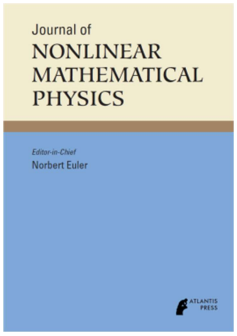

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Reduction by λ–symmetries and σ–symmetries: a Frobenius approach

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Different kinds of reduction for ordinary differential equations, such as λ–symmetry and σ–symmetry reductions, are recovered as particular cases of Frobenius reduction theorem for distribution of vector fields. This general approach provides some hints to tackle the reconstruction problem and to solve it under suitable assumptions on the distribution involved in the reduction process.

Keywords: Solvable Structures; Reduction by Symmetries; Ordinary Differential Equations.

2000 Mathematics Subject Classification: 34C14, 37C10, 34A26

1. Introduction

It is well known that the knowledge of a Lie algebra of symmetries for a system of ordinary differential equations (ODEs) can be used to reduce its order at least locally. This procedure is particularly effective when also the reconstruction problem can be solved, *i.e.*, when we are able to get the solution to a given system of ODEs starting from the knowledge of the solution to the reduced system and integrating by quadratures a set of closed one-forms. This is actually what happens when we find a solvable k–dimensional symmetry algebra for a scalar k^{th} order equation. Indeed, in this case the equation can be completely integrated by quadratures [4, 13, 17, 20]. However, in general, it is not possible to compute the complete symmetry algebra of a system of equations. On the other hand, equations solvable by quadratures but with a lack of local symmetries are quite common (see e.g. [5–7, 19]). This fact lead to various generalizations of the notion of reduction by symmetry such as reduction by λ–symmetries, σ–symmetries and solvable structures (see e.g. [2, 3, 5–11, 14–16, 19, 21, 22]).

The aim of this paper is to show that all this different kind of reductions can be seen as particular cases of the original idea of Frobenius integrability for a distribution of vector fields. The main advantage of this general approach is that it provides a useful starting point for dealing with the reconstruction problem. In the first part of the paper we consider a system $\mathscr{E}$ of first order ODEs on a manifold M, which can be naturally associated with a vector field Z on M such that the integral lines of Z are the solutions to $\mathscr{E}$. In this framework we recast Frobenius reduction theorem showing that the existence of an integrable distribution $\mathscr{D}_0$ transversal to Z and such that the distribution obtained by adding Z to the generators of $\mathscr{D}_0$ is still integrable, ensures that it is possible to perform a dimensional reduction for $\mathscr{E}$. In fact, under suitable regularity hypotheses, we get a new system $\overline{\mathscr{E}}$ of ODEs in a quotient space $\overline{M}$ such that solutions to the original system $\mathscr{E}$ project onto solutions to $\overline{\mathscr{E}}$. Of course, in general, we cannot ensure that the reduced system is easier to solve than the original

one. Moreover, even if the general solution to the reduced system is known, we may not be able to use it to reconstruct the general solution to $\mathscr{E}$. In fact, the reconstruction problem is in general a nontrivial task, depending on the nature of the distribution $\mathscr{D}_0$ we used in the reduction process. In this paper we show that, if the distribution $\mathscr{D}_0$ is a partial solvable structure for the vector field Z corresponding to $\mathscr{E}$, it is possible to reconstruct the solution to $\mathscr{E}$ starting from the knowledge of the solution to $\overline{\mathscr{E}}$ and integrating a given system of closed one-forms.

In the second part of the paper this general reduction method is applied to (systems of) higher order ODEs. In this framework the invariance by differentiation property plays an essential role to achieve order reduction, instead of general dimensional reduction. In particular, in the scalar case, Frobenius reduction method naturally leads to definition of λ–symmetries (reducing to standard symmetries for $\lambda = 0$) as the natural vector fields to be used to reduce the order of a single k^{th} order ODE.

On the other hand, when we deal with higher order systems of ODEs, σ–symmetries arise as a natural tool to obtain suitable dimensional reductions corresponding to order reductions: if we consider a system of n ODEs of order k admitting an n–dimensional distribution of σ–symmetries, the invariance by differentiation property ensures that the proposed reduction procedure provide us with a reduced system of n ODEs of order $(k-1)$.

Finally we address the reconstruction problem for a system of n ODEs of order k and we define a new class of σ–symmetries, which we call triangular σ–symmetries, characterized by the property that the distribution generated by the σ–prolonged vector fields is a partial solvable structure for the vector field associated with the system $\mathscr{E}$. Once again, under these hypotheses, the solution to $\mathscr{E}$ can be obtained starting from the knowledge of the solution to the reduced system $\overline{\mathscr{E}}$ and integrating by quadrature a given system of closed one–forms.

The plan of the paper is the following: in Section 2 we collect some basic definitions about the geometry of differential equations and distributions of vector fields. In Section 3 we describe the dimensional reduction for a vector field and we address the reconstruction problem in terms of partial solvable structures. Finally, in Section 4 we apply previous results to higher order ODEs.

