

Reduction of a wave packet in quantum Brownian motion

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The effect of the environment on a quantum system is studied on an exactly solvable model: a harmonic oscillator interacting with a one-dimensional massless scalar field. We show that in an open quantum system, dissipation can cause decorrelation on a time scale significantly shorter than the relaxation time which characterizes the approach of the system to thermodynamic equilibrium. In particular, we demonstrate that the density matrix decays rapidly toward a mixture of "approximate eigenstates" of the "pointer observable," which commutes with the system-environment interaction Hamiltonian. This observable can be regarded as continuously, if inaccurately, monitored by the scalar field environment. Both because in a harmonic oscillator the state of the system rotates in the phase space and because the effective environment "measurement" is weak, the system, on the short "collision" time scale ($1/\Gamma$), maintains a coherence in this pointer observable on time scales of order $[\gamma/\Omega \ln(\Gamma/\Omega)]^{1/2}$ and on longer time scales settles into a mixture of coherent states with a dispersion approximately consistent with the vacuum state. The master equation satisfied by the exact solution differs from the other master equations derived both for the high-temperature limit and for $T=0$. We discuss these differences and study the transition region between the high- and low-temperature regimes. We also consider the behavior of the system in the short-time "transient" regime. For $T=0$, we find that, in the long-time limit, the system behaves as if it were subject to "1/f noise." The generality of our model is considered and its predictions are compared with previous treatments of related problems. Some of the possible applications of the results to experimentally realizable situations are outlined. The significance of the environment-induced reduction of the wave packet for cosmological models is also briefly considered.

I. INTRODUCTION

The purpose of this paper is to discuss one of the few exactly solvable problems in the quantum theory of open systems. We shall consider a linear system—a harmonic oscillator, a free particle, and an "upside-down" harmonic oscillator interacting with a simple model of a quantum thermal reservoir. As is the case with all such "classic" problems, this one had been treated quite a few years ago through a variety of approaches.¹⁻⁷ Reviews of these early discussions are also available.⁸⁻¹⁰

The recent resurgence of interest in this as well as other related issues was motivated in part by the realization that many of the experimentally accessible systems fall in the category of "open" quantum systems.¹¹⁻¹⁵ Moreover, recent experimental observations of "macroscopic" quantum systems give urgency to a better understanding of such open quantum systems.

The other motivation has to do with the revival of interest in the problem of quantum measurement. Calculations of simple quantum systems have already demonstrated that their interaction with the environment may "bring out" classical features at the expense of quantum correlations.¹⁶⁻²⁰ We shall investigate the extent to which the "openness" of macroscopic objects may serve as an explanation of their classical behavior.

The plan of this paper is as follows. In the next section we describe our model—a quantum harmonic oscillator interacting with a scalar field—and give the exact solution for time evolution of its density matrix. In addition we introduce a new representation of the density function, the " (k, Δ) " representation (actually double Fourier transform of the usual Wigner-function representation). In this representation the master equation has a very simple first-order form and the Green's function is a simple algebraic expression, unlike in other representations of the density matrix. After a brief discussion of its properties we proceed, in Sec. III, to obtain a nonstationary master equation (ME) satisfied by the density matrix of Sec. II, which in the Wigner representation has a Fokker-Planck form. This ME differs from the usual high- and low-temperature limits. Additional terms make it more accurate both in the short-time transient regime and in the transition range where neither a high- nor very-low-temperature approximation can be justified.

Section IV is devoted to the discussion of this master equation. In particular, we obtain the closed-form expression for its coefficients in the high-temperature as well as in the $T=0$ case. Section V illustrates behavior of the "open" harmonic oscillator under a variety of assumptions. In particular, we use there entropy of the system to quantify the extent of decoherence in the course of

the system-environment interaction. Section VI considers the significance of our results for the issues of quantum theory of measurement and for the transition between quantum and classical behaviors. We also comment briefly there on the quantum-classical transition in the cosmological context.

The second part of the following section as well as much of Sec. III are somewhat more technical, and, consequently, more difficult to follow than the remainder of the paper. Some readers may find it easier to scan through on this part of the paper on the first reading, and return to it after going through Secs. IV, V, and VI.

II. THE MODEL

There are two largely equivalent ways to model a quantum "heat bath." The first one is to consider a reservoir composed of a large ensemble of noninteracting quantum systems such as harmonic oscillators.¹⁻¹⁵ The second is to use a free field φ (Refs. 21 and 22). In our treatment we shall take advantage of this latter possibility. The complete system under consideration consists therefore of a single "particle" described by its coordinate q and a heat bath modeled by the field φ .

The Lagrangian action for the system will be taken to be

$$I = \int \left\{ \frac{1}{2} \left[\dot{\varphi}^2 - \left(\frac{\partial \varphi}{\partial x} \right)^2 \right] + \delta(x) (\dot{q}^2 - \Omega_0^2 q^2 - \epsilon q \dot{\varphi}) \right\} dt dx . \quad (2.1)$$

The time derivative coupling between the field and the oscillator is taken in order to ensure simple damping behavior for the oscillator in the coupled system. In addition this form of the coupling ensures that "runaway" solutions do not exist since the energy

$$E = \int \left\{ \frac{1}{2} \left[\dot{\varphi}^2 + \left(\frac{\partial \varphi}{\partial x} \right)^2 \right] \right\} dx + (\dot{q}^2 + \Omega_0^2 q^2)$$

is a positive-definite quantity.

In the Heisenberg picture, the evolution of the harmonic oscillator is generated by

$$m(\ddot{q} + \Omega_0^2 q) = -\epsilon \dot{\varphi} \quad (2.2a)$$

$$q = [q_0 \cos \Omega t + (p_0 + \gamma q_0) \sin \Omega t / \Omega] e^{-\gamma t} - (\epsilon / \Omega) \int_0^t [\sin \Omega(t-t') e^{-\gamma(t-t')} \dot{\varphi}_0(t')] dt' , \quad (2.5)$$

where the damping coefficient γ equals

$$\gamma = \epsilon^2 / 4 \quad (2.6)$$

and the angular frequency of the damped harmonic oscillator is

$$\Omega = \sqrt{\Omega_0^2 - \gamma^2} . \quad (2.7)$$

The momentum of the oscillator particle ($m = 1$) is then

while the field φ obeys

$$\ddot{\varphi} - \partial_x^2 \varphi = \epsilon \dot{q} \delta(x) . \quad (2.2b)$$

Above, Ω_0 is the frequency of the undamped harmonic oscillator.

The scalar field φ interacts with the harmonic oscillator at $x = 0$ in the space in which the scalar field travels. It is important to note that by writing $\delta(x)$ rather than $\delta(x - q)$ in Eq. (2.1), we distinguish between the space q of the harmonic-oscillator coordinate and the space x in which the field φ propagates. Coordinate q (or, alternatively, coordinate x) may be regarded as characterizing a distinct "internal" space. For instance, one could regard Eqs. (2.1) and (2.2) as modeling the internal degree of freedom q of a particle permanently located at $x = 0$ in physical space.

The constant ϵ characterizes the strength of coupling between the field and the system. The mass m can be incorporated into the definition of ϵ and q through the change of variables $\epsilon' = \epsilon / \sqrt{m}$, $q' = q \sqrt{m}$. We shall assume below that this simplifying substitution was introduced—e.g., we shall set $m = 1$ —but we shall not carry cumbersome primes to indicate this in our notation.

Equation (2.2b) is solved by

$$\varphi = \varphi_0 + (\epsilon / 2) [q(t-x)\Theta(x) + q(t+x)\Theta(-x)] . \quad (2.3)$$

Here φ_0 is the "unperturbed" solution of Eq. (2.2b). The term in square brackets shows that the effect of the perturbation is restricted to the "light cone," $t \leq |x|$. This solution can now be substituted to Eq. (2.2a). One obtains

$$\ddot{q} + (\epsilon^2 / 2) \dot{q} + \Omega_0^2 q = -\epsilon \dot{\varphi}_0 . \quad (2.4)$$

It is the assumption of retarded interaction in Eq. (2.3) which leads to the damping term $\epsilon^2 / 2$ rather than to the "antidamping" with the opposite sign. Equation (2.4) can be regarded as a generalized Langevin equation for the coordinate q . In the usual Langevin equation the fluctuating force on the right-hand side would have the time dependence of the white noise: $\langle \dot{\varphi}_0(t) \dot{\varphi}_0(0) \rangle \sim \delta(t)$. In the case considered here this will be true only in the high-temperature limit, when

$$\langle \dot{\varphi}_0(t) \dot{\varphi}_0(0) \rangle \approx (kT / 2) \delta(t) ,$$

where terms independent of T have been neglected.

Equation (2.4) is solved by

$$p = \dot{q} = \{ p_0 \cos \Omega t - [(\gamma / \Omega)(p_0 + \gamma q_0) + \Omega q_0] \sin \Omega t \} e^{-\gamma t} - \frac{\epsilon}{\Omega} \frac{d}{dt} \int_0^t [\sin \Omega(t-t') e^{-\gamma(t-t')} \dot{\varphi}_0(t')] dt' . \quad (2.8)$$

The harmonic oscillator is underdamped when Ω , defined in Eq. (2.7), is real. The case of imaginary Ω corresponds to the overdamped harmonic oscillator as long as Ω_0 is

real. For a free particle we have $\Omega = i\gamma$ or $\Omega_0 = 0$. Imaginary Ω_0 , or equivalently $\Omega^2 < -\gamma^2$, corresponds to the particle moving in an "upside down" unstable harmonic-oscillator potential. Equations (2.5) and (2.8) can be readily adopted to all the cases when Ω is imaginary through the substitution $i\Omega \rightarrow \Omega$ and the usual formulas relating trigonometric and hyperbolic functions ($\sinh \Omega = -i \sin i\Omega$, $\cosh \Omega = \cos i\Omega$).

Note that for this linear system the above set of equations applies equally well to classical and quantum systems. From now on we shall often have to regard q , p , and φ as operators. This will be indicated by the use of capital Q and P in equations where the operator nature of position and momentum is essential. The state of the complete system, oscillator plus field, is in general described by a density matrix ρ_{OF} . Our attention will, however, focus on the harmonic oscillator. Therefore, we shall be interested only in the evolution of the reduced oscillator density matrix ρ :

$$\rho \equiv \rho_O = \text{Tr}_F \rho_{OF}, \quad (2.9)$$

where the trace is taken over the degrees of freedom of the field.

We have discovered that the calculations become more tractable when the density matrix ρ is given not in the po-

sition representation, $\rho(q, q')$, but, rather, in a new " (k, Δ) " representation:

$$\rho(k, \Delta) = \int e^{ikq'} \rho(q' - \Delta/2, q' + \Delta/2) dq'. \quad (2.10)$$

Here, Δ measures the distance from the diagonal in the position representation while k is the wave number in the direction parallel to the diagonal. Note that this is just the double Fourier transform of the usual Wigner function:

$$W(p, q) = (2\pi)^{-1} \int \rho(q - \Delta/2, q + \Delta/2) e^{ip\Delta} d\Delta.$$

The density matrix operator in Eq. (2.10) can now be expressed as

$$\rho(q - \frac{1}{2}\Delta, q + \frac{1}{2}\Delta) = \text{Tr} \delta(Q - q) e^{iP\Delta/2} \rho e^{iP\Delta/2} \quad (2.11)$$

by means of the shift operator $e^{iP\Delta/2}$. Using Eqs. (2.10) and (2.11), as well as the commutation relation $[P, Q] = -i$, one obtains

$$\rho(k, \Delta) = \text{Tr} e^{i(kQ + \Delta P)} \rho. \quad (2.12)$$

We are working in the Heisenberg representation. Therefore, ρ remains constant while P, Q are functions of time given by Eqs. (2.5) and (2.8). We can therefore write a more explicit form of Eq. (2.12):

$$\begin{aligned} \rho(k, \Delta) = & \text{Tr} \left\{ \rho^0 \exp i \left[\Delta \left(P_0 \cos \Omega t - \frac{\gamma}{\Omega} P_0 \sin \Omega t - \frac{\gamma^2 + \Omega^2}{\Omega} Q_0 \sin \Omega t \right) e^{-\gamma t} + k \left(Q_0 \cos \Omega t + \frac{P_0 + \gamma Q_0}{\Omega} \sin \Omega t \right) e^{-\gamma t} \right] \right\} \\ & \times \text{Tr}_F \left\{ \rho_F^0 \exp \left[-i \left[\Delta \frac{d}{dt} + k \right] \frac{\epsilon}{\Omega} \int_0^t \sin[\Omega(t-t')] e^{-\gamma(t-t')} \varphi_0(t') dt' \right] \right\}. \end{aligned} \quad (2.13)$$

Here $\rho^0 \equiv \rho_O^0$ and ρ_F^0 are initial density matrices of the system and of the field. We have assumed that they are not correlated at $t=0$, that is, initially we have

$$\rho_{OF}^0 = \rho^0 \rho_F^0.$$

Indeed, if this assumption is satisfied one can write, at any time,

$$\rho(k, \Delta) = \rho^0(k(t), \Delta(t)) E(k, \Delta; t), \quad (2.14a)$$

where

$$E(k, \Delta; t) = \text{Tr}_F \left\{ \rho_F^0 \exp \left[-i \left[\Delta \frac{d}{dt} + k \right] \frac{\epsilon}{\Omega} \int_0^t \sin[\Omega(t-t')] e^{-\gamma(t-t')} \varphi_0(t') dt' \right] \right\}. \quad (2.14b)$$

and

$$\Delta(t) = \left\{ \Delta [\cos \Omega t - (\gamma/\Omega) \sin \Omega t] + k \Omega^{-1} \sin \Omega t \right\} \exp(-\gamma t),$$

$$k(t) = \left\{ k [\cos \Omega t + (\gamma/\Omega) \sin \Omega t] - \Delta (\gamma^2 + \Omega^2) \Omega^{-1} \sin \Omega t \right\} \exp(-\gamma t).$$

It is the great simplicity of this result for the time development of the reduced density matrix in this (k, Δ) representation which makes it convenient. The Green's function for the density matrix is just

$$G(k', \Delta', k, \Delta, t) = \delta(k' - k(t)) \delta(\Delta' - \Delta(t)) E(k', \Delta'; t).$$

In other representations the Green's function is a very complicated and singular function.

The first term in Eq. (2.14a) is evaluated already in Eq. (2.13). Obtaining the second term in a more explicit form is cumbersome. Here we provide only the resulting formula:

$$E(k, \Delta; t) = \exp \left\{ -\frac{1}{\pi} \left[\frac{\epsilon}{\Omega} \right]^2 \int_0^\Gamma \left[|kz_\omega(t) + \Delta \dot{z}_\omega(t)|^2 \coth \frac{\beta\omega}{2} \right] d\omega \right\} \quad (2.15)$$

calculated under the assumption that the field is in thermodynamic equilibrium at the temperature $T = \beta^{-1}$. The time- and frequency-dependent function $z_\omega(t)$ in Eq. (2.15) is given by

$$z_\omega(t) = \frac{e^{-i\omega t} - e^{i(\Omega - \gamma)t}}{\gamma - i\Omega - i\omega} - \frac{e^{-i\omega t} - e^{-i(\Omega + \gamma)t}}{\gamma + i\Omega - i\omega}. \quad (2.16)$$

The cutoff Γ in Eq. (2.15) is introduced to regularize certain logarithmic divergences. It is somewhat easier to express the results of the calculations in terms of elementary functions when the cutoff Γ is "abrupt," as in Eq. (2.15). However, a sharp cutoff introduces oscillations with characteristic frequency Γ into the solutions. To get rid of this unphysical feature we shall replace $\int_0^\Gamma d\omega$ with an exponential cutoff $\int_0^\infty e^{-\omega/\Gamma} d\omega$ in the calculations illustrating the behavior of the system in Secs. IV and V. Explicit evaluation of the integrals in Eq. (2.15) is straightforward, but cumbersome. Here we consider only the case of $T = 0$, and only for times $t \gg \Gamma^{-1}$. The logarithmic dependences on the cutoff, which we shall some-

times refer to as the logarithmic divergence, that appear in this situation are symptomatic of the generic behavior.

