REDUCTION MAPS AND MINIMAL MODEL THEORY

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Abstract. We use reduction maps to study the minimal model program. Our main result is that the existence of a good minimal model for a klt pair (X, Δ) can be detected on the base of the $(K_X + \Delta)$-trivial reduction map. Thus we show that the main conjectures of the minimal model program can be interpreted as a natural statement on the existence of curves on X.

Contents

1. Introduction

Suppose that X is a smooth projective variety over $\mathbb{C}$. The minimal model program predicts that there is a birational model of X that satisfies particularly nice properties. More precisely, if X has non-negative Kodaira dimension then X should admit a good minimal model: a birational model X' with mild singularities such that some multiple of $K_{X'}$ is basepoint free. In this paper we use reduction maps to study the existence of good minimal models for pairs (X, Δ).

We first interpret the existence of good minimal models in terms of the numerical dimension of [N] and [BDPP]. Using results of [Lai], we show that for a kawamata log terminal pair (X, Δ) the existence of a good minimal minimal model is equivalent to the abundance of $K_X + \Delta$. Thus we will focus on the following conjecture:

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Conjecture 1.1. *Let (X, Δ) be a kawamata log terminal pair. Then $K_X + \Delta$ is abundant, that is, $\kappa(K_X + \Delta) = \nu(K_X + \Delta)$.*

Our main goal is to show that the abundance of $K_X + \Delta$ can be detected on the base of certain morphisms:

Theorem 1.2 (=Corollary 3.5)**.** *Let (X, Δ) be a kawamata log terminal pair. Suppose that $f : X \to Z$ is a morphism with connected fibers to a variety Z such that $\nu((K_X + \Delta)|_F) = 0$ for a general fiber F of f. Then there exist a smooth birational model Z' of Z and a kawamata log terminal pair $(Z', \Delta_{Z'})$ such that $K_X + \Delta$ is abundant if and only if $K_{Z'} + \Delta_{Z'}$ is abundant.*

In order to apply Theorem 1.2 in practice, the key question is whether one can find a map such that the numerical dimension of $(K_X + \Delta)|_F$ vanishes for a general fiber F. The $(K_X + \Delta)$-trivial reduction map constructed in [Leh1] satisfies precisely this property. We will consider a birational version of this map better suited for the study of adjoint divisors; we then define $\widetilde{\tau}(X, \Delta)$ to be the dimension of the image of this birational version. Thus we obtain the following:

Theorem 1.3 (Corollary of Theorem 5.1)**.** *Let (X, Δ) be a kawamata log terminal pair. If $0 \leq \widetilde{\tau}(K_X + \Delta) \leq 3$ then (X, Δ) has a good minimal model.*

The $(K_X + \Delta)$-trivial reduction map is constructed by quotienting by curves with $(K_X + \Delta) \cdot C = 0$. Thus, another way to approach the problem is to focus on the properties of curves on X. Recall that an irreducible curve C is said to be movable if it is a member of a family of curves dominating X. Conjecture 1.1 yields the following prediction:

Conjecture 1.4. *Let (X, Δ) be a kawamata log terminal pair. Suppose that $(K_X + \Delta).C > 0$ for every movable curve C on X. Then $K_X + \Delta$ is big.*

Our final goal is to show that the two conjectures are equivalent:

Theorem 1.5 (=Corollary 5.2)**.** *Conjecture 1.4 holds up to dimension n iff Conjecture 1.1 holds up to dimension n.*

The use of reduction maps to study the minimal model program was initiated by [A1]. Our work relies on Ambro's techniques. Note that our main theorem generalizes [A1] and [Fu]. Related topics have been considered in [BDPP]. The method of using fibrations to study the abundance conjecture seems to appear first in [Ka1]. Finally, similar ideas have appeared independently in the recent preprint [Siu].

We summarize the contents of this paper. In Section 2, we prepare for the proof of main results. In Section 3, we prove Theorem 1.2 and show the equivalence of the abundance conjecture and the existence of good minimal models. In Section 4, we introduce the D-trivial reduction map of [Leh1] and define $\widetilde{\tau}(X, \Delta)$ for a log pair (X, Δ). In Section 5, we show Theorem 1.3 and Theorem 1.5.

Notation and Definition 1.6. Throughout this paper, we work over $\mathbb{C}$. All varieties X are assumed to be normal and projective unless otherwise stated. The term "divisor" always refers to a $\mathbb{R}$-Cartier $\mathbb{R}$-Weil divisor.

