\begin{aligned} S_{-\delta_1}(\Theta(\varrho(\Delta_1))) &= S_{-\delta_1}[\operatorname{Ind}_{P_1}^G \Theta_{M_1}(\Psi, \varrho(\Psi)) \otimes (\varrho|_{(\mathfrak{a}_1)_0}) \otimes 1] \\ &= \operatorname{Ind}_{P_1}^G (S_{-\delta_1} \Theta_{M_1}(\Psi, \varrho(\Psi))) \otimes (\delta_2|_{(\mathfrak{a}_1)_0}) \otimes 1 \\ &= \operatorname{Ind}_{P_1}^G \Theta_{M_1}(\Psi, 0) \otimes (\delta_2|_{(\mathfrak{a}_1)_0}) \otimes 1. \end{aligned}$$

If we recall that $s_{\beta_1} \in W(M_1/T_1)$, and apply formula (7b) of [12], we see that $\Theta_{M_1}(\Psi, 0)$ is the character of a principal series representation of M_1. Hence $S_{-\delta_1}(\Theta(\varrho(\Delta_1)))$ is the character of a principal series representation of G. Since $S_{-\delta_1}(\Theta(\varrho(\mathfrak{n}_0)))$ is the character of a principal series representation containing $S_{-\delta_1}(\Theta(\varrho(\Delta_1)))$, it follows that $S_{-\delta_1}(\Theta(\varrho(\mathfrak{n}_0))) - \Theta(\varrho(\Delta_1)) = 0$. By Theorem 5.15 (compare the proof of Lemma 6.4) each irreducible constituent of $\Theta(\varrho(\mathfrak{n}_0)) - \Theta(\varrho(\Delta_1))$ is α_1-singular. Q.E.D.

LEMMA 7.2. *$\bar{\pi}(P_i, \varrho(\Delta_i))$ satisfies the assumptions of Lemma 7.1 with respect to P_j, $j \neq i$.*

Proof. For definiteness assume again H_1 and H_2 connected, and $i = 1$. We must show that $\bar{\pi}(P_1, \varrho(\Delta_1))$ is not a constituent of $\pi(P_2, \varrho(\Delta_2))$. Write

$$\varrho(\Delta_1) = (\lambda_1, \nu_1)$$

$$\varrho(\Delta_2) = (\lambda_2, \nu_2)$$

If $\langle\lambda_1, \lambda_1\rangle \leqslant \langle\lambda_2, \lambda_2\rangle$, then Proposition 2.10 implies the lemma. Otherwise we may shift the parameter ϱ to γ, with $\langle\gamma, \alpha_1\rangle$ small and $\langle\gamma, \alpha_2\rangle$ large. If we use primes to denote the parameters of the shifted representations, then we will have

$$\langle\lambda_1', \lambda_1'\rangle = \frac{\langle\alpha_1, \gamma\rangle^2}{\langle\alpha_1, \alpha_1\rangle} < \frac{\langle\alpha_2, \gamma\rangle^2}{\langle\alpha_2, \alpha_2\rangle} = \langle\lambda_2', \lambda_2'\rangle.$$

Now Proposition 2.10 applied to the shifted representations gives the desired result. Q.E.D.

LEMMA 7.3. *Let $H = TA$ be a θ-invariant Cartan with* $\dim \mathfrak{a}_0 = 1$, $\gamma \in \hat{H}'$, α_1, α_2 *the simple roots of* Δ_γ^+. *Assume that α_1 is simple imaginary and $2\langle\alpha_2, \gamma\rangle/\langle\alpha_2, \alpha_2\rangle = n$ is a positive integer. If H is connected, $\bar{\pi}(\gamma)$ is α_2-singular. If H is not connected, at least one representation attached to a pseudocharacter γ_1 with $\bar{\gamma}_1 = \bar{\gamma} \in \mathfrak{h}^*$ is α_2-singular.*

Proof. After a shift we can assume that $\bar{\gamma} = \varrho(\Delta_j)$. If H is connected, Lemmas 7.2 and 7.1 imply Lemma 7.3. If H is not connected, there is a pseudocharacter γ_1 trivial on the $\mathbf{Z}_2$ factor, with $\bar{\gamma}_1 = \varrho(\Delta_j)$. In this case Lemmas 7.2 and 7.1 imply Lemma 7.3 for the corresponding representation. Q.E.D.