To simplify notation we introduce

$$A(t) = \int_0^\Gamma |z_\omega(t)|^2 \coth(\beta\omega/2) d\omega, \quad (2.17a)$$

$$B(t) = \int_0^\Gamma |z_\omega(t)\dot{z}_\omega(t)| \coth(\beta\omega/2) \omega d\omega = \dot{A}, \quad (2.17b)$$

$$C(t) = \int_0^\Gamma |\dot{z}_\omega(t)|^2 \coth(\beta\omega/2) \omega d\omega. \quad (2.17c)$$

Equation (2.15) can now be written as

$$E(t) = \text{Tr}_F \{ \dots \} \\ = \exp \left[-\frac{1}{\pi} \left(\frac{\epsilon}{\Omega} \right)^2 (Ak^2 + 2Bk\Delta + C\Delta^2) \right]. \quad (2.18)$$

This Gaussian form will simplify transformation to representations other than (k, Δ) .

In the limit of $\beta \rightarrow \infty$ ($\coth \beta\omega/2 \rightarrow 1$) and for real Ω we get

$$A(t) = 4 \left[e^{-2\gamma t} \sin^2 \Omega t (\Omega^2 + \gamma^2)^2 - 2\Omega e^{-\gamma t} \int_0^\Gamma \frac{\sin \Omega t \omega^2 \sin \omega t \omega \cos \omega t}{\omega^4 + 2(\gamma^2 - \Omega^2)\omega^2 + (\Omega^2 + \gamma^2)^2} d\omega \right. \\ \left. + [(\gamma \sin \Omega t + \Omega \cos \Omega t)^2 e^{-2\gamma t} + \Omega^2] \int_0^\Gamma \frac{\omega}{\omega^4 + 2(\gamma^2 - \Omega^2)\omega^2 + (\Omega^2 + \gamma^2)^2} d\omega \right]. \quad (2.19)$$

In the limit of very short times, $t < \Gamma^{-1}$, the first two integrals are both logarithmically divergent. We shall leave closer inspection of this "transient" regime for Sec. IV. In the regime of intermediate times $t \gg \Gamma^{-1}$ the dominant term in the expression for $A(t)$ is

$$A(t) \simeq 4e^{-2\gamma t} \sin^2 \Omega t \left[\frac{1}{2} \ln \frac{\Gamma^2}{\Omega^2 + \gamma^2} + \frac{\Omega^2 - \gamma^2}{4\Omega\gamma} \left[\arctan \frac{\Omega^2 - \gamma^2}{2\Omega\gamma} + \frac{\pi}{2} \right] \right]. \quad (2.20)$$

For very long times the last term of Eq. (2.19) becomes dominant:

$$A(\infty) = \frac{\Omega}{\gamma} \left[\arctan \frac{\Omega^2 - \gamma^2}{2\Omega\gamma} - \frac{\pi}{2} \right]. \quad (2.21)$$

Similar discussion of the "mixed term" B gives for the intermediate times

$$B(t) = 4e^{-2\gamma t} (\Omega \cos \Omega t \sin \Omega t - \gamma \sin^2 \Omega t) \left[\frac{1}{2} \ln \frac{\Gamma^2}{\Omega^2 + \gamma^2} + \frac{\Omega^2 - \gamma^2}{4\Omega\gamma} \left[\arctan \frac{\Omega^2 - \gamma^2}{2\Omega\gamma} + \frac{\pi}{2} \right] \right]. \quad (2.22)$$

For very long times

$$B(\infty) = 0. \quad (2.23)$$

The last term becomes

$$C(t) = 4[\Omega^2 + (\gamma \sin \Omega t - \Omega \cos \Omega t)^2 e^{-2\gamma t}] \left[\frac{1}{2} \ln \frac{\Gamma^2}{\Omega^2 + \gamma^2} + \frac{\Omega^2 - \gamma^2}{4\Omega\gamma} \left[\arctan \frac{\Omega^2 - \gamma^2}{2\Omega\gamma} + \frac{\pi}{2} \right] \right] \quad (2.24)$$

in the intermediate times, and tends to

$$C(\infty) = 4\Omega^2 \left[\frac{1}{2} \ln \frac{\Gamma^2}{\Omega^2 + \gamma^2} \right. \\ \left. + \frac{\Omega^2 - \gamma^2}{4\Omega\gamma} \left[\arctan \frac{\Omega^2 - \gamma^2}{2\Omega\gamma} - \frac{\pi}{2} \right] \right] \quad (2.25)$$

when $t \rightarrow \infty$.

It is interesting to note the difference between the behavior of the diagonal and off-diagonal (e.g., governed by $\sim k^2$ and $\sim \Delta^2$ terms) elements of the density matrix in the position representation. Logarithmically divergent terms are present in all of the coefficients A , B , and C in the intermediate times. However, they enter into A and

B multiplied by either $\sin\Omega t$ or $\sin^2\Omega t$. Hence, the effect of the logarithmic divergence sets in on a dynamical time scale Ω^{-1} . By contrast, logarithmic divergence enters into $C(t)$ almost instantly, on a very much shorter "collision" time scale of Γ^{-1} . By the similar token, the spread of the wave packet in the spatial direction is given by Eq. (2.21), and contains no logarithmic divergences while the spread of the wave packet in Δ is small for $t \rightarrow \infty$, as it is inversely proportional to the logarithmically divergent quantity $\sqrt{C(\infty)}$.

These preliminary results contain trends which will be confirmed by more detailed case-by-case discussion in Secs. IV and V. They show that off-diagonal elements of the density matrix expressed in the preferred "pointer basis" are damped much faster than the elements on the diagonal.

III. DERIVATION OF THE MASTER EQUATION

The equation which generates time evolution of the density matrix,

$$\dot{\rho} = L\rho, \quad (3.1)$$

is known as a master equation. Above, L is an evolution operator which takes into account both the self-Hamiltonian of the system and its interaction with the environment. Consequently, L is typically non-Hermitian. The goal of this section is to derive and discuss the master equation which governs the evolution of a harmonic oscillator, a free particle, or an "upside down" harmonic oscillator in the heat bath, Eqs. (2.1) and (2.2). This may appear to be an unnecessary exercise. After all, we have already obtained in the preceding section a general solution of this problem.

In spite of that, the master equation is, in a sense, worth more than a solution. To begin with, it can be written more compactly than the solution. This compactness often allows one to notice physically interesting features which would remain hidden in the intricacies of the solution. More importantly, it is easier to compare results of our approach with the other discussions of the same problem as the usual goal in the treatment of open systems is the derivation of the appropriate master equation. Finally, in this case it is easier to numerically integrate the master equation than to evaluate the solution.

To obtain the expression for the operator L we must calculate the time derivative of the time-dependent density matrix $\rho(k, \Delta)$, Eq. (2.13):

$$L = \frac{\dot{\rho}(k, \Delta)}{\rho(k, \Delta)}. \quad (3.2)$$

The density matrix $\rho(k, \Delta)$ has the form $\rho = \text{Tr}\rho^0 e^{i(kQ + \Delta P)}$. To evaluate its time derivative, we need the formula

$$\frac{d e^{A(t)}}{dt} = \int_0^1 e^{\lambda A(t)} \dot{A} e^{(1-\lambda)A(t)} d\lambda. \quad (3.3)$$

Above A is an operator which depends on a parameter t . We leave the proof of Eq. (3.3) to the reader.

The time derivative of the density matrix is then given by

$$\dot{\rho} = \text{Tr}\rho^0 \int_0^1 e^{i\lambda(kQ + \Delta P)} [i(k\dot{Q} + \Delta\dot{P})] e^{i(1-\lambda)(kQ + \Delta P)} d\lambda. \quad (3.4)$$

Moreover, using $\dot{Q} = P$ and employing Eq. (2.4) to evaluate $\dot{P} = \dot{Q}$ we arrive at

$$\dot{\rho} = \text{Tr}\rho^0 \int_0^1 e^{i\lambda(kQ + \Delta P)} i [kP + \Delta(-\Omega_0^2 Q - \epsilon^2 P/2 - \epsilon\dot{\varphi}_0)] e^{i(1-\lambda)(kQ + \Delta P)} d\lambda. \quad (3.5)$$

We can now employ Eq. (3.3) again (this time in the "reverse" direction) to obtain

$$\dot{\rho} = \left[k \frac{\partial}{\partial \Delta} - \Delta \Omega_0^2 \frac{\partial}{\partial k} - \frac{\Delta \epsilon^2}{2} \frac{\partial}{\partial \Delta} \right] \rho(k, \Delta) - i \Delta \epsilon \text{Tr}\rho^0 \int_0^1 e^{i\lambda(kQ + \Delta P)} \dot{\varphi}_0 e^{i(1-\lambda)(kQ + \Delta P)} d\lambda. \quad (3.6)$$

We still have to obtain a more explicit form of the second part of the second part of the above expression:

$$D(t) \equiv -i \Delta \epsilon \text{Tr}\rho_0 \int_0^1 e^{i\lambda(kQ + \Delta P)} \dot{\varphi}_0 e^{i(1-\lambda)(kQ + \Delta P)} d\lambda \quad (3.7)$$

before we will have a useful master equation. To this end we calculate

$$kQ + \Delta P = k [Q_0 \cos\Omega t + (P_0 + \gamma Q_0) \sin\Omega t] e^{-\gamma t} + \Delta \{ P_0 \cos\Omega t - [(\gamma/\Omega)(P_0 + \gamma Q_0) + \Omega Q_0] \sin\Omega t \} e^{-\gamma t} \\ - (\Delta d/dt + k) \left[(\epsilon/\Omega) \int_0^t \sin\Omega(t-t') e^{-\gamma(t-t')} \dot{\varphi}_0 dt' \right]. \quad (3.8)$$

Let us denote the first, φ -independent part of the above expression by $S(k, \Delta, P_0, Q_0)$. Then $D(t)$ can be written as a product:

$$D(t) = -i \Delta \epsilon (\text{Tr}\rho^0 e^{iS(k, \Delta, P_0, Q_0)}) \text{Tr}\rho_F^0 \int_0^1 d\lambda \\ \times \left\{ \exp \left[-i\lambda(k + \Delta d/dt) \left[(\epsilon/\Omega) \int_0^t \sin\Omega(t-t') e^{-\gamma(t-t')} \dot{\varphi}_0(t') dt' \right] \right] \right\} \dot{\varphi} \\ \times \exp \left[-i(1-\lambda)(k + \Delta d/dt) \left[(\epsilon/\Omega) \int_0^t \sin\Omega(t-t') e^{-\gamma(t-t')} \dot{\varphi}_0(t') dt' \right] \right] \Bigg\}. \quad (3.9)$$

Now the density matrix $\rho(k, \Delta)$ is given by Eq. (2.13):

$$\rho(k, \Delta) = \text{Tr} \rho^0 e^{iS(k, \Delta, P_0, Q_0)} \text{Tr} \left\{ \rho_F^0 \exp \left[-i(k + \Delta d/dt) \left[(\epsilon/\Omega) \int_0^t \sin \Omega(t-t') e^{-\gamma(t-t')} \dot{\varphi}_0(t') dt' \right] \right] \right\}. \quad (3.10)$$

Above, as in Sec. II, we have again assumed that the density matrices $\rho^0 \equiv \rho_O^0$ and ρ_F^0 are initially uncorrelated.

Now it follows that $D(t)$, Eq. (3.9), can be written as

$$D(t) = -i\Delta\epsilon\rho(k, \Delta) \left\{ -i \frac{\partial}{\partial \alpha} \ln \text{Tr} \rho_F^0 \exp \left[i(k + \Delta d/dt) \left[-(\epsilon/\Omega) \int_0^t \sin \Omega(t-t') e^{-\gamma(t-t')} \dot{\varphi}_0(t') dt' \right] + \alpha \varphi_0(t) \right] \right\} \Big|_{\alpha=0}. \quad (3.11)$$

To proceed further we use the formula

$$\text{Tr} \rho_F^0 \exp(iO\varphi_0) = \exp[-\frac{1}{2} \text{Tr} \rho_F^0 (O\varphi_0)^2] \quad (3.12)$$

valid for an arbitrary linear operator O and Gaussian density matrix ρ_F^0 . In our case

$$O\varphi_0 = -(k + \Delta d/dt) \left[(\epsilon/\Omega) \int_0^t \sin \Omega(t-t') e^{-\gamma(t-t')} \dot{\varphi}_0(t') dt' \right] + \alpha \dot{\varphi}_0(t) \quad (3.13)$$

and we will always take the field to be in either a thermal or vacuum state, either of which have Gaussian ρ_F^0 .

We can now evaluate the derivative $\partial/\partial\alpha$ in Eq. (3.11) and arrive, after a few additional manipulations, at the expression

$$D(t) = (-\Delta\epsilon^2/2)\rho(k, \Delta) \left\{ (k - \gamma\Delta) \int_0^t (\sin \Omega\tau/\Omega) e^{-\gamma\tau} \langle \dot{\varphi}_0(t)\varphi_0(t') + \dot{\varphi}_0(t')\dot{\varphi}_0(t) \rangle dt' \right. \\ \left. + \Delta \int_0^t \cos \Omega\tau e^{-\gamma\tau} \langle \dot{\varphi}_0(t)\dot{\varphi}_0(t') + \dot{\varphi}_0(t')\dot{\varphi}_0(t) \rangle dt' \right\}, \quad (3.14)$$

where $\tau = t - t'$. Moreover, the correlation function of the quantum noise is given by

$$\langle \dot{\varphi}_0(t)\dot{\varphi}_0(t') + \dot{\varphi}_0(t')\dot{\varphi}_0(t) \rangle = 2\pi^{-1} \int_0^\Gamma \omega \coth(\beta\omega/2) \cos(\omega\tau) d\omega. \quad (3.15)$$

In the above, we have assumed that the incoming field, φ_0 is in a thermal state at a temperature given by $T = \beta^{-1}$. Therefore, $D(t)$ can now be explicitly evaluated:

$$D(t) = (-\Delta\epsilon^2)\rho(k, \Delta) \left\{ \frac{k - \gamma\Delta}{\Omega\pi} \int_0^\Gamma \left[\int_0^t \cos \omega\tau \sin \Omega\tau e^{-\gamma\tau} d\tau \right] \omega \coth(\beta\omega/2) d\omega \right. \\ \left. + \frac{\Delta}{\pi} \int_0^\Gamma \left[\int_0^t \cos \omega\tau \cos \Omega\tau e^{-\gamma\tau} d\tau \right] \omega \coth(\beta\omega/2) d\omega \right\}. \quad (3.16)$$

We can now finally write down the master equation in the (k, Δ) representation:

$$\dot{\rho} = \left[k \frac{\partial}{\partial \Delta} - \Omega_0^2 \Delta \frac{\partial}{\partial k} - 2\gamma\Delta \frac{\partial}{\partial \Delta} + 4\gamma\Delta^2 h(t, \Gamma, \beta) - 4\gamma\Delta k f(t, \Gamma, \beta) \right] \rho. \quad (3.17)$$

Above we have used the notation

$$f(t, \Gamma, \beta) = \frac{1}{\Omega\pi} \int_0^\Gamma \left[\int_0^t \cos \omega\tau \sin \Omega\tau e^{-\gamma\tau} d\tau \right] \omega \coth(\beta\omega/2) d\omega, \quad (3.18a)$$

$$g(t, \Gamma, \beta) = \frac{1}{\pi} \int_0^\Gamma \left[\int_0^t \cos \omega\tau \cos \Omega\tau e^{-\gamma\tau} d\tau \right] \omega \coth(\beta\omega/2) d\omega, \quad (3.18b)$$

$$h(t, \Gamma, \beta) = g - \gamma f. \quad (3.18c)$$

These time-dependent functions determine the evolution of the density matrix and will be discussed below. However, in order to appreciate their physical significance it is useful to rewrite the master equation in the more familiar position representation, and in the Wigner representation.