Let X be a normal variety and let Δ be an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that (X, Δ) is *kawamata log terminal* (sometimes abbreviated by *klt*) if the discrepancy $a(E, X, \Delta) > -1$ for every prime divisor E over X.

For an $\mathbb{R}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$ (resp. the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$), where for every real number x, $\ulcorner x \urcorner$ (resp. $\llcorner x \lrcorner$) is the integer defined by $x \leq \ulcorner x \urcorner < x + 1$ (resp. $x - 1 < \llcorner x \lrcorner \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$.

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2. Preliminaries

In this section, we introduce definitions and collect some lemmas for the proof of main results.

Definition 2.1. Let (X, Δ) be a kawamata log terminal pair and $\varphi : W \to X$ a log resolution of (X, Δ). Choose Δ_W so that

$$K_W + \Delta_W = \phi^*(K_X + \Delta) + E$$

where Δ_W and E are effective $\mathbb{Q}$-divisors that have no common component. We call (W, Δ_W) a log smooth model of (X, Δ). Note that a minimal model of (W, Δ_W) may not be a minimal model of (X, Δ). To correct this deficiency, define

$$F = \sum_{F_i:\varphi\text{-exceptional}} F_i \text{ and } \Delta_W^\epsilon = \Delta_W + \epsilon F$$

for a positive number ϵ. We call (W, Δ_W^ϵ) an *ϵ-log smooth model*.

Remark 2.2. Note that our definition of a log smooth model differs from that of Birkar and Shokurov (cf. [Bi]).

Definition 2.3 (cf. [ELMNP]). Let $\pi : X \to Y$ be a projective morphism of normal quasi-projective varieties and D an $\mathbb{R}$-Cartier divisor on X. We set

$$\mathbf{B}(D/Y) = \bigcap_{D\sim_{Y,\mathbb{R}}E\geq 0} \operatorname{Supp} E, \text{ and } \mathbf{B}_-(D/Y) = \bigcup_{\epsilon\in\mathbb{R}_{>0}} \mathbf{B}(D + \epsilon A/Y),$$

where A is a π-ample divisor. We remark that these definitions are independent of the choice of A. When $Y = \operatorname{Spec}\mathbb{C}$, we write simply $\mathbf{B}(D)$ and $\mathbf{B}_-(D)$.

We next turn to numerical properties of pseudo-effective divisors described in [N].

Definition 2.4. Let X be a smooth variety and D a pseudo-effective $\mathbb{R}$-divisor on X. Fix an ample divisor A on X. Given a prime divisor Γ on X, we define

$$\sigma_\Gamma(D) = \min\{\mathrm{mult}_\Gamma(D') | D' \sim_{\mathbb{Q}} D + \epsilon A \text{ for some } \epsilon > 0\}.$$

Note that this definition is independent of the choice of A.

It is shown in [N] that for any pseudo-effective divisor D there are only finitely many prime divisors Γ such that $\sigma_\Gamma(D) > 0$.

Definition 2.5. Let X be a smooth variety and D a pseudo-effective $\mathbb{R}$-divisor on X. We define the $\mathbb{R}$-divisors $N_\sigma(D) = \sum_\Gamma \sigma_\Gamma(D)\Gamma$ and $P_\sigma(D) = D - N_\sigma(D)$. The decomposition

$$D = P_\sigma(D) + N_\sigma(D)$$

is known as the *σ-decomposition*. It is also known as the *sectional decomposition* ([Ka2]), the *divisorial Zariski decomposition* ([Bo]), and the *numerical Zariski decomposition* ([Ka3]).

The basic properties of the σ-decomposition are:

Lemma 2.6 ([N, III.1.4 Lemma] and [N, V.1.3 Lemma]). *Let X be a smooth projective variety and D a pseudo-effective $\mathbb{R}$-Cartier divisor. Then*

(1) $\kappa(X, D) = \kappa(X, P_\sigma(D))$ *and*
(2) $\mathrm{Supp}(N_\sigma(D)) \subset \mathbf{B}_-(D)$.

There is a relative version of the σ-decomposition satisfying similar properties; we refer to [N] for the details of the construction. Closely related to the σ-decomposition is the numerical dimension, a numerical measure of the positivity of a divisor.