2. Preliminaries

2.1. *ODEs as submanifolds of jets spaces*

Let $(\mathbb{R}\times Q,\pi,\mathbb{R})$ be a fiber manifold, with Q is an n–dimensional smooth manifold. We assume that the local coordinates on $\mathbb{R}\times Q$ are (t,x^a), where x^a denote the dependent variables and $a=1,\dots,n$. Let us consider the k^{th} order jet bundle $J^k(\pi)$ with local coordinates (t,x^a,x^a_i), where x^a_i denote the derivatives of x^a with respect to t up to a fixed order i. It is well known that the k^{th} order jet space $J^k(\pi)$ is naturally equipped with the *contact distribution* $\mathscr{C}^k$ that, in local coordinates, is spanned by

$$\omega^a_i = dx^a_i - x^a_{i+1}dt, \qquad 0\le i\le k-1 \quad 1\le a\le n.$$

Given a system of n ODEs $\Delta^b(t,x^a,x^a_i)=0$ of order k which can be put in the normal form

$$x^a_k = F^a(t,x^b,x^b_1,\dots,x^b_{k-1}) \qquad (a,b=1,\dots,n), \tag{2.1}$$

it is possible to identify (2.1) with a submanifold $\mathscr{E}$ of the jet space $J^k(\pi)$. In this framework a solution to (2.1) is a section of π whose k^{th} order prolongation is an integral manifold of the restriction $\mathscr{C}^k\,|_{\mathscr{E}}$ of the contact distribution to $\mathscr{E}$.

Let Z be the vector field on $\mathscr{E}$ obtained by restricting the total derivative operator $D_t := \partial_t + x^a_1\partial_{x^a} + \ldots + x^a_k\partial_{x^a_{k-1}}$ to $\mathscr{E}$. In local coordinates we have

$$Z = \frac{\partial}{\partial t} + x^a_1\frac{\partial}{\partial x^a} + \ldots + F^a\frac{\partial}{\partial x^a_{k-1}}\,, \tag{2.2}$$

and the integral lines of Z are the solutions to the following system of first order ODEs

$$\begin{cases} \dot{x}^a = x^a_1 \\ \dot{x}^a_1 = x^a_2 \\ \vdots \\ \dot{x}^a_{k-1} = F^a \end{cases} \tag{2.3}$$

where $x^a, x^a_1, \ldots, x^a_{k-1}$ are considered as new dependent variables.
A vector field on $\mathbb{R}\times Q$ of the form

$$X = \xi(t,x^b)\frac{\partial}{\partial t} + \phi^a(t,x^b)\frac{\partial}{\partial x^a}$$

is a point symmetry of (2.1) if its k^{th} order prolongation

$$X^{(k)} = \xi\frac{\partial}{\partial t} + \phi^a\frac{\partial}{\partial x^a} + \Phi^a_{(s)}\frac{\partial}{\partial x^a_s} \qquad s = 1,\ldots,k$$

satisfies

$$X^{(k)}\Delta^b(t,x^a,x^a_i) = 0 \quad \text{whenever} \quad \Delta^b(t,x^a,x^a_i) = 0.$$

Here the coefficients $\Phi^a_{(s)}$ are given by the standard prolongation formula

$$\Phi^a_{(s)} = D_t\Phi^a_{(s-1)} - x^a_s D_t\xi, \qquad \Phi^a_0 = \phi^a, \qquad s = 1,\ldots,k. \tag{2.4}$$

In terms of the vector field Z defined by (2.2) a vector field X on $\mathbb{R}\times Q$ is a symmetry of (2.1) if $[X^{(k)},Z] = hZ$ where h is a suitable smooth function on $\mathscr{E}$.

2.2. *Distributions of vector fields*

Let M be a n–dimensional smooth manifold: we denote by $\mathscr{X}(M)$ the Lie algebra of smooth vector fields on M, by $\Lambda(M)$ the graded algebra of differential forms on M and by $\Lambda^k(M)$ the set of k–forms on M.

Given a set of vector fields $\{Y_1,\ldots,Y_r\}$ on M, we denote by $\mathscr{D} := \langle Y_1,\ldots,Y_r\rangle$ the distribution generated by Y_i $(i = 1,\ldots,r)$. If the vector fields Y_i are pointwise linearly independent in an open domain $\mathscr{U}$ of M, we say that $\mathscr{D}$ is a distribution of *maximal constant rank* r on $\mathscr{U}$. A distribution $\mathscr{D}$ is said to be *integrable* (in Frobenious sense) if $[X,Y]\in\mathscr{D}$, $\forall X,Y\in\mathscr{D}$. It is well known that an integrable distribution $\mathscr{D}$ of constant rank r in $\mathscr{U}$ generates an r–dimensional foliation of $\mathscr{U}$ whose leaves are described by some level manifolds of $n-r$ functions $I_i = c_i$, where $i = 1,\ldots,n-r$ and c_i are constants. Under this hypothesis, we can define a projection map

$$q : \mathscr{U} \to \mathscr{U}/\mathscr{D} := \overline{\mathscr{U}} \tag{2.5}$$

as $q(x) = S_x$, where S_x denotes the connected maximal integral manifold of $\mathscr{D}$ through x. Here and throughout the paper we suppose that the projection map q is a smooth submersion.

Given a distribution $\mathscr{D}$, a vector field X is a symmetry of $\mathscr{D}$ if $[X,Y] \in \mathscr{D}$, $\forall Y \in \mathscr{D}$. Let $\mathscr{D}$ and $\mathscr{F}$ be two distributions on M. We say that $\mathscr{D}$ and $\mathscr{F}$ are *transversal*, at $p \in M$, iff they do not vanish at p and $\mathscr{D}(p) \cap \mathscr{F}(p) = \{0\}$. Analogously, $\mathscr{D}$ and $\mathscr{F}$ are transversal iff they are transversal at any point.