To transform Eq. (3.17) into the position representa-

tion we employ the identities

$$\Delta = q - q', \quad (3.19a)$$

$$k = i \left[\frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right], \quad (3.19b)$$

$$\frac{\partial}{\partial k} = -i(q+q'), \quad (3.19c)$$

$$\frac{\partial}{\partial \Delta} = \frac{\partial}{\partial q} - \frac{\partial}{\partial q'}. \quad (3.19d)$$

This results in the equation

$$\begin{aligned} \dot{\rho} = & \left[i \left[\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial q'^2} - \Omega_0^2(q^2 - q'^2) \right] \right. \\ & \left. - 2\gamma(q - q') \left[\frac{\partial}{\partial q} - \frac{\partial}{\partial q'} \right] - 4\gamma(q - q')^2 h \right] \rho \\ & + 4i\gamma(q - q') \left[\frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right] f \rho. \end{aligned} \quad (3.20)$$

Our master equation consists of the von Neumann equation [the term in the first line of (3.20)], the damping term (with $\gamma = \epsilon^2/4$), and the "quantum diffusion" term [both in the second line of (3.20)]. All of these appear in other master equations.¹⁻¹² However, in our case the diffusion coefficient $h = g - \gamma f$ is still time dependent. Therefore, the term in large square brackets is nonstationary. Moreover, Eq. (3.20) contains a correction term proportional to f . This is the last term in Eq. (3.20). It arises from the last term in the (k, Δ) version of the master equation, Eq. (3.17). We shall discuss their behavior and significance in the following two sections.

In the Wigner representation of the density matrix, the equation takes the Fokker-Planck form

$$\begin{aligned} \dot{W} = & \left[-\frac{\partial}{\partial q} \rho + \Omega_0^2 \frac{\partial}{\partial p} q + 2\gamma \frac{\partial}{\partial p} p + 4\gamma h \frac{\partial^2}{\partial p^2} \right. \\ & \left. - 4\gamma f \frac{\partial}{\partial q} \frac{\partial}{\partial p} \right] W. \end{aligned} \quad (3.21)$$

The unusual features of this Fokker-Planck equation are that the diffusion coefficients are time dependent and that the diffusion coefficient matrix is not positive definite.

Equations very similar to (3.20) and (3.21) were obtained by Dekker^{7(a)} from a purely phenomenological point of view, in which he quantized the damped harmonic oscillator and introduced *ad hoc* quantum white-noise terms to preserve the commutation relations of his oscillator phase-space variables. These extra terms were for him a manifestation of the fluctuation dissipation relations. It is interesting to note that our exact model produces such a quantum noise source, which, however, is not white. Furthermore, the diffusion coefficients [terms multiplying the second derivatives in (3.21)] are not constants.

IV. MASTER EQUATION AND THE EVOLUTION OF THE DENSITY MATRIX

Again, the unusual feature of Eqs. (3.17), (3.20), and (3.21) is both the time dependence of the source (diffusion) terms, and the presence of the term proportional to $k \Delta \rho$ in (3.17), which is absent in the usual high-

temperature master equation (see, e.g., Ref. 12 and references therein). To understand the physical origin of these terms, let us look at the time dependence of the expectation values of P^2 , Q^2 , and $PQ + QP$. One can evaluate these using the equations of motion for the P and Q operators obtained from Eqs. (2.4) and (2.8), or one can obtain these from Eq. (3.17) for the density matrix in the (k, Δ) representation, using the relations

$$\langle P^2 \rangle = - \frac{\partial^2 \rho}{\partial \Delta^2} \Big|_{k=\Delta=0}, \quad (4.1a)$$

$$\langle PQ + QP \rangle = - \frac{\partial^2 \rho}{\partial k \partial \Delta} \Big|_{k=\Delta=0}, \quad (4.1b)$$

$$\langle Q^2 \rangle = - \frac{\partial^2 \rho}{\partial k^2} \Big|_{k=\Delta=0}, \quad (4.1c)$$

by taking derivatives with respect to k, Δ at $k=0, \Delta=0$, and assuming $\rho(k, \Delta) \simeq 1 + O(k^2, k\Delta, \Delta^2)$. We thus obtain

$$8\gamma h = - \langle [P, \epsilon \dot{\phi}_0]_+ \rangle, \quad (4.2a)$$

$$4\gamma f = \langle [Q, \epsilon \dot{\phi}_0]_+ \rangle. \quad (4.2b)$$

Here $[,]_+$ is defined as the anticommutator of the enclosed operators. We thus see that the f term arises from the correlations between Q and the heat-bath forcing term $\dot{\phi}_0$, while the $h = g - \gamma f$ comes from the correlations between P and $\dot{\phi}_0$. If the heat bath were pure white noise, that is, if the correlation function of the heat-bath forcing term had the form

$$\langle [\dot{\phi}_0(t), \dot{\phi}_0(t')]_+ \rangle \sim \delta(t - t'), \quad (4.3a)$$

there would be no correlation with Q and the correlation with P would be a constant due to the dependence of P and Q on $\dot{\phi}_0$. Thus it is the nonwhite nature of the noise from the heat bath which leads both to the time dependence of the coefficients and to the unusual term $k \Delta \rho$ in the master equation.

In particular, the correlation function of the noise induced by a field in its vacuum state is given by

$$\langle [\dot{\phi}_0(t), \dot{\phi}_0(t')]_+ \rangle \sim 1/(t - t') \quad (4.3b)$$

and we encounter the divergence at $t = t'$ which forces us to introduce a cutoff Γ . (In the plots we will use an exponential cutoff.)

To get a feeling for these two unusual terms as well as for the diffusion term h we have plotted f and h as a function of time for various values of γ and T (with $\Omega_0 = 1$) in Fig. 1.

The temporal behavior can be divided into three regimes. Initially, over the cutoff time scales of the order of $1/\Gamma$ the diffusion term h becomes very large (of order $\Gamma/2\pi$) and then falls off as t^{-1} . This transient can be approximated by a function

$$\zeta(t) = (\Gamma/\pi) \Gamma t / (\Gamma^2 t^2 + 1). \quad (4.4)$$

To illustrate the universality of this initial transient, we have plotted h for a wide variety of values for γ, Ω_0 , and

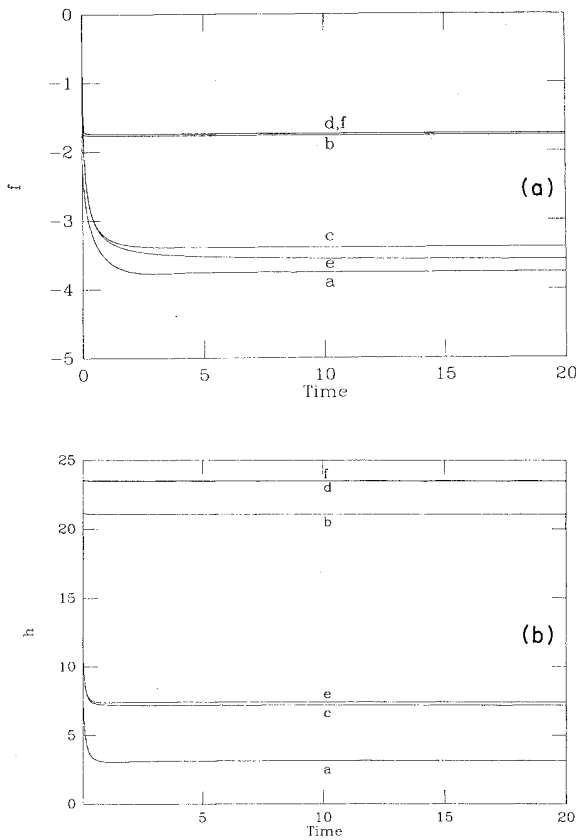


FIG. 1. Time dependence of f and h coefficients appearing in Eqs. (3.17) and (3.20) for a variety of cases, all with cutoff $\Gamma=1000$: (a) harmonic oscillator ($\Omega=1$, $\gamma=0.3$) at $T=0$; (b) the same harmonic oscillator as in (a), but at $T=10$; (c) critically damped harmonic oscillator ($\Omega=0$, $\gamma=1.0$) at $T=0$; (d) the same critically damped harmonic oscillator at $T=10$; (e) free particle ($\gamma=1.0$) at $T=0$ ("1/ f noise" diffusion); (f) free particle ($\gamma=1.0$) at $t=10$. Coefficient f arises from the correlation between the position and the heat-bath-forcing term $\dot{\varphi}_0$, and tends to disappear ($f \sim T/\Gamma$) for harmonic oscillator in the classical (high-temperature) limit. Coefficient h is responsible for the diffusion term in the master equation. It arises from the correlation between $\dot{\varphi}_0$ and the momentum of the particle. It tends to a constant ($h \sim T$) for a harmonic oscillator in the classical limit. In the vacuum ($T=0$) both terms are important.

T in Fig. 2, corresponding to the range of these values in the remainder of the figures. Notice that the difference between the various cases is barely observable on this graph. The magnitude of this initial transient drops to a value smaller than the magnitude of the other terms contributing to h on a time scale of the order of the minimum of $(\gamma \ln \Gamma / \Omega_0)^{-1}$, $1/\Omega$, or $1/T$. From the point of view of dynamical time scales $1/\Omega$, the initial transient is so short that its effect can be almost always approximated by a δ function of amplitude $\sim \pi^{-1} \ln(\Gamma t_c)$ where t_c is the above transition time. This unusual behavior is missed in these derivations of the master equation which

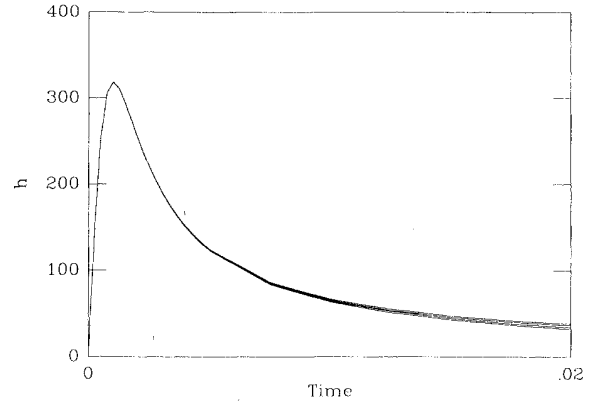


FIG. 2. Universality of the transient behavior of the diffusion coefficient; $h(t)$, Eq. (4.3), is plotted as a function of t for all the cases listed in Fig. 1. Note the difference in scale from Fig. 1.

from the outset assume the high-temperature limit.

The initial evolution caused by this transient is nearly instantaneous and has an enormous impact on the coherence of the quantum state of the system. It practically "wipes out" off-diagonal part of the density matrix in the position representation. Because it is so rapid, and because it affects quantum coherence rather than expectation values, it has not been treated in the literature with the care it deserves in the context of the discussion of quantum theory of measurement. We shall explore it in further detail in the next section.

In contrast with the first regime of the evolution, the stage following the rapid transient has been thoroughly studied.¹⁻¹⁰ It occurs on the dynamical and dissipative time scales, and, especially in the high-temperature limit, where it has a well-defined classical analog—damped Brownian motion—it offers few surprises. The only aspect of the evolution which has not been appreciated until recently, and which is present already in the high-temperature limit of the master equation, is the efficiency with which the quantum correlations are damped out.¹³⁻²⁰

The third and last stage of the evolution of the system is the asymptotic ($t \rightarrow \infty$) regime. Coefficients of the diffusive (h) and correction (f) terms behave differently in the high temperature and vacuum ($T=0$) environments. At finite temperatures, as Γ becomes large in comparison with the other characteristic frequencies of the problem (defined by Ω and γ), the diffusive term approaches the familiar constant value of T . For instance, for an underdamped oscillator,

$$h = \frac{1}{\pi\beta} \left[\arctan \frac{\Gamma + \Omega}{\gamma} + \arctan \frac{\Gamma - \Omega}{\gamma} \right] - \gamma f \approx T. \quad (4.5a)$$

The corrective term becomes negligible in the same limit. For a damped harmonic oscillator and a discrete cutoff we get

$$f = (2\pi\beta\Omega)^{-1} \ln \frac{(\Gamma + \Omega)^2 + \gamma^2}{(\Gamma - \Omega)^2 + \gamma^2} \sim \frac{T}{\Gamma}. \quad (4.6a)$$

The coefficient depends on the exact cutoff assumed. Note that f is negligible when $T \ll \Gamma$. Similar exact results and identical high-temperature limits can be obtained for all other cases of the harmonic oscillator (including the unstable one) as well as for the free particle.

The situation is somewhat different in vacuum. For all the stable cases of the harmonic oscillator the diffusion coefficient approaches the limiting constant value given by

$$h = \frac{2\gamma}{\pi} \ln \frac{\Gamma}{\Omega_0} + \left[\frac{\Omega^2 - \gamma^2}{\Omega} \right] \left[\frac{1}{2} - \frac{\arg(\Omega + i\gamma)}{\pi} \right]. \quad (4.5b)$$

The correction term is no longer small compared to h . Now

$$f = \frac{-1}{\pi} \ln \frac{\Gamma}{\Omega_0} + \frac{\gamma}{\Omega} \left[\frac{1}{2} - \frac{\arg(\Omega + i\gamma)}{\pi} \right]. \quad (4.6b)$$

The case of the free particle is still different. At times of the order of the relaxation time γ^{-1} the coefficient h with exponential cutoff is given by

$$h = \frac{\exp(-2\gamma t)}{\pi} \left[\frac{t}{t^2 + \Gamma^{-2}} + \gamma [\text{Ei}(2i\gamma/\Gamma) + \text{Ei}(-2i\gamma/\Gamma)] \right], \quad (4.5c)$$

while the coefficient f equals

$$f = \frac{1}{2\pi\gamma} \left[\frac{t}{t^2 + \Gamma^{-2}} - \exp(-2\gamma t) \left[\frac{t}{t^2 + \Gamma^{-2}} + \gamma [\text{Ei}(2i\gamma/\Gamma) + \text{Ei}(-2i\gamma/\Gamma)] \right] \right]. \quad (4.6c)$$

Above $\text{Ei}(u) = -\int_{-u}^{\infty} (e^{-s}/s) ds$ is the usual exponential integral. At times long compared to the relaxation time, f still has a non-negligible effect on the behavior of the free particle, as it decreases only with the inverse of time:

$$f \approx \frac{1}{2\pi\gamma} \frac{1}{t}.$$

This time dependence results in an effective “ $1/f$ ” noise, as it increases the spread of the wave packet in the position with the logarithm of time.¹⁴

Both h and f diverge for the unstable case. In the long-time limit the leading term of the diffusion coefficient is

$$h = \frac{1}{\pi} \left[1 - \frac{\gamma}{\Omega} \right] \left[-e^{(\Omega-\gamma)t} + (\Omega-\gamma) \text{Ei}((\Omega-\gamma)t) \right], \quad (4.5d)$$

while the corrective term is

$$f = \frac{1}{2\pi\Omega} \left[-e^{(\Omega-\gamma)t} + (\Omega-\gamma) \text{Ei}((\Omega-\gamma)t) \right]. \quad (4.6d)$$

The high-temperature master equation

$$\dot{\rho} = \left[i \left[\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial q'^2} - \Omega^2 (q^2 - q'^2) \right] - 2\gamma (q - q') \left[\frac{\partial}{\partial q} \frac{\partial}{\partial q'} \right] - 4\gamma T (q - q')^2 \right] \quad (4.7)$$

can be rigorously justified only in the high-temperature limit:

$$T \gg \Gamma \gg \max(\gamma, \Omega). \quad (4.8)$$

It has nevertheless been successfully applied in a variety of situations on which the inequality $T \gg \Gamma$ was clearly

violated. In particular, T is at best comparable, and often much less than Γ in many of the applications of Eq. (4.7) in quantum optics²³⁻²⁵ and to the “original” Brownian motion of a free particle.²⁶ Yet Eq. (4.7) yielded no obvious contradictions with the experiments performed so far.

