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REDUCTION IN PRINCIPAL FIBER BUNDLES: COVARIANT EULER-POINCARÉ EQUATIONS

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ABSTRACT. Let $\pi : P \to M^n$ be a principal G-bundle, and let $\mathcal{L} : J^1P \to \Lambda^n(M)$ be a G-invariant Lagrangian density. We obtain the Euler-Poincaré equations for the reduced Lagrangian l defined on $\mathcal{C}(P)$, the bundle of connections on P.

1. Introduction

Classical Euler-Poincaré equations arise through a reduction of the variational principal $\int_a^b L(\dot{g}(t))dt$ where $L : TG \to \mathbb{R}$ is a G-invariant Lagrangian defined on the tangent bundle of a Lie group G. In this setting, one defines the reduced Lagrangian $l : TG/G \cong \mathfrak{g} \to \mathbb{R}$ by $l(\xi) = L(R_{g^{-1}}\dot{g})$ (or by left-translation depending on the Lagrangian), and proves that with a restricted class of variations, the extremal ξ of $\int_a^b l(\xi(t))dt$ is equivalent to the extremal of the original variational problem for L.

The purpose of this note is to extend the variational reduction program to the setting of a principle fiber bundle $\pi : P \to M$, using the fact that $J^1P/G \cong \mathcal{C}(P)$, where J^1P is the first jet bundle of P, and $\mathcal{C}(P)$ denotes the bundle of connections on P. The reduced equations obtained can be seen as generalized Euler-Poincaré equations for field theory. A remarkable fact is that these reduced equations on $\mathcal{C}(P)$ are not enough for the reconstruction of the original problem for $\dim M > 1$. In classical mechanics, direct integration of the Euler-Poincaré equations gives solutions of the variational problem, but for field theory a set of compatibility equations are needed, and they arise as the vanishing of the curvature of the reduced solution. This paper is the first in a series. Herein, we establish the covariant reduction process in the case that G is a matrix group. In the following notes, we shall make the extension to more general Lie groups, as well as to the very interesting setting of homogeneous spaces.

2. Preliminaries and notations

Throughout this paper, differentiable will mean C^∞ and if $E \to M$ is a fiber bundle, $C^\infty(E)$ will denote the space of differentiable sections of E over M.

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2.1. **The bundle of connections.** Let $\pi\colon P \to M$ be a principal G-bundle. The right group action of G on TP is given by the lifted action

$$X \cdot g = (R_g)_*(X), \quad \forall X \in TP,\ g \in G.$$

The quotient TP/G is a differentiable manifold and is endowed with a vector bundle structure over M. Let $\mathrm{ad}P := (P \times \mathfrak{g})/G$, the bundle associated to P by the adjoint representation of G on $\mathfrak{g}$. With VP the vertical subbundle of TP, the map $h\colon \mathrm{ad}P \to VP/G$ given by

$$h((p,\xi)_G) = (\hat{\xi}_p)_G$$

is a vector bundle diffeomorphism, where $\hat{\xi}_p = (d/dt)|_0 p \cdot \exp(t\xi)$. Let

$$\ddagger : VP \to \mathrm{ad}P \overset{h}{\cong} VP/G \tag{2.1}$$

be the projection induced by the diffeomorphism h. The fibers $(\mathrm{ad}P)_x$ of the adjoint bundle are endowed with a Lie algebra structure determined by the following condition:

$$[(p,\xi)_G, (p,\eta)_G] = (p, [\xi,\eta])_G, \qquad \forall p \in \pi^{-1}(x), \forall \xi, \eta \in \mathfrak{g}, \tag{2.2}$$

where $[\cdot,\cdot]$ denotes the bracket on $\mathfrak{g}$.

The quotient modulo G of the following exact sequence of vector bundles over P,

$$0 \to VP \to TP \xrightarrow{\pi_*} \pi^* TM \to 0,$$

becomes the exact sequence of vector bundles over M,

$$0 \to \mathrm{ad}P \to TP/G \xrightarrow{\pi_*} TM \to 0,$$

which is called the *Atiyah sequence* (see, for example, [1]).

Definition 2.1. A connection on P is a distribution $\mathcal{H}$ complementary to VP, such that $\pi_{*p} : \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism for all $p \in P$. The horizontal lift of a vector field X on M is the vector field $\tilde{X}$ on P defined by $\tilde{X}(p) := (\pi_*|_{\mathcal{H}_p})^{-1} X(\pi(p))$.