3. Reduction and reconstruction problem

In this section we present a reduction method for a vector field Z on a manifold M, based on the knowledge of a suitable integrable distribution $\mathscr{D}_0$ transversal to Z. In particular, we are interested in a dimensional reduction of Z as explained by the following

Definition 3.1. Given a vector field Z on a n–dimensional manifold M, we call reduction map for Z a smooth submersion $q : M \to \overline{M}$ such that

- $dim(\overline{M}) < dim(M)$
- there exists a vector field $\overline{Z}$ on $\overline{M}$ such that the integral lines of Z project onto the integral lines of $\overline{Z}$.

We recall that Definition 3.1 said that the vector field Z is projectable with respect to q (see [18]), and the following Theorem can be seen as a recasting of Frobenious Theorem leading to orbital reduction for Z.

Theorem 3.1. *Let Z be a vector field on a smooth n–dimensional manifold M and $\mathscr{D}_0 = \langle Y_1, \dots, Y_r \rangle$ be an integrable distribution of maximal constant rank r in an open domain $\mathscr{U} \subseteq M$ such that the projection map q defined by (2.5) is a smooth submersion. If $\mathscr{D}_0$ is transversal to Z and $\mathscr{D} = \langle Z, Y_1, \dots, Y_r \rangle$ is an integrable distribution, then it is possible to define a reduction map $q : \mathscr{U} \to \overline{\mathscr{U}}$ for Z, where $\overline{\mathscr{U}} = \mathscr{U}/\mathscr{D}_0$.*

Proof. Given $x \in \mathscr{U}$, let S_x denote the connected maximal integral manifold of $\mathscr{D}_0$ through x. We define the quotient map

$$q : \mathscr{U} \to \mathscr{U}/\mathscr{D}_0 = \overline{\mathscr{U}}$$

so that $q(x) = S_x$. Let $\mathscr{I}$ be the submodule of $\Lambda^1(\mathscr{U})$ defined by

$$\mathscr{I} = \{\beta \in \Lambda^1(\mathscr{U}) |\ Z \lrcorner \beta = 0, Y_i \lrcorner \beta = 0, \qquad i = 1, \dots r\}.$$

As $\mathscr{D}_0$ and $\mathscr{D}$ are both integrable distributions, $\forall Y \in \mathscr{D}_0$ we have

$$L_Y(\mathscr{I}) \subseteq \mathscr{I}$$

and, following [1, 12], we can consider the submodule of $\Lambda^1(\overline{\mathscr{U}})$ given by

$$\overline{\mathscr{I}} = \{\beta \in \Lambda^1(\overline{\mathscr{U}}) |\ q^*(\beta) \in \mathscr{I}\}.$$

The annihilator of $\overline{\mathscr{I}}$ is a one–dimensional distribution in $\overline{\mathscr{U}}$ generated by a vector field $\overline{Z}$ such that the integral lines of Z project onto integral lines of $\overline{Z}$. Therefore the projection

$$q : \mathscr{U} \to \mathscr{U}/\mathscr{D}_0 = \overline{\mathscr{U}}$$

turns out to be a reduction map for Z. □

Remark 3.1. Given a vector field Z, the knowledge of an r–dimensional algebra $\mathscr{G}$ of non trivial symmetries for Z allows us to perform above reduction. In fact, under this hypotheses, $\mathscr{D}_0 = \mathscr{G} = \langle Y_1, \ldots, Y_r \rangle$ is an integrable distribution and the symmetry conditions $[Y_i, Z] = h_i Z$ ensure that the distribution $\mathscr{D} = \langle Z, Y_1, \ldots, Y_r \rangle$ is integrable as well. Hence, standard symmetry reduction turns out to be a particular case of this general reduction procedure recasting classical Frobenious theorem in the language of Lie symmetries.

The general reduction procedure described in Theorem 3.1 allows us to address the reconstruction problem in terms of the distribution $\mathscr{D}_0$. In particular, in the following we discuss when and how it is possible to reconstruct the general solutions to the system $\mathscr{E}$ (associated with the original vector field Z) starting from the knowledge of the general solutions to the reduced system associated with $\overline{Z}$. For the convenience of the reader we start by recalling some basic definitions and facts about solvable structures for a given vector field (see [2,3,7,8,16] for a more general discussion on solvable structures).

Definition 3.2. Given a vector field Z on a n–dimensional manifold M, a set of vector fields $\{Y_1, Y_2, \ldots, Y_{n-1}\}$ is a *solvable structure* for Z on an open domain $\mathscr{U} \subseteq M$ if and only if, denoting by

$$\mathscr{D}_r = \langle Z, Y_1, \ldots, Y_r \rangle \qquad (r \leq n-1)$$

the following conditions are satisfied:

(1) $\langle Y_1, Y_2, \ldots, Y_r \rangle$ is an r–dimensional distribution transversal to Z on $\mathscr{U}$, for any $r \leq n-1$;
(2) $\mathscr{D}_r$ is an $(r+1)$–dimensional distribution on $\mathscr{U}$;
(3) $L_{Y_1}(Z) = hZ$ and $L_{Y_r}\mathscr{D}_{r-1} \subseteq \mathscr{D}_{r-1}$, $\forall r \in \{1, \ldots, n-1\}$.

