Reduction of Homogeneous Riemannian structures

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Outline

1. Preliminaries
   - Homogeneous Riemannian structures
   - The mechanical connection

2. Reduction
   - Reduction by a normal subgroup of isometries
   - Reduction in a principal bundle
   - Reduction of homogeneous classes and examples

3. An application: Sasakian-Kähler reduction

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Homogeneous Riemannian structures

- A Riemannian manifold (M, g) is called *homogeneous* if there is a Lie group G of isometries acting transitively on it.

Theorem (Ambrose-Singer)

A connected, simply connected and complete Riemannian manifold (M, g) is homogeneous if and only if it admits a $(1, 2)$-tensor field S such that, if $\widetilde{\nabla} = \nabla - S$ where ∇ is the Levi-Civita connection, then

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0. \tag{1}$$

- A tensor field S satisfying (1) is called a homogeneous Riemannian structure.

Homogeneous Riemannian structures

Homogeneous Riemannian structures are classified in eight invariant classes:

- The class of symmetric spaces ($S = 0$).
- Three primitive classes

$$\begin{aligned}
\mathcal{S}_1 &= \{S \in \mathcal{S} \,/\, S_{XYZ} = g(X,Y)\varphi(Z) - g(X,Z)\varphi(Y),\ \varphi \in \Gamma(T^*M)\} \\
\mathcal{S}_2 &= \{S \in \mathcal{S} / \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0, \quad c_{12}(S) = 0\} \\
\mathcal{S}_3 &= \{S \in \mathcal{S} \,/\, S_{XYZ} + S_{YXZ} = 0\}.
\end{aligned}$$

- Their direct sums $\mathcal{S}_1 \oplus \mathcal{S}_2$, $\mathcal{S}_1 \oplus \mathcal{S}_3$, $\mathcal{S}_2 \oplus \mathcal{S}_3$.
- And the generic class $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$.

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The mechanical connection

- Let $(P, \bar{g})$ be a Riemannian manifold and $\pi : P \to M$ be an H-principal bundle with H acting on P by isometries. Let $V_{\bar{x}}P$ denote the vertical subspace at a point $\bar{x} \in P$.

- The mechanical connection is defined by the (H-invariant) horizontal distribution

$$H_{\bar{x}}P = (V_{\bar{x}}P)^{\perp}, \quad \bar{x} \in P$$

- In this situation there is a unique Riemannian metric g in M such that $\pi_* : H_{\bar{x}}P \to T_{\pi(\bar{x})}M$ is an isometry, $\forall \bar{x} \in P$ (we called g the reduced metric).

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Problem

$$\begin{array}{ccc} (P, \bar{g}) & & \bar{S} \\ H \big\downarrow & & \big\downarrow \\ (M, g) & & ? \end{array}$$

Outline

Reduction by a normal subgroup of isometries

- Let $(P, \bar{g})$ be a connected homogeneous Riemannian manifold and $\bar{G}$ a Lie group of isometries acting transitively on it. Suppose $\bar{G}$ has a normal subgroup H acting freely on P.
- We consider the quotient manifold $M = P/H$, the (left) H-principal bundle $\pi : P \to M$ endowed with the mechanical connection, and the reduced metric g in M.

- In this situation the quotient group $G = \bar{G}/H$ acts on (M, g) by

$$\pi \circ \Phi_{\bar{a}} = \Phi_a \circ \pi$$

where $\bar{a} \in \bar{G}$, $a = [\bar{a}] \in G$, and Φ denotes the corresponding maps for the actions. This action is transitive and by isometries.

Reduction by a normal subgroup of isometries

This means that (M, g) is homogeneous Riemannian with G a Lie group of isometries acting transitively.

Question

How homogeneous Riemannian structure tensors associated to the action of G on M are related to that of the action of $\bar{G}$ on $\bar{M}$?

$$\begin{array}{ccccc} \bar{G} & \circlearrowright & P & \rightsquigarrow & \bar{S} \\ \downarrow H & & \downarrow H & & \downarrow ? \\ G = \bar{G}/H & \circlearrowright & M = P/H & \rightsquigarrow & S \end{array}$$

Reduction by a normal subgroup of isometries

Proposition

In the previous situation, every homogeneous Riemannian structure $\bar{S}$ associated to the action of $\bar{G}$ induces a homogeneous Riemannian structure S in M associated to the action of G given by

$$S_X Y = \pi_* \left(\bar{S}_{X^H} Y^H \right) \quad X, Y \in \mathfrak{X}(M),$$

where X^H denotes the horizontal lift with respect to the mechanical connection.

Reduction by a normal subgroup of isometries

Proof.