Definition 2.7 (Numerical dimension)**.** Let X be a normal projective variety, D an $\mathbb{R}$-Cartier divisor and A an ample divisor on X. We set

$$\nu(D) = \max\{k \in \mathbb{Z}_{\geq 0} | \limsup_{m\to\infty} m^{-k} \dim H^0(X, \llcorner mD \lrcorner + A) > 0\}$$

if $H^0(X, \llcorner mD \lrcorner + A) \neq 0$ for infinitely many $m \in \mathbb{N}$ or $\sigma(D, A) = -\infty$ otherwise.

Remark 2.8. By the results of [Leh2], this definition coincides with the notions of $\kappa_\nu(L)$ from [N, V, 2.20, Definition] and $\nu(L)$ from [BDPP, 3.6, Definition].

The numerical dimension satisfies a number of natural properties.

Lemma 2.9 ([N, V, 2.7], [Leh2, 6.7])**.** *Let X be a normal variety, D an $\mathbb{R}$-Cartier divisor on X, and $\phi : W \to X$ a birational map from a normal variety W. Then:*
(1) $\nu(D) = \nu(D')$ *for any $\mathbb{R}$-divisor D' such that $D \equiv D$,*
(2) $\nu(\varphi^* D) = \nu(D)$, *and*
(3) *if X is smooth then $\nu(D) = \nu(P_\sigma(D))$.*

Finally, we identify several different ways a divisor can be "exceptional" for a morphism.

Definition 2.10 ([Ft], [N, III, 5.a], [Lai, Definition 2.9] and [Ta])**.** Let $f : X \to Y$ be a surjective projective morphism of normal quasi-projective varieties with connected fibers and D an effective f-vertical $\mathbb{R}$-Cartier divisor. We say that D is *f-exceptional* if

$$\mathrm{codim}\, f(\mathrm{Supp}\, D) \geq 2.$$

We say that D is *of insufficient fiber type* with respect to f if

$$\mathrm{codim}\, f(\mathrm{Supp}\, D) = 1$$

and there exist a codimension 1 irreducible component P of $f(\mathrm{Supp}\, D)$ and a prime divisor Γ such that $f(\Gamma) = P$ and $\Gamma \not\subset \mathrm{Supp}(D)$.

We call D *f-degenerate* if for any prime divisor P on Y there is some prime divisor $\Gamma \subset \mathrm{Supp}(f^*P)$ such that $f(\Gamma) = P$ and $\Gamma \not\subset \mathrm{Supp}(D)$. Note that the components of a degenerate divisor can be either f-exceptional or of insufficient fiber type with respect to f.

Lemma 2.11. *Let $f : X \to Y$ be a surjective projective morphism of normal quasi-projective varieties. Suppose that D is an effective f-vertical $\mathbb{Q}$-Cartier divisor such that $f_*\mathcal{O}_X(\llcorner kD \lrcorner)^{**} \cong \mathcal{O}_Y$ for every positive integer k. Then D is f-degenerate.*

Proof. If D were not f-degenerate, there would be an effective f-exceptional divisor E on X and an effective $\mathbb{Q}$-divisor T on Y such that $f^*T \leq D + E$. But since E is f-exceptional it is still true that $f_*\mathcal{O}_X(\llcorner k(D+E) \lrcorner)^{**} \cong \mathcal{O}_Y$, yielding a contradiction. □

Degenerate divisors behave well with respect to the σ-decomposition.

Lemma 2.12 (cf. [Ft, (1.9)], [N, III.5.7 Proposition])**.** *Let $f : X \to Y$ be a surjective projective morphism from a smooth quasi-projective variety to a normal quasi-projective variety and let D be an effective f-degenerate divisor. For any pseudo-effective divisor L on Y we have $D \leq N_\sigma(f^*L + D/Y)$.*

Proof. [N, III.5.1 Proposition] and [N, III.5.2 Proposition] together show that for a degenerate divisor D there is some component $\Gamma \subset \mathrm{Supp}(D)$ such that $D|_\Gamma$ is not $f|_\Gamma$-pseudo-effective. Since $P_\sigma(f^*L + D/Y)|_\Gamma$ is pseudo-effective, we see that Γ must occur in $N_\sigma(f^*L+D/Y)$ with positive coefficient.

Set D' to be the coefficient-wise minimum

$$D' = \min\{N_\sigma(f^*L + D/Y), D\}.$$

If $D' < D$, then $D - D'$ is still f-degenerate. Thus, there is some component of $D - D'$ contained in $\mathrm{Supp}(N_\sigma(f^*L + (D-D')/Y))$ with positive coefficient, a contradiction. □

Corollary 2.13 ([Lai, Lemma 2.10])**.** *Let $f : X \to Y$ be a surjective projective morphism of normal quasi-projective varieties and D an f-degenerate divisor. Suppose that L is a pseudo-effective divisor on Y. Then there are codimension 1 components of $\mathbf{B}_-(f^*L + D/Y)$.*

3. The existence of minimal models and abundance

In this section, we show that the abundance conjecture is equivalent to the existence of good minimal models. We also prove Theorem 1.2, the main technical tool for the inductive arguments of Section 5.

Lemma 3.1 ([N, V.4.2 Corollary])**.** *Let (X, Δ) be a kawamata log terminal pair with $\kappa(K_X + \Delta) \geq 0$. Then the following are equivalent:*

(1) $\kappa(K_X + \Delta) = \nu(K_X + \Delta)$.

(2) *Let $\mu : X' \to X$ be a birational morphism and $f : X' \to Z'$ a morphism resolving the Iitaka fibration for $K_X + \Delta$. Then*

$$\nu(\mu^*(K_X + \Delta)|_F) = 0$$

for a general fiber F of f.

If either of these equivalent conditions hold, we say that L is abundant.

To relate abundance to the existence of minimal models, we will use the following special case.

Theorem 3.2 ([N, V.4.9 Corollary] and [D, Corollaire 3.4])**.** *Let (X, Δ) be a klt pair such that $\nu(K_X + \Delta) = 0$. Then $K_X + \Delta$ is abundant and (X, Δ) admits a good minimal model.*

The following theorem is known to experts; for example, see [DHP, Remark 2.6]. The main ideas of the proof are from [Lai].

Theorem 3.3 (cf. [DHP])**.** *Let (X, Δ) be a klt pair. Then $K_X + \Delta$ is abundant if and only if (X, Δ) has a good minimal model.*

Proof. First suppose that (X, Δ) has a good minimal model (X', Δ'). Let Y be a common resolution of X and X' (with morphisms f and g respectively) and write

$$f^*(K_X + \Delta) = g^*(K_{X'} + \Delta') + E$$

where E is an effective exceptional $\mathbb{Q}$-divisor. Thus

$$P_\sigma(f^*(K_X + \Delta)) = P_\sigma(g^*(K_{X'} + \Delta'))$$

and since the latter divisor is semi-ample, the abundance of $K_X + \Delta$ follows.

Conversely, suppose that $K_X + \Delta$ is abundant. Let $f : (X, \Delta) \dashrightarrow Z$ be the Iitaka fibration of $K_X + \Delta$. Choose an ϵ-log smooth model $\varphi : (W, \Delta_W^\epsilon) \to X$ with sufficiently small $\epsilon > 0$ so that f is resolved on W. By [BCHM, Lemma 3.6.10] we can find a minimal model for (X, Δ) by constructing a minimal model of (W, Δ_W^ϵ) . Moreover we see that $f \circ \varphi$ is also the Iitaka fibration of $K_W + \Delta_W^\epsilon$ and $\nu(K_W + \Delta_W^\epsilon) = \nu(K_X + \Delta)$. Replacing (X, Δ) by (W, Δ_W^ϵ), we may suppose that the Iitaka fibration f is a morphism on X.

By [N, V.4.2 Corollary], $\nu(K_F + \Delta_F) = 0$ where F is a general fiber of f and $K_F + \Delta_F = (K_X + \Delta)|_F$. Therefore (F, Δ_F) has a good minimal model by [D, Corollaire 3.4].

The arguments of [Lai, Theorem 4.4] for (X, Δ) now show that (X, Δ) has a good minimal model. □

The following lemma is key for proving our main results. It is a consequence of Ambro's work on LC-trivial fibrations (cf. [A2]).

Lemma 3.4. *Let (X,Δ) be a projective klt pair. Suppose that $f : X \to Z$ is a projective morphism with connected fibers to a smooth projective variety Z such that $\nu((K_X+\Delta)|_F)=0$ for a general fiber. Then there exists a log resolution $\mu : X' \to X$ of (X,Δ), a klt pair $(Z',\Delta_{Z'})$, and a projective morphism $f' : X' \to Z'$ birationally equivalent to f such that*

$$P_\sigma(\mu^*(K_X+\Delta)) \sim_{\mathbb{Q}} P_\sigma(f'^*(K_{Z'}+\Delta_{Z'})).$$

We may assume that Z' is any sufficiently high birational model of Z.