The next Theorem, proved in [2,3,16], shows how the knowledge of a solvable structure for a given vector field Z allows us to obtain the integral lines of Z by quadratures.

Theorem 3.2. *Let Z be a vector field on an orientable n–dimensional manifold M and $\{Y_1, \ldots, Y_{n-1}\}$ be a solvable structure for Z on an open domain $\mathscr{U} \subseteq M$. Then, $\forall x \in \mathscr{U}$, the integral lines of Z passing through x can be found by quadratures.*

Proof. Denoting by Ω a volume form on M, the one–dimensional distribution generated by Z can be described as the annihilator of the submodule of $\Lambda^1(\mathscr{U})$ generated by the one–forms

$$\Omega_k = \frac{1}{\Delta}(Y_1 \lrcorner \ldots \lrcorner \widehat{Y}_k \lrcorner \ldots \lrcorner Y_{n-1} \lrcorner Z \lrcorner \Omega), \qquad (k = 1, \ldots, n-1)$$

where the hat denotes omission of the corresponding vector field and Δ is the function on $\mathscr{U}$ defined by $\Delta = Y_1 \lrcorner Y_2 \lrcorner \ldots \lrcorner Y_{n-1} \lrcorner Z \lrcorner \Omega$. Moreover the one–forms Ω_i satisfy

$$\begin{aligned} &d\Omega_{n-1} = 0, \\ &d\Omega_k = 0 \mod\{\Omega_{k+1}, \ldots, \Omega_{n-1}\} \end{aligned}$$

for any $k \in \{1, \ldots, n-2\}$ (see [2,3,16]) for a detailed proof of this fact). Thus the integral lines of Z can be described in implicit form as level manifolds $\Gamma_c = \{I_1 = c_1,\ I_2 = c_2,\ \ldots,\ I_{n-1} = c_{n-1}\}$, where each I_k is defined as an integral of the one–form Ω_k restricted to level manifolds $\{I_{n-1} = c_{n-1}, \ldots, I_{k+1} = c_{k+1}\}$. □

It is clear by the Definition 3.2 that, in principle, a maximal solvable structure for Z always exists, in a neighborhood of a non-singular point for Z. Nevertheless, for a given vector field Z, it may be difficult to find such a structure explicitly. For this reason we introduce the following weaker definition, that turns out to be useful when we deal with the reconstruction problem (see Theorem 3.3).

Definition 3.3. Given a vector field Z on a n–dimensional orientable manifold M, a set of vector fields $\{Y_1, Y_2, \ldots, Y_h\}$ (with $h < n-1$) is a *partial solvable structure* for Z on an open domain $\mathscr{U} \subseteq M$ if and only if, denoting by

$$\mathscr{D}_r = \langle Z, Y_1, \ldots, Y_r \rangle \qquad (r \leq h),$$

the following conditions are satisfied:

(1) $\langle Y_1, Y_2, \ldots, Y_r \rangle$ is an r–dimensional distribution transversal to Z on $\mathscr{U}$, for any $r \leq h$;
(2) $\mathscr{D}_r$ is an $(r+1)$–dimensional distribution on $\mathscr{U}$;
(3) $L_{Y_1}(Z) = hZ$ and $L_{Y_r}\mathscr{D}_{r-1} \subset \mathscr{D}_{r-1}$, $\forall r \in \{1, \ldots, h\}$.

Theorem 3.3. *Let $\{Y_1, Y_2, \ldots, Y_r\}$ be a partial solvable structure for the vector field Z on an open domain $\mathscr{U} \subseteq M$ such that the distribution $\mathscr{D}_0 = \langle Y_1, Y_2, \ldots, Y_r \rangle$ is integrable. If we consider the reduction map associated with $\mathscr{D}_0$ and the corresponding reduced vector field $\overline{Z}$, then the integral lines of Z can be obtained by quadratures starting from the knowledge of the integral lines of $\overline{Z}$.*

Proof. The knowledge of integral lines for the reduced vector field $\overline{Z}$ implies the knowledge of a set of generators for $\overline{\mathscr{I}}$ given by exact forms dF_i, $(i = 1, \ldots, n-r-1)$. If we consider the pullback of these forms along the projection $q : \mathscr{U} \to \overline{\mathscr{U}}$ we get $(n-r-1)$ exact one–forms on $\mathscr{U}$

$$q^*(dF_i) = d(q^*(F_i)) = dG_i$$

such that $Z(G_i) = 0$ and $Y_k(G_i) = 0$. Hence we can restrict Z and Y_k to the $(r+1)$–dimensional manifold

$$\Sigma_c : \{G_i = c_i\}$$

and previous hypotheses ensure that the restriction of $\mathscr{D}_0 = \langle Y_1, Y_2, \ldots, Y_r \rangle$ to Σ_c is a solvable structure for the restriction of Z to Σ_c. Therefore the reconstruction problem can be solved by quadratures by using Theorem 3.2. □

The following simple explicit example illustrates how theorem 3.3 works in practise.