- Let $\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ be a reductive decomposition of $\bar{\mathfrak{k}}$ at $\bar{x} \in P$ corresponding to $\bar{S}$. From the isomorphism $\bar{\mathfrak{m}} \to T_{\bar{x}}P$ (given by the infinitesimal action) the mechanical connection induces an $\mathrm{Ad}(\bar{K})$-invariant decomposition
$$\bar{\mathfrak{m}} = \bar{\mathfrak{m}}^h \oplus \bar{\mathfrak{m}}^v.$$
- Let $\tau : \bar{G} \to G$ be the quotient homomorphism. One proves that
$$\mathfrak{g} = \tau_*(\bar{\mathfrak{m}}^h) \oplus \mathfrak{k}$$
is a reductive decomposition of $\mathfrak{g}$ at $\pi(\bar{x}) \in M$.
- One proves that the *reduced homogeneous Riemannian structure tensor* S corresponding to this decomposition is
$$S_X Y = \pi_* \left(\bar{S}_{X^H} Y^H \right) \quad X, Y \in \mathfrak{X}(M).$$

□

Reduction by a normal subgroup of isometries

Proposition

The set of homogeneous Riemannian structures associated to the action of $\bar{G}$ in P reducing to a given homogeneous Riemannian structure S associated to the action of G on M, is in one to one correspondence with the space of $\mathrm{Ad}(\bar{K})$-equivariant maps

$$\varphi : \mathfrak{h} \to \bar{\mathfrak{k}},$$

where $\bar{K}$ is the isotropy group and $\bar{\mathfrak{k}}$ is its Lie algebra.

(One can obtain the explicit expression for those tensor fields)

Outline

Reduction in a principal bundle

$$\begin{array}{ccc} (P,\bar{g}) & \rightsquigarrow & \bar{S} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle ?} \\ (M,g) & \rightsquigarrow & S \end{array}$$

- We ask under which conditions the tensor field

$$S_X Y = \pi_* \left(\bar{S}_{X^H} Y^H \right) \qquad X, Y \in \mathfrak{X}(M)$$

defines a homogeneous Riemannian structure in (M, g).

Reduction in a principal bundle

Remark

- *In the previous section, $H \lhd \bar{G} \Rightarrow$ the mechanical connection is $\bar{G}$-invariant $\Rightarrow$ the connection form ω is $\mathrm{Ad}(\bar{G})$-equivariant.*
- *Infinitesimally this becomes*

$$(\tilde{\bar{\nabla}}_{\bar{X}}\omega)(\bar{Y}) = \mathrm{ad}(\mu^{-1}(\bar{X}))(\omega(\bar{Y})) \quad \bar{X}, \bar{Y} \in \mathfrak{X}(P)$$

  where $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$, and $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$.
- *So the covariant derivative of the connection form ω with respect to $\tilde{\bar{\nabla}}$ is "proportional" to itself by a suitable linear operator.*

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Reduction in a principal bundle

Theorem

Let $(P, \bar{g})$ be a Riemannian manifold. Let $\pi : P \to M$ be a principal bundle with structure group H acting by isometries, and endowed with mechanical connection ω. For every H-invariant homogeneous Riemannian structure $\bar{S}$ in P with $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$, if

$$\tilde{\bar{\nabla}}\omega = \alpha \cdot \omega$$

for a certain 1-form α in P taking values in $\mathrm{End}(\mathfrak{h})$. Then the tensor field defined by

$$S_X Y = \pi_*(\bar{S}_{X^H} Y^H) \quad X, Y \in \mathfrak{X}(M)$$

is a homogeneous Riemannian structure in (M, g).

Reduction in a principal bundle

Proof.

- The tensor field S is well-defined by H-invariance of $\bar{S}$.
- For all $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned} \omega(\tilde{\bar{\nabla}}_{X^H} Y^H) &= X^H\left(\omega(Y^H)\right) - \left(\tilde{\bar{\nabla}}_{X^H}\omega\right)(Y^H) \\ &= -\alpha(X^H)(\omega(Y^H)) = 0, \end{aligned}$$

thus $\left(\tilde{\nabla}_X Y\right)^H = \tilde{\bar{\nabla}}_{X^H} Y^H$ (where $\tilde{\nabla} = \nabla - S$).

- With this one proves that S satisfies Ambrose-Singer equations: equations for g and S are easy, but equation for the curvature R is much more delicate (the curvature form of the mechanical connection appears).

$\square$

Outline

Reduction of homogeneous classes

Proposition

- *The classes $\mathcal{S}_1$, $\mathcal{S}_3$, $\mathcal{S}_1 \oplus \mathcal{S}_2$ and $\mathcal{S}_1 \oplus \mathcal{S}_3$ are invariant under the reduction procedure.*
- *For the classes $\mathcal{S}_2$ and $\mathcal{S}_2 \oplus \mathcal{S}_3$ (for which the trace c_{12} of the homogeneous structure must vanish) one has*

$$c_{12}(S)(X) = c_{12}(\bar{S})(X^H) - \bar{g}(\mathrm{H}, X^H) \quad X \in \mathfrak{X}(M),$$

  where H is the mean curvature of the fibre (as a sub-Riemannian manifold of $(P, \bar{g})$) at each point.
- *In particular the classes $\mathcal{S}_2$ and $\mathcal{S}_2 \oplus \mathcal{S}_3$ are invariant under reduction if and only if the fibres are minimal sub-Riemannian manifolds of $(P, \bar{g})$.*