Let $\mathcal{H}$ be a connection on P, and let $\tilde{X} \in \mathfrak{X}(P)$ be the horizontal lift with respect to $\mathcal{H}$ of a vector field $X \in \mathfrak{X}(M)$. The horizontal lift is a G-invariant vector field on P projecting onto X. Namely,

$$R_{g_*}\mathcal{H}_p = \mathcal{H}_{pg} \quad \forall g \in G \text{ and } p \in P. \tag{2.3}$$

Hence, there exists a splitting of the Atiyah sequence

$$\sigma\colon TM \to TP/G, \qquad \sigma(X) = \tilde{X}.$$

Conversely, any splitting $\sigma\colon TM \to TP/G$ induces a unique connection: let $\xi \in \mathfrak{g}$ and $\psi \in \mathcal{H}$, and define the $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on P by

$$\mathcal{A}\langle \hat{\xi} + \psi \rangle = \xi.$$

It follows that

$$\mathcal{H}_p = \mathrm{Ker}\mathcal{A}_p,$$

so there is a natural bijective correspondence between connections on P and splittings of the Atiyah sequence.

In the case that $P = M \times G$ is trivial, condition (2.3) implies that the horizontal lift of a vector field X on M is given by

$$\tilde{X} = (X_x, -(\mathcal{A}_x\langle X \rangle g)_g) = (X_x, -(R_g)_* \mathcal{A}_x \langle X \rangle), \quad p = (x,g). \tag{2.4}$$

Definition 2.2. We denote by $p : \mathcal{C}(P) \to M$ the subbundle of $\mathrm{Hom}(TM, TP/G)$ determined by all linear mappings (see [2], [4])

$$\sigma_x : T_xM \to (TP/G)_x \text{ such that } \pi_* \circ \sigma_x = \mathrm{Id}_{T_xM}.$$

An element $\sigma_x \in \mathcal{C}(P)_x$ is a distribution at x; that is, σ_x induces a complementary subspace $\mathcal{H}_p$ of the vertical subspace V_pP for any $p \in \pi^{-1}(x)$. Addition of a linear mapping $l_x \colon T_xM \to (\mathrm{ad}P)_x \in \mathrm{Ker}\pi_*$ to σ_x produces another element $\sigma'_x = l_x + \sigma_x \in (\mathcal{C}(P))_x$, so that $\mathcal{C}(P)$ is an affine bundle modeled over the vector bundle $\mathrm{Hom}(TM, \mathrm{ad}P) \simeq T^*M \otimes \mathrm{ad}P$.

Accordingly, any global section $\sigma \in C^\infty(\mathcal{C}(P))$ can be identified with a global connection on P. Similarly, the difference of the two global sections σ and $\mathcal{H}$ may be identified with a section of the bundle $T^*M \otimes \mathrm{ad}P$. If we fix a connection $\mathcal{H}$, the map

$$\Phi_{\mathcal{H}} : \mathcal{C}(P) \to T^*M \otimes \mathrm{ad}P \text{ given by } \Phi_{\mathcal{H}}(\sigma) = \sigma - \mathcal{H} \tag{2.5}$$

is a fibered diffeomorphism. Note, however, that although $\mathcal{C}(P) \simeq T^*M \otimes \mathrm{ad}P$, the diffeomorphism is not canonical; it depends on the choice of the connection $\mathcal{H}$. We will denote by $\sigma^{\mathcal{H}}$ the image of σ under $\Phi_{\mathcal{H}}$.

2.2. **The identification** $J^1P/G \simeq \mathcal{C}(P)$.

Definition 2.3. Let $\pi : P \to M$ be a principal G-bundle and denote the 1-jet bundle of a local section of π by $\pi_1 \colon J^1P \to M$. This is the affine bundle of all linear mappings $\lambda_x : T_xM \to T_pP$ such that $\pi_{*p} \circ \lambda_x = \mathrm{Id}_{T_xM}$ for any $p \in \pi^{-1}(x)$. If s is a local section of P, its first jet extension j^1s is identified with the tangent map of s, i.e. $j^1_xs = T_xs$, $x \in M$.

The group G acts on J^1P in a natural way by

$$j^1_xs \cdot g = j^1_x(R_g \circ s), \tag{2.6}$$

where R_g is the right action of G on P. The quotient J^1P/G exists as a differentiable manifold and can be identified with the bundle of connections in the following way. We have $j^1_xs \cdot g = j^1_x(R_g \circ s) = T_x(R_g \circ s) = (R_g)_*T_xs$; then a coset $(j^1_xs)_G \in J^1P/G$ can be seen as a G-invariant horizontal distribution over M, that is, an element in $\mathcal{C}(P)_x$. Let

$$q : J^1P \to \mathcal{C}(P) \simeq J^1(P)/G \tag{2.7}$$

be the projection. Let $U \subset M$ be a local neighborhood of $x \in M$. If $s \in C^\infty(P|_U)$, we obtain a local section $\sigma : U \to \mathcal{C}(P)$ as $\sigma(x) = q(j^1_xs)$.

3. Euler-Poincaré reduction

Let $\pi : P \to M$ be a fiber bundle. A *Lagrangian density* is a bundle map $\mathcal{L} \colon J^1P \to \Lambda^nM$, where $n = \dim M$.