The reason for this somewhat surprising agreement between the approximate high-temperature equation (4.7) and the more exact equation (3.20) in the experimentally explored regime can be traced back to the manner in which the discrepancies between them are manifested. On the dynamical time scales and at a finite temperature the corrective term is negligible in comparison with the diffusive one whenever $\Gamma \gg \max(\gamma, \Omega)$, as is shown in Fig. 3. This inequality is quite hard to violate. Moreover, the size of the vacuum contribution to the diffusive term is only $\sim \gamma \ln \Gamma / \Omega_0$ for a damped harmonic oscillator, which is again typically less than T . In other words, on the dynamical or relaxation time scale Eq. (4.7) is a fair approximation of Eq. (3.20) whenever

$$\Gamma / \Omega_0 \ll \exp(T/\gamma). \quad (4.9)$$

This inequality and the corresponding inequalities for the free particle and the unstable case are clearly much less restrictive than the rigorous high-temperature limit, Eq. (4.8). We are thus still left with the question: So what is the price paid for allowing $T \leq \Gamma$?

The hint of the answer has already appeared in our comments concerning the transient regime. The enormous discrepancy between Eq. (3.20) and Eq. (4.7) occurs on the cutoff time scales, when the transient form of the diffusion coefficient, Eq. (4.4) is valid. Its dramatic effect is, however, limited to the off-diagonal part of the density matrix in the position representation on these short-time scales. Moreover, the impact of the transient is most pronounced when the system starts from the pure state un-

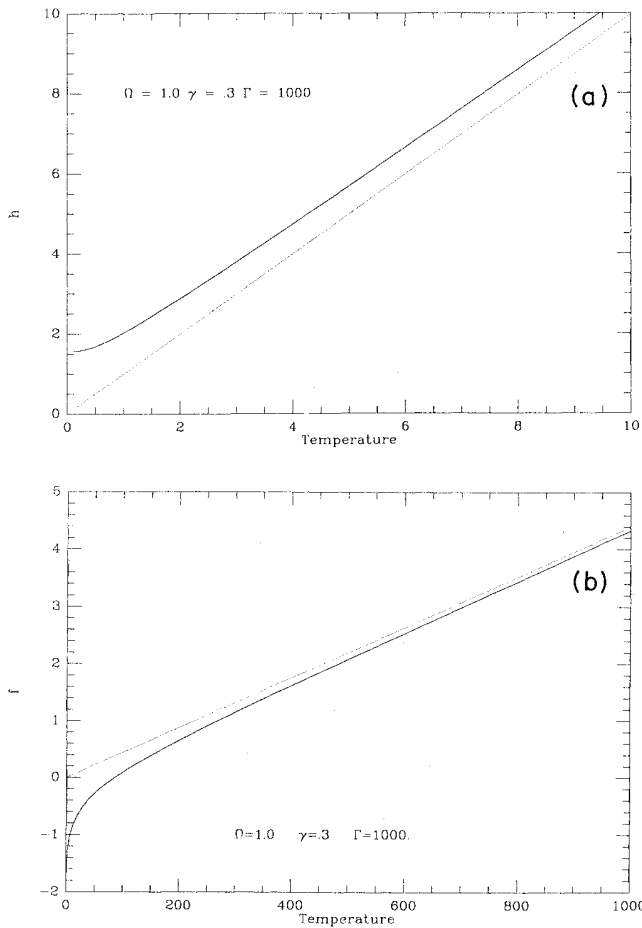


FIG. 3. Temperature dependence of h and f for a damped harmonic oscillator ($\Omega = 1$, $\gamma = 0.3$, $\Gamma = 1000$) near the long-time limit $t = 40$. Dotted lines indicate the high-temperature limit. The agreement between the exact and the asymptotic values sets in for temperatures much smaller than Γ for large times.

correlated with the environment degrees of freedom. It is not clear to what extent this condition can be enforced in the experiments. The system must thus be prepared on a very-short-time scale, and in a way which is independent

of the heat bath. If, during the preparation time, the system remains in contact with the bath, and if that preparation time is of the order of the dynamical time scale of the system, the transient regime will not be present.

Finally, when the initial state is a Gaussian "coherent" or vacuum state the impact of the transient is not as dramatic as for states which are coherent superpositions of more distant positions. Thus we are led to conclude that the most interesting tests of the nonclassical nonstationary aspects of Eq. (3.17) require low temperatures, fast system preparation, and "nonstandard" (widely correlated) quantum states. We shall show in the next sections that the "squeezed states" may offer such an opportunity.

V. LOSS OF COHERENCE IN QUANTUM BROWNIAN MOTION

In this section we will gain some familiarity with the effects of the heat bath on the oscillator by looking at the reduced density matrix for the oscillator in various representations. Although the (k, Δ) representation is by far the easiest in which to derive the equations of motion, it is a sufficiently unfamiliar representation to make the interpretation difficult. We will therefore look at the density matrix in the Wigner (p, q) representation, in the position space (q, q') representation, and in the momentum (p, p') representation.

In all cases we will use Gaussian density matrices for the initial states, primarily because of the simplicity of representation. Note that this means that they remain Gaussian at all times [see, e.g., Eqs. (2.14) and (2.18) for the reason]. Gaussian density matrices also have the advantage that their values everywhere can be represented by the 1σ contour. The changes in the 1σ contour therefore immediately show the changes caused by the combination of damping plus heat-bath fluctuations on the oscillator. Furthermore, we can easily calculate the entropy of a Gaussian density matrix. Such a density matrix can always be brought into the form

$$\rho = N^{-1} \exp \left\{ -[\alpha(Q - \langle Q \rangle)^2 + \beta(P - \langle P \rangle)(Q - \langle Q \rangle) + (Q - \langle Q \rangle)(P - \langle P \rangle) + \lambda(P - \langle P \rangle)^2] \right\}, \quad (5.1)$$

where $\langle P \rangle$ and $\langle Q \rangle$ are the expectation values for P and Q in the state, P and Q are the momentum and position operators and N is a normalization factor. The entropy

$$\mathcal{H} = -\text{Tr} \rho \ln \rho \quad (5.2)$$

can be calculated by writing

$$\rho = N^{-1} e^{-2\sqrt{\alpha\lambda - \beta^2} \times (\bar{P}^2 + \bar{Q}^2)/2}, \quad (5.3)$$

where

$$\begin{bmatrix} \bar{Q} \\ \bar{P} \end{bmatrix} = \Lambda^{1/2} \begin{bmatrix} Q - \langle Q \rangle \\ P - \langle P \rangle \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} / \sqrt{(\alpha\lambda - \beta^2)}.$$

Note that

$$[\tilde{P}, \tilde{Q}] = \sqrt{\det \Lambda} [P, Q] = i,$$

and $\frac{1}{2}(\tilde{P}^2 + \tilde{Q}^2)$ is an effective harmonic-oscillator Hamiltonian with frequency unity. The entropy is therefore given by the usual thermal oscillator expression

$$\mathcal{H} = -\ln(1 - e^{-Y}) + \frac{Y}{e^Y - 1}, \quad (5.4)$$

where

$$Y = 2\sqrt{\alpha\lambda - \beta^2}. \quad (5.5)$$

When the density matrix in (k, Δ) representation is

$$\rho(k, \Delta) \propto \exp[-(Ak^2 + 2Bk\Delta + C\Delta^2)], \quad (5.6)$$

we can relate the coefficients A , B , and C to the values α , β , γ by

$$\begin{aligned} 2A &= -\left. \frac{\partial^2 \rho}{\partial k^2} \right|_{k, \Delta=0} \\ &= \langle Q^2 \rangle \\ &= -\frac{\partial}{\partial \alpha} \ln \text{Tr} e^{-[\alpha Q^2 + \beta(PQ + QP) + \lambda P^2]}, \end{aligned} \quad (5.7a)$$

$$\begin{aligned} 2B &= -\left. \frac{\partial^2 \rho}{\partial k \partial \Delta} \right|_{k, \Delta=0} \\ &= \langle PQ + QP \rangle \\ &= -\frac{\partial}{\partial \beta} \ln \text{Tr} e^{-[\alpha Q^2 + \beta(PQ + QP) + \lambda P^2]}, \end{aligned} \quad (5.7b)$$

$$\begin{aligned} 2C &= -\left. \frac{\partial^2 \rho}{\partial \Delta^2} \right|_{k, \Delta=0} \\ &= \langle P^2 \rangle \\ &= -\frac{\partial}{\partial \lambda} \ln \text{Tr} e^{-[\alpha Q^2 + \beta(PQ + QP) + \lambda P^2]}. \end{aligned} \quad (5.7c)$$

We finally find that

$$4\sqrt{AC - B^2} = \coth \sqrt{\alpha\lambda - \beta^2} \quad (5.8a)$$

or that

$$e^Y = \frac{4\sqrt{AC - B^2} + 1}{4\sqrt{AC - B^2} - 1}, \quad (5.8b)$$

where Y is the inverse of the "equivalent temperature" defined as in Eq. (5.5).

One of the most interesting features of the behavior of such an oscillator coupled to a heat bath is the loss of quantum coherence caused by the heat bath. This loss of coherence can be regarded as the result of the measurement performed by the external environment on the system. If one begins with the system in some pure state, the transfer of information into the system-environment correlations will convert that pure state into a mixture of states and result in an increase of entropy.

In our case, the external field is coupled to the oscillator via the position variable q . As a result we would expect the system to lose its quantum coherence in the q representation more rapidly than in the p representation.

In order to investigate this effect we set up states which are coherent over a wide range of q values and of p values. To do this we use "squeezed" states, with a squeezing factor of r , i.e.,

$$\begin{aligned} \langle \Delta q^2 \rangle_{\text{squeezed}} &= r \langle \Delta q^2 \rangle_{\text{vac}}, \\ \langle \Delta p^2 \rangle_{\text{squeezed}} &= \langle \Delta p^2 \rangle_{\text{vac}} / r, \end{aligned}$$

where we display the results for $r = 1$, $r = 8$, and $r = \frac{1}{8}$.

We examine in some detail the case in which the frequency Ω of the oscillator is unity, and the damping coefficient γ is .3. We will examine this for various initial states, and for various temperatures in Figs. 4, 5, 6, and 8. On the rest of Figs. 4–11 we chose other values of Ω and γ to study. In each of the figures, graph (a) is a plot of the $1 - \sigma$ contour of the Wigner function [the density matrix in (q/p) representation] at a selected set of times. (The $1 - \sigma$ contour are the values of the arguments where the density matrix falls to $e^{-1/2}$ of its peak value.) In this graph we have put the oscillator into an initial state which includes a nonzero value for the expectation value for the momentum P , $\langle P \rangle = 10$. In addition we have plotted the path that the expectation values $\langle P \rangle$ and $\langle Q \rangle$ follow. The $1 - \sigma$ contours for the Wigner function at the selected times are plotted around that point on that path that the particle has reached at that selected time. The center of the ellipse is thus the expectation value of Q , P at that time. The ellipses are labeled from 1 to 5 or 6 so that one can follow the structure of the density matrix at those selected times in the various representations.

Graph (b) in each figure is a plot of the density matrix in the position representation $q - q'$. Again we plot the $1 - \sigma$ contour of the modulus of the density matrix at the same selected times as in graph (a). Here the value of the density matrix along the diagonal ($q = q'$) represent the probability of finding the particle with that position, while the off-diagonal values represent the correlation in the density matrix between the points q and q' . Note that the phase, which is not plotted, is in general nontrivial. We have used exactly the same initial conditions as in graph (a) except that the "center-of-mass" motion has been removed so as to center all of the ellipses at $q = q' = 0$.

Graph (c) in each set is a similar plot of the $1 - \sigma$ contours of density matrix in the momentum $p - p'$ representation. Again the plots are for the same selected times as in (a) and (b). Again the "center-of-mass" motion has been removed.

Finally, graph (d) is a plot of the entropy of the density matrix as a function of time (up to $t = 20$) with an insert showing the time dependence of the entropy over the time from 0 to 0.2. (Remember that the scale is set by the values of the dynamic quantities such as Ω_0 and γ . For the oscillator examples, Ω_0 is taken to be unity, while for the free particle, γ is 1.) Along the time axis the numbered crosses mark the times at which the $1 - \sigma$ contours are plotted in graphs (a)–(d). In all cases, the crosses for the first two times completely overlap with the origin on the main diagram. Time 1 is the initial time, while time 2 is only about 0.01, and represents a time slightly larger than the cutoff time $1/\Gamma = 0.001$. Thus this point

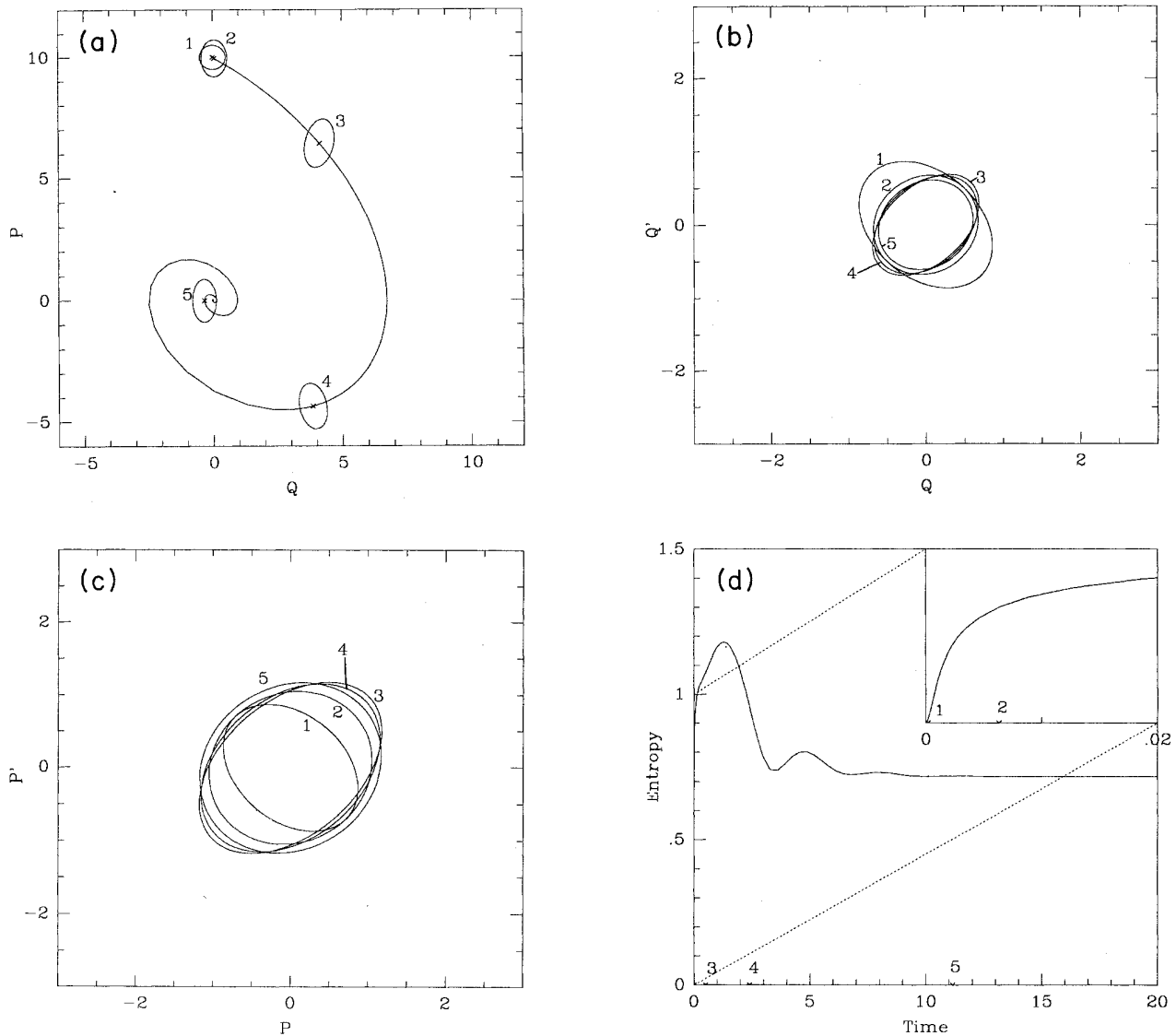


FIG. 4. Evolution of (a) $1\text{-}\sigma$ contour of the Wigner distribution; (b) density matrix in $q-q'$ representation; (c) density matrix in $p-p'$ representation; (d) entropy $\mathcal{H} = -\text{Tr} \rho \ln \rho$ for a damped harmonic oscillator ($\Omega=1$, $\gamma=0.3$, $\Gamma=1000$) at $T=0$. Initial state is a Gaussian coherent state ($\Delta q = \Delta p$).

represents a time which is about 10^{-3} of the period of the oscillator.

In Fig. 4, the first of the series of pictures, the initial state is for the oscillator in its ground state ($r=1$) with $\gamma=0.3$. Even here, the density matrix has changed slightly but significantly by the second time, and in particular, the off-diagonal strength in the $q-q'$ representation [Fig. 4(a)] has already decreased slightly.

Note that the entropy first increases dramatically (over at time scale of order $1/\Gamma$) and then settles down to a nonzero value (~ 0.8) even though the temperature of the heat bath is zero. This is an indication of the fact that the usual statement—that the temperature of the oscillator comes to an equilibrium value at the temperature of

the heat bath—applies only in the limit in which the coupling between the bath and the oscillator goes to zero (weak coupling). At all nonzero values of ϵ , there will exist significant correlations between the oscillator and the heat bath which result in a nonzero entropy for the reduced density matrix of the oscillator. The equilibrium value for the entropy can be derived via Eqs. (5.4) and (5.8). At equilibrium, $\dot{\rho}=0$, and we use Eq. (5.6) with

$$A_e = \Omega_0^2 g(t = \infty, \Gamma, \beta),$$

$$B_e = 0,$$

$$C_e = f(t = \infty, \Gamma, \beta),$$

where the subscript e refers to the equilibrium value. In the high-temperature limit $Y = \Omega_0/T$, and we find that the entropy is given by the familiar

$$\mathcal{H}_e \approx \ln(T/\Omega_0),$$

but at lower temperatures ($T \leq \Omega$) this is not true. Note also that the equilibrium entropy depends on the asymptotic values of both f and g . Also, at equilibrium, $\langle P^2 \rangle$ and $\langle Q^2 \rangle$ are not equal to each other as can be seen in (a) graphs. The difference between them is directly proportional to the equilibrium value of f .

The behavior of the entropy can be understood qualitatively as follows. Initially the entropy increases rapidly because of the "measurement" of the position Q which the field ϕ performs. Notice that the off-diagonal terms in the position representation decrease much more rapid-

ly than in the p representation. In all of the graphs, the spread of position (length of the $1-\sigma$ contour) along the diagonal remains constant between times 1 and 2 while the off-diagonal strength decreases. In the momentum representation, the on-diagonal spread increases (because of the effective measurement of q) while the off-diagonal coherence remains unchanged. Then, as the system comes to equilibrium the high initial entropy produced by the initial "measurement" decays away into the heat bath.

In this first Fig. 4 with $r=1$, the effects described above are not very dramatic. In Figs. 5 and 6 we look at the same oscillator, but with the initial state squeezed in the p and q directions ($r=8$ and $\frac{1}{8}$, respectively). In Fig. 5 we have set up an initial pure, squeezed state which is very broad in the position representation. As expected

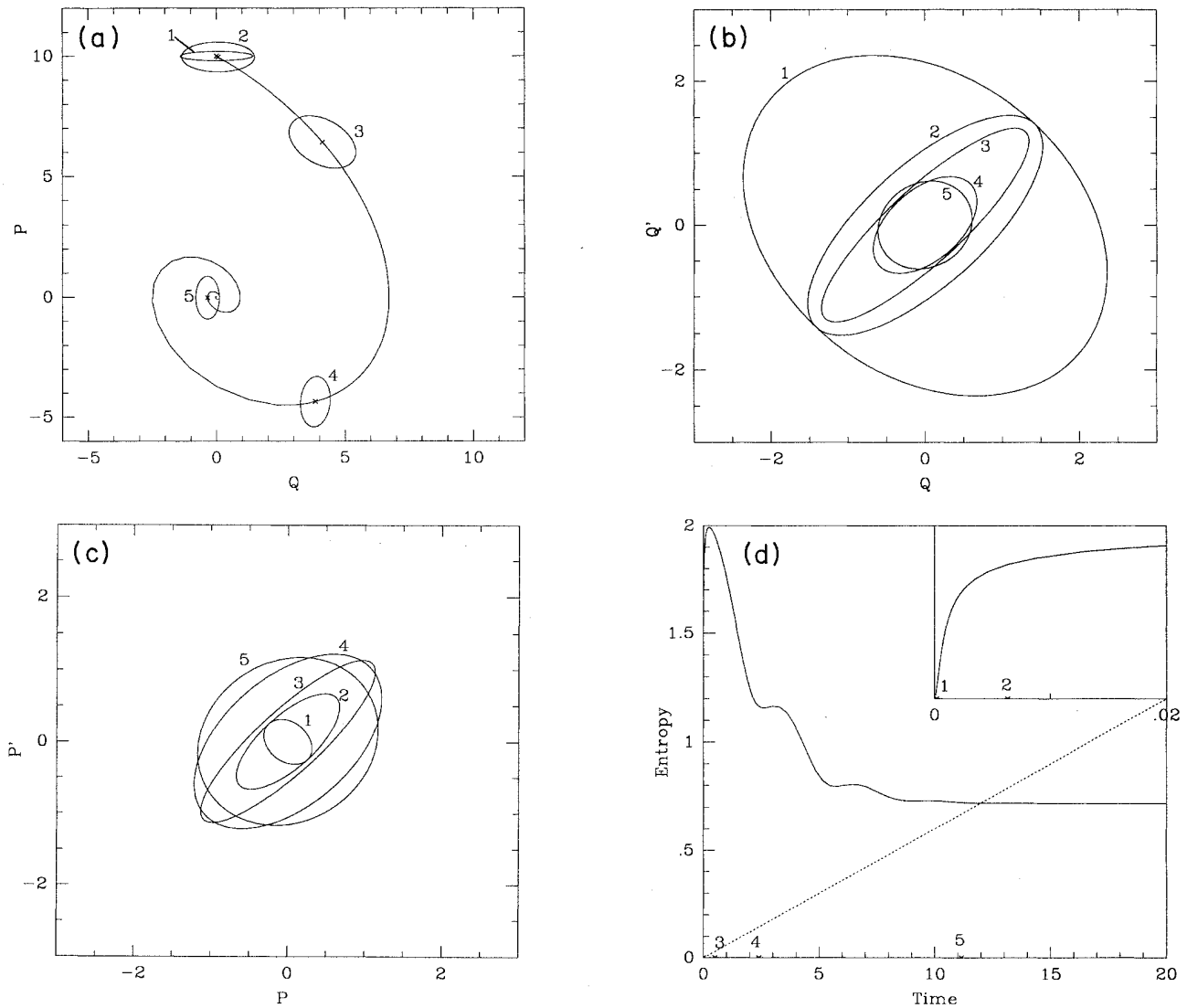


FIG. 5. Same as Fig. 4, but for an initial state squeezed in momentum ($\Delta q = 8\Delta p$). Note the rapid initial decoherence, manifested in the change of shape of the Wigner distribution (a) and by the decay of the off-diagonal elements of the density matrix in the position representation (b). See text for further discussion.

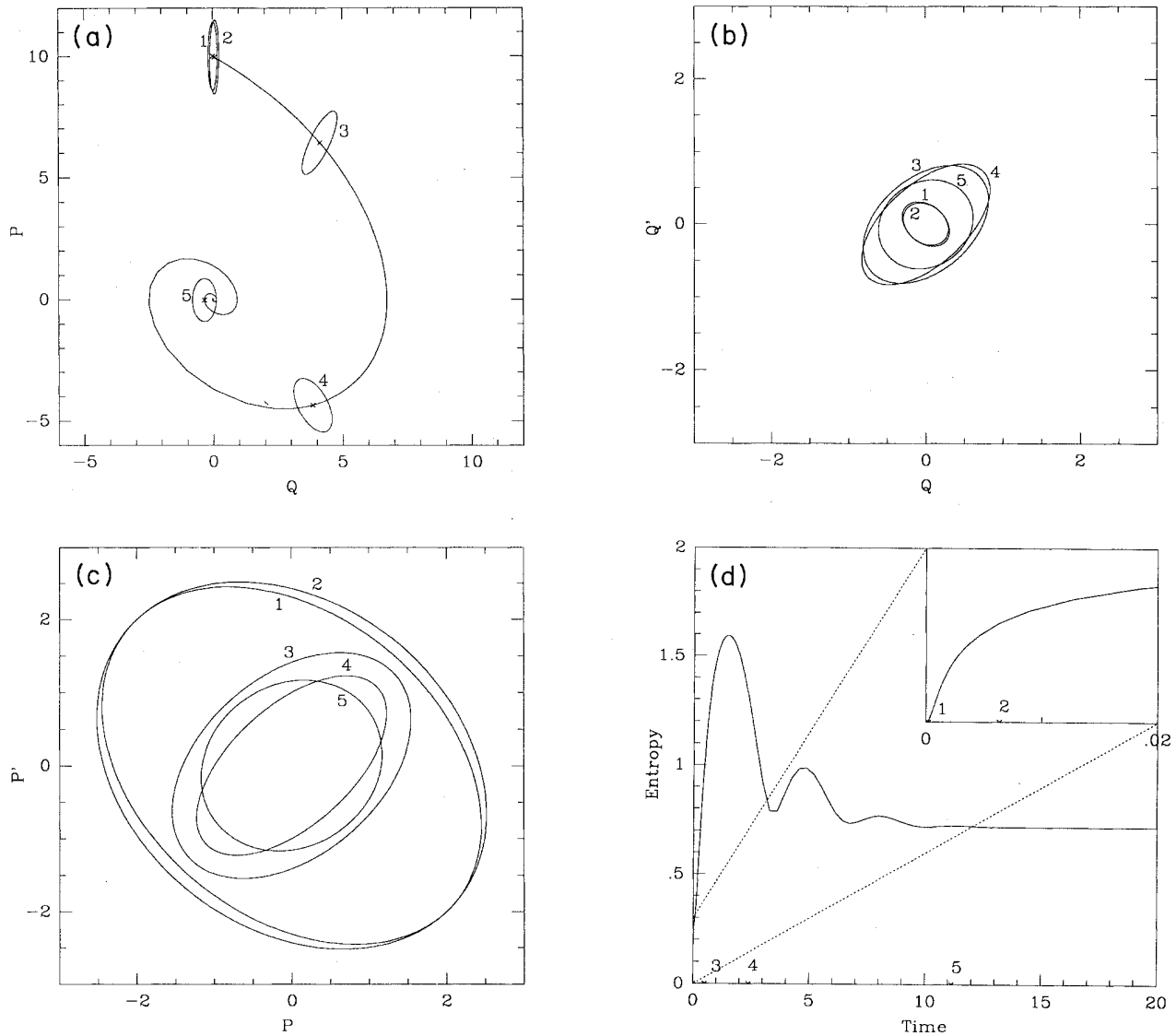


FIG. 6. Same as previous figures, but for an initial state squeezed in position ($\Delta q = \Delta p / 8$). Now the initial wave packet is a much better approximation of the position eigenstate. As the system-environment interaction Hamiltonian commutes with q (and, therefore, “measures” it), decoherence is rather inefficient: Wigner distribution function, $\rho(q, q')$ and $\rho(p, p')$ evolve on dynamical and damping time scales.

for a pure state, its off-diagonal correlations are large, as can be seen in Fig. 5(b). However, over a time period of order $1/\Gamma$, these off-diagonal terms have been reduced dramatically. The density matrix has suddenly become one which can be represented as an incoherent sum of density matrices each of which has a coherence width of approximately that for a vacuum state. The probability of finding the particle at any position has of course not changed over such a short period (so the diagonal values have remained the same), but the coherence of the density matrix between different values of the position has been dramatically reduced over an incredibly short time scale $1/\Gamma$.

On the other hand, in the P representation, the off-diagonal coherence has remained unchanged. (If one were to look at the phases of the off-diagonal terms, one would have found dramatic changes.) The on-diagonal spread of the matrix has however *increased*. The probability of the particles having some momentum p has changed because the “measurement” of the position performed by the external field.

This dramatic difference between the density matrix in position and momentum representation illustrates the idea of *pointer basis*^{16,17,20,27} which has been developed by one of us. In this case, because of the coupling of the oscillator to the external field via the position of the parti-

cle, the position variable is the “pointer” variable. The environment (heat bath of the field) rapidly reduces the density matrix to approximately diagonal form in this pointer basis representation, but not in the conjugate momentum representation. Note that it does not reduce to position eigenstates—the “measurement” by the heat bath is not an ideal exact measurement. Rather the reduction takes place with an accuracy approximately given by the uncertainty of the vacuum state. As we will see in Fig. 7, this reduction also depends on the strength of the coupling with the heat bath.

This creation of an incoherent density matrix is reflected in the plot of the entropy in Fig. 5(d). The entropy shoots up to the high value of 2.0 on the cutoff time

scale (see the insert), and then decays to the final equilibrium value.

These effects are highlighted by an examination of Figs. 6(a)–6(d). Here we have initiated the system in a pure state with a large spread in momenta, but a small spread in position. Recall that the original oscillator Hamiltonian was completely symmetric with respect to P and Q . Only the interaction selects out the Q coordinate. As expected, the change between times 1 and 2 is no longer dramatic at all. The momentum spread has increased slightly, but the off-diagonal coherence components of the density matrix in neither position nor momentum representation have changed dramatically. Furthermore, the entropy between these two times has

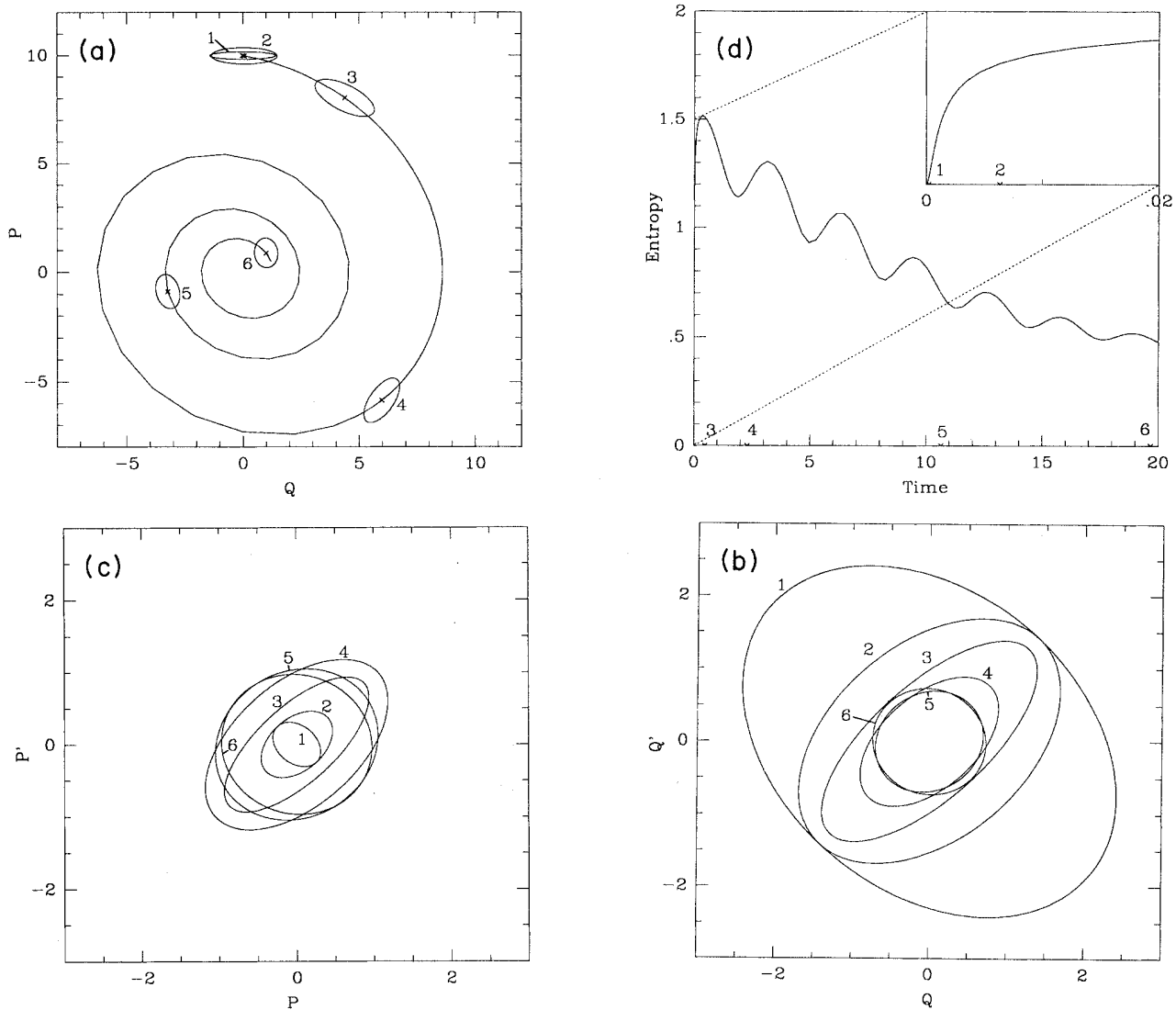


FIG. 7. Same as Fig. 5—the initial state is again squeezed in momentum and extended in position, $\Delta q = 8\Delta p$ —but now the damping coefficient is smaller ($\gamma = 0.1$) than in the previous cases. This slows down the rate of “spiraling down,” but does little to slow down the process of decoherence. Both the Wigner distribution function and the density matrices quickly evolve under the “measuring interaction” with the environment into the form corresponding to the mixture of wave packets with definite positions and limited extent. As in Fig. 5, entropy increases on a rapid (Γ^{-1}) time scale as a consequence of such decoherence.

changed little (it is only 0.16 by time 2 rather than the value of about 1.6 it had in the previous case at time 2). The entropy does finally increase to a value of about 1.6 in this case, but now over a quarter of an oscillation period rather than “instantly.” This is the time period over which the large initial spread in momentum creates a large spread in position because of the dynamics of the oscillator. The field then “measures” the position of the oscillator again producing an incoherent density matrix with a large entropy.

Figure 7 illustrates behavior of an oscillator initiated in a state identical with the one shown in Fig. 4, but with a weaker coupling to the heat bath ($\gamma=0.1$). This difference in damping is most apparent in Fig. 7(a), where the trajectory takes many more revolutions on its way to

the equilibrium state. This difference is also reflected in the appearance of the density matrix in the position as well as in the momentum representation. Coherence loss is now less rapid, and the entropy of the oscillator state does not reach as high an intermediate value as for the damped case shown in Fig. 5. Moreover, in accord with expectations, a less strongly coupled oscillator approaches a ground state which is less “mixed,” i.e., has smaller entropy, at $T=0$. While the envelope of the oscillations is still decaying at the last instant illustrated in Fig. 7, there can be no question that the final value of entropy will be, in this case, significantly less than for the one of Figs. 4–6.

We can also see what difference a nonzero temperature makes. In Fig. 8 we look at the same oscillator as in Figs.

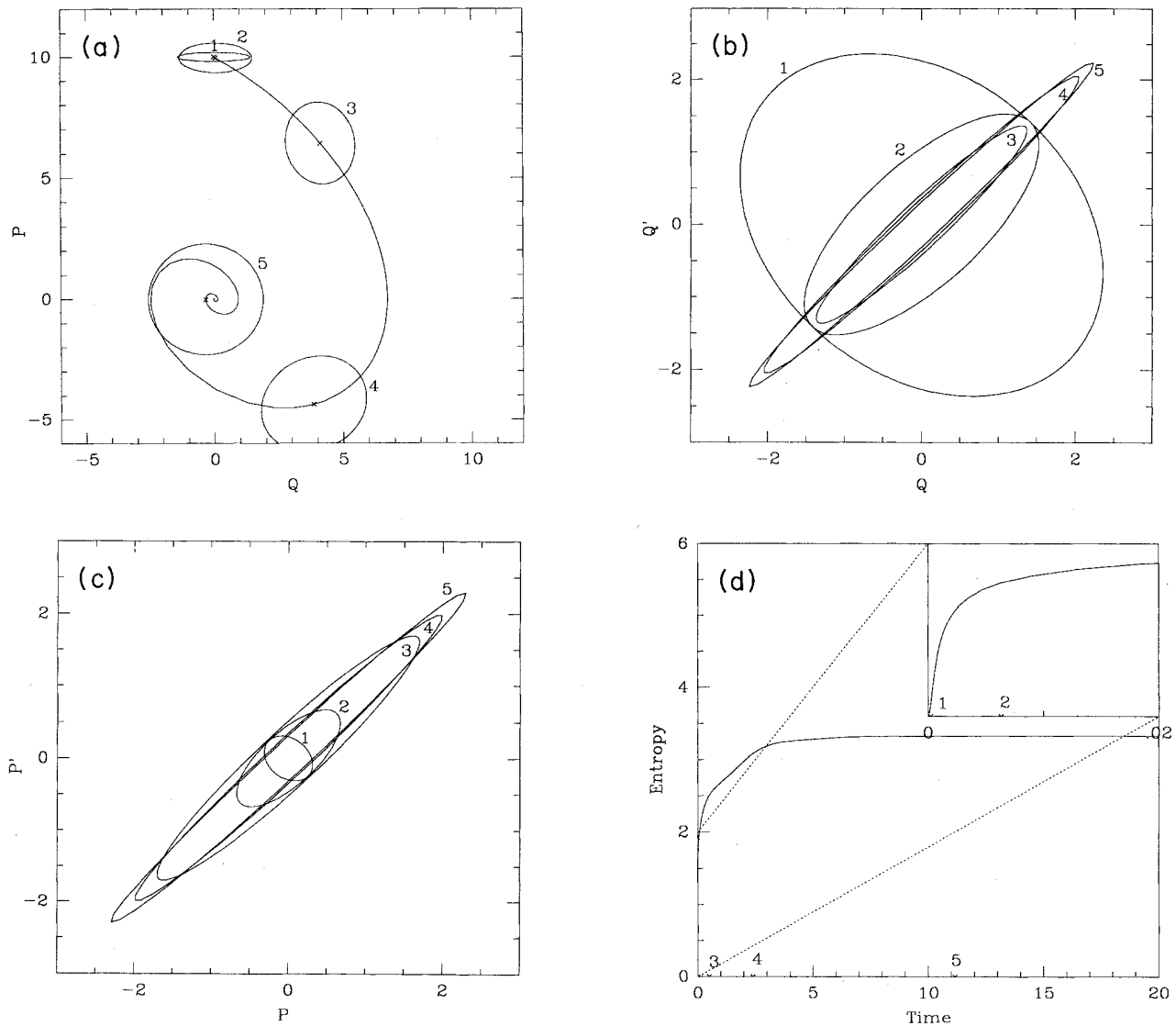


FIG. 8. As in Fig. 5, $\gamma=0.3$ and the initial state is squeezed in the direction of momentum and spread in the position, but the environment is at a finite temperature ($T=10$). Temperature has little influence on the rapid initial coherence [$T \ll \Gamma$, so that we are far from the usual high-temperature limit, Eq. (4.8)] but it does alter the long-time behavior in the expected manner.

4–6 ($\Omega_0=1$, $\gamma=0.3$, $\Gamma=1000$) at a temperature of $10(\hbar\Omega_0/k_B)$ and with $r=8$. It thus has the same initial condition as in Fig. 5. Over the very-short-time scale between times 1 and 2, there is essentially no difference in the behavior of the oscillator. By time 3 however, which is of the order of a few “thermal” times ($t_B \simeq \hbar/k_B T$) the effect of the nonzero temperature has begun to be felt. We still have the sudden increase in entropy in Fig. 7(d) due to the initial “measurement” of the position, but now the equilibrium entropy is much higher (of order 2.0) and is just what one would expect for the given temperature. Over short times of the order of a few oscillation time scales, the entropy again has gone to values significantly higher than equilibrium, and then settled down to the equilibrium value.

Figure 9 shows the behavior of a squeezed state for a critically damped case, $\gamma=\Omega_0=1$. On short time scales

the behavior is clearly dominated by the same effects as those illustrated in previous figures. As expected, the long-time limit is approached without any oscillations because of critical damping. Decoherence occurs now even more rapidly than for the case of Fig. 5, as can be clearly seen in the plots of density matrix in the position representation [Fig. 9(b)]. Consequently, the intermediate as well as equilibrium values of entropy are also in excess of those seen in Fig. 5(d).

The next case considered in this section, Fig. 10, illustrates a free particle interacting with the heat bath. The short-term behavior is again similar to the one illustrated in Figs. 4–9. Apart from the obvious differences caused by different dynamics, the only difference we would like to point out occurs at late times, when the spread of the wave packet increases logarithmically with time:¹⁴

$$\langle \Delta q^2 \rangle = (2\gamma/\pi) \ln(\sqrt{\Gamma\gamma} t).$$

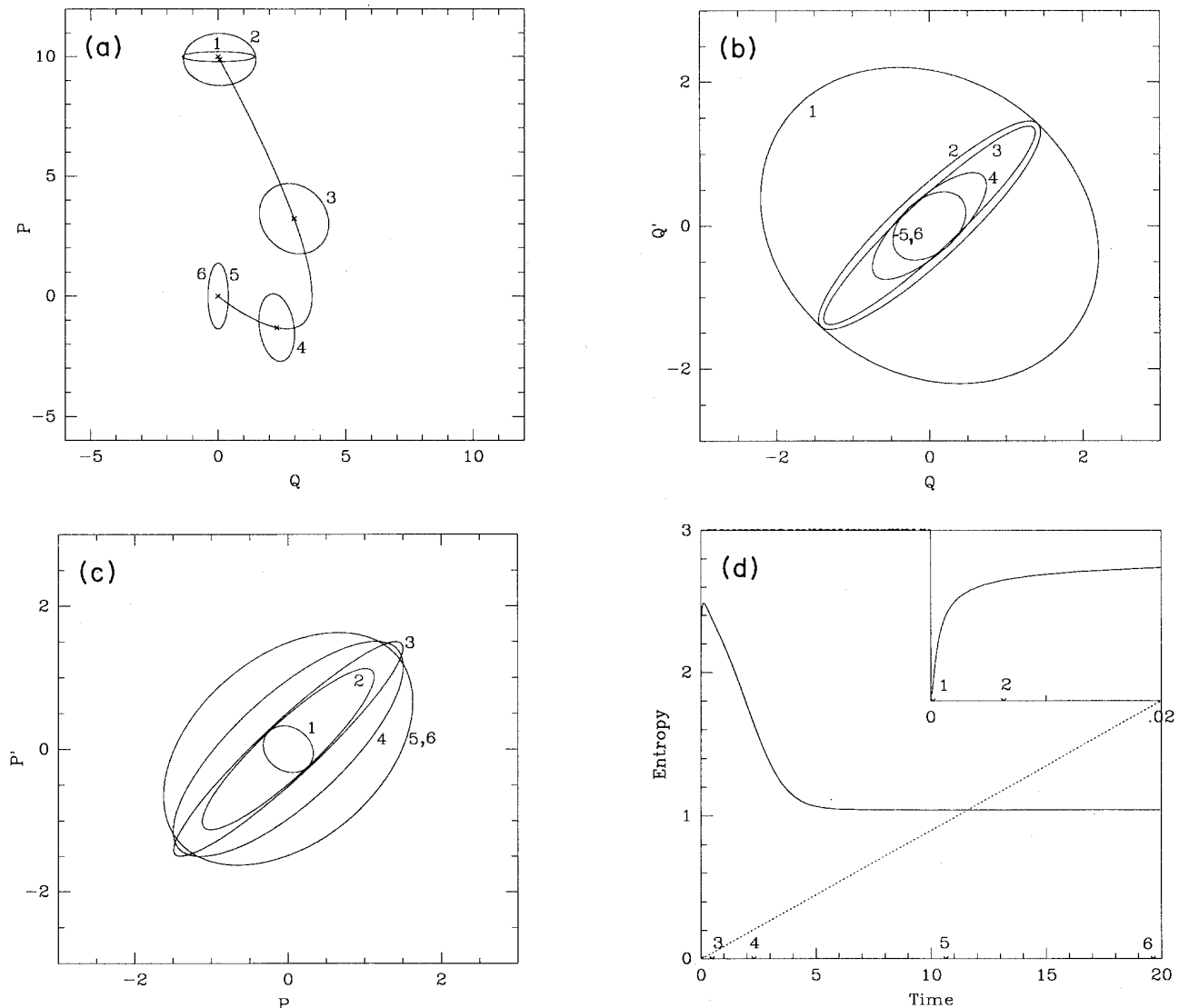


FIG. 9. Critically damped harmonic oscillator $\Omega=0$, $\gamma=1.0$) initiated in a squeezed state ($\Delta x = 8\Delta p$). Initial coherence is again very rapid. Entropy evolves on a collision time scale Γ^{-1} to a large value.

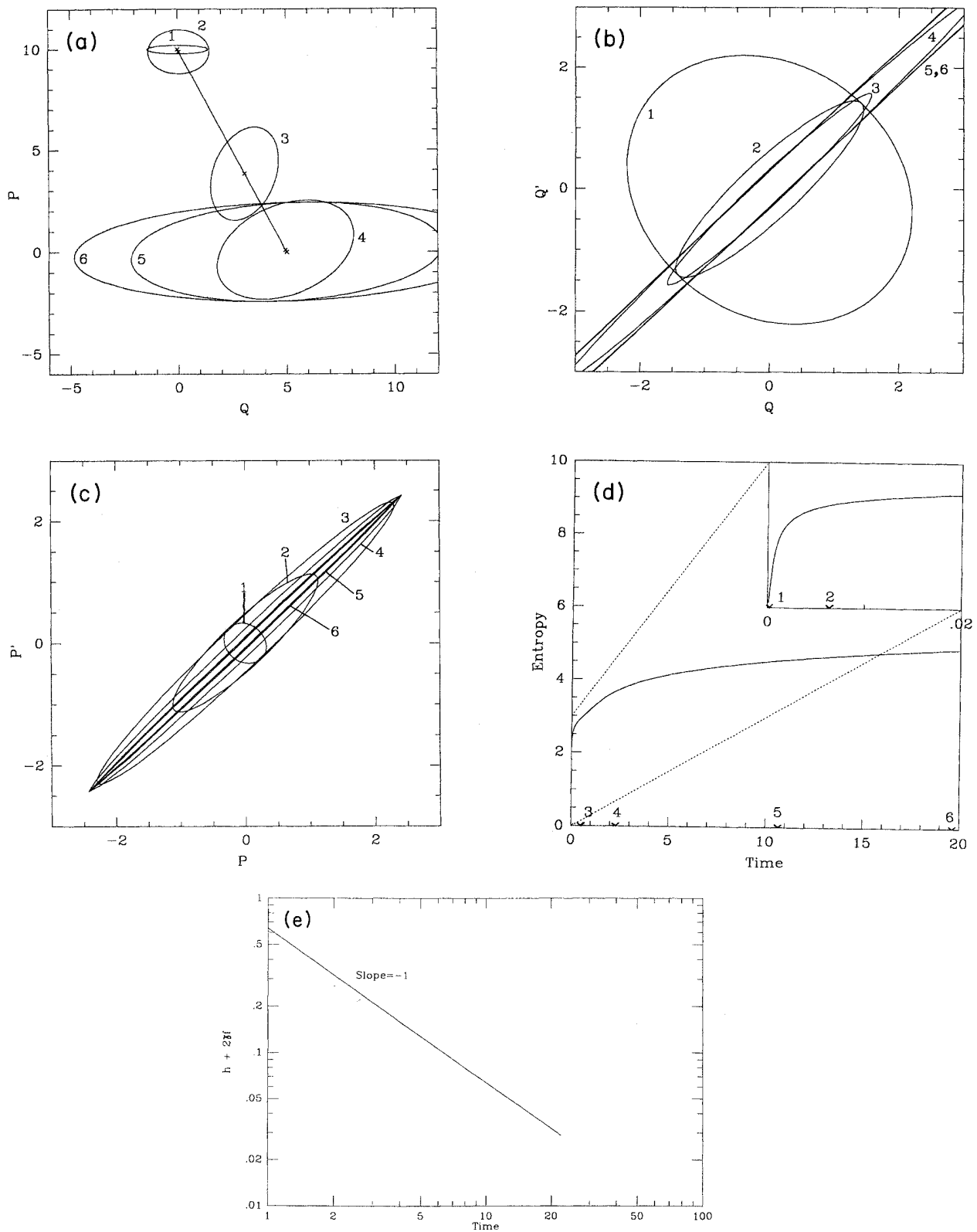


FIG. 10. Free particle at $t = 0$. Plots of the Wigner distribution (a) as well as density matrix $\rho(q, q')$; (b) shows that the spatial extent of the wave packet increases to that $\langle |q^2| \rangle^{1/2} \sim 2\gamma \ln(t\sqrt{\gamma\Gamma})$. The $p-p'$ representation (c) exhibits a gradual narrowing of the off-diagonal correlations, while the entropy (d) exhibits its logarithmic increase with time. The coefficient $h + 2\gamma f$ responsible for these effects is almost exactly proportional to the inverse of time, as (e) demonstrates.

This slow spreading is a manifestation of the “ $1/f$ ” noise, and can be traced back to the form of the autocorrelation function of the quantum noise. It is not clear what, if any, relation this noise has with the experimentally observed manifestations of the logarithmic drift of measured quantities.²⁸

The behavior of an unstable damped oscillator is shown in Fig. 11, where we have chosen $\Omega_0=i$, $\gamma=0.8$. As in Fig. 4, the initial ground state is rapidly decohered in the position pointer observable. This is clearly seen in the plot of the Wigner distribution function (decoherence in position results in the increase of the uncertainty in momentum). An even better illustration of this effect is

again provided by the plot of the density matrix in the position representation, Fig. 11(b).

The competition of damping with the destabilizing force $F\sim q$ is clearly visible in the evolution of the Wigner distribution, Fig. 11(a). The particle starts at $q=0$ with a positive momentum. Large damping and initially small force cause it to slow down (momentum decreases), until it reaches $q\sim 6$, at which state it begins to accelerate “down the hill” (see the insert). Moreover, as the force increases for large q , the wave packet is being exponentially stretched in the q direction [Fig. 11(b)], and in the p direction [Fig. 11(c)]. This increase in the on-diagonal extent is not accompanied by either an increase

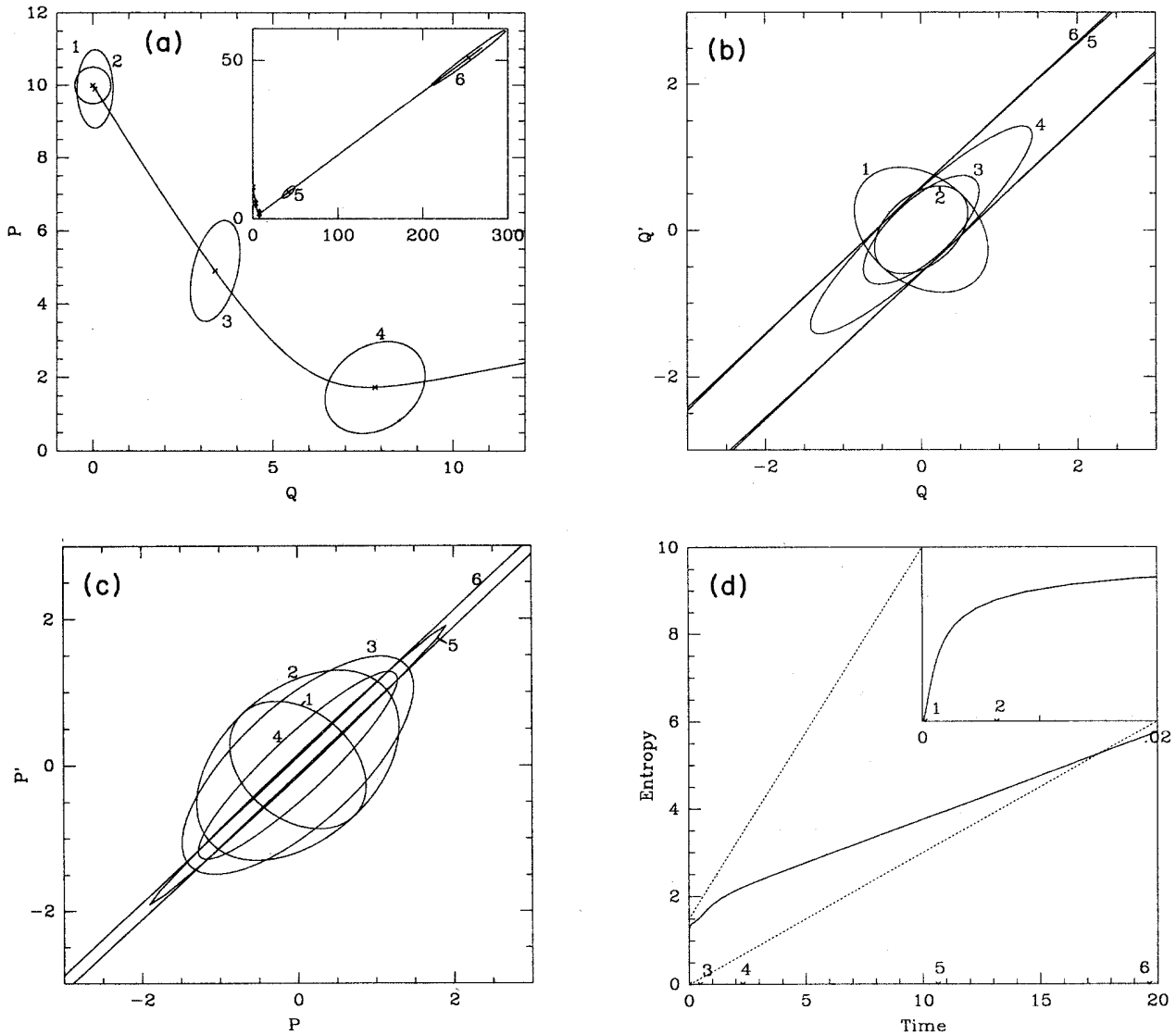


FIG. 11. Unstable damped oscillator ($\Omega=i$, $\gamma=0.8$) in a vacuum ($T=0$), initiated in a minimum uncertain Gaussian ($\Delta x = \Delta p$), but with finite momentum. After a period of slowing down [the plot of Wigner distribution (a)] the evolution is accelerated by a force increasing with distance from the origin [inset in (a)]. The same force “stretches” the wave packet along the direction of motion in phase space [5 and 6 in the inset of (a)]. However, interaction with the environment does not allow it to maintain coherence. Therefore the wave packet has a finite extent in the off-diagonal direction. Since the number of individual, locally coherent contributions increases exponentially with time ($\mathcal{N} \sim \langle \Delta q^2 \rangle^{1/2}$), entropy $\mathcal{H} \sim \ln \mathcal{N}$ grown linearly.

or a decrease of its off-diagonal width: Interaction with the environment controls the off-diagonal terms. Thus, at late times the system is in an approximate mixture of Gaussian coherent states, each of which has spatial extent Δq controlled by the interaction with the environment and corresponding spread in momentum $\Delta p \cong \hbar/\Delta q$ resulting from the principle of indeterminacy. The entropy [Fig. 11(d)] should be regarded as the natural logarithm of the number of such coherent pieces of the wave packet. Since the spatial extent of the wave packet increases exponentially at late time, entropy grows linearly with time.

It is tempting to apply this model to understand the origins of the transition from quantum to classical in the inflationary phase of the early Universe, where the effective potential of the inflation field is thought to have the qualitative features of either an “upside down” or the usual harmonic oscillator (“new” or “chaotic” inflation). We shall comment on this issue in more detail in the next section.

VI. DISCUSSION

The transition between quantum and classical is the principal motivation of the study of open quantum systems we have carried out in this paper. The key idea—environment-induced superselection^{16–20,27}—relates quantum or classical nature of a system to its ability to retain quantum coherence. To be effectively quantum, the system must be able to remain pure for the time intervals over which experiments are carried out. Quantum systems interacting with other systems, which constitute their “environments,” lose quantum coherence as a consequence of such interaction.

A. Quantum, classical, and the measurement

It is important to understand the nature of the decoherence mechanism and the extent to which it is modeled by various approximate methods. The interaction between the system and the environment is dynamically perfectly reversible. In a transparent notation and under an artificial (and unnecessary, but very convenient) assumption of the purity of initial state $|\sigma\rangle$ of the system and $|\varepsilon\rangle$ of the environment, the chain of events proceeds as

$$|\sigma\rangle \otimes |\varepsilon\rangle \rightarrow \sum \alpha_i |\sigma_i\rangle \otimes |\varepsilon_i\rangle = |\Phi\rangle. \quad (6.1)$$

The final state of the combined system $|\Phi\rangle$ is still pure. Hence the total entropy of the whole system has not increased. Yet the information an observer has about any of the subsystems will usually decrease as a result of the correlation between the subsystems.

The system and the environment are described by the reduced density matrices:

$$\rho_S = \text{Tr}_E |\Phi\rangle \langle \Phi|, \quad (6.2a)$$

$$\rho_E = \text{Tr}_S |\Phi\rangle \langle \Phi|. \quad (6.2b)$$

Both ρ_S and ρ_E are, generally, mixed. The entropy of each of the subsystems is now positive:

$$\mathcal{H}_S = -\text{Tr} \rho_S \ln \rho_S, \quad (6.3a)$$

$$\mathcal{H}_E = -\text{Tr} \rho_E \ln \rho_E. \quad (6.3b)$$

The “lost” information is still present in the correlation between them. The mutual information measuring the strength of this correlation

$$\mathcal{H}_{(S,E)} = \mathcal{H}_{(S,E)} - \mathcal{H}_S - \mathcal{H}_E \quad (6.4)$$

has increased in course of the interaction by the amount given by the total increase of the entropy of subsystems, so that the joint entropy of the system and the environment, $\mathcal{H}_{(S,E)}$, remains constant. This transfer of information that previously resided in the two systems separately into the correlation between them is responsible for the loss of the quantum coherence in open quantum systems. It is also responsible for the establishment of quantum correlation between the apparatus and the observable of the measured system in course of quantum measurements.²⁹ In a sense, mutual information enters the discussion of quantum measurements twice: first, in the process of correlating the apparatus and the measured system, and subsequently in the process assuring that the outcome of the measurement is indelibly recorded in the pointer observable of the apparatus which is effectively classical.