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Examples

- In the fibration $\mathbb{R}H(n) \to \mathbb{R}H(n-1)$ ($\bar{G} = \mathbb{R}H(n), H = \mathbb{R}$), the standard $\mathcal{S}_1$ structure reduces to the standard $\mathcal{S}_1$ structure. For $\bar{G} = SO(n-2)\mathbb{R}H(n), H = \mathbb{R}$, a family of structures in the generic class $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$ reduces to a family of structures in the generic class, and for one value of the family parameter to the standard $\mathcal{S}_1$ structure.
- In the Hopf fibrations $S^3 \to S^2$ ($\bar{G} = U(2), H = U(1)$) and $S^7 \to \mathbb{C}P(3)$ ($\bar{G} = U(4), H = U(1)$) (fibres are totally geodesic), a family of $\mathcal{S}_2 \oplus \mathcal{S}_3$ structures reduces to the symmetric case $S = 0$.
- In the Hopf fibration $S^7 \to \mathbb{C}P(3)$ ($\bar{G} = Sp(2)U(1), H = U(1)$), a 2-parameter family of $\mathcal{S}_2 \oplus \mathcal{S}_3$ structures reduce to a 1-parameter family of $\mathcal{S}_2 \oplus \mathcal{S}_3$ structures.

Outline

Extra Geometry

Theorem (Kiričenko)

Let (M, g) be a connected, simply connected and complete Riemannian manifold, and $T_1, ..., T_n$ tensor fields in M. Then (M, g) is Riemannian homogeneous with $T_1, ..., T_n$ invariant if and only if it admits a homogeneous Riemannian structure S such that $\widetilde{\nabla} T_i = 0$, $i = 1, ..., n$.

- A homogeneous Riemannian structure S in an almost contact metric manifold (M, ϕ, η, ξ, g) is called an homogeneous Riemannian almost contact metric structure if $\widetilde{\nabla}\phi = 0$. If (M, ϕ, η, ξ, g) is moreover (almost) Sasakian then S is called (almost) Sasakian.
- A homogeneous Riemannian structure S in an almost Hermitian manifold (M, J, g) is called an homogeneous Riemannian almost Hermitian structure if $\widetilde{\nabla} J = 0$. If (M, J, g) is moreover (almost) Kähler then S is called (almost) Kähler.

Fiberings of almost contact manifolds

- **Theorem** (Ogiue): Let (P, ϕ, ξ, η) be an invariant strictly regular almost contact manifold and M the space of orbits given by ξ. Then $\pi : P \to M$ is a principal bundle, η is a connection form, and $J(X) = \pi_*(\phi(X^H))$, $X \in \mathfrak{X}(M)$ is an almost complex structure in M.

- When the almost contact structure $(P, \phi, \xi, \eta, \bar{g})$ is metric the connection η is the mechanical connection.
- If moreover $(P, \phi, \xi, \eta, \bar{g})$ is (almost) Sasakian then (M, J, g) is (almost) Kähler.
- If $\bar{S}$ is an homogeneous almost contact metric structure then $(\tilde{\bar{\nabla}}\phi = 0 \Rightarrow)$ $\tilde{\bar{\nabla}}\eta = 0$, so we are in the situation of the Reduction Theorem above with $\alpha = 0$, $\omega = \eta$.

Homogeneous Almost contact-Hermitian and Sasakian-Kähler reduction

Proposition

If $\bar{S}$ is a homogeneous almost contact metric structure on $(P, \phi, \xi, \eta, \bar{g})$, then it can be reduced to a homogeneous almost Hermitian structure S on (M, J, g). If moreover $(P, \phi, \xi, \eta, \bar{g})$ is (almost) Sasakian then S is a homogeneous (almost) Kähler structure.

Examples:

- In the Hopf fibrations $S^3 \to S^2$ and $S^7 \to \mathbb{C}P(3)$ a family of homogeneous Sasakian structures reduces to the unique homogeneous Kähler structures $S = 0$ in S^2 and $\mathbb{C}P(3)$ respectively.
- A homogeneous Sasakian structure in the trivial bundle $\mathbb{C}H(n) \times \mathbb{R} \to \mathbb{C}H(n)$ reduce to a nontrivial homogeneous Kähler structure.

Future work

- Reduction the other way around

$$\begin{array}{ccc} (P, \bar{g}) & \rightsquigarrow & ? \\ {\scriptstyle H}\downarrow & & \uparrow \\ (M, g) & \rightsquigarrow & S \end{array}$$

- Geometric study of the condition $\tilde{\bar{\bar{\nabla}}}\omega = \alpha \cdot \omega$ (which leads to an "equivariant" version of Kiričenko's Theorem).
- Application to symplectic/Kähler reduction (resp. hyper Kähler reduction).