Example 3.1. Let M be $\mathbb{R}^5$ with local coordinate (t, x, y, z, w) and let $\mathscr{U}$ be the open subset of $\mathbb{R}^5$ given by $\mathscr{U} = \{(t, x, y, z, w) \in \mathbb{R}^5 \mid yzw \neq 0\}$. Given the vector field

$$Z = \frac{\partial}{\partial t} + yz\frac{\partial}{\partial x} + \frac{1}{yz}e^{\frac{1}{2}y^2}\frac{\partial}{\partial y} + e^{y^2+\frac{1}{w}}\frac{\partial}{\partial z} + \frac{w^2}{ze^{\frac{1}{w}}}\frac{\partial}{\partial w}, \tag{3.1}$$

it is easy to check that the vector fields

$$Y_1 = \frac{\partial}{\partial x}, \qquad Y_2 = \frac{1}{y}\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}$$

provide a partial solvable structure for Z, being

$$[Y_1, Z] = 0, \;\; [Y_2, Y_1] = 0, \;\; [Y_2, Z] = \frac{z}{y}(1+y^2)Y_1.$$

If we consider the reduction map $q : \mathscr{U} \to \overline{\mathscr{U}}$ associated with the distribution $\mathscr{D}_0 = \langle Y_1, Y_2 \rangle$, we can take as natural coordinate on $\overline{\mathscr{U}}$ the joint invariants of the distribution $\mathscr{D}_0$

$$\xi = t, \;\; \eta_1 = \frac{1}{z} e^{\frac{1}{2}y^2}, \;\; \eta_2 = z e^{\frac{1}{w}}.$$

Then, following Theorem 3.1, we get a reduced vector field on $\overline{\mathscr{U}}$ given by

$$\overline{Z} = \frac{\partial}{\partial \xi} + \eta_1^2(1 - \eta_1\eta_2)\frac{\partial}{\partial \eta_1} + (\eta_1^2\eta_2^2 - 1)\frac{\partial}{\partial \eta_2}. \tag{3.2}$$

In order to show how the reconstruction procedure works, let us denote by $\eta_i = \varphi_i(\xi)$ $(i = 1, 2)$ the general solution to the system associated with the vector field $\overline{Z}$ given by (3.2). If we rewrite this solution in the implicit form $F_i(\xi, \eta_1, \eta_2) = c_i$ with $(i = 1, 2)$, we can consider the submanifold Σ_c of $\mathscr{U}$ defined by

$$\Sigma_c := \{G_i = c_i\}, \tag{3.3}$$

where $G_i := q^*(F_i)$ denotes the pullback of F_i along the projection q. Choosing coordinate (t, x, y) on Σ_c and $\Omega = dt \wedge dx \wedge dy$ as a volume form over there, we can use the partial solvable structure $\{Y_1, Y_2\}$ to compute the 1–forms

$$\begin{aligned} \Omega_2 &= \tfrac{1}{\Delta}(Y_1 \lrcorner Z \lrcorner \Omega) = y dy - \tfrac{1}{z} e^{\frac{1}{2}y^2} dt \\ \Omega_1 &= \tfrac{1}{\Delta}(Y_2 \lrcorner Z \lrcorner \Omega) = dx - yz dt. \end{aligned}$$

It is easy to check that the one–form Ω_2 is closed, being

$$\Omega_2 = y dy - \eta_1 dt = y dy - \varphi_1(t) dt = d\left[\frac{1}{2}y^2 - \int_{t_0}^{t} \varphi_1(t) dt\right] = dI_2.$$

Therefore we can find the explicit solution $y = \psi(t)$ by solving $I_2 = k_2$. Moreover, the restriction $\overline{\Omega}_1$ of Ω_1 to the submanifold $\Sigma_c \cap \{I_2 = k_2\}$ is a closed form, being

$$\overline{\Omega}_1 = dx - \psi(t)\frac{e^{\frac{1}{2}\psi^2(t)}}{\varphi_1(t)} dt$$

and the integration of this closed one–form provides the general solution for the integral lines of the vector field Z given by (3.1).

4. Higher order ODEs

When we consider a (system of) higher order ODEs it is quite natural to look at previous results from a slightly different perspective. In this case, in fact, we are in general more interested in reducing the order of the ODEs than in obtaining a general dimensional reduction. In the following we show that this order reduction arises as a particular case of previous dimensional reduction when we choose a distribution $\mathscr{D}_0$ suitably adapted to the jet bundle structure.

4.1. *Reduction of order*

Let $\mathscr{E}$ be a system of n ODEs of order k in the normal form (2.1) and Z be the vector field on $\mathscr{E}$ defined by (2.2). If we apply the reduction procedure proposed in Section 3 by using a general completely integrable distribution $\mathscr{D}_0$ of rank r on $\mathscr{E}$, we get a reduced vector field $\overline{Z}$ on $\mathscr{U}/\mathscr{D}_0$ (with $\mathscr{U} \subseteq \mathscr{E}$) encoding a system of $nk-r$ first order equations which may not be equivalent to a system of higher order ODEs. On the other hand, if we are interested in order reduction instead of dimensional reduction, we have to preserve the jet bundle structure choosing a distribution $\mathscr{D}_0$ suitably adapted to $J^k(\pi)$. Let $\pi : \mathbb{R} \times Q \to \mathbb{R}$ be a fibred manifold, with Q an n–dimensional manifold, and let $\mathscr{B} = \langle X_1, \ldots, X_n \rangle$ be a completely integrable distribution of constant rank n on $\mathbb{R} \times Q$. If $\mathscr{B}^r$ denotes the distribution on $J^r(\pi)$ generated by the r^{th} order prolongations of the vector fields X_i and $\mathscr{B}^r$ has constant rank n on $\mathscr{U} \subseteq J^r(\pi)$, then the following diagram commutes:

$$\begin{array}{ccc} J^k(\pi) & \xrightarrow{\sigma_k} & J^k(\pi)/\mathscr{B}^k \simeq J^{k-1}(\overline{\pi}_0) \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \overline{\pi}_{k-1}} \\ J^1(\pi) & \xrightarrow[\sigma_1]{} & J^1(\pi)/\mathscr{B}^1 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \overline{\pi}_0} \\ \mathbb{R} \times Q & \xrightarrow[\sigma_0]{} & (\mathbb{R} \times Q)/\mathscr{B} \end{array} \tag{4.1}$$

