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REDUCTION OF PAIRWISE COMPARISONS IN DECISION MAKING VIA A DUALITY APPROACH

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ABSTRACT: Although pairwise comparisons have been seen by many as an effective and intuitive way for eliciting qualitative data for multi-criteria decision making problems, a major drawback is that the number of the required comparisons increases quadratically with the number of the entities to be compared. Thus, