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Reduction and reconstruction for self-similar dynamical systems

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Abstract. We present a general method for analyzing and numerically solving partial differential equations with self-similar solutions. The method employs ideas from symmetry reduction in geometric mechanics, and involves separating the dynamics on the *shape space* (which determines the overall shape of the solution) from those on the *group space* (which determines the size and scale of the solution). The method is computationally tractable as well, allowing one to compute self-similar solutions by evolving a dynamical system to a steady state, in a scaled reference frame where the self similarity has been factored out. More generally, bifurcation techniques can be used to find self-similar solutions, and determine their behavior as parameters in the equations are varied.

The method is given for an arbitrary Lie group, providing equations for the dynamics on the reduced space, for reconstructing the full dynamics, and for determining the resulting scaling laws for self-similar solutions. We illustrate the technique with a numerical example, computing self-similar solutions of the Burgers equation.

1. Introduction

Self similarity occurs in a great variety of applications, in fields as varied as fluid mechanics [2], biology [16], and optics [14, 23, 21]. Symmetry reduction has long been recognized as a useful tool for finding and analyzing self-similar solutions of partial differential equations [15, 17]. The usual approach

involves assuming a particular form of the solution, and introducing a change of variables that simplifies the governing equations. For instance, given a partial differential equation for a quantity $u(x, y, t)$, one might choose new independent variables $\xi(x, y, t)$ and $\eta(x, y, t)$ (in an intelligent way, using the underlying symmetry), and assume a solution of the form $u(\xi, \eta)$. For instance, for a traveling wave, one might define $\xi = x - \alpha t$, and for a diffusing wave, one might define $\xi = x/t^\alpha$. The number of independent variables has thus been reduced from three to two.

The method of symmetry reduction discussed in this paper is quite different from this standard approach. We treat the governing equations explicitly as a dynamical system, and the goal is not to reduce the number of independent variables, but rather to express the equations in a scaled coordinate system in which the underlying symmetry has been factored out. The central feature of the method is that the scaling of the coordinate system is determined *dynamically*, not by any a priori assumption of the group invariance of the solution. As a result, the method may be used not only to find self-similar solutions, but also to describe how an arbitrary initial shape *evolves* into a self-similar solution. Furthermore, tools for finding fixed points and studying bifurcations of dynamical systems may be used to find self-similar solutions and study their "bifurcations." The method can be used (and has been used in [21]) to investigate self-similar solutions of both the first and the second kind [2], and is closely related to the renormalization scheme described in [7].

Symmetry considerations are especially important for model reduction of large finite- or infinite-dimensional systems, such as fluid flows. For instance, techniques such as Proper Orthogonal Decomposition are often used to educe coherent structures from data, and to obtain optimal basis functions on which to project the governing equations. In the presence of certain symmetries, such as translational invariance, the optimal basis functions are Fourier modes. These say nothing about coherent structures in the data, and furthermore, usually a large number of Fourier modes must be retained in order to capture the dynamics. When symmetry reduction is used, the optimal modes are no longer Fourier modes, so coherent structures may be extracted, and often reduced-order models of much lower dimension may be obtained [18].

Even if reduced-order models are not desired, proper treatment of symmetry is valuable in numerical methods. For instance, by dynamically rescaling an adaptive mesh, one can ensure that the discrete system retains the original symmetries of the continuous equations, and thus ensure that the

discrete system retains the same conservation laws as the original system [3]. The method given in [3] is in fact quite similar in philosophy to the present method, in that the time and spatial scales are rescaled dynamically, although the methods for choosing these scales are quite different.

History and context. The foundations of the method are standard tools for symmetry reduction in geometric mechanics [11, 12]. Motivated by low-dimensional models of systems with traveling waves, these ideas were recently applied to equivariant partial differential equations (PDEs) in [18]. That paper introduced a method for removing translational invariance by shifting the solution to line up with a pre-selected *template function*, as inspired by [8]. An important contribution was to generalize this template fitting method to other symmetry groups in a natural way. Factoring out the group invariance (through so-called *pinning conditions*) is also an important component of many numerical methods for systems with symmetry. For instance, a similar template-based phase condition is often used in the computation of periodic solutions of autonomous ordinary differential equations [4], and a related method was recently used in [19] for bifurcation analysis of systems where the governing equations are not explicitly known, but only a numerical timestepper is available.

These techniques were subsequently adapted to a class of self-similar problems in [1]. Though the method is similar in spirit to that presented in [18], it is not identical: the meaning of the template function in [1] is different, and the method requires an additional pinning condition (needed to pin down one of the scale factors in the self-similar anzatz—see [1], following equation (11) and §3 below), which does not readily generalize to arbitrary symmetry groups.

The goal of this paper is to unify these ideas, developing a systematic framework for studying and numerically solving equations with self-similar solutions. We describe the method of reduction and reconstruction in §2, illustrate the method with a simple PDE example in §3, discuss the resulting scaling laws and neutral directions in §4 and §5, and finally give two more involved numerical examples in §6, using the Kuramoto-Sivishinsky equation and the Burgers equation.

2. Reduction and reconstruction

Consider a dynamical system on a manifold M whose evolution equation is denoted

$$\dot{u} = X(u), \tag{1}$$

where $u(t)$ is a curve in M and X is a vector field on M. Let G be a Lie group which acts on M by $\Phi_g : M \to M$ and on TM by $\Psi_g : TM \to TM$ (not necessarily the tangent action; this is discussed below). Suppose that the vector field X is *equivariant* with respect to these actions: that is, for all $g \in G$,

$$X \circ \Phi_g = \Psi_g \circ X, \tag{2}$$

where we regard X as a map of M to TM.

In the usual context of equivariant dynamical systems, one usually assumes that the action on TM is just the tangent of the action on M; that is, one usually takes $\Psi_g = T\Phi_g$. In this case, the flow of X is also equivariant—that is, if $u(t) \in M$ satisfies (1), then $\Phi_g(u(t))$ also satisfies (1), for any fixed $g \in G$. This is the case treated in [18]. Here, we consider a slightly more general case, where the actions Φ_g and Ψ_g have some independence. In particular, self similar solutions will arise when these actions are related by

$$T\Phi_g = m(g)\Psi_g \tag{3}$$

where m is a homomorphism of Lie groups $(G, \cdot) \to (\mathbb{R}^+, \times)$; that is, m is multiplicative: $m(g_1 g_2) = m(g_1)m(g_2)$. Henceforth, we will assume the actions are related by (3), and we will denote both the actions of Φ and $T\Phi$ by concatenation:

$$g \cdot x := \Phi_g(x), \qquad x \in M \tag{4}$$

$$g \cdot v := T\Phi_g(v), \qquad v \in TM, \tag{5}$$

and equivariance of X implies

$$X(g \cdot x) = \frac{1}{m(g)} g \cdot X(x).$$

2.1. Separation of dynamics

The main idea of symmetry reduction is to separate the dynamics in the group direction from the dynamics in the remaining directions of phase space. Such a separation is always possible for certain equivariant dynamical systems. In particular, whenever the flow of a vector field on a manifold M is equivariant,

one gets well-defined dynamics on the quotient space M/G, defined as the set of equivalence classes: the equivalence relation is that elements of M related by the group action are identified. Throughout, we assume that the action on M is proper and free, which guarantees that the quotient space is a smooth manifold. In symmetry reduction, one writes explicitly the dynamics induced on this quotient space, and these dynamics are called the *reduced dynamics*.

Here, the situation is not as simple: because the actions on M and TM are different, the *flows* are not equivariant (e.g., if $u(t)$ is a solution, then $g\cdot u(t)$ is not a solution), and in general one does *not* have well-defined reduced dynamics in the usual sense. However, one can obtain a version of reduced dynamics if one rescales time as well. Two such methods are considered here: in the *method of slices*, the dynamics evolve on a subspace of M that is locally isomorphic to M/G; in the *method of connections*, we use the structure of a principal connection to obtain the *horizontal dynamics*, which are evolution equations for the horizontal lift of a path in M/G. We discuss the abstract reduction further in section 2.5. The relation of these methods to "pinning techniques" arising in the numerical analysis of periodic solutions is elaborated in [10].

Let $g(\tau)$ be a curve in G, $r(\tau)$ a curve in M, and consider a solution of (1) of the form

$$u(t) = g(\tau)\cdot r(\tau) \tag{6}$$

where τ is a function of t, as yet unspecified. Eventually, we will restrict r so that it lies in (or is tangent to) a subspace of M locally isomorphic to M/G, and its evolution will represent the reduced dynamics, but for now, we allow arbitrary curves in M. Differentiating (6) with respect to t gives

$$u_t = g\cdot(r_\tau + \xi_M(r))\frac{d\tau}{dt}, \tag{7}$$

where $\xi = g^{-1}\dot{g}$, which is a curve in the Lie algebra $\mathfrak{g}$ of G, and ξ_M denotes the infinitesimal generator of the action Φ in the direction of ξ (thus, for each $\xi\in\mathfrak{g}$, ξ_M is a vector field on M). See, e.g., Chapter 9 of [13] for a derivation of formulas such as (7).

Inserting the expressions (6) and (7) into equation (1) and using equivariance gives

$$(r_\tau + \xi_M(r))\frac{d\tau}{dt} = \frac{1}{m(g)}X(r). \tag{8}$$

If we now define $\tau(t)$ by

$$dt = m(g)d\tau, \tag{9}$$

then (8) becomes

$$r_\tau = X(r) - \xi_M(r), \tag{10}$$

which is independent of g (though of course it depends on $\xi = g^{-1}\dot{g}$).

Note that equation (10) is precisely equivalent to the original equation (1), and is now underconstrained, as we have not placed any conditions on the evolution of g and r. That is, g and r are not uniquely specified by (10). The methods of slices and connections correspond to two different choices for ξ, which will specify the dynamics of r and g uniquely.

2.2. Method of slices

Assume that M is an inner product space, with inner product denoted $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, and choose an element $r_0 \in M$, called the *template*. The *slice* S_{r_0} passing through r_0 is defined by

$$S_{r_0} = \{r \in M \mid \langle\!\langle r - r_0, \xi_M(r_0) \rangle\!\rangle = 0, \text{ for all } \xi \in \mathfrak{g}\}. \tag{11}$$

The geometric interpretation of the slice is as follows: let $G{\cdot}r_0 = \{g{\cdot}r_0 \mid g \in G\}$ denote the group orbit through r_0, and $\mathfrak{g} \cdot r_0 = \{\xi_M(r_0) \mid \xi \in \mathfrak{g}\}$ denote the tangent space to this group orbit at the point r_0. Then the slice S_{r_0} is the affine space orthogonal to $\mathfrak{g} \cdot r_0$, through the point r_0. If the action of G on M is proper and free, then the slice S_{r_0} is locally isomorphic to the quotient space M/G [5].

The physical interpretation of the slice reveals why r_0 is called the template: consider the set of functions r that are (locally) *aligned* with the template r_0. That is, if $g(s)$ is a curve in G with $g(0) = \mathrm{Id}$, r satisfies

$$\frac{d}{ds}\Big|_{s=0} \|r - g(s) \cdot r_0\|^2 = 0.$$

That is, the "distance" between r and $g \cdot r_0$ is a (local) minimum when $g = \mathrm{Id}$. Letting $\xi = \dot{g}(0) \in \mathfrak{g}$, this is equivalent to

$$-2 \langle\!\langle r - r_0, \xi_M(r_0) \rangle\!\rangle = 0.$$

Thus, the set of functions locally aligned with the template is precisely the set S_{r_0}.

Given an arbitrary function $u \in M$, one can find a scaled version $g \cdot u$ that lies in the slice by choosing $g \in G$ to minimize $\|g \cdot u - r_0\|$. This procedure was used in [18] to preprocess data for computing POD modes. The resulting POD modes form an optimal basis for the slice, and are useful in forming reduced-order models.

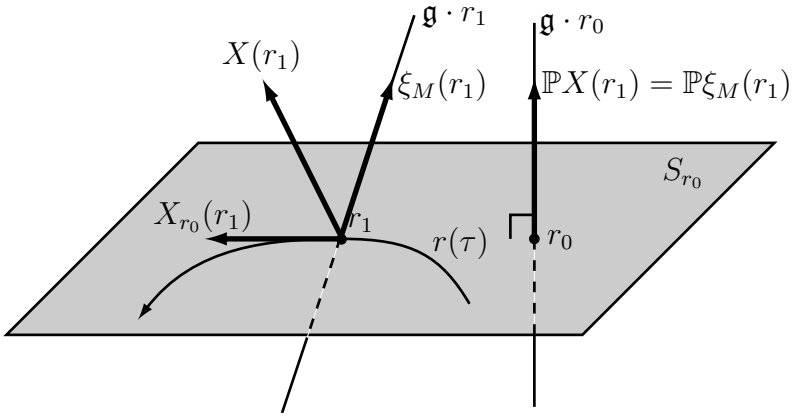

Figure 1. Geometry of the method of slices.

Slice dynamics. We wish to determine dynamics for r by constraining $r(\tau)$ to lie in the slice. Let $\mathbb{P} : M \to M$ be the orthogonal projection onto $\mathfrak{g} \cdot r_0$ (see figure 1). Applying the projection to each side of (10), we have

$$\mathbb{P}\xi_M(r) = \mathbb{P}X(r), \tag{12}$$

since $\mathbb{P}r_\tau = 0$ if $r(\tau)$ is constrained to lie in the slice. Equation (12) is an algebraic equation which may be solved for ξ. The dynamics for r are then given by

$$r_\tau = X_{r_0}(r) := X(r) - (\xi(r))_M(r) \tag{13}$$

where $\xi(r)$ denotes the solution of (12) for ξ, regarded as a function of r. Figure 1 illustrates the corresponding geometry.

Equation (13) determines the *slice dynamics*, which may be viewed as a locally embedded version of the reduced dynamics (since S_{r_0} is locally isomorphic to M/G). A similar equation was obtained in [1], and has been referred to as the MN-dynamics. However, note that though the equations are similar in spirit, the MN dynamics in [1] are not literally a special case of (13). The similarities and differences are illustrated by the example in §3.

Once the evolution of $r(\tau)$ has been obtained (by integrating (13)), we may reconstruct the full solution $u(t)$ from (6). To do this, we must first determine $g(\tau)$. Since $r(\tau)$ is known, we have

$$g(\tau)^{-1}\dot{g}(\tau) = \xi(r(\tau)), \tag{14}$$

where again $\xi(r)$ denotes the solution to (12). The equation above is called the *reconstruction equation*, and may be integrated to find $g(\tau)$. Then (9) may be integrated to give $\tau(t)$, and finally the solution $u(t)$ is found from (6).

Note that the only difference between this procedure and that given in [18] is that time has been rescaled according to $dt = m(g)d\tau$. In particular, if $m(g) = 1$ for all $g \in G$, then $t = \tau$ and the procedure here is identical to that in [18].

2.3. Method of connections

For this method, M does not need to be an inner product space, but we do make use of some additional structure, namely that of a principal connection. Recall that whenever the action of G on M is proper and free, the quotient space M/G is a smooth manifold with a special structure, namely that of a principal fiber bundle [5]. A *connection* on this bundle is a Lie algebra valued one form $\mathcal{A} : TM \to \mathfrak{g}$ with the following properties:

(i) $\mathcal{A}(\xi_M(u)) = \xi$ for all $\xi \in \mathfrak{g}$ and $u \in M$,

(ii) $\mathcal{A}$ is equivariant with respect to the action of G on M and the adjoint representation of G in $\mathfrak{g}$; that is, for $v \in TM$,

$$\mathcal{A}(g \cdot v) = \mathrm{Ad}_g(\mathcal{A}(v)), \tag{15}$$

(iii) The horizontal space $\mathrm{Hor}_u = \ker \mathcal{A}|T_uM$ is a complement to the vertical space $\mathfrak{g} \cdot u$.