In fact, the prolongation formula (2.4) ensures that $[X_i^{(r)}, X_k^{(r)}] = [X_i, X_k]^{(r)}$ and the distribution $\mathscr{B}^r$ is an integrable distribution of constant rank n on $\mathscr{U} \subseteq J^r(\pi)$. Moreover, a natural coordinates system in the reduced space $\overline{\mathscr{U}} = \mathscr{U}/\mathscr{B}^r$ is given by the invariants of the distribution $\mathscr{B}^r$. We recall that, due to the prolongation formula (2.4), given two invariants η and ζ for $\mathscr{B}^r$, the quotient $\eta_1 := D_t(\eta)/D_t(\zeta)$ provides a new invariant for $\mathscr{B}^r$. This property is often called "invariance by differentiation property" (IBDP) and ensures that it is possible to obtain higher order invariants for $\mathscr{B}^r$ starting from the knowledge of lower order ones. In particular, the knowledge of a zero order invariant ζ and of a complete set of first order invariants η^a $(a = 1, \ldots, n)$ for $\mathscr{B}^r$ allows us to find a natural coordinate system in $\overline{\mathscr{U}} = \mathscr{U}/\mathscr{B}^r$ given by $\eta_i^a := D_t^i(\eta^a)/D_t^i(\zeta)$, $(a = 1, \ldots, n, \quad i = 1, \ldots r-1)$. Therefore we can consider the contact forms $\overline{\omega}_i^a$

$$\overline{\omega}_i^a = d\eta_i^a - \eta_{i+1}^a d\zeta.$$

on $J^k(\pi)/\mathscr{B}^k$ providing an isomorphism between $J^k(\pi)/\mathscr{B}^k$ and the $(k-1)$ order jet bundle $J^{k-1}(\overline{\pi}_0)$.

This result obviously applies to the reduction of a system of n ODEs of order k admitting an n–dimensional algebra $\mathscr{G}$ of Lie point symmetries, but can be extended to vector fields more general than prolonged ones, providing a geometrical interpretation of reduction by λ–symmetries for scalar k^{th} order ODEs and of reduction by σ–symmetries for systems of k^{th} order ODEs. For the convenience of the reader we recall here some definitions and facts about λ and σ–symmetries. The interested reader is referred to [6, 9, 10, 14, 19] for more details on these topics.

Definition 4.1. Given a vector field

$$X = \xi(t,x)\partial_t + \phi(x,t)\partial_x$$

on a fibred manifold $\pi : \mathbb{R} \times Q \to \mathbb{R}$, where Q is a one–dimensional manifold, we say that the vector field

$$Y = \xi(t,x)\partial_t + \phi(x,t)\partial_x + \Psi_1\partial_{x_1} + \ldots + \Psi_k\partial_{x_k}$$

on $J^k(\pi)$ is the λ–prolongation of X if and only if

$$\Psi_i = (D_t + \lambda)\Psi_{i-1} - x_i(D_t + \lambda)\xi \qquad \Psi_0 = \phi \ ,$$

where D_t denotes the total derivative operator.

It is easy to prove that a λ–prolonged vector field Y can be characterized by the condition $[D_t, Y] = hD_t + \lambda Y$ where h is a suitable smooth function on $J^k(\pi)$.

Given a k^{th} order ODE $\mathscr{E}$ written in normal form

$$x_k = F(t,x,x_1,\ldots,x_{k-1})$$

we can consider the vector field

$$Z = \partial_t + x_1\partial_x + \ldots + F\partial_{x_{k-1}} \tag{4.2}$$

whose integral lines correspond to the solutions to $\mathscr{E}$.

Definition 4.2. A vector field Y is a *λ–symmetry* for $\mathscr{E}$ if Y is a λ–prolonged vector field satisfying

$$[Y,Z] = hZ + \lambda Y \tag{4.3}$$

where Z is given by (4.2) and h is a suitable smooth function on $\mathscr{E}$.

The following Theorem recasts in our framework the result of [19], showing that λ–symmetries are as effective as standard ones in order to achieve reduction for a given ODE.

Theorem 4.1. *Let $\mathscr{E}$ be a k^{th} order ODE and let Y be a λ–symmetry for the vector field Z given by (4.2). Then it is possible to obtain a $(k-1)^{th}$ order equation $\overline{\mathscr{E}}$ such that the the solutions to $\mathscr{E}$ project onto the solution to $\overline{\mathscr{E}}$.*

We remark that, if we deal with non-trivial λ–symmetries, the reconstruction problem cannot be solved by quadrature. In fact, the only possibility for the vector field Y to provide a solvable structure for Z is that Y is a standard Lie point symmetry for Z.

In order to generalize λ–symmetry reduction to systems of n ODEs of order k we have to face up to the fact that, in general, λ–symmetries form neither a Lie algebra nor an integrable distribution, preventing us to use more than one λ–symmetry in the reduction process. For this reason, when we deal with systems of ODEs, the following generalization of Definition 4.2 is quite natural (see [9–11]).