Definition 3.1. A variation of $s \in C^\infty(P)$ is a curve $s_\epsilon = \phi_\epsilon \circ s$, where ϕ_ϵ is the flow of a vertical vector field V on P which is compactly supported in M. One says that s is a fixed point of the variational problem associated with $\mathcal{L}$ if

$$\delta \int_M \mathcal{L}(j^1s) := \frac{d}{d\epsilon}\left[\int_M \mathcal{L}(j^1s_\epsilon)\right]\Big|_{\epsilon=0} = 0 \tag{3.1}$$

for all variations s_ϵ of s.

For a fixed volume form dx on M, we define the *Lagrangian* associated to $\mathcal{L}$ as the mapping $L : J^1P \to \mathbb{R}$ which verifies $\mathcal{L}(j^1_x s) = L(j^1_x s)dx$, $\forall j^1_x s \in J^1P$. Then, formula (3.1) becomes

$$\delta \int_M L(j^1_x s)dx = 0.$$

Henceforth, we shall restrict attention to the principal G-bundle $\pi : P \to M$ with $\dim M = n$ and with volume form dx.

Definition 3.2. A Lagrangian $L : J^1P \to \mathbb{R}$ is G-invariant if

$$L(j^1_x s \cdot g) = L(j^1_x s), \qquad \forall j^1_x s \in J^1P, \forall g \in G,$$

where the action on J^1P is defined in formula (2.6).

If L is a G-invariant Lagrangian, it defines a mapping

$$l : J^1P/G \simeq \mathcal{C}(P) \to \mathbb{R}$$

in a natural way. With $\delta s := (d/d\epsilon)|_0 s_\epsilon \in C^\infty(VP)$, define $\eta \in C^\infty(\mathrm{ad}P)$ by

$$\eta(x) = \ddagger\delta s(x).$$

Proposition 3.1. *Let $\pi : P \to M$ be a principal G-bundle, G a matrix group, with a fixed connection $\mathcal{H}$, and consider the curve $\epsilon \mapsto s_\epsilon = \phi_\epsilon \circ s$, where ϕ_ϵ is the flow of a π-vertical vector field V. Define $\sigma_\epsilon = q(j^1 s_\epsilon)$ and $\sigma^{\mathcal{H}} = \Phi_{\mathcal{H}}(\sigma)$. Then*

$$\delta\sigma := (d/d\epsilon)|_0 \sigma_\epsilon = \nabla^{\mathcal{H}}\eta - [\sigma^{\mathcal{H}}\langle\cdot\rangle, \eta],$$

where $[\cdot,\cdot]$ is given by (2.2), and $\nabla^{\mathcal{H}} : C^\infty(\mathrm{ad}P) \to C^\infty(T^*M \otimes \mathrm{ad}P)$ is the covariant derivative induced by $\mathcal{H}$ in the associated bundle ad$P$ *defined in a trivialization by*

$$\nabla^{\mathcal{H}}\eta = T\xi + [\mathcal{A}\langle\cdot\rangle, \xi], \tag{3.2}$$

where $\eta(x) = (x, \xi(x))$.

Remark 3.1. If $\mathcal{H}'$ is another connection on P, then

$$\nabla^{\mathcal{H}'}\eta - [\sigma^{\mathcal{H}'}\langle\cdot\rangle, \eta] = \delta\sigma = \nabla^{\mathcal{H}}\eta - [\sigma^{\mathcal{H}}\langle\cdot\rangle, \eta].$$

Remark 3.2. If we consider a principal fiber bundle with a left group action instead of a right action, then the expression for the infinitesimal variation is

$$\delta\sigma = \nabla^{\mathcal{H}}\eta + [\sigma^{\mathcal{H}}\langle\cdot\rangle, \eta].$$

Proof. Since this is a local statement, we may assume that $P = U \times G$, where $U \subset M$ is open, with π the projection onto the first factor, and with right action $R_{g'}$ given by

$$R_{g'}(x, g) = (x, g) \cdot g' = (x, gg').$$

Hence, $\mathrm{ad}P \simeq M \times \mathfrak{g}$ via the map $((x, e), \xi)_G \mapsto (x, \xi)$ and the projection $\ddagger : V_{(x,g)}P \to (\mathrm{ad}P)_x \simeq \mathfrak{g}$ is given explicitly by right translation

$$\ddagger(0_x, v) = (R_{g^{-1}})_* v = vg^{-1}, \qquad \forall v \in T_gG. \tag{3.3}$$