Now we begin a case by case analysis to prove Theorem 6.9. With notation as there, we need to show that $\bar{\pi}(\gamma - n\alpha)$ is a constituent of $\pi(\gamma)$, or that $\bar{\pi}(\gamma)$ is α-singular (by Lemma 6.8). If $G = \mathrm{SL}(3, \mathbf{R})$, the fundamental Cartan subgroup $H = TA$ is connected. If $\gamma \in \hat{H}'$ satisfies the conditions of Theorem 6.9, then Δ_γ^+ has a simple imaginary root and hence Lemma 7.3 implies Theorem 6.9.

If $G = \mathrm{Sp}(2, \mathbf{R})$ we write α_1, α_2 for the long simple root and the short simple root respectively, and H_1, H_2 for the corresponding θ–invariant Cartan subgroups. H_1 is disconnected, H_2 connected. Let $\gamma \in \hat{H}_2'$ satisfy the conditions of Theorem 6.9. Then Δ_γ^+ has an imaginary simple root, and hence Lemma 7.3 implies Theorem 6.9. Let $\gamma \in \hat{H}_1'$. Then Δ_γ^+ has a simple imaginary root. After a shift we may assume that the weight of γ is $\varrho(\Delta_\gamma^+)$. If γ is trivial on the $\mathbf{Z}_2$ factor Theorem 6.9 follows from Lemma 7.2.

Now assume γ is non-trivial on m_{α_2}. By Proposition 5.22 and induction by stages we can assume $\pi(\gamma)$ is a subrepresentation of $\pi(\varrho(\mathfrak{n}_0)(+, -))$. Here $\varrho(\mathfrak{n}_0)(+, -)$ is a pseudocharacter of the split Cartan with weight $\varrho(\mathfrak{n}_0)$ and

$$\varrho(\mathfrak{n}_0)(+, -)(m_{\alpha_1}) = 1$$

$$\varrho(\mathfrak{n}_0)(+, -)(m_{\alpha_2}) = -1$$

By Proposition 5.22 and induction by stages, one computes easily that $\bar{\pi}(\gamma - \alpha_2)$ is also a composition factor of $\pi(\varrho(\mathfrak{n}_0)(+, -))$.

Assume now that $\bar{\pi}(\gamma-\alpha_2)$ is not a composition factor of $\pi(\gamma)$. Then by Lemma 6.8, $S_{-\delta_2}\overline{\Theta}(\gamma)=S_{-\delta_2}\overline{\Theta}(\gamma-\alpha_2)\neq 0$, and thus the composition factor $S_{-\delta_2}\overline{\Theta}(\gamma-\alpha_2)$ has multiplicity two in $S_{-\delta_2}\Theta(\varrho(\mathfrak{n}_0)(+,-))=\Theta(\delta)$. But $\gamma-\delta_2$ is singular with respect to the imaginary root α_2, so $\pi(\gamma-\delta_2)$ has the same restriction to K as a constituent of some tempered principal series representation. In particular the lowest K-type of $\bar{\pi}(\gamma-\delta_2)$ is fine, and hence has multiplicity one in the representation $\pi(\delta)$. Thus $\bar{\pi}(\gamma-\delta_2)$ has at most multiplicity one in $\pi(\delta)$, and hence $S_{-\delta_2}\overline{\Theta}(\gamma)=0$.

Now assume G is of type G_2. We write α_1, α_2 for the long simple root and for the short simple root respectively. The corresponding θ-invariant Cartan subgroups are denoted by H_1, H_2. Both Cartan subgroups are connected.

If $\gamma\in\hat{H}_i'$ satisfies the conditions of Theorem 6.9, then either γ is integral with respect to all roots, or $\gamma\in\hat{H}_2'$ and γ is integral only with respect to the short roots.