Definition 4.3. Let $\mathscr{B} = \langle X_1, \dots, X_r \rangle$ be an integrable distribution on a fibred manifold $\mathbb{R} \times Q \to \mathbb{R}$, where Q is a n–dimensional manifold and $n > 1$. If the vector fields X_i are written in local coordinates as

$$X_i = \xi_i(t,x)\partial_t + \phi_i^a(x,t)\partial_{x^a},$$

we say that the vector fields

$$Y_i = \xi_i(t,x)\partial_t + \phi_i^a(x,t)\partial_{x^a} + \Psi_{i1}^a \partial_{x_1^a} + \ldots + \Psi_{ik}^a \partial_{x_k^a}$$

on $J^k(\pi)$ are σ–*prolongation* of X_i if and only if they satisfy the following conditions:

$$[D_t, Y_i] = h_i D_t + \sigma_{ij} Y_j \,. \tag{4.4}$$

It is easy to prove that if $\mathscr{D}_0 = \langle Y_1, \dots, Y_r \rangle$ is the σ–prolongation of a distribution $\mathscr{B}$ on $Q \times \mathbb{R}$, the IBDP property for $\mathscr{D}_0$ holds. In particular, if $r = n$, given a set of joint invariants ζ, η^a for $\mathscr{D}_0$ of order zero and one respectively, the higher order joint invariants of $\mathscr{D}_0$ are given by $\eta_i^a := D_t^i(\eta^a)/D_t^i(\zeta)$. This ensure that, if we consider σ–prolongations instead of standard prolongations, diagram (4.1) still commutes. In order to use this σ–prolonged vector fields to reduce a system of ODEs, we give the following

Definition 4.4. Given a system of ODEs $\mathscr{E}$ of the form (2.1), an integrable distribution $\mathscr{B} = \langle X_1, \dots, X_r \rangle$ is a σ–*symmetry* of $\mathscr{E}$ if the distribution $\mathscr{D}_0 =< Y_1, \dots, Y_r >$ generated by the σ–prolonged vector fields Y_i is integrable and the following condition holds

$$[Y_i, Z] = h_i Z + \sigma_{ij} Y_j$$

where Z is given by (2.2) and h_i are suitable smooth functions on $\mathscr{E}$.

The following Theorem recasts the result of [9–11] in order to show that the reductions derived by σ–symmetries fit in the scheme of dimensional reduction given by Theorem 3.1.

Theorem 4.2. *Let $\mathscr{E}$ be a system of n ODEs of order k of the form (2.1) and let Z be the vector field (2.2). The knowledge of a σ–symmetry $\mathscr{B} = \langle X_1, \dots, X_n \rangle$ of (2.1) such that $\mathscr{D}_0$ is of constant rank on an open domain $\mathscr{U} \subseteq J^k(\pi)$ allows us to define a reduction map $q : \mathscr{U} \to \overline{\mathscr{U}} = \mathscr{U}/\mathscr{D}_0$. Moreover, the reduced vector field corresponds to a system $\overline{\mathscr{E}}$ of n equations of order $(k-1)$ such that the solutions to $\mathscr{E}$ project onto the solution to $\overline{\mathscr{E}}$.*

4.2. *Reconstruction problem*

As we have already pointed out in Section 3, the reconstruction problem depends on the nature of the distribution $\mathscr{D}_0$ involved in the reduction procedure. In this section we define a particular class of σ–symmetries (that we call triangular σ–symmetries) allowing us to solve the reconstruction problem, *i.e.* to obtain the solution to the initial system of ODEs by quadratures starting from the knowledge of the solution to the reduced one. We recall that one of the limitations of σ–symmetries reduction method is the difficulty of actually determining σ–symmetries of a given system, due to the number of unknown coefficients involved in the determining equations ([9–11]). An advantage of the following definition is that the number of non zero coefficients to be found is reduced, so that triangular σ–symmetries for a given system might be easier to find than general ones.

Definition 4.5. Given a system of ODEs $\mathscr{E}$ of the form (2.1) and the corresponding vector field Z (2.2), an integrable distribution $\mathscr{B} = \langle X_1, \dots, X_r \rangle$ is a *triangular σ–symmetry* of $\mathscr{E}$ if $\mathscr{B}$ is a σ-symmetry and the σ–prolonged vector fields $\{Y_1, \dots, Y_r\}$ provide a partial solvable structure for Z.

Theorem 4.3. *Let $\mathscr{B} = \langle X_1, \dots, X_n \rangle$ be a triangular σ–symmetry for a given a system of n ODEs $\mathscr{E}$ of order k. The integrable distribution $\mathscr{D}_0 = \langle Y_1, Y_2, \dots, Y_n \rangle$ generated by the σ–prolongations of $\mathscr{B}$ allows us to define a reduction map $q : \mathscr{U} \to \mathscr{U}/\mathscr{D}_0$ such that the general solution to $\mathscr{E}$ can be obtained from the general solution to the reduced system $\overline{\mathscr{E}}$ integrating by quadratures a system of closed one–forms.*