We identify the map $g \in C^\infty(U, G)$ with $s \in C^\infty(U \times G)$ by $s(x) = (x, g(x))$ and the map $\xi \in C^\infty(U, \mathfrak{g})$ with $\eta \in C^\infty(\mathrm{ad}P)$ by $\eta(x) = (x, \xi(x))$. We have the following identifications:

$$(TP/G)_x \simeq T_{(x,e)}P \simeq T_xM \times T_eG \simeq T_xM \times \mathfrak{g},$$

so that

$$\sigma_x = q(T_x s) = (\mathrm{Id}_{T_xM}, (R_{g^{-1}})_* T_x g) = (\mathrm{Id}_{T_xM}, T_x g \cdot g^{-1}).$$

Then,

$$\begin{aligned} \delta\sigma(x) &= (d/d\epsilon)|_0 \sigma_\epsilon(x) = (d/d\epsilon)|_0 (\mathrm{Id}_{T_xM}, T_x g_\epsilon \cdot g_\epsilon^{-1}) \\ &= (0_x, \{(d/d\epsilon)|_0 T_x g_\epsilon\} \cdot g^{-1} - T_x g \cdot g^{-1} \cdot \delta g \cdot g^{-1}) \\ &= (0_x, T_x \delta g \cdot g^{-1} - T_x g \cdot g^{-1} \cdot \delta g \cdot g^{-1}) \\ &= \left(0_x, [\delta g \cdot g^{-1}, T_x g \cdot g^{-1}\langle\cdot\rangle] - \delta g \cdot g^{-1} \cdot T_x g \cdot g^{-1} + T_x \delta g \cdot g^{-1}\right) \\ &= \left(0_x, [\delta g \cdot g^{-1}, T_x g \cdot g^{-1}\langle\cdot\rangle] + T_x(\delta g \cdot g^{-1})\right), \end{aligned}$$

where the bracket is the commutator of matrices as G is a matrix group. Hence, $\delta\sigma(x) \in T_x^*M \otimes (\mathrm{ad}P)_x \simeq T_x^*M \otimes \mathfrak{g}$, a $\mathfrak{g}$-valued (vertical-valued) 1-form. Now, $\eta = \ddagger\delta s$, so using (3.3), $\xi = \delta g g^{-1}$. (We make the identification $T_{\xi(x)}\mathfrak{g} \simeq \mathfrak{g}$.)

So for any vector field X on M,

$$\begin{aligned} \delta\sigma\langle X\rangle &= [\xi, \sigma\langle X\rangle] + T\xi \\ &= [\xi, \sigma\langle X\rangle - \tilde{X}] + T\xi + [\xi, \tilde{X}]. \end{aligned}$$

Let $\mathcal{A}$ be the local connection 1-form associated to $\mathcal{H}$. Then using (2.4),

$$\delta\sigma = [\xi, \sigma\langle\cdot\rangle + \mathcal{A}\langle\cdot\rangle] + T\xi - [\xi, \mathcal{A}\langle\cdot\rangle] = -[\sigma^{\mathcal{H}}\langle\cdot\rangle, \xi] + T\xi - [\xi, \mathcal{A}\langle\cdot\rangle].$$

Now to obtain the formula for $\nabla^{\mathcal{H}}$, we use the injective correspondence between $C^\infty(\mathrm{ad}P)$ and $\{f_\eta \in C^\infty(P, \mathfrak{g}) | f_\eta(pg) = \mathrm{Ad}_{g^{-1}} f_\eta(p),\ p \in P, g \in G\}$. Hence,

$$f_\eta(x, g) = \mathrm{Ad}_{g^{-1}}\xi(x).$$

It is standard (see [5]) that $\nabla^{\mathcal{H}}_X \eta$ is given by $(f_\eta)_*\langle\tilde{X}\rangle$, so we need only use (2.4) to compute the horizontal lift $\tilde{X}_{(x,e)}$. We have that

$$\begin{aligned} (f_\eta)_*\langle\tilde{X}\rangle &= (f_\eta)_*\langle X\rangle + (d/dt)|_0 f_\eta(x, \exp(-t\mathcal{A}\langle X\rangle)) \\ &= T\xi\langle X\rangle + (d/dt)|_0 \mathrm{Ad}_{\exp(t\mathcal{A}\langle X\rangle)}\xi(x) = T\xi\langle X\rangle + [\mathcal{A}\langle X\rangle, \xi], \end{aligned}$$

so that $\nabla^{\mathcal{H}}\eta = T\xi + \mathrm{ad}_{\mathcal{A}\langle\cdot\rangle}\xi$, and this completes the proof. □