Proof. By [N, V.4.9 Corollary] $\kappa((K_X+\Delta)|_F)=0$. By [A2, Theorem 3.3] and [FM, 4.4], there exist a log resolution $\mu : X' \to X$ of (X,Δ), a morphism $f' : X' \to Z'$, an effective divisor $\Delta_{Z'}$ on Z', and a (not necessarily effective) $\mathbb{Q}$-divisor $B = B^+ - B^-$ that satisfy:

(1) (X',Δ') is a log smooth model and $(Z',\Delta_{Z'})$ is klt,
(2) $K_{X'}+\Delta' \sim_{\mathbb{Q}} f'^*(K_{Z'}+\Delta_{Z'})+B$,
(3) there exist positive integers m_1 and m_2 such that

$$H^0(X', mm_1(K_{X'}+\Delta')) = H^0(Z', mm_2(K_{Z'}+\Delta_{Z'})),$$

for any positive integers m,
(4) B^- is f'-exceptional and
(5) $f_*\mathcal{O}_X(\llcorner lB^+ \lrcorner) = \mathcal{O}_Y$ for every sufficiently divisible integer l.

Moreover f' is the resolution of a flattening by the Fujino–Mori construction (cf. [FM, 4.4]). Thus B^- is μ-exceptional so that

$$P_\sigma(\mu^*(K_X+\Delta)) = P_\sigma(K_{X'}+\Delta'+B^-).$$

We now turn our attention to B^+. Note that $\nu((K_{X'}+\Delta')|_{F'})=0$ for a general fiber F' of f' (since the same is true on the general fiber of f). In particular $P_\sigma(K_{X'}+\Delta')|_{F'} \equiv 0$. Thus

$$B^+|_{F'} \leq N_\sigma((K_{X'}+\Delta')|_{F'}) = N_\sigma(K_{X'}+\Delta')|_{F'}.$$

This implies that $B^+_h \leq N_\sigma(K_{X'}+\Delta')$, where B^+_h is the horizontal part of B^+. Therefore

$$\begin{aligned} P_\sigma(K_{X'}+\Delta'+B^-) &\sim_{\mathbb{Q}} P_\sigma(f'^*(K_{Z'}+\Delta_{Z'})+B^+_v+B^+_h) \\ &= P_\sigma(f'^*(K_{Z'}+\Delta_{Z'})+B^+_v) \\ &= P_\sigma(f'^*(K_{Z'}+\Delta_{Z'})) \end{aligned}$$

where the last step follows from the fact that B^+_v is f'-degenerate by Lemma 2.11.

□

Corollary 3.5. *Let (X,Δ) be a kawamata log terminal pair. Suppose that $f : X \to Z$ is a projective morphism with connected fibers to a normal projective variety Z such that $\nu((K_X+\Delta)|_F)=0$ for a general* *fiber F of f. Then there exists a higher birational model Z' of Z and a kawamata log terminal pair $(Z', \Delta_{Z'})$ such that $K_X + \Delta$ is abundant if and only if $K_{Z'} + \Delta_{Z'}$ is abundant.*

Proof. This follows from Lemma 3.4 and the fact that the numerical dimension is invariant under pull-back and under passing to the positive part $P_\sigma(L)$. □

4. Reduction maps and $\widetilde{\tau}(X, \Delta)$

The results of the previous section are most useful when combined with the theory of numerical reduction maps. We will focus on the D-trivial reduction map as defined in [Leh1]:

Theorem 4.1 ([Leh1, Theorem 1.1])**.** *Let X be a normal variety and D a pseudo-effective $\mathbb{R}$-Cartier divisor on X. Then there exist a projective birational morphism $\varphi : W \to X$ and a surjective projective morphism $f : W \to Y$ with connected fibers such that*

(0) *W is smooth,*
(1) *$\nu(\varphi^* D|_F) = 0$ for a general fiber F of f,*
(2) *if $w \in W$ is very general and C is an irreducible curve through w with $\dim f(C) = 1$, then $\varphi^* L.C > 0$, and*
(3) *If there exist a projective birational morphism $\varphi' : W' \to X$ and a dominant projective morphism $f' : W' \to Y'$ with connected fibers satisfying condition (2), then f' factors birationally through f.*

We call the composition $f \circ \phi^{-1} : X \dashrightarrow Y$ the D-trivial reduction map. Note that it is only unique up to birational equivalence.

Remark 4.2. The D-trivial reduction map is different from the pseudo-effective reduction map (cf. [E2] and [Leh1]), the partial nef reduction map (cf. [BDPP]), and Tsuji's numerically trivial fibration with minimal singular metrics (cf. [Ts] and [E1]).

Definition 4.3. Let X be a normal variety and D a pseudo-effective $\mathbb{R}$-Cartier divisor on X. If $f : X \dashrightarrow Y$ denotes the D-trivial reduction map, we define

$$\tau(D) := \dim Y.$$

The following properties follow immediately from the definition.

Lemma 4.4. *Let X be a normal projective variety and D a pseudo-effective $\mathbb{R}$-Cartier divisor on X. Then*

(1) *$\nu(D) = 0$ if $\tau(D) = 0$,*

(2) *if D' is a pseudo-effective $\mathbb{R}$-Cartier divisor on X such that $D' \geq D$, then $\tau(D') \geq \tau(D)$, and*
(3) *$\tau(f^*D) = \tau(D)$ for every surjective morphism $f : Y \to X$ from a normal variety.*

Since the canonical divisor is not a birational invariant, we need to introduce a slight variant of this construction that accounts for every ϵ-log smooth model.

Remark 4.5. Let (X, Δ) be a kamawata log terminal pair with $K_X+\Delta$ pseudo-effective. Suppose that $\phi : W \to X$ is a log resolution of (X, Δ). Then the value of $\tau(K_W + \Delta_W^\epsilon)$ for the ϵ-log smooth model (W, Δ_W^ϵ) is independent of the choice of $\epsilon > 0$, since if C is a movable curve with $(K_W + \Delta_W^\epsilon).C = 0$ then $E.C = 0$ for any ϕ-exceptional divisor E.

Definition 4.6. Let (X, Δ) be a projective kawamata log terminal pair such that $K_X + \Delta$ is pseudo-effective. We define

$$\begin{aligned}\widetilde{\tau}(X, \Delta) \quad = \quad & \max\{\tau(K_W + \Delta_W^\epsilon) | (W, \Delta_W^\epsilon) \text{ is an } \epsilon\text{-log smooth model} \\ & \text{of } (X, \Delta) \text{ with } 0 < \epsilon \ll 1\}.\end{aligned}$$

Lemma 4.7. *Let (X, Δ) be a kawamata log terminal pair such that $K_X + \Delta$ is pseudo-effective. Then there exists an ϵ-log smooth model $\varphi : (W, \Delta_W^\epsilon) \to (X, \Delta)$ such that the $(K_W + \Delta_W^\epsilon)$-trivial reduction map is a morphism on W whose image has dimension $\widetilde{\tau}(X, \Delta)$ for a sufficiently small positive number ϵ.*

Proof. We may certainly assume that $\tau(K_W + \Delta_W^\epsilon) = \widetilde{\tau}(X, \Delta)$. Suppose W' resolves the $(K_W + \Delta_W^\epsilon)$-trivial reduction map. By the maximality of the definition, the $(K_{W'} + \Delta_{W'}^{\epsilon'})$-trivial reduction map is birationally equivalent to the $(K_W + \Delta_W^\epsilon)$-trivial reduction map for any sufficiently small $\epsilon' > 0$. Thus it can be realized as the (resolved) morphism on W'. □

Remark 4.8. If D is a nef divisor, the D-trivial reduction map is birationally equivalent to the nef reduction map of D (see [BCEK+]). Thus $n(D) = \tau(D)$. Moreover, for a klt pair (X, Δ) such that $K_X + \Delta$ is nef, $\tau(K_X + \Delta) = \widetilde{\tau}(X, \Delta)$ since the nef reduction map is almost holomorphic.

Next, we prove that $\widetilde{\tau}(X, \Delta)$ is preserved under flips and divisorial contractions. Although we do not need this property to prove our main results, we include it for completeness.

Definition 4.9. Let X be an n-dimensional normal projective variety and $T \subset \mathrm{Chow}(X)$ an irreducible and compact normal covering family of 1-cycles in the sense of Camapana (cf. [C]). Let D be a $\mathbb{R}$-Cartier divisor on X. A covering family $\{C_t\}_{t\in T}$ is *D-trivial* if $D.C_t = 0$ for all $t \in T$. A covering family $\{C_t\}_{t\in T}$ is *1-connected* if for general x and $y \in X$ there is $t \in T$ such that C_t is an irreducible curve containing x and y.

Proposition 4.10 (cf. [Leh1, Proposition 4.8])**.** *Let X be a normal variety and D an $\mathbb{R}$-divisor on X. Suppose that there exists a D-trivial 1-connected covering family $\{C_t\}_{t\in T}$. Then $\nu(D) = 0$.*

Proof. For any birational map $\varphi : W \to X$, the strict transforms of the curves C_t are still 1-connecting. Thus, the generic quotient (in the sense of [Leh1, Construction 3.2]) of X by the family $\{C_t\}_{t\in T}$ contracts X to a point. Thus $\nu(D) = \nu(f^*D) = 0$ by [Leh1, Theorem 1.1]. □

Proposition 4.11. *Let (X, Δ) be a kawamata log terminal pair. Then $\nu(K_X+\Delta) = 0$ if and only if there exists a $(K_X+\Delta)$-trivial 1-connected covering family $\{C_t\}_{t\in T}$ such that $C_t \cap \mathbf{B}_-(K_X + \Delta) = \emptyset$ for general $t \in T$.*

Proof. The reverse implication follows from Proposition 4.10. Now assume that $\nu(K_X + \Delta) = 0$. By [D, Corollaire 3.4], we get a good minimal model (Y, Γ) of (X, Δ) with $K_Y + \Gamma \sim_{\mathbb{Q}} 0$. Take the following log resolutions:

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \dashrightarrow & Y. \end{array}$$

Set

$$p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E,$$

where E is an effective q-exceptional divisor. Now, since $K_Y + \Gamma \sim_{\mathbb{Q}} 0$, it holds that

$$p^*(K_X + \Delta) \sim_{\mathbb{Q}} E.$$

Because $\operatorname{codim} q(\operatorname{Supp} E) \geq 2$, there exists a complete intersection irreducible curve C with respect to very ample divisors $H_1, \ldots, H_{n-1}$ containing two general points x, y such that $C \cap q(\operatorname{Supp} E) = \emptyset$. Let $\bar{C}$ be the strict transform of C on X. Then

$$(K_X + \Delta).\bar{C} = 0.$$

Since $p(\operatorname{Supp} E) = \mathbf{B}_-(K_X+\Delta)$, the desired family can be constructed by deforming C. □

Proposition 4.12. *Let (X,Δ) be a klt pair such that $K_X+\Delta$ is pseudo-effective. Suppose that*

$$\varphi : (X,\Delta) \dashrightarrow (X',\Delta')$$

is a flip or a divisorial contraction. Then $\widetilde{\tau}(X,\Delta) = \widetilde{\tau}(X',\Delta')$.

Proof. Consider a log resolution

$$\begin{array}{ccc} & W & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & \dashrightarrow & X'. \end{array}$$

Write

$$K_W + \Delta_W^{\epsilon} = p^*(K_X+\Delta) + G$$

for the ϵ-log smooth structure induced by (X,Δ) and

$$K_W + \Delta_W'^{\epsilon} = q^*(K_{X'}+\Delta') + G',$$

for the ϵ-log smooth structure induced by (X',Δ') for a sufficiently small positive number ϵ. (Note that these structures might differ, if for example ϕ is centered in a locus along which the discrepancy is negative.) Using Lemma 4.7, we may find a log resolution W so that

(1) the $(K_W+\Delta_W^{\epsilon})$-trivial reduction map is a morphism $f : W \to Y$ with dim $Y = \widetilde{\tau}(X,\Delta)$, and
(2) the $(K_W+\Delta_W'^{\epsilon})$-trivial reduction map is a morphism $f' : W \to Y'$ with dim $Y' = \widetilde{\tau}(X',\Delta')$ and Y' is smooth.

Since ϕ is a $(K_X+\Delta)$-negative contraction, there is some effective q-exceptional divisor E' such that $K_W + \Delta_W^{\epsilon} = K_W + \Delta_W'^{\epsilon} + E'$. From Lemma 4.4 (2), it holds that $\widetilde{\tau}(X,\Delta) \geq \widetilde{\tau}(X',\Delta')$. Note that as E' is q-exceptional and $(W,\Delta_W'^{\epsilon})$ is an ϵ-log smooth model with $\epsilon > 0$, we have $\tau E' \leq N_\sigma(K_W + \Delta_W'^{\epsilon})$ for some $\tau > 0$.