Proof. It is just a particular case of Theorem 3.3. □

4.3. *Example*

In this section we provide a simple explicit example in order to show how previous reduction and reconstruction procedure can be performed in practise.
Given the second order system

$$\begin{cases} \ddot{x} = y + \dot{x} \\ \ddot{y} = \dot{x} - \dot{y} \end{cases} \tag{4.5}$$

corresponding tho the vector field

$$Z = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + (y + \dot{x})\frac{\partial}{\partial \dot{x}} + (\dot{x} - \dot{y})\frac{\partial}{\partial \dot{y}} ,$$

the integrable distribution $\mathscr{B}$ generated by $X_1 = \partial_x$ and $X_2 = y\partial_y$ is a σ–symmetry for (4.5) with

$$\sigma = \begin{pmatrix} 0 & 0 \\ \dot{x} & 0 \end{pmatrix} . \tag{4.6}$$

In particular, the σ–prolongation of $\mathscr{B} = \langle X_1, X_2 \rangle$ is generated by the vector fields

$$Y_1 = \frac{\partial}{\partial x}, \qquad Y_2 = y\frac{\partial}{\partial y} + \dot{x}\frac{\partial}{\partial \dot{x}} + \dot{y}\frac{\partial}{\partial \dot{y}}$$

satisfying

$$[Y_1, Y_2] = 0 \qquad [Y_1, Z] = 0, \qquad [Y_2, Z] = \dot{x}Y_1 ,$$

so that the distribution $\mathscr{D}_0 = \langle Y_1, Y_2 \rangle$ is integrable and defines a solvable structure for Z. Hence, we can reduce system (4.5) considering the joint invariants of $\mathscr{D}_0$

$$\xi = t, \qquad \eta_1 = \frac{\dot{x}}{y}, \qquad \eta_2 = \frac{\dot{y}}{y}$$

and choosing a section s of the projection map q defined, for example, by

$$t = \xi, x = 1, y = 1, \dot{x} = \eta_1, \dot{y} = \eta_2.$$

The submodule of the one forms annihilated by Z, Y_1 and Y_2 is generated by the two one–forms

$$\begin{aligned} \beta_1 &= y d\dot{x} - \dot{x} dy + (\dot{x}\dot{y} - y^2 - y\dot{x}) dt \\ \beta_2 &= y d\dot{y} - \dot{y} dy + (\dot{y}^2 - y\dot{x} + y\dot{y}) dt \end{aligned} \tag{4.7}$$

whose pullback along the section s are

$$\begin{array}{l} s^*(\beta_1) = d\eta_1 + (\eta_1\eta_2 - 1 - \eta_1)d\xi \\ s^*(\beta_2) = d\eta_2 + (\eta_2^2 - \eta_1 + \eta_2)d\xi \ , \end{array} \tag{4.8}$$

corresponding to the reduced system

$$\begin{cases} \dot{\eta}_1 = 1 + \eta_1 - \eta_1\eta_2 \\ \dot{\eta}_2 = \eta_1 - \eta_2 - \eta_2^2 \ . \end{cases} \tag{4.9}$$

In this case, the triangular structure of the σ–symmetry $\mathscr{B}$ allows us to solve the reconstruction problem. In fact, denoting by $\eta_i = \varphi_i(\xi)$ $(i = 1,2)$ the general solution to the reduced system (4.9) and rewriting this solution in the implicit form $F_i(\xi, \eta_1, \eta_2) = c_i$ with $(i = 1,2)$, we can consider the submanifold Σ_c of $\mathscr{U}$ defined by

$$\Sigma_c := \{G_i = c_i\}, \tag{4.10}$$

where $G_i := q^*(F_i)$ denotes the pullback of F_i along the projection q. Then, choosing (t,x,y) as coordinates on Σ and $\Omega = dt \wedge dx \wedge dy$ as a volume form over there, we can compute $\Delta = Z \lrcorner X_2 \lrcorner X_1 \lrcorner \Omega = y$ and Theorem 3.2 ensures that the one–form $\Omega_2 = (1/\Delta)(Z \lrcorner Y_1 \lrcorner \Omega)$ is closed on Σ. In fact, by explicit computation we find

$$\Omega_2 = -\frac{1}{y}dy + \frac{\dot{y}}{y}dt$$

and the knowledge of the general solution $\eta_i = \varphi_i(\xi)$ $(i = 1,2)$ to the reduced system (4.9) ensures that $\eta_2 = \frac{\dot{y}}{y} = \varphi_2(t)$. Hence

$$\Omega_2 = -\frac{1}{y}dy + \varphi_2(t)dt = -d\left(\log y - \rho_2(t)\right) = dI_2 \ ,$$

where $\rho_2(t)$ satisfies $\rho_2'(t) = \varphi_2(t)$. Therefore, the one–form Ω_2 is (locally) exact and we can explicitly compute the solution $y(t) = K_1 e^{\rho_2(t)}$, with $K_1 \in \mathbb{R}$. The next step of the reconstruction process consists in considering the one–form

$$\Omega_1 := (1/\Delta)(Z \lrcorner Y_2 \lrcorner \Omega) = dx - \dot{x}dt$$

which, as expected, is not closed on Σ. Nevertheless Ω_1 turns out to be closed when we restrict to $\Sigma_c \cap \{I_2 = k_2\}$. In fact

$$\dot{x} = y\eta_1 \ = \ K_1 e^{\rho_2(t)}\varphi_1(t)$$

and we can find the general solution

$$\begin{cases} x(t) = \int_{t_0}^t K_1 e^{\rho_2(t)}\varphi_1(t)dt \\ y(t) = K_1 e^{\rho_2(t)}. \end{cases}$$

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