Remark 3.3. The dual of the adjoint bundle $(\mathrm{ad}P)^*$ can also be seen as the bundle associated to P by the dual adjoint representation of G on $\mathfrak{g}^*$; *i.e.*, $g \mapsto \mathrm{Ad}^*_{g^{-1}}$. There is a similar injective correspondence between $C^\infty((\mathrm{ad}P)^*)$ and the set $\{f_\eta \in C^\infty(P, \mathfrak{g}) | f_\eta(pg) = \mathrm{Ad}^*_g f_\eta(p),\ p \in P, g \in G\}$. Hence, if $\nu \in C^\infty((\mathrm{ad}P)^*)$ and $\nu(x) = (x, \psi(x))$, then the covariant derivative induced by the connection $\mathcal{H}$ in $(\mathrm{ad}P)^*$ is defined by

$$\tilde{\nabla}^{\mathcal{H}}\nu = T\psi - \mathrm{ad}^*_{\mathcal{A}\langle\cdot\rangle}\psi, \tag{3.4}$$

or equivalently

$$(\tilde{\nabla}_X \nu)\eta = X(\langle\nu, \eta\rangle) - \langle\nu, \nabla_X \eta\rangle, \quad \forall \eta \in C^\infty(\mathrm{ad}P), X \in \mathfrak{X}(M).$$

Given a section $\sigma \in \mathcal{C}(P)$, the mapping $l : \mathcal{C}(P) \to \mathbb{R}$ defines a linear operator

$$\frac{\delta l}{\delta\sigma} : T^*M \otimes \mathrm{ad}P \to \mathbb{R}$$

by

$$\frac{\delta l}{\delta\sigma}(\zeta_x) = \lim_{\epsilon\to 0} \frac{l(\sigma(x) + \epsilon\zeta_x) - l(\sigma(x))}{\epsilon}, \quad \forall \zeta_x \in (T^*M \otimes \mathrm{ad}P)_x.$$

The operator $\delta l/\delta\sigma$ can be seen as a section of the dual bundle $(T^*M \otimes \mathrm{ad}P)^* \simeq TM \otimes (\mathrm{ad}P)^*$.

Lemma 3.1. *For a fixed connection $\mathcal{H}$ on $\pi : P \to M$, there exists an associated divergence operator $\mathrm{div}^{\mathcal{H}} : C^\infty(TM \otimes (\mathrm{ad}P)^*) \to C^\infty((\mathrm{ad}P)^*)$ which satisfies the following conditions. Let $\mathcal{X}, \mathcal{X}' \in C^\infty(TM \otimes (\mathrm{ad}P)^*)$, $\eta \in C^\infty(\mathrm{ad}P)$, and $f \in C^\infty(M)$. Then*

i) $\mathrm{div}^{\mathcal{H}}(\mathcal{X} + \mathcal{X}') = \mathrm{div}^{\mathcal{H}}(\mathcal{X}) + \mathrm{div}^{\mathcal{H}}(\mathcal{X})$,
ii) $\mathrm{div}^{\mathcal{H}}(f\mathcal{X}) = \mathcal{X}\cdot \mathrm{d}f + f\mathrm{div}^{\mathcal{H}}(\mathcal{X})$,
iii) $\mathrm{div}(\mathcal{X}\cdot\eta) = (\mathrm{div}^{\mathcal{H}}\mathcal{X})\cdot\eta + \mathcal{X}\cdot\nabla^{\mathcal{H}}\eta$.

Furthermore, if $\{E^1, ..., E^m\}$ is a basis of local sections of the bundle $(\mathrm{ad}P)^*$ for which any element $\mathcal{X} \in C^\infty(TM \otimes (\mathrm{ad}P)^*)$ may be expressed as $\mathcal{X} = \sum X_i \otimes E^i$, $X_i \in \mathfrak{X}(M)$, then

$$\mathrm{div}^{\mathcal{H}}(\mathcal{X}) = \sum_{i=1}^{m} \left(\mathrm{div}(X_i) \otimes E^i + \tilde{\nabla}^{\mathcal{H}}_{X_i} E^i \right). \tag{3.5}$$

Remark 3.4. In the case $P = M \times G$ and $\mathcal{H}$ is the trivial connection, then $\mathrm{div}^{\mathcal{H}}$ is the usual divergence operator.

Proof. We use the same notation as in the proof of Proposition 3.1. Let $\{E_1,\dots,E_m\}$ be a basis of $\mathfrak{g}$ and $\{E^1, \dots, E^m\}$ its dual basis. Let $\mathcal{X} = \sum X_i \otimes E^i$ be any section of $TM \otimes (\mathrm{ad}P)^* \cong TM \otimes \mathfrak{g}^*$, and let $\xi = \sum f^i \otimes E_i$ be any section of $\mathrm{ad}P \cong M \times \mathfrak{g}$. Using (3.2) and (3.4), we have that