It is important at this point to differentiate between the model we have presented here, in which the field φ irreversibly carries off the information, and other interacting systems in which the information remains in the “measuring” system, to be undone by subsequent interaction with the “measured” system of interest. For example, the interactions of the electromagnetic waves with dielectric medium do not result in an effective measurement of the state of the electromagnetic wave. The change in the dielectric medium does not persist, is not “remembered,” and thus no effective measurement takes place. On the other hand, in this case, the field φ immediately moves away from the interaction region, never to return. It “remembers” the outcome of the interaction. This distinction is an explanation for the possibility of quantum interference in many solid-state systems where there is often a sufficiently strong interaction with other components of the system to apparently suffer “measurement” and thus loss of coherence.³⁰

B. Master equations and pointer observables

The master equations employed in the discussion of the measurement process and, more generally, in the discussion of the transition between quantum and classical, are usually Markoffian and stationary.^{13,20,24–26} This assumption facilitates detailed calculation of the consequences of the environment-induced superselection, and allows one to demonstrate the effectiveness with which the environment forces macroscopic systems to behave classically. In particular, a number of interesting models inspired by quantum optics have been constructed^{23–25} to demonstrate the facility with which the coupling with the environment destroys correlations between eigenstates of the preferred pointer observable $\hat{\Lambda}$ which commutes with the interaction Hamiltonian:

$$[\hat{\Lambda}, H_{\mathcal{S}\mathcal{E}}] = 0. \quad (6.5)$$

This condition is also satisfied by “nondemolition observables” of quantum systems³¹ used as sensitive detectors of weak forces (due, for instance, to gravity waves). One can therefore say that in some cases the environment performs a “nondemolition measurement” of the observable which, as a result, becomes classical.¹⁶

The position of a particle satisfies the condition Eq. (6.5) for many interaction Hamiltonians. (Whenever the force is a function of distance $H_{\mathcal{S}\mathcal{E}} = H_{\mathcal{S}\mathcal{E}}(q)$, as is the case for all the “potentials” $[q, H_{\mathcal{S}\mathcal{E}}] = 0$.) As a result, for the usual high-temperature Markoffian equation, one can express decoherence time θ in terms of the relaxation time scale ${}^{20}\tau$ (see also Ref. 13 for a different form of this relation):

$$\theta = \tau(\lambda_{dB} / \Delta q)^2. \quad (6.6)$$

Here Δq is a separation between parts of the wave function and λ_{dB} is the thermal de Broglie wavelength:

$$\lambda_{dB} = \hbar / \sqrt{4mk_B T}. \quad (6.7)$$

For a “canonical” classical system (mass $m = 1$ g at room temperature $T = 300$ K) and a macroscopic separation $\Delta q = 1$ cm, Eq. (6.6) predicts $\theta/\tau \simeq 10^{-40}$. This enormous disparity of the decoherence and relaxation time scales demonstrates how effective the environment-induced superselection can be in suppressing quantum behavior even in systems which, from the point of view of their dynamics, evolve almost completely reversibly and are, therefore, by the usual standards extremely well isolated from the environment.²⁰

Joos and Zeh¹⁹ have made a similar point in a careful discussion of the reduction of the wave packet using the master equation suggested by Wigner.¹⁸ They demonstrate, in particular, that even scattering of photons at a relatively low temperature (e.g., of the cosmic-microwave background) can extract enough information about the location of the macroscopic object to induce classical behavior.

The possibility of an experimental study of environment-induced superselection has been also raised.^{13,19,24} Partly in spite of and partly because of the effectiveness of the environment-induced superselection, finding the right subject of such an experiment is a difficult task. Microscopic quantum systems, such as photons, electrons, and atoms, turn out to be hard to decohere, while macroscopic systems are already too classical, presumably because they are very open, to detect any quantum features. The recent development of intermediate scale systems, such as Josephson junctions,³² or the Weber bars gravity wave detectors,³¹ hold out some hope of being able to see some of these effects.

An ideal subject for an experiment may also emerge from the nonlinear quantum optics. As discussed by Milburn,³³ an anharmonic oscillator with quartic correction (Hamiltonian given by $H = H_0 + \lambda H_0^2$ where H_0 is the standard harmonic-oscillator Hamiltonian) evolves from an initial Gaussian wave packet through a sequence of “nonclassical” states to—eventually, after the Poincaré cycle is completed—reassemble as a Gaussian. These

nonclassical states include, for instance, two wave packets with Gaussian envelopes placed on the opposite sides of the oscillator. Quantum coherence between them must be maintained if the Poincaré cycle is to be successfully completed. In presence of the environment, as Milburn and Holmes³³ point out, coherence will be destroyed and the evolution of the phase-space density distribution will quickly begin to look like that of the classical system with the analogous Hamiltonian. Hence, the degree to which the destruction of coherence will occur in course of the Poincaré cycle appears to determine whether the system will exhibit quantum or classical behavior. This general criterion was originally suggested as the key factor distinguishing quantum and classical regimes.¹⁷ Moreover, as noted by Yurke and Stoler,³⁴ systems which exhibit quartic nonlinearity can be realized in practice and may allow one to generate nonclassical wave packets ideal for the experimental study of decoherence.

Our analysis here is based on a linear system. Yet, most of the real life systems, and, especially, most of the systems in which environment-induced reduction of the wave packet could be studied experimentally are nonlinear. Calculations of even idealized nonlinear interactions are prohibitively difficult. Calculations of more realistic examples are unlikely to be tractable. How can one be confident that general features of the environment-induced superselection will extend beyond the domain of the linear systems?

The first point is that the initial transient, which plays such an important role in our analysis, is really independent of the dynamics of the system. The forces of the oscillator have not had time to act by the time this initial “jolt,” delivered by the heat bath, is felt. This feature we would expect to retain even in a strongly nonlinear system. It is, however, problematic experimentally, as pointed out earlier, as it depends on the initial decoherence of heat bath and system on a very-short-time scale.

Another feature of the problem studied here is the rapid transition into an ordinary Markoffian-Fokker-Planck-type master equation. The effective noise source for the vacuum fluctuations is not white. It has a nontrivial non-Markoffian correlation function. It is unclear to what extent this feature would be visible in the evolution of the density matrix for a nonlinear system. If the system moves slowly compared with the damping time scale, one would expect that the harmonic approximation to the potential during the damping time period would be a good approximation, and the analysis presented here should be valid. On the other hand, for a system interacting weakly with the bath, where the nonlinear dynamic time scale is shorter than the damping time scale, one might expect to find significant non-Markoffian effects due to the nonwhite correlations in the quantum noise from the heat bath. It is, however, surprising in the case of the oscillator how little of these effects are seen (except in the case of the free particle).

C. Markoffian approximation

Most of the treatments mentioned above have been carried out in the context of Markoffian master equa-

tions, which are explicitly irreversible, and are derived on the basis of assumptions the validity of which could be questioned in many of the applications. The most notorious of such assumptions is the condition which demands that the temperature of the heat bath be greater than the energy corresponding to the "collision frequency" Γ , the cutoff of the spectrum of the environment oscillators. Recently, however, several more detailed treatments of specific models have been published.^{11,12,14,35,36} They are usually concerned with models even more specific than the harmonic oscillator^{14,35,36} and only rarely provide an adequate treatment of the transition from the reversible, dynamical evolution to the irreversible master equation.

Some of these shortcomings have been avoided in the present treatment. The master equation (3.20) is, as a result, more cumbersome, but its solutions are still relatively easy to obtain, especially in the (k, Δ) representation. Time and temperature dependence of the coefficients is an inevitable consequence of the underlying dynamics. For example, the diffusion term h can be approximated, for short times by

$$\xi(t) = (\Gamma/\pi)\Gamma t / (\Gamma^2 t^2 + 1), \quad (6.8)$$

which is an odd function of time. Thus for $t < 0$ (the time at which the initial state of the oscillator was coupled with the heat bath), Eq. (3.20) would actually predict that the state of the system becomes spontaneously "pure," uncorrelated with its environment. This observation can be dismissed by insisting that the coupling with the environment was established at $t = 0$ and, hence, discussion of the evolution for negative times makes no sense. This would, however, miss the key point. Evolution of the complete (oscillator+environment) system is reversible. Initiating the oscillator in a pure state is, in effect, equivalent to imposing a condition on the evolution of the underlying dynamical system, which must be satisfied by both the advanced and retarded solutions. In other words, the system demonstrates that the rapid coherence change is consistent with its dynamics in both time directions. "Initial condition" is the only reason for the prevalence of the irreversible diffusion with the "right" sign.

In spite of the underlying reversibility, solutions of the exact master equation turn into mixtures on the fastest time scales available. The initial "jolt" which takes away most of the coherence happens on the collision time scale Γ^{-1} . Figures presented in the last section demonstrate this quite conclusively. Such effectiveness of the environment-induced superselection raises a perplexing question: How can one understand, in view of the rapid decoherence predicted even for the environments at $T=0$, manifestations of quantum physics on not only microscopic, but also much larger scales?^{15,37} Much of this apparent problem can be understood as the consequence of the idealized assumptions about the nature of the environment and its coupling with the system. In particular, the coupling used in Eqs. (2.1) and (2.2) does not respect Lorentz invariance.²⁷ Thus, it can be regarded as an adequate model only for these kinds of a heat bath which do have a preferred reference frame. For example, a gas at a finite temperature does constitute such an environment, while the electromagnetic vacuum at $T=0$ does not.

This remark is especially important in view of the fact that the exact treatment leads to a master equation which, ostensibly, holds for all temperatures, that is, both in the high-temperature regime and when $T \approx 0$. Clearly, conclusions regarding the heat bath at a finite temperature do imply a particular reference frame and are not affected by the caveat brought up in the previous paragraph. By contrast, conclusions which refer to low temperatures and short time scales should be reevaluated with some care within the context of a specific physical situation. The sharp onset of decoherence is a feature, in particular, associated with the "vacuum."

Availability of the master equation (3.17) applicable at all temperatures (providing that the underlying dynamical model is satisfied) offers one an opportunity to study the range of validity of the high-temperature Markoffian approximation. In particular, for a harmonic oscillator coupled with the environment at a finite temperature T , the time-dependent part of the diffusion coefficient, Eq. (6.8), becomes comparable with its long-time value $h \approx T$, at a time given approximately by

$$t_\beta = \pi/T. \quad (6.9)$$

For times appreciably in excess of t_β behavior of the system should be consistent with the predictions of the Markoffian versions of Eq. (3.17). This suggests a condition for the validity of the high-temperature approximation which is much less restrictive than was demanded:

$$T \gg \Gamma \quad (6.10)$$

usually assumed in the derivations. For, it appears that the high-temperature master equation will be obeyed whenever the system has evolved for a time span

$$t \gg t_\beta \quad (6.11)$$

and for $T \geq \max(\gamma, \Omega)$ regardless of the actual relation between T and Γ . These observations explain the remarkable success of the Markoffian master equation in the variety of situations.²³⁻²⁶ It would be, however, interesting to devise experiments to test the non-Markoffian transient regime.

D. Cosmological applications of environment-induced superselection

The model explored in this paper has, apart from the issues of quantum theory of measurement, a variety of interesting, practical applications. Master equations have been extensively employed in physical chemistry, quantum optics, and transport theory. We believe that quantum cosmology should be added to this list. An isolated unstable quantum oscillator was considered by Guth and Pi in their discussion of the transition between quantum and classical in the inflationary era of the Universe.³⁸ They have calculated the wave function ψ of the state which obtains from the initially narrow Gaussian. After a sufficiently long evolution position and momentum of the unstable oscillator become correlated, so that

$$QP\psi \cong \Omega_0 Q^2\psi, \quad (6.12a)$$

$$PQ\psi \cong \Omega_0 Q^2\psi - i\hbar\psi. \quad (6.12b)$$

For a sufficiently spread-out wave packet, the systematic correlation of position and momentum is thus much in excess of the uncertainty:

$$\Omega_0 Q^2 \gg \hbar. \quad (6.13)$$

Guth and Pi suggested that whenever condition (6.13) is satisfied the pure state can be in effect regarded as a collection of individual, narrow wave packets following approximately classical trajectories.

In our view, this interpretation fails to address the key issue of the actual transition between the spatially extended but nevertheless still pure wave function and the mixed state with the envelope proportional to $|\psi(x)|^2$. Note in particular that if one were to examine such an unstable oscillator with no damping, one would have an exponential growth of width in both the q - q' and the p - p' representations in both the on-diagonal and off-diagonal terms. The system remains coherent over the whole of the wave function in both q and p representations. In the Wigner representation, the ellipse would exponentially stretch in the direction of motion and exponentially shrink in the direction perpendicular to the motion, but the shrinkage is not indicative of any loss of coherence. However, it is precisely this aspect of the evolution that has been used to claim that the undamped oscillator is approaching "classical" behavior. In our opinion this claim cannot be justified. This can be seen by noticing that in the q , q' representation both the diagonal and the off-diagonal spread increases exponentially for the undamped oscillator. All points in the stretched-out density matrix are still strongly, quantum-mechanically correlated even at very late times.

Moreover, the question of the transition from quantum to classical in the context of inflation is not "academic" as the value of the cosmological constant which determines the rate of the inflationary expansion depends on the manner in which inflation field couples via the vacuum energy with the gravity. Thus, while the conclusions of Guth and Pi with regard to the approximately classical nature of the dynamical evolution of fragments of the wave packet are (approximately) correct, they are of little help in clarifying the transition from quantum to classical in the new inflationary model. In particular, the following problems need clarification.

(i) What value of the cosmological constant should appear on the right-hand side of Einstein's equations?

(ii) Moreover, all the coherent wave packets sliding down the side of the unstable part of the potential will eventually come together with the new vacuum. What is the guarantee that quantum correlations will remain unimportant at this time in the course of the reheating process?

Perhaps even more important than these criticisms, applicable specifically to the new inflationary scenario, is the simple observation that the mechanism suggested by Guth and Pi cannot be implemented in the chaotic inflationary model of Linde.³⁹ There the effective poten-

tial is either a regular (that is, not "upside down") harmonic oscillator or a quartic oscillator. The inflation occurs while the inflation field slides down from a large ($\gtrsim M_p$) initial fluctuation. Hence, the wave packet does not spread, and the correlation between q and p does not follow the pattern of Eqs. (6.12). Consequently, inequality (6.13) is not valid and one cannot appeal to it in search of a justification of a transition from quantum to classical.

Of course, some may take the above paragraph as an argument for new and against chaotic inflationary scenario. We believe, however, that a very different conclusion is in order. The scenario for a transition from quantum to classical cannot be based on Eq. (6.13) for neither new nor chaotic variant of inflation. Moreover, there are several additional reasons, including the ability to initiate inflationary phase and the size of the perturbations, which appear to favor chaotic inflation.⁴⁰

The issue is then to find how one can induce the transition from quantum to classical without relying on the features specific to the new inflation. The answer we would like to suggest calls on the environment, i.e., all of the fields which couple to the inflation, to force the environment-induced collapse of the wave packet. It is only the introduction of damping, of the heat bath, which leads to the loss of the off-diagonal coherence. Notice in Fig. 11 that the off-diagonal "width" approaches a constant at late times, even as the on-diagonal spread is increasing exponentially. Moreover, both Q and P are almost equally good "classical" observables with negligible off-diagonal coherence in presence of damping. This mechanism is clearly applicable to the simple one-dimensional analogs of the model, as previous sections amply demonstrate: the coherent quantum wave packet "falls apart" into a mixture of distinct states, each of which evolves independently of the others. It is not hard to see how this mechanism solves difficulties (i) and (ii). Moreover, it applies equally well to the "upside down" as well as to the "regular" oscillator. Hence, it can be invoked in both the new and chaotic scenarios.

The same simple model of the new inflationary epoch was recently investigated by Cornwall and Bruinsma,⁴¹ who have calculated Feynman-Vernon influence functional for an "upside down" oscillator in a thermal heat bath. The two approaches lead to the same qualitative results, although Ref. 41 adopts the high-temperature limit early on which leads to very different decoherence rates.

The "wave function of the Universe" approach pioneered by Hawking, Hartle, and Vilenkin⁴² poses an even greater challenge in terms of delineating the transition between quantum and classical.⁴³ It has been suggested that the general approach based on the environment-induced superselection¹⁶⁻²⁰ may be applicable also in this context.⁴⁴ For, while the Universe has no environment "outside" it, certain degrees of freedom in it will be able to serve as the environment for the other degrees of freedom, and may force them to behave in an effectively classical manner. We shall pursue cosmological applications of environment-induced superselection elsewhere.

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