As in the method of slices, we consider a solution of the form (6), but here instead of constraining r to lie in the slice, we constrain r_τ to be *horizontal*; that is, $\mathcal{A}(r_\tau) = 0$. Applying the connection to (10), one obtains

$$\xi = \mathcal{A}(X(r)), \tag{16}$$

which defines ξ in terms of r. Substituting this expression into (10) then gives the *horizontal dynamics*

$$r_\tau = X(r) - \mathcal{A}(X(r))_M(r). \tag{17}$$

The geometry of the method is illustrated in Figure 2. To reconstruct the solution $u(t)$ once $r(\tau)$ has been found, one first solves the reconstruction equation

$$g(\tau)^{-1}\dot{g}(\tau) = \mathcal{A}(X(r(\tau)))$$

and then finds $\tau(t)$ and $u(t)$ as in the method of slices. Note that the horizontal spaces in the method of connections play the same role as the slice in the method of slices.

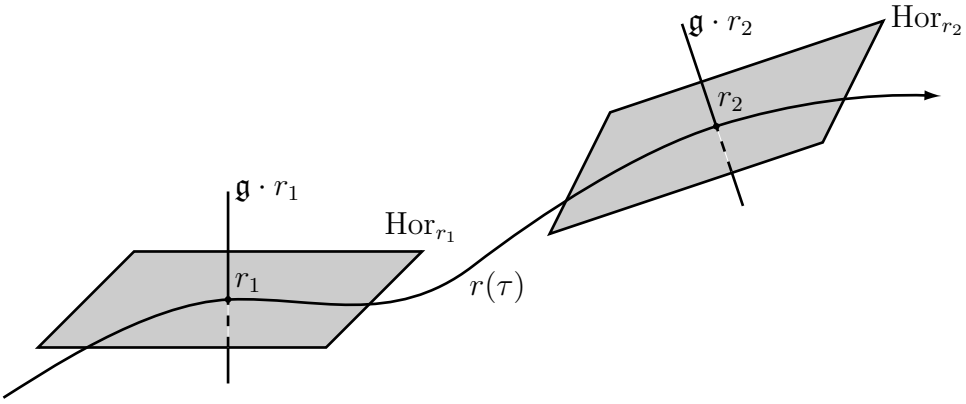

Figure 2. Geometry of the method of connections.

Mechanical connection. In certain commonly occurring cases, it is possible to construct a particular connection called the *mechanical connection* [11]. The construction holds whenever one has the additional structure of a Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on M, and under certain conditions on the group action (for instance, when the action is by isometries).

First, for each $u \in M$, one defines the *locked inertia tensor* $I(u) : \mathfrak{g} \to \mathfrak{g}^*$ by

$$\langle I(u)\xi, \eta \rangle = \langle\langle \xi_M(u), \eta_M(u) \rangle\rangle \,, \tag{18}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing. Next, one constructs the *momentum map* $J : TM \to \mathfrak{g}^*$ by

$$\langle J(v_u), \eta \rangle = \langle\langle v_u, \eta_M(u) \rangle\rangle \,, \tag{19}$$

where $v_u \in T_u M$. Finally, the connection $\mathcal{A} : TQ \to \mathfrak{g}$ is given by

$$\mathcal{A}(v_u) = I(u)^{-1} \cdot J(v_u). \tag{20}$$

One easily verifies that this connection satisfies properties (i) and (iii) above, and furthermore that the horizontal space Hor_u is the orthogonal complement to $\mathfrak{g} \cdot u$. For equivariance, however, one needs an additional condition on the group action. It is straightforward to check that (ii) holds whenever the action satisfies

$$\langle\langle g \cdot v_u, g \cdot w_u \rangle\rangle = f(g) \, \langle\langle v_u, w_u \rangle\rangle \tag{21}$$

where $f : G \to \mathbb{R}^+$ is a homomorphism. Usually, one considers the case where the action is by isometries, and so $f(g) = 1$, but for the actions that arise in self-similarity, the more general condition (21) is appropriate.

It is insightful to notice that using (20) with the method of connections is precisely equivalent to the method of slices, where the template function r_0 is taken to be the time-varying function $r(\tau)$—that is, using the most recent copy of the solution itself as the template. The intuition is immediately apparent on comparing figures 1 and 2. To see this precisely, take $r_0 = r(\tau)$ in equation (12), to obtain

$$\langle\!\langle \xi_M(r), \eta_M(r) \rangle\!\rangle = \langle\!\langle X(r), \eta_M(r) \rangle\!\rangle \,,$$

for all $\eta \in \mathfrak{g}$, which is equivalent to $\xi = \mathcal{A}(X(r))$ with $\mathcal{A}$ given by (20).

2.4. Relation between slices and connections

The horizontal dynamics (17) are related to the slice dynamics (13), but the geometric interpretation of the two is somewhat different. For $r \in M$, let $[r] = G{\cdot}r \in M/G$ denote its equivalence class, and suppose we are given a curve $[r](t) \in M/G$. A *horizontal lift* is a curve $r(t) \in M$ such that $[r(t)] = [r](t)$, and such that $\dot{r}(t) \in \mathrm{Hor}_{r(t)}$, the horizontal space of the connection. If the initial point $r(0)$ is given, then the horizontal lift is uniquely specified, and the dynamics for its evolution are those given by (17). The horizontal lift is a useful construction for studying geometric phases [12].

By contrast, in the method of slices, the slice may be viewed as a local coordinate representation for M/G, so equation (13) is actually a local version of the reduced dynamics, written in this coordinate system. The distinction is an important one: a solution which projects to a loop in M/G will be a periodic solution of (13), but may not be a periodic solution of the horizontal dynamics (17), as there may be a net phase change around the loop (this is called the *holonomy*, or *geometric phase* of the path).

Despite these differences, we have the following important result:

Theorem 2.1. *A point $r \in M$ is a fixed point of the slice dynamics (13) $\iff$ r is a fixed point of the horizontal dynamics (17). Furthermore, for such a fixed point, a solution of (12) is given by $\xi(r) = \mathcal{A}(X(r))$. The fixed point r is a relative equilibrium of (1), corresponding to a self-similar solution.*

Proof. If r is a fixed point of (13), then $X(r) = (\xi(r))_M(r)$, so $\mathcal{A}(X(r)) = \xi(r)$, and r is a fixed point of (17). Conversely, if r is a fixed point of (17), then $X(r) = \mathcal{A}(X(r))_M(r)$, so $\xi = \mathcal{A}(X(r))$ is a solution of (12), and r is a fixed point of (13).

If r is such a fixed point, then the solution $u(t)$ of (1) is given by (6) as $u(t) = g(\tau) \cdot r$, and hence is a relative equilibrium, or in other words, a self-similar solution. □

2.5. Reduction in a space-time setting

As mentioned in section 2.1, in the more general setting of self similarity, where the actions on M and TM are different, one does not get well defined reduced dynamics in the usual sense. However, one may define reduced dynamics in a space-time setting, in which one allows rescalings of time as well.

Let $F_t : M \to M$ denote the flow of the vector field X. Using equivariance of X, one obtains

$$\Phi_g \circ F_t = F_{m(g)t} \circ \Phi_g.$$

If $m(g) = 1$, then the flow is equivariant, and one obtains reduced dynamics in the usual way, by projecting F_t to the quotient space, and obtaining the corresponding vector field on M/G. However, if $m(g) \neq 1$, then the flow is not equivariant, and so does not induce a well-defined flow on M/G. In this case, we allow rescalings of time as well. Define the space-time flow by

$$F : M \times \mathbb{R} \to M \times \mathbb{R} : (u, t) \mapsto (F_t(u), t),$$

and define the action on $M \times \mathbb{R}$ by

$$\Theta_g : (u, t) \mapsto (\Phi_g(u), m(g)t).$$

It is straightforward to check that $\Theta_g \circ F = F \circ \Theta_g$, and thus F induces a space-time flow on the quotient $(M \times \mathbb{R})/G$. Denoting this quotient flow by $\tilde{F}$, the following diagram commutes:

$$\begin{array}{ccc} M \times \mathbb{R} & \xrightarrow{F} & M \times \mathbb{R} \\ \downarrow{\scriptstyle \pi_\Theta} & & \downarrow{\scriptstyle \pi_\Theta} \\ (M \times \mathbb{R})/G & \xrightarrow{\tilde{F}} & (M \times \mathbb{R})/G \end{array} \tag{22}$$

The flow on the reduced space is defined in this space-time sense, but in general one does not get evolution equations on the quotient space M/G. However, the *integral curves* in M/G are well defined objects: if $u(t)$ is a curve in M that satisfies the dynamics (1), then $g \cdot u(t/m(g))$ is another curve that satisfies the dynamics. These two curves project to the same curve in M/G, but with different parameterizations of time.

One may view the slice dynamics of section 2.2 as a way of choosing a "reference clock" that defines which time parameterization to use. More precisely, a slice S may be used to define a diffeomorphism between $(M \times \mathbb{R})/G$ and $(M/G) \times \mathbb{R}$, which makes $\tilde{F}$ into a flow on M/G. Letting $[u]_\Phi \in M/G$ denote the equivalence class of $u \in M$, and $[(u, t)]_\Theta$ denote the equivalence

class in $M\times\mathbb{R}$, we define a diffeomorphism between $(M\times\mathbb{R})/G$ and $(M/G)\times\mathbb{R}$ by

$$[(u,t)]_\Theta \mapsto ([u]_\Phi, m(g)t), \qquad \text{where } g \text{ is such that } g\cdot u \in S.$$

This map is well defined since

$$\begin{aligned}[(\Phi_h(u), m(h)t)]_\Theta &\mapsto ([\Phi_h(u)]_\Phi, m(g)m(h)t), && g\cdot(h\cdot u)\in S\\ &= ([u]_\Phi, m(gh)t), && (gh)\cdot u \in S.\end{aligned}$$

Intuitively, this corresponds to choosing the time scale that corresponds to the particular choice of $u \in M$ that lies in the slice S at time $t = 0$. This map then establishes a flow on the quotient space M/G, which can then be used to define a vector field on M/G.

3. Example

We now illustrate the two methods of reduction using an example treated in [1], a partial differential equation

$$u_t = D(u)$$

where $u(t) \in M$, a function space, and D is a differential operator that satisfies the scaling relation

$$D_x\left(Bf\left(x/A\right)\right) = A^a B^b D_y(f(y)), \qquad \text{where } y = x/A. \tag{23}$$

That is, the operator D is equivariant with respect to scalings in amplitude (B) and spatial scale (A). Here, the subscript x or y denotes the variable which the differential operator D acts with respect to. In the terminology of the previous sections, (23) means that the operator D is equivariant with respect to actions Φ_g and Ψ_g of the multiplication group $G = \mathbb{R}^+ \times \mathbb{R}^+$. For $g = (A, B) \in G$, the group actions are given by

$$\Phi_g(u)(x) = Bu(x/A) \tag{24}$$

$$\Psi_g(u)(x) = A^a B^b u(x/A). \tag{25}$$

Since Φ_g is linear, $T\Phi_g = \Phi_g$, and so we have

$$T\Phi_g = A^{-a}B^{1-b}\Psi_g = m(g)\Psi_g,$$

where $m(g) = A^{-a}B^{1-b}$.

The Lie algebra of G is $\mathfrak{g} = \mathbb{R}^2$, and the infinitesimal generator $\xi_M(u)$ is found by differentiating the action $g(t)\cdot u$ at the identity $g(0) = (1, 1)$. For $\xi = (\xi_1, \xi_2) \in \mathfrak{g}$, we have

$$\xi_M(u)(x) = -\xi_1 x u_x(x) + \xi_2 u(x). \tag{26}$$

Writing $u(x,t) = g(\tau) \cdot r(x,\tau)$, equation (10) becomes

$$r_\tau = D(r) + \xi_1 x r_x - \xi_2 r, \tag{27}$$

where $\xi(r)$ will be determined below, either by the method of slices or the method of connections.

3.1. Reduction using slices

Choosing a template function $\bar{r} \in M$, the slice $S_{\bar{r}}$ is given by (11), which (using (26)) consists of all functions r satisfying

$$\int (r - \bar{r}) x \bar{r}_x \, dx = 0 \quad \text{and} \quad \int (r - \bar{r}) \bar{r} \, dx = 0 \tag{28}$$

(in section 2.2, the template $\bar{r}$ was denoted r_0). We now project (27) onto the group orbit directions by multiplying by $x\bar{r}_x$ and $\bar{r}$ respectively, integrating, and using (28), to get

$$-\xi_1 \int x^2 r_x \bar{r}_x \, dx + \xi_2 \int x r \bar{r}_x \, dx = \int D(r) x \bar{r}_x \, dx \tag{29}$$

$$-\xi_1 \int x r_x \bar{r} \, dx + \xi_2 \int r \bar{r} \, dx = \int D(r) \bar{r} \, dx. \tag{30}$$

These equations correspond to (12), and have the form

$$I_{\bar{r}}(r) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = J_{\bar{r}}(D(r)),$$

where

$$I_{\bar{r}}(r) = \begin{pmatrix} \int x^2 r_x \bar{r}_x \, dx & -\int x r \bar{r}_x \, dx \\ -\int x r_x \bar{r} \, dx & \int r \bar{r} \, dx \end{pmatrix}$$

$$J_{\bar{r}}(v) = \begin{pmatrix} -\int v x \bar{r}_x \, dx \\ \int v \bar{r} \, dx \end{pmatrix}.$$

Solving for $\xi = (\xi_1, \xi_2)$, we obtain

$$\xi(r) = I_{\bar{r}}(r)^{-1} J_{\bar{r}}(D(r)), \tag{31}$$

which defines ξ_1 and ξ_2 in equation (27), to give the slice dynamics.

Note that equations (27) and (31) are analogous to equation (13) in [1], which is referred to as the "MN-dynamics." The procedure used in [1] to derive the MN-dynamics is similar in spirit to the procedure used here, but is not identical. In particular, the values of ξ_1 and ξ_2 are different, as two different pinning conditions were used in place of the single template used here.

3.2. Reduction using connections

We begin by constucting the mechanical connection $\mathcal{A}$ for the example, as described in §2.3. Using (26), with $\xi = (\xi_1, \xi_2)$, the locked inertia tensor defined by (18) becomes

$$I(u)\xi = (\xi_1 \;\; \xi_2) \begin{pmatrix} \langle\langle xu_x, xu_x \rangle\rangle & -\langle\langle u, xu_x \rangle\rangle \\ -\langle\langle xu_x, u \rangle\rangle & \langle\langle u, u \rangle\rangle \end{pmatrix}.$$

The momentum map defined by (19) is then

$$J(v_u) = (-\langle\langle v_u, xu_x \rangle\rangle \;\; \langle\langle v_u, u \rangle\rangle),$$

and the connection $\mathcal{A} : TQ \to \mathfrak{g}$ is given by

$$\begin{aligned} \mathcal{A}(v_u) &= I(u)^{-1} \cdot J(v_u) \\ &= \begin{pmatrix} \langle\langle xu_x, xu_x \rangle\rangle & -\langle\langle u, xu_x \rangle\rangle \\ -\langle\langle xu_x, u \rangle\rangle & \langle\langle u, u \rangle\rangle \end{pmatrix}^{-1} \begin{pmatrix} -\langle\langle v_u, xu_x \rangle\rangle \\ \langle\langle v_u, u \rangle\rangle \end{pmatrix}. \end{aligned}$$

The horizontal dynamics for the example are then given by (27), where $(\xi_1, \xi_2) = \xi(r) = \mathcal{A}(D(r))$.