$$\begin{aligned}
\mathcal{X}\cdot\nabla^{\mathcal{H}}\eta &= \sum_{i=1}^{m}\left(Tf^i\langle X_i\rangle + f^i \sum_{j=1}^{m} \langle [\mathcal{A}\langle X_j\rangle, E_i], E^j\rangle \right) \\
&= \sum_{i=1}^{m}\left(X_i(f^i) + f^i \sum_{j=1}^{m} \langle \mathrm{ad}^*_{\mathcal{A}\langle X_j\rangle} E^j, E_i\rangle \right) \\
&= \sum_{i=1}^{m}\left(\mathrm{div}(f^i X_i) - f^i \mathrm{div} X_i \right) + \sum_{j=1}^{m} \langle \mathrm{ad}^*_{\mathcal{A}\langle X_j\rangle} E^j, \eta\rangle \\
&= \mathrm{div}(\mathcal{X}\cdot\eta) - \left(\langle \sum_{j=1}^{m} \mathrm{div} X_j \otimes E^j, \eta\rangle + \langle \sum_{j=1}^{m} \tilde{\nabla}^{\mathcal{H}}_{X_j} E^j, \eta\rangle \right),
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Hence, the operator $\mathrm{div}^{\mathcal{H}}$ satisfying iii) is

$$\mathrm{div}^{\mathcal{H}}\left(\sum_{j=1}^{m} X_j \otimes E^j \right) = \sum_{j=1}^{m} (\mathrm{div} X_j \otimes E^i + \tilde{\nabla}^{\mathcal{H}}_{X_j} E^j).$$

This expression can be defined globally, and it is straightforward to verify items i) and ii). ∎

3.1. **Reduction.**

Theorem 3.1. *Let $\pi : P \to M$ be a principal G-fiber bundle over a manifold M with a volume form dx and let $L : J^1P \to \mathbb{R}$ be a G-invariant Lagrangian. Let $l : \mathcal{C}(P) \to \mathbb{R}$ be the mapping defined by L in the quotient. For a section $s : U \to P$*

of π defined in a neighborhood $U \subset P$, let $\sigma : U \to \mathcal{C}(P)$ be defined by $\sigma(x) = q(j^1_x s)$. Then, for every connection $\mathcal{H}$ of the bundle $\pi|_U$, the following are equivalent:

1) *s satisfies the Euler-Lagrange equations for L,*
2) *the variational principle*
$$\delta \int_M L(j^1_x s)dx = 0$$
holds, for variations with compact support,
3) *the Euler-Poincaré equations hold:*
$$\mathrm{div}^{\mathcal{H}} \frac{\delta l}{\delta \sigma} = -\mathrm{ad}^*_{\sigma^{\mathcal{H}}\langle\cdot\rangle} \frac{\delta l}{\delta \sigma},$$
4) *the variational principle*
$$\delta \int_M l(\sigma(x))dx = 0$$
holds, using variations of the form
$$\delta\sigma = \nabla^{\mathcal{H}}\eta - [\sigma^{\mathcal{H}}\langle\cdot\rangle, \eta],$$
where $\eta : U \to \mathrm{ad}P$ is a section with compact support.

Proof. 1) $\Leftrightarrow$ 2) is a standard argument in the calculus of variations. For 2) $\Leftrightarrow$ 4), we use
$$\delta \int_M L(j^1_x s)dx = \delta \int_M l(\sigma(x))dx$$
with Proposition 3.1.

For 3) $\Leftrightarrow$ 4), we have that
$$0 = \delta \int_M l(\sigma(x))dx = \int_M \frac{\delta l}{\delta \sigma}\delta\sigma dx = \int_M \frac{\delta l}{\delta \sigma}(\nabla^{\mathcal{H}}\eta - [\sigma^{\mathcal{H}}\langle\cdot\rangle, \eta])dx.$$

Item iii) of Lemma 3.1 shows that
$$\frac{\delta l}{\delta \sigma}\nabla^{\mathcal{H}}\eta = \mathrm{div}(\frac{\delta l}{\delta \sigma}\eta) - \mathrm{div}^{\mathcal{H}}(\frac{\delta l}{\delta \sigma})\eta,$$
so that
$$0 = \int_M (\mathrm{div}(\frac{\delta l}{\delta \sigma}\eta) - \mathrm{div}^{\mathcal{H}}(\frac{\delta l}{\delta \sigma})\eta - \mathrm{ad}^*_{\sigma^{\mathcal{H}}\langle\cdot\rangle}\frac{\delta l}{\delta \sigma}\eta)dx.$$
As η has compact support, by the Stokes theorem, $\int_M \mathrm{div}(\frac{\delta l}{\delta \sigma}\eta)dx = 0$, so we conclude that
$$0 = \int_M (\mathrm{ad}^*_{\sigma^{\mathcal{H}}\langle\cdot\rangle}\frac{\delta l}{\delta \sigma} + \mathrm{div}^{\mathcal{H}}(\frac{\delta l}{\delta \sigma}))\eta dx,$$
for all sections η of $\mathrm{ad}P$ with compact support. Thus, we obtain the Euler-Poincaré equations. □