Assume first γ is integral with respect to all roots. If there is a simple imaginary root in Δ_γ^+, Lemma 7.3 implies Theorem 6.9. For the remaining cases we only sketch a proof. Most of the details are left to the reader. After a shift we may assume that $\pi(\gamma)$ has infinitesimal character $\varrho(\mathfrak{n}_0)$. The Weyl groups for each Cartan have order 4, and the complex Weyl group has order 12. Thus there are 3 generalized principal series representations with infinitesimal character $\varrho(\mathfrak{n}_0)$ associated to each Cartan. Write a, b, c for those generalized principal series representations associated to H_1, ordered according to decreasing length of the $\mathfrak{a}$ parameter. Write d, e, f, for those generalized principal series representations associated to H_2, also ordered according to decreasing lengths of the $\mathfrak{a}$ parameter. Write capital letters for the corresponding Langlands subquotients. Write G, H, I for the discrete series representations with infinitesimal character ϱ; there is a short simple compact root in the root system associated to G, a long simple compact root for I, and no simple compact root for H.

By Proposition 5.22 and the remark after its proof,

$$f=F+H+G$$
$$c=C+H+I.$$

It follows by Proposition 5.14 that $S_{-\alpha_1}H=H+F$, etc. (All formulas here should be understood as character identities; the representations in question do not decompose as direct sums.)

The positive root systems associated to c and f contains no simple roots satisfying the conditions of Theorem 6.9. So the only remaining cases are b and e; since these are completely similar, we consider only b. In the set of positive roots associated to b the long simple root α_1 is complex, and $\theta\alpha_1$ is negative. It is easy to compute that $S_{-\alpha_1}b = c$; so we must show that b contains C as a composition factor. Suppose not. By Lemma 6.8,

$$S_{-\alpha_1}C = B + \Theta_0$$
$$S_{-\alpha_1}B = C + \Theta_1,$$

with Θ_0 and Θ_1 the characters of α_1-singular representations. We claim that $\Theta_0 = 0$; the following proof was originally found by G. Zuckerman. (Compare the proof of Lemma 6.8.) Let δ_1 be the fundamental weight with $2\langle\alpha_i, \delta_1\rangle / \langle\alpha_i, \alpha_i\rangle = \delta_{i1}$. Recalling the proof of Theorem 5.20, we must show that

$$\varphi_{\varrho}^{\varrho-\delta_1}\psi_{\varrho-\delta_1}^{\varrho}(C) \cong B \oplus C.$$

Write X for the left-hand-side. The argument of Theorem 5.20 produces maps

$$B \oplus C \hookrightarrow X, \quad X \twoheadrightarrow B \oplus C.$$

If K is the kernel of the second map, then (since B and C have multiplicity one in X) $X \cong K \oplus B \oplus C$. If $K \neq 0$, we have Hom $(K, X) \neq 0$. A formal argument like that given for Theorem 5.20 implies that $\psi_{\varrho-\delta_1}^{\varrho} X$ contains at least three copies of $\psi_{\varrho-\delta_1}^{\varrho} B \cong \psi_{\varrho-\delta_1}^{\varrho} C$. But Zuckerman has shown ([21], Lemma 3.1) that $\Theta(\psi_{\varrho-\delta_1}^{\varrho} X) = 2\Theta(\psi_{\varrho-\delta_1}^{\varrho} C)$. This contradiction proves that $K = 0$. In particular $S_{-\alpha_1}C = B$.

By the remarks above, $c = C + H + I$; and

$$S_{-\alpha_1}H = F + H$$
$$S_{-\alpha_1}I = -I.$$

Therefore

$$b = S_{-\alpha_1}c = B + F + H - I,$$

which is impossible since b is a representation. This contradiction proves that b must in fact contain C as a composition factor.