Note the similarity between the above, and the corresponding definition for slices, equation (31). As mentioned in §2.3, if $\langle\langle \cdot, \cdot \rangle\rangle$ is the L^2 inner product, then the two definitions are in fact identical, if the template function $\bar{r}$ is taken to be the (time-varying) function $r(\tau)$.

4. Scaling laws and exponents

As summarized in Theorem 2.1, fixed points of either the slice dynamics or the horizontal dynamics are self-similar solutions of the original equation. Given a fixed point of these dynamics, one would like to find the scaling laws that describe how the full (reconstructed) solution changes with time. Typically, these scaling laws will be power laws, and we wish to compute the values of the exponents from knowledge of the fixed point.

4.1. Example

First, we illustrate with the example from the previous section. Suppose that r is a fixed point of the slice dynamics, and let $(\xi_1, \xi_2) = \xi(r)$. Then the scaling laws for the self-similar solution are given by the reconstruction equations

$$\frac{1}{A}\frac{dA}{d\tau} = \xi_1, \qquad \frac{1}{B}\frac{dB}{d\tau} = \xi_2, \tag{32}$$

obtained from (14), with $g^{-1}\dot{g} = (\dot{A}/A, \dot{B}/B)$. These are easily solved to give

$$A(\tau) = \exp(\xi_1\tau), \qquad B(\tau) = \exp(\xi_2\tau). \tag{33}$$

We wish to rewrite this solution in terms of the physical time t, where $dt = A^{-a}B^{1-b}d\tau$. We have

$$\begin{aligned}\frac{dt}{d\tau} &= A^{-a}B^{1-b}\\ &= \exp(\xi_1\tau)^{-a}\exp(\xi_2\tau)^{1-b}\\ &= \exp((-a\xi_1 + (1-b)\xi_2)\tau)\\ &= \exp(\mu\tau),\end{aligned}$$

where

$$\mu = -a\xi_1 + (1-b)\xi_2 \tag{34}$$

is a constant. The above equation is easily integrated to give

$$1 + \mu t = \exp(\mu\tau). \tag{35}$$

Writing (33) in terms of t, we obtain

$$A(t) = \exp(\xi_1/\mu \log(1+\mu t)) = (1+\mu t)^{\xi_1/\mu} \tag{36}$$

$$B(t) = \exp(\xi_2/\mu \log(1+\mu t)) = (1+\mu t)^{\xi_2/\mu}. \tag{37}$$

Thus, the scaling exponents are given by ξ_1/μ and ξ_2/μ. Furthermore, from the definition of μ, we see that the exponents satisfy

$$-a\frac{\xi_1}{\mu} + (1-b)\frac{\xi_2}{\mu} = 1,$$

a well known scaling condition (see equation (6) of [1]). In the next subsection, we show that one can generalize this procedure to an arbitrary Lie group.

4.2. Abstract setting

Let $dt = m(g)d\tau$, where $m : (G, \cdot) \to (\mathbb{R}^+, \times)$ is the homomorphism from §2. Also, let $\mu = T_1 m : \mathfrak{g} \to \mathbb{R}$, the tangent map of m at the identity. An elementary result from the theory of Lie groups [5] is that if $\Phi : G \to H$ is a homomorphism of Lie groups, then $\Phi(\exp_G \xi) = \exp_H(T_1\Phi(\xi))$, for all $\xi \in \mathfrak{g}$. That is, the following diagram commutes:

$$\begin{array}{ccc}\mathfrak{g} & \xrightarrow{\exp_G} & G\\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle m}\\ \mathbb{R} & \xrightarrow{\exp} & \mathbb{R}^+\end{array} \tag{38}$$

The function μ will play an important role in determining the scaling laws, and also determining whether solutions have finite-time singularities.

Suppose $r \in M$ is a fixed point of the slice dynamics, and let $\xi = \xi(r) \in \mathfrak{g}$. Then the reconstruction equation (14) gives

$$g(\tau) = \exp_G(\xi\tau). \tag{39}$$

We wish to obtain $g(t)$, where $dt = m(g)d\tau$. We have

$$dt = m(\exp_G(\xi\tau))d\tau = \exp(\mu(\xi)\tau)\, d\tau. \tag{40}$$

Since ξ is independent of time, let $\mu = \mu(\xi) \in \mathbb{R}$. If $\mu \neq 0$, then the solution of (40) is

$$1 + \mu t = \exp(\mu\tau),$$

and hence

$$\begin{aligned} g(t) &= \exp_G(\xi/\mu \log(1 + \mu t)) \\ &= (1 + \mu t)^{\xi/\mu} \end{aligned} \tag{41}$$

where for $t \in \mathbb{R}^+$, $\xi \in \mathfrak{g}$, we define

$$t^\xi := \exp_G(\xi \log(t)).$$

To summarize: given a fixed point r of the reduced dynamics, the exponents of the power laws are given by ξ/μ, where $\xi = \xi(r) \in \mathfrak{g}$, and $\mu = \mu(\xi(r)) \in \mathbb{R}$.

Furthermore, notice from (41) that a singularity exists when $1 + \mu t = 0$. If $\mu < 0$, this corresponds to an explosion or implosion at time $t = -1/\mu > 0$. If $\mu > 0$, the singularity corresponds to the "virtual origin" of the self-similar solution, at time $t = -1/\mu < 0$, and the solution exists for all positive time (see Figure 3). The virtual origin corresponds to a finite-time singularity in reverse time, as in a diffusion problem where a Gaussian initial condition will approach a Dirac measure in reverse time.

5. Neutral directions

Since self-similar solutions are now fixed points of the slice dynamics (or horizontal dynamics), one can now numerically search for self-similar solutions, and discuss their stability, using conventional techniques. Some numerical methods for finding fixed points (e.g., Newton iteration) have difficulties whenever one has a *manifold* of fixed points rather than an isolated fixed point. In that case, "neutral directions" exist—i.e., the linearized equations have one or more zero eigenvalues at the fixed point. Those numerical methods

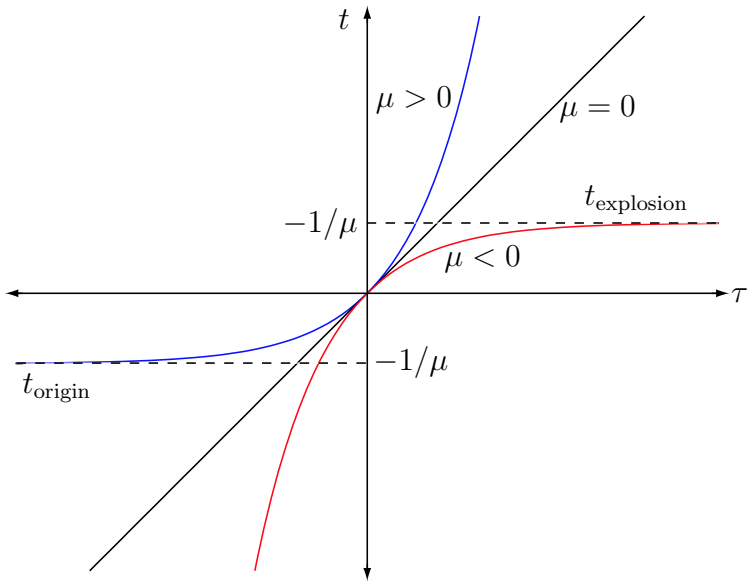

Figure 3. Qualitative behavior of scaling laws. For $\mu > 0$, the self-similar solution exists for all positive time, and $t = -1/\mu$ is the "virtual origin." For $\mu < 0$ there is a finite-time singularity (either explosion or implosion) at $t = -1/\mu > 0$. In the scaled time τ, self-similar solutions exist for all time, regardless of the value of μ.

require that the neutral directions be identified, and the iteration constrained accordingly.