Remark 3.5. If we consider a principal fiber bundle with a left action, instead of a right action, and a left invariant Lagrangian L, the Euler-Poincaré equations are
$$\mathrm{div}^{\mathcal{H}} \frac{\delta l}{\delta \sigma} = \mathrm{ad}^*_{\sigma^{\mathcal{H}}\langle\cdot\rangle} \frac{\delta l}{\delta \sigma}.$$

3.2. **Reconstruction.** Let $s : U \to P$ be a solution of the variational problem defined by a G-invariant Lagrangian L. Then, the section $\sigma = q(j^1 s)$ of the bundle of connections is a solution of the Euler-Poincaré equations (Theorem 3.2). This new section is a connection which verifies that $s(U)$ is an integral manifold; that is, σ is a flat connection. Conversely, given a flat connection σ which verifies the Euler-Poincaré equations, the integral submanifolds in P of σ are the image of the sections of the solution of the original variational problem. In other words

Theorem 3.2. *The following systems of equations are equivalent:*

i) *Euler-Lagrange equations of L, and*
ii) *Euler-Poincaré equations of l together with vanishing curvature.*

The projection of a solution s of i) *gives a solution $\sigma = q(j^1\sigma)$ of* ii)*, and the integral manifolds of a solution σ of* ii) *provides a solution of* i)*.*

That is, the Euler-Poincaré equations are not sufficient for reconstructing the solution of the original variational problem. One must impose an additional compatibility condition given by the vanishing of the curvature. See [6] for additional discussion.

4. Examples of reduction in a principal fiber bundle

4.1. **Classical Euler-Poincaré equations.** For a Lie group G, we consider the principal fiber bundle $\pi : \mathbb{R}\times G \to \mathbb{R}$, where π is the projection onto the first factor. Let $L : J^1P \simeq \mathbb{R} \times TG \to \mathbb{R}$ be a right G-invariant Lagrangian. We fix the trivial connection and obtain the following identifications:

$$\mathcal{C}(P) \simeq T^*\mathbb{R} \otimes \mathrm{ad}P \simeq (\mathbb{R}\times)\mathbb{R} \otimes (\mathbb{R} \times \mathfrak{g}) \simeq \mathbb{R} \times \mathfrak{g}.$$

Again, we identify $s \in C^\infty(P)$, $\eta \in C^\infty(\mathrm{ad}P)$ and $\sigma \in C^\infty(\mathcal{C}(P))$ with the maps $g \in C^\infty(\mathbb{R}, G)$, $\eta \in C^\infty(\mathbb{R}, \mathfrak{g})$ and $\sigma \in C^\infty(\mathbb{R}, \mathfrak{g})$, respectively. Because of the trivial connection, $\mathrm{div}^{\mathcal{H}}$ is simply the usual divergence operator satisfying $\mathrm{div}(f\frac{\partial}{\partial t}) = \frac{df}{dt}$.

We recover the classical Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta l}{\delta \sigma} = -\mathrm{ad}^*_\sigma \frac{\delta l}{\delta \sigma}$$

for a right invariant Lagrangian (see [7]).

4.2. **Harmonic maps.** Let (M, g) be a compact oriented C^∞ n-dimensional Riemannian manifold, and let (G, h) be an m-dimensional Riemannian matrix Lie group. With $P = M\times G$, we denote the principal fiber bundle by $\pi : P \to M$, and by triviality, identify $C^\infty(P)$ with $C^\infty(M, G)$. For each $\phi \in C^\infty(M, G)$, the Riemannian metrics on M and G naturally induce a metric $\langle\cdot,\cdot\rangle$ on $C^\infty(T^*M \otimes \phi^*(TG))$, and so we may define the energy $\mathcal{E}$ on $C^\infty(M, G)$ by

$$\mathcal{E}(\phi) = \int_M L(j^1\phi)dx, \text{ where } L(j^1\phi) = \frac{1}{2}\langle T\phi, T\phi\rangle. \tag{4.1}$$

The Euler-Lagrange equations for (4.1) are given by

$$\mathrm{Tr}(\nabla T\phi) = 0, \tag{4.2}$$

where ∇ is the induced Riemannian covariant derivative on $C^\infty(T^*M \otimes \phi^*(TG))$ and Tr is the trace defined by g (see, for example, [3]). By definition, the set of harmonic maps from M to G is the subset of $C^\infty(P)$ whose elements solve

(4.2). Using Einstein's summation convention, we have the following coordinate expressions:

$$L(j^1\phi) = \frac{1}{2} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} h_{\alpha\beta}, \tag{4.3}$$

and for (4.2)

$$g^{ij}\left(\frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \phi^\gamma}{\partial x^k} + \tilde{\Gamma}^\gamma_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \right) = 0, \quad 1 \le \gamma \le m, \tag{4.4}$$

where Γ^k_{ij} , $\tilde{\Gamma}^\gamma_{\alpha\beta}$ denote the Christoffel symbols of the Levi-Civita connections of g and h. We shall derive the reduced form of (4.2) for two specific cases: $G = \mathbb{R}$, and $G = \mathbb{S}^3 \cong SU(2) \overset{2:1}{\cong} SO(3)$.