Now assume that $\gamma \in \hat{H}_2'$. and γ is integral only with respect to the short roots. Let β_1 and β_2 be the simple roots of the subsystem of short roots. After a shift we may assume that $2\langle\beta_i, \gamma\rangle / \langle\beta_i, \beta_i\rangle = i$. There are three inequivalent generalized principal series representations $\pi(\gamma_1)$, $\pi(\gamma_2)$, and $\pi(\gamma_3)$, $\gamma_i \hat{H}_2'$, with infinitesimal character γ; we assume they are ordered by decreasing length of the $\mathfrak{a}$ parameter. Only the positive system defined by γ_2 contains a root satisfying the hypotheses of Theorem 6.9; it is β_1, and

$$S_{-\beta_1}\pi(\gamma_2) = \pi(\gamma_3).$$

So we must show (by Lemma 6.8) that $\bar{\pi}(\gamma_2)$ is β_1-singular. This is established exactly as in the case of SL $(3, \mathbf{R})$, by showing that $\bar{\pi}(\gamma_2)$ occurs in the representation induced from a certain finite dimensional representation. Details are left to the reader. This completes the proof of Theorem 6.9.

References

[1] BOREL, A. & WALLACH, N. R., *Seminar on the cohomology of discrete subgroups of semisimple groups.* Mimeographed notes, Institute for Advanced Study, 1976.

[2] ENRIGHT, T. J. & WALLACH, N. R., The fundamental series of semisimple Lie algebras and semisimple Lie groups. Preprint.

[3] HARISH-CHANDRA, Harmonic analysis on real reductive groups I. *J. Functional Analysis*, 19 (1975), 104–204.

[4] —— Harmonic analysis on real reductive groups III. *Ann. of Math.*, 104 (1976), 117–201.

[5] —— Representations of semisimple Lie groups III. *Trans. Amer. Math. Soc.*, 76 (1954), 234–253.

[6] HECHT H. & SCHMID, W., A proof of Blattner's conjecture. *Invent. Math.*, 31 (1975), 129–154.

[7] HIRAI, T., On irreducible representations of the Lorentz group of n-th order. *Proc. Japan Acad.*, 38 (1962), 83–87.

[8] HUMPHREYS, J. E., *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag, New York, 1972.

[9] JANTZEN, J. C., Zur Charakterformel gewisser Darstellungen halbeinfacher Gruppen und Lie-Algebren. *Math. Z.*, 140 (1974), 127–149.

[10] KOSTANT, B., On the existence and irreducibility of certain series of representations. *Bull. Amer. Math. Soc.*, 75 (1969), 627–642.

[11] KNAPP, A. W. & STEIN, E. M., Singular integrals and the principal series IV. *Proc. Nat. Acad. Sci. U.S.A.*, 72 (1975), 2459–2461.

[12] KNAPP, A. W. & ZUCKERMAN, G., Classification of irreducible tempered representations of semisimple Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 73 (1976), 2178–2180.

[13] LANGLANDS, R. P., *On the classification of irreducible representations of real algebraic groups.* Mimeographed notes. Institute for Advanced Study, 1973.

[14] LEPOWSKY, J., Algebraic results on representations of semisimple Lie groups. *Trans. Amer. Math. Soc.*, 176 (1973), 1–44.

[15] SCHIFFMAN, G., Intégrales d'entrelacement et fonctions de Whittaker. *Bull. Soc. Math. France*, 99 (1971), 3–72.

[16] SCHMID, W., On the characters of the discrete series (the Hermitian symmetric case). *Invent. Math.*, 30 (1975), 47–144.

[17] —— Two character identities for semisimple Lie groups, in *Non-commutative Harmonic Analysis*. Lecture Notes in Mathematics 587, Springer-Verlag, Berlin, 1977.

[18] SPEH, B., *Some results on principal series for* GL(n, **R**). Ph.D. Dissertation, Massachusetts Institute of Technology, June, 1977.

[19] VOGAN, D., The algebraic structure of the representations of semisimple Lie groups. *Ann. of Math.*, 109 (1979), 1–60.

[20] WARNER, G., *Harmonic Analysis on Semisimple Lie Groups I.* Springer-Verlag, New York, 1972.

[21] ZUCKERMAN, G., Tensor products of finite and infinite dimensional representations of semisimple Lie groups. *Ann. of Math.*, 106 (1977), 295–308.

Received March 31, 1978