For an equivariant dynamical system, if u_0 is a fixed point, then $g \cdot u_0$ is also a fixed point, for any $g \in G$, so equilibria always come in orbit-fulls. Accordingly, directions of the group action are obviously neutral directions for the original dynamical system. Note that though self-similar solutions are not equilibria of the original dynamics, they are equilibria of the slice dynamics, and may be viewed as *relative* equilibria of the original dynamics, as stated in Theorem 2.1. These relative equilibria also occur in orbit-fulls, but time must be rescaled, as in Theorem 5.3 below. The main result of this section is the theorem below, which shows that fixed points of the horizontal dynamics also occur in orbit-fulls.

Theorem 5.1. *If r is a fixed point of the horizontal dynamics (17), then $g \cdot r$ is also a fixed point, for any fixed $g \in G$.*

Proof. Let $\xi = \mathcal{A}(X(r))$, and note that since r is a fixed point of (17), we have

$$X(r) - \xi_M(r) = 0.$$

Letting

$$\eta = \mathcal{A}(X(g \cdot r)) = \mathcal{A}(m(g)^{-1} g \cdot X(r)) = m(g)^{-1} \operatorname{Ad}_g \xi,$$

we have

$$\begin{aligned} X(g \cdot r) - \eta_M(g \cdot r) &= m(g)^{-1} g \cdot X(r) - g \cdot (\operatorname{Ad}_{g^{-1}} \eta)_M(r) \\ &= m(g)^{-1} g \cdot [X(r) - \xi_M(r)] = 0, \end{aligned}$$

and so $g \cdot r$ is also a fixed point. □

Corollary 5.2. *If r is a fixed point of the slice dynamics (13), then $g \cdot r$ is also a fixed point, for any fixed $g \in G$.*

Proof. Combine Theorems 2.1 and 5.1. □

Since the group orbit $G \cdot r$ is a smooth manifold of fixed points whenever r is a fixed point, the neutral directions lie in the tangent space to the group orbit, and are therefore the group directions $\xi_M(r)$. In other words, if the slice dynamics (or horizontal dynamics) are linearized about the fixed point r, then the linearized equations will have a zero eigenvalue (with multiplicity), with eigenspace $\mathfrak{g} \cdot r$, the tangent space to the group orbit through r. Note, however, that for the method of slices, these are not neutral directions if the domain is restricted to the slice (i.e., those functions aligned with the template), as the group does not act on this space.

For connections, we also have a stronger result, that holds for arbitrary trajectories $r(t)$.

Theorem 5.3. *If $r(t)$ satisfies the horizontal dynamics (17), then $g \cdot r(t/m(g))$ satisfies (17), for any fixed $g \in G$.*

Proof. As before, for any $g \in G$, $r \in M$, we have

$$X(g \cdot r) - \mathcal{A}(X(g \cdot r))_M(g \cdot r) = \frac{1}{m(g)} g \cdot [X(r) - \mathcal{A}(X(r))_M(r)].$$

Noticing that

$$\frac{d}{dt} g \cdot r(t/m(g)) = \frac{1}{m(g)} g \cdot \dot{r}(m(g)t)$$

establishes the result. □

6. Numerical examples

We present two examples: first, we illustrate the difference between the methods of slices and connections for a traveling solution of the Kuramoto-Sivashinsky equation, as considered in [8, 18]. Next, we use the method to find a self-similar solution of the Burgers equation.

6.1. Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky equation is simplified model of phenomena in flame dynamics and turbulence, and may be written

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0, \qquad x \in [0, 2\pi],$$

with periodic boundary conditions. This system admits a wide variety of solutions, for different values of ν, and here we consider a particular value $\nu = 4/87$, as considered in [8, 18]. For this value, the equation admits solutions that are traveling, beating waves. The equation is invariant to translations, so we consider the additive group $G = \mathbb{R}$, with group action defined by $g \cdot u(x) = u(x - g)$. The infinitesimal generator is then $\xi_M(u) = -\xi u_x$, and $\xi = \dot{g}$ represents the propagation speed. Writing the equation as $u_t = X(u)$, with

$$X(u) = -uu_x - u_{xx} - \nu u_{xxxx},$$

equivariance becomes $X(g \cdot u) = g \cdot X(u)$, so the function $m : G \to \mathbb{R}^+$ from section 2.1 is just $m(g) = 1$, and time does not need to be rescaled. The modified dynamics are then

$$u_t = X(u) + \xi u_x, \tag{42}$$

where if ξ is determined by slices (equation (12)), we call this the reduced dynamics, and if ξ is determined by connections (equation (16)) this is the horizontally lifted dynamics. Choosing a template function $\bar{u}(x)$, (12) becomes

$$\xi(u) = -\frac{\langle X(u), \bar{u}_x \rangle}{\langle u_x, \bar{u}_x \rangle}, \tag{43}$$

and using the mechanical connection, (16) becomes

$$\xi(u) = \mathcal{A}(X(u)) = -\frac{\langle X(u), u_x \rangle}{\langle u_x, u_x \rangle}. \tag{44}$$

We solve (42) using a spectral collocation method, with 32 modes in x, using a Crank-Nicolson scheme to advance the linear terms in time, and 2nd-order Adams-Bashforth for the nonlinear term, with a timestep of $\Delta t = 10^{-4}$.

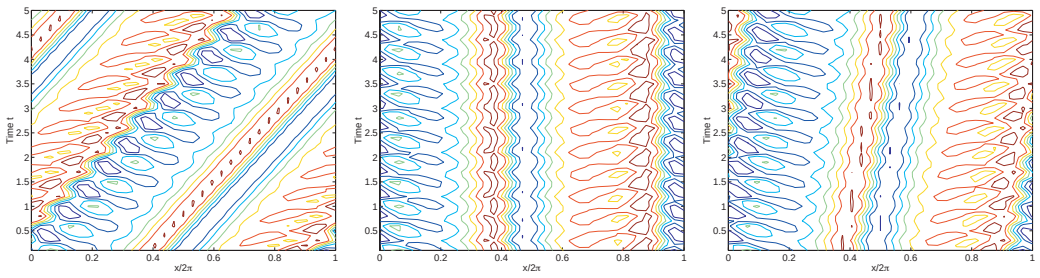

Figure 4. Contour plot of solutions of the Kuramoto-Sivashinsky equation, with $\nu = 4/87$. *Left*: original dynamics ($\xi = 0$); *Middle*: slice dynamics (ξ determined by (43)); *Right*: horizontal dynamics (ξ determined by (44))

A typical solution is shown in Figure 4, for an initial condition with spatial mean equal to 1. The left figure shows the solution of the original equation, and has the form of a traveling, beating wave. The middle figure shows the solution of the slice dynamics, with the template function taken to be the initial condition. Note that the traveling component has been removed. The right figure shows the horizontal dynamics, and note that some traveling remains: this is the geometric phase of the solution. It is interesting to note that for the nearby parameter value $\nu = 4/84$, the geometric phase apparently disappears, and the method of connections removes all of the traveling component, although the shape of the solution looks qualitatively very similar. This is consistent with behavior others have observed for these parameter values [8].