For the case that $G = \mathbb{R}$, the abelian group of translations, we choose the trivial connection for P. The divergence operator div^M is naturally defined by the metric g and its associated Riemannian connection. In this case, $(TP/G)_x \simeq T_xM \times \mathbb{R}$ and $(\operatorname{ad}P)_x \simeq \mathbb{R}$. Let $\sigma = q(T\phi)$, so that $\sigma_x : T_xM \to T_xM \times \mathbb{R}$, acting as the identity on the first factor. Then, σ can be considered as a 1-form with local expression $\sigma = p_i \mathrm{d}x^i$, $p_i = \partial\phi/\partial x^i$.

The Lagrangian L is clearly $\mathbb{R}$-invariant. Denoting by l the projection of L to $\mathcal{C}(P)$, Theorem 3.1 asserts that σ satisfies

$$\operatorname{div}^M \frac{\delta l}{\delta \sigma} = 0,$$

or, in coordinates,

$$\operatorname{div}^M \big(g^{kj} p_j \frac{\partial}{\partial x^k}\big) = \frac{\partial (g^{ij} p_j)}{\partial x^i} + \Gamma^i_{ik} g^{kj} p_j = 0,$$

since $l(\sigma) = \frac{1}{2} g^{ij} p_i p_j$. It is straightforward to check that the above equation, together with vanishing curvature

$$\frac{\partial p_i}{\partial x^j} = \frac{\partial p_j}{\partial x^i}$$

and $p_i = \partial\phi/\partial x^i$, is equivalent to formula (4.4) for $\gamma = 1$ and $\tilde{\Gamma}^\gamma_{\alpha\beta} = 0$, as is stated in Theorem (3.2).

For the case $G = \mathbb{S}^3$, we make the identifications $(TP/G)_x \cong T_xM \times \mathfrak{su}(2)$, $(\operatorname{ad}P)_x \cong \mathfrak{su}(2)$ and $\mathcal{C}(P) \simeq T^*M \otimes \mathfrak{su}(2)$. Then, $\sigma = q(T\phi)$ can be considered as a 1-form taking values in $\mathfrak{su}(2)$. Let $\{E_1, E_2, E_3\}$ be a basis of $\mathfrak{su}(2)$. Then σ can be written as $\sigma(x) = p^\alpha_i \mathrm{d}x^i \otimes E_\alpha$ with $p^\alpha_i \otimes E_\alpha = \partial\phi/\partial x^i = T\phi(\partial/\partial x^i)$.

The Lagrangian L is $SU(2)$-invariant and its projection to $\mathcal{C}(P)$ is

$$l(\sigma) = \frac{1}{2} g^{ij} p^\alpha_i p^\beta_j h_{\alpha\beta}.$$

Then

$$\frac{\delta l}{\delta \sigma} = g^{ij} p^\alpha_i h_{\alpha\beta} \frac{\partial}{\partial x^j} \otimes E^\beta,$$

and its usual divergence is

$$\operatorname{div} \frac{\delta l}{\delta \sigma} = \left(\frac{\partial}{\partial x^j} \left(g^{ij} p^\alpha_i h_{\alpha\beta} \right) + \Gamma^k_{kj} g^{ij} p^\alpha_i h_{\alpha\beta} \right) \otimes E^\beta.$$

The coadjoint map can be written in coordinates as

$$\langle \operatorname{ad}^*_\sigma \frac{\delta l}{\delta \sigma}, E_\beta \rangle = \langle \frac{\delta l}{\delta \sigma}, [\sigma, E_\beta] \rangle = g^{ij} p^\alpha_i p^\rho_j c^\gamma_{\rho\beta} h_{\alpha\gamma}.$$

Then, Euler-Poincaré equations for the trivial connection on $M \times SU(2)$ are (Theorem (3.1))

$$\frac{\partial}{\partial x^j}\left(g^{ij}p_i^\alpha h_{\alpha\beta}\right) + \Gamma^k_{kj} g^{ij} p_i^\alpha h_{\alpha\beta} = -g^{ij} p_i^\alpha p_j^\rho c^\gamma_{\rho\beta} h_{\alpha\gamma}.$$

The above system of equations, together with vanishing curvature

$$\frac{\partial p_i^\gamma}{\partial x^j} - \frac{\partial p_j^\gamma}{\partial x^i} + p_i^\alpha p_j^\beta c^\gamma_{\alpha\beta} = 0 \quad \forall i, j = 1, ..., n; \ \ \gamma = 1, 2, 3,$$

and $p_i^\alpha \otimes E_\alpha = \partial\phi/\partial x^i$, are equivalent to equations (4.4), as is asserted in Theorem (3.2).

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