Every movable curve C with $(K_W+\Delta_W^{\epsilon}).C = 0$ also satisfies $(K_W + \Delta_W'^{\epsilon}).C = 0$. Conversely, by Proposition 4.11 a very general fiber F' of f' admits a 1-connecting covering family of $K_W + \Delta_W'^{\epsilon}$-trivial curves $\{C_t\}_{t\in T}$ such that $C_t \cap \mathbf{B}_-((K_W+\Delta_W'^{\epsilon})|_{F'}) = \emptyset$ for general $t \in T$.

Since $E'|_{F'}$ is effective and $\nu((K_W+\Delta_W'^{\epsilon})|_{F'}) = 0$, we know that $\tau E'|_{F'} \leq N_\sigma((K_W+\Delta_W'^{\epsilon})|_{F'})$. Thus $E'.C_t = 0$ for general t since C_t avoids $\mathbf{B}_-((K_W+\Delta_W'^{\epsilon})|_{F'})$. So

$$\begin{aligned}(K_W+\Delta_W^{\epsilon}).C_t &= (K_W+\Delta_W'^{\epsilon}+E').C_t \\ &= 0.\end{aligned}$$

By the universal property of the D-trivial reduction map, f and f' are birationally equivalent. $\square$

Corollary 4.13. *Let (X,Δ) be a klt pair such that $K_X+\Delta$ is pseudo-effective. Suppose that (X,Δ) has a good minimal model. Then*

$$\widetilde{\tau}(X,\Delta)=\kappa(X,\Delta).$$

5. The MMP and the $(K_X+\Delta)$-trivial reduction map

In this section we use our main technical result Theorem 3.4 to analyze the main conjectures of the minimal model program inductively.

Theorem 5.1. *Assume that the existence of good minimal models for klt pairs in dimension d. Let (X,Δ) be a kawamata log terminal pair such that $K_X+\Delta$ is pseudo-effective and $\widetilde{\tau}(X,\Delta)=d$. Then there exists a good log minimal model of (X,Δ).*

Proof. Using Lemma 4.7, we can find a birational morphism $\varphi: W\to X$ from an ϵ-log smooth model (W,Δ_W^ϵ) of (X,Δ) for a sufficiently small positive number ϵ and a projective morphism $f: W\to Y$ with connected fibers such that

(i) $\nu((K_W+\Delta_W^\epsilon)|_F)=0$ for the general fiber F of f and
(ii) $\dim Y=\widetilde{\tau}(X,\Delta)$.

Theorem 3.3 and Corollary 3.5 imply that (W,Δ_W^ϵ) has a good minimal model. (X,Δ) then has a good minimal model by [BCHM, Lemma 3.6.10]. □

Corollary 5.2. *Conjecture 1.4 holds up to dimension n if and only Conjecture 1.1 holds up to dimension n.*

Proof. Assume that Conjecture 1.4 holds up to dimension n. By induction on dimension, we may assume that Conjecture 1.1 holds up to dimension $n-1$. Let (X,Δ) be a kawamata log terminal pair of dimension n. If $\widetilde{\tau}(X,\Delta)<\dim X$ then $K_X+\Delta$ is abundant by Theorem 5.1. If $\widetilde{\tau}(X,\Delta)=\dim X$ then the abundance of $(K_X+\Delta)$ follows by assumption from Conjecture 1.4.

Conversely, assume that Conjecture 1.1 holds up to dimension n. By Theorem 3.3 we obtain the existence of good minimal models up to dimension n. Let (X,Δ) be a kawamata log terminal pair of dimension $k\le n$ such that $\widetilde{\tau}(X,\Delta)=k$. By Corollary 4.13 X is covered by irreducible curves C such that $(K_X+\Delta).C=0$ unless $\kappa(K_X+\Delta)=n$. □

Remark 5.3. It seems likely that one could formulate a stronger version of Theorem 5.2 using the pseudo-effective reduction map for $K_X+\Delta$ (cf. [E2] and [Leh1]). The difficulty is that the pseudo-effective reduction map only satisfies the weaker condition $\nu(P_\sigma(K_X+\Delta)|_F)=0$

on a general fiber F, so it is unclear how to use the inductive hypothesis to relate F with X.

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