6.2. Burgers equation

Consider the Burgers equation

$$u_t + uu_x = \nu u_{xx}, \qquad x \in \mathbb{R} \tag{45}$$

and now let $X(u) = -uu_x + \nu u_{xx}$, so that this equation may be written $u_t = X(u)$. First, we investigate the equivariance of X. Consider transformations of u of the form

$$u(x) \mapsto bu\Big(\frac{x-c}{a}\Big).$$

We wish to express this transformation as a group action. The relevant group is not completely obvious, but has an interesting structure. Let $(a, b, c) \in G = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, with group composition defined by

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1a_2, b_1b_2, c_1 + a_1c_2).$$

The inverse is given by $(a,b,c)^{-1} = (1/a, 1/b, -c/a)$, and the tangent to left translation by $(a,b,c)(\dot{a},\dot{b},\dot{c}) = (a\dot{a}, b\dot{b}, a\dot{c})$. Abstractly, G has the structure of a *semidirect product*, and may be denoted $\mathbb{R}^+ \times (\mathbb{R}^+ \circledS \mathbb{R})$. With the (left) group action defined by

$$(g \cdot u)(x) = bu\Big(\frac{x-c}{a}\Big)$$

then letting $y = (x-c)/a$, we have

$$(g \cdot X(u))(x) = -buu_y + b\nu u_{yy}$$

and

$$X(g \cdot u)(x) = -\frac{b^2}{a} uu_y + \nu \frac{b}{a^2} u_{yy}.$$

For self-similarity, we require

$$m(g)X(g \cdot u) = g \cdot X(u)$$

and so we require

$$b = m(g)b^2/a, \qquad b = m(g)b/a^2$$

which together imply $b = 1/a$, $m(g) = a^2$. Since b is now superfluous, henceforth we shall denote elements of G by $(a,c) \in \mathbb{R}^+ \circledS \mathbb{R}$.

Reduced dynamics. The infinitesimal generator of the action is found as follows. Consider a curve $g(t) \in G$ with $g(0) = (1,0)$, the identity element. Then

$$\frac{d}{dt}\Big|_{t=0} g(t) \cdot u = \frac{d}{dt}\left[\frac{1}{a} u\Big(\frac{x-c}{a}\Big)\right]_{a=1,c=0} = -\dot{a}(u + xu_x) - \dot{c}u_x.$$

Thus, the infinitesimal generator is given by

$$\xi_M(u) = -\xi_1(u + xu_x) - \xi_2 u_x$$

where $(\xi_1, \xi_2) \in \mathfrak{g}$, the Lie algebra of G. Notice the linear combination of the convective term u_x (as for a traveling wave) with the self-similar scaling terms u and xu_x.

The slice dynamics are then

$$\begin{aligned} r_\tau &= X(r) - \xi_M(r) \\ &= -rr_x + \nu r_{xx} + \xi_1(r + xr_x) + \xi_2 r_x \end{aligned} \tag{46}$$

where (ξ_1, ξ_2) are given by the reconstruction equation below: if $\bar{r}(x)$ is a template function, then $\xi(r)$ is defined implicitly by

$$\langle \xi_M(r), \eta_M(\bar{r}) \rangle = \langle X(r), \eta_M(\bar{r}) \rangle, \qquad \forall \eta \in \mathfrak{g}.$$

This becomes

$$\begin{pmatrix} \int (r + xr_x)(\bar{r} + x\bar{r}_x)\,dx & \int r_x(\bar{r} + x\bar{r}_x)\,dx \\ \int (r + xr_x)\bar{r}_x\,dx & \int r_x\bar{r}_x\,dx \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -\int X(r)(\bar{r} + x\bar{r}_x)\,dx \\ -\int X(r)\bar{r}_x\,dx \end{pmatrix}. \quad (47)$$

Equation (47) determines the scaling rate ξ_1 and propagation speed ξ_2 from the wave shape r.

Scaling laws To find the scaling laws, first we find the exponential map $\exp_G : \mathfrak{g} \to G$. To find $\exp_G$, we solve the differential equation $g^{-1}\dot{g} = \xi$ for g, with $g(0) = 1_G$, the identity, and then $\exp_G(\xi t) = g(t)$. (One must be careful here, because of the semidirect product structure of G.) In particular, we have

$$g^{-1}\dot{g} = \left(\frac{1}{a}, -\frac{c}{a}\right)(\dot{a}, \dot{c}) = \left(\frac{\dot{a}}{a}, \frac{\dot{c}}{a}\right).$$

Note that here, concatenation represents the *tangent* to left translation. Next, writing $g^{-1}\dot{g} = (\xi_1, \xi_2)$, we have

$$\dot{a} = \xi_1 a, \qquad \dot{c} = \xi_2 a$$

and solving, we obtain

$$a(\tau) = \exp(\xi_1\tau), \qquad c(\tau) = \frac{\xi_2}{\xi_1}(\exp(\xi_1\tau) - 1).$$

To determine scaling laws, we require the function $\mu : \mathfrak{g} \to \mathbb{R}$ from § 4. Since $m(g) = a^2$, we have $\mu = T_1 m$, so

$$\mu(\xi) = 2\xi_1.$$

Finally, the scaling laws are then

$$\begin{aligned} g(t) &= (1 + \mu t)^{\xi/\mu} \\ &= \exp_G(\xi_1/\mu \log(1 + \mu t), \xi_2/\mu \log(1 + \mu t)) \\ &= ((1 + \mu t)^{\xi_1/\mu}, \xi_2/\xi_1((1 + \mu t)^{\xi_1/\mu} - 1)) \end{aligned}$$

Defining $\alpha = \xi_1/\mu$, $\beta = \xi_2/\mu$, we have $\alpha = 1/2$, and so

$$a(t) = \sqrt{1 + \mu t}, \qquad c(t) = 2\beta(\sqrt{1 + \mu t} - 1).$$

For typical solutions of the Burgers equation, the amplitude decreases, so $\dot{a} > 0 \Longrightarrow \mu > 0$, and there is no finite-time singularity.

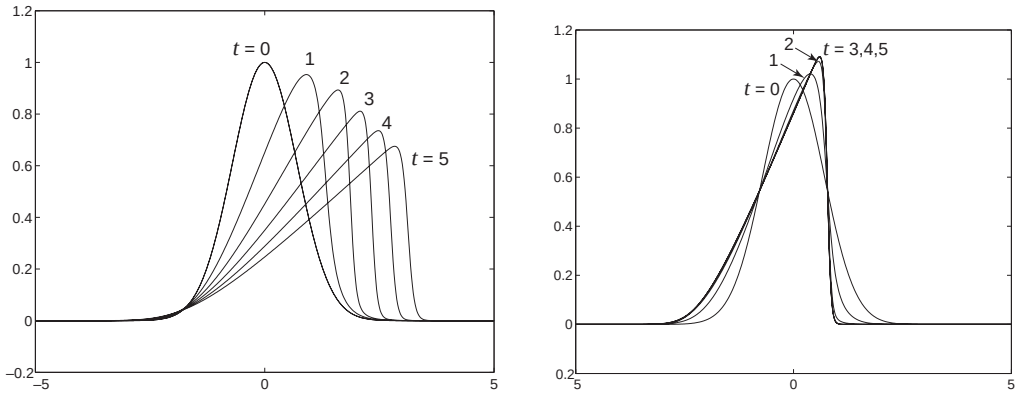

Figure 5. Solution of the Burgers equation (left), and the Burgers equation with symmetry reduction (right), for $\nu = 0.025$, at $t = 0, 1, 2, 3, 4, 5$, for the initial condition $u(x, 0) = \exp(-x^2)$.

Simulation results The results of a simulation are shown in Figure 5. A very simple numerical scheme was used, with second-order finite differences in space, and explicit Euler for the time march. The figure on the left shows the solution of the Burgers equation (45) with a Gaussian initial condition, with $\nu = 0.025$. For this simulation, 501 gridpoints were used for $-5 \leq x \leq 5$, with a timestep of $\Delta t = 10^{-3}$. Solutions are plotted for $t = 0, 1, 2, 3, 4, 5$, and one sees the initial data steepen into a sharp front (a viscous shock), propagate to the right, and spread out as it decreases in amplitude.

The right figure shows the evolution of the horizontal dynamics, that is, equations (46) and (47), with the template function $\bar{r}$ chosen to be the current solution $r(t)$. The parameters are identical to those of the earlier simulation. Here, the initial data also steepens into a sharp front, but the front does not propagate, or grow in amplitude or spatial scale. Instead, the solution approaches a steady state after about time $t = 3$ which corresponds, of course, to a self-similar solution.

For this steady state, we compute the values ξ_1, ξ_2 from (47), and then compute $\mu = 2\xi_1$, $\alpha = \xi_1/\mu$, $\beta = \xi_2/\mu$, to obtain $\alpha = 0.5$, $\beta = 0.993$, $\mu = 0.422$. Thus, for this self-similar solution, the "virtual origin" occurs at time $t = -1/\mu = -2.37$, and the amplitude and position evolve as

$$a(t) = \sqrt{1 + 0.422t}, \qquad c(t) = 1.98(\sqrt{1 + 0.422t} - 1).$$

The propagation speed is therefore $\dot{c} = 0.42/\sqrt{1 + 0.422t}$, or about $0.42/a$. Note, however, that this speed $\dot{c}$ does not correspond to the speed of the front, since the spatial scale $a(t)$ also contributes to the motion of the front.

An exact self-similar solution of the Burgers equation is well known [22], and is given by

$$u(x,t) = \sqrt{\frac{\nu}{\pi t}} \frac{(e^{A/(2\nu)} - 1)\exp(-x^2/(4\nu t))}{1 + (e^{A/(2\nu)} - 1)/2 \cdot \operatorname{erfc}(x/\sqrt{4\nu t})},$$

where A is the initial amplitude of the initial condition, a Dirac measure. This analytic solution agrees almost exactly with the steady state shown in Figure 5, with $A = \sqrt{\pi}$ (the area under the curve $u(x)$, a conserved quantity), and with x shifted by the amount $x_0 = \beta/\alpha = 1.98$, and time shifted by the amount $t_0 = 1/\mu = 2.37$.

This exact solution is easily found by more conventional methods, such as reduction to ODEs. However, the present method has several advantages: one may computationally study the stability of self-similar solutions; continuation methods may be used to find nearby self-similar solutions (or bifurcations) as parameters are varied; and furthermore, the present method can be useful in the study of modulated self-similar solutions, analogous to the modulated traveling solutions shown in Figure 4.

7. Conclusions

We have presented a method for analyzing and numerically solving equations with self-similar solutions. The method unifies ideas for traveling waves, given in [18], with those for self-similar solutions given in [1], and is presented for an arbitrary continuous symmetry group. The only difference between the method presented here and that in [18] is that for the self-similar equations considered here, time must be rescaled appropriately.

Scaling laws are obtained, and are typically power laws, as is well known for self-similar problems. A function $\mu : \mathfrak{g} \to \mathbb{R}$ was introduced in § 4, and plays an important role in determining the exponents of the power laws, and determining the existence of finite-time singularities.

The method also has some desirable characteristics from the numerical point of view. First, the reduced dynamics are identical to the original dynamics, with the addition of some extra terms (and the number of additional terms is the same as the number of group variables). Thus, it is relatively simple to convert an exisiting code for solving a self-similar PDE into a code to solve the symmetry-reduced PDE: one need only add the extra terms to the routine that computes the right-hand-side of the PDE. Because the scales and position of the solution are fixed, there is less need for time-adaptive meshes than in the original equation. Second, since the neutral directions

of the symmetry-reduced equations are known (up to some discretization error), it is possible to modify the Recursive Projection Method [20] or the Newton-Picard method [9] to compute steady-states or periodic solutions of these equations. If it is impossible to adapt an existing code to integrate the symmetry-reduced equations, it should still be possible to compute and analyse the unknown symmetry-reduced system using the approach of [19], which does not explicitly need the symmetry-reduced equations but relies only on a black-box time integrator, combined with symmetry transformations of the state. Combining this discrete-time approach with the coarse integration and bifurcation techniques we have been recently developing [6] may help with computer-assisted analysis of PDE-level, "coarse" self-similar solutions for problems for which only microscopic or stochastic descriptions are available.

Acknowledgements

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References

[1] D. G. Aronson, S. I. Betelu, and I. G. Kevrekidis. Going with the flow: a Lagrangian approach to self-similar dynamics and its consequences. Submitted to *PNAS*, 2001. (Also preprint nlin.A0/0111055 at arXiv.org.)

[2] G. I. Barenblatt. *Scaling, self-similarity, and intermediate asymptotics*. Cambridge University Press, 1996.

[3] C. J. Budd and M. D. Piggott. The geometric integration of scale-invariant ordinary and partial differential equations. *J. Comp. and Appl. Math.* **128**:399–422, 2001.

[4] E. Doedel, H. B. Keller and J. P. Kernevez. Numerical analysis and control of bifurcation problems, (II) Bifurcation in infinite dimensions. *Int. J. Bifurcation and Chaos* **1**(4):745–772, 1991.

[5] J. J. Duistermaat and J. A. C. Kolk. *Lie Groups*. Springer-Verlag, 1999.

[6] C. W. Gear, I. G. Kevrekidis, and C. Theodoropoulos. Coarse integration/bifurcation analysis via microscopic simulators: micro-Galerkin methods. To appear in *Computers and Chemical Engineering*, also NEC Institute Technical Report NECI TR 2001-106, 3 Oct 2001.

[7] N. Goldenfeld. *Lectures on phase transitions and the renormalization group*. Addison-Wesley, 1992.

[8] M. Kirby and D. Armbruster. Reconstructing phase space from PDE simulations. *Z. Angew. Math. Phys.*, **43**, 999–1022, 1992.

[9] K. Lust, D. Roose, A. Spence, and A. R. Champneys. An adaptive Newton-Picard algorithm with subspace iteration for computing periodic solutions. *SIAM J. Sci. Comput.*, **19**(4):1188-1209, 1998.

[10] K. Lust and C. W. Rowley and I. G. Kevrekidis. On the computation and stability analysis of self-similar solutions. In preparation.

[11] J. E. Marsden. *Lectures on Mechanics*, volume 174 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1990.

[12] J. E. Marsden, R. Montgomery, and T. S. Ratiu. Reduction, symmetry, and phases in mechanics. *Mem. Amer. Math. Soc.*, **88**(436), Nov. 1990.

[13] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, second edition, 1994.

[14] D. W. McLaughlin, G. C. Papanicolaou, C. Sulem, P. L. Sulem. Focusing singularity of the cubic Schrödinger equation. *Phys. Rev. A*, **34**(2):1200–1210, Aug. 1986.

[15] P. J. Olver. *Applications of Lie Groups to Differential Equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, 1986.

[16] H. G. Othmer and A. Stevens Aggregation, blowup, and collapse: The ABC's of taxis in reinforced random walks. *SIAM J. Appl. Math.*, **57**(4), 1044–1081, Aug. 1997.

[17] L. V. Ovsiannikov. *Group Analysis of Differential Equations*. Academic Press, 1982.

[18] C. W. Rowley and J. E. Marsden. Reconstruction equations and the Karhunen-Loève expansion for systems with symmetry. *Phys. D*, **42**:1–19, 2000.

[19] O. Runborg, C. Theodoropoulos, and I. G. Kevrekidis. Effective bifurcation analysis: a time-stepper-based approach. *Nonlinearity*, **15**(2):491–511, Mar 2002.

[20] G. Shroff and H. B. Keller. Stabilization of unstable procedures: the recursive projection method. *SIAM J. Numer. Anal.*, **30**(4):1099–1120, 1993.

[21] C. Siettos, I. G. Kevrekidis, and P. G. Kevrekidis. Focusing revisited: a renormalization/bifurcation approach. *Nonlinearity*, **16**:497–506, 2003.

[22] G. B. Whitham. *Linear and Nonlinear Waves*. Wiley, 1974.

[23] V. E. Zakharov and V. F. Shvets. Nature of wave collapse in the critical case. *JETP Lett.*, **47**(4):275–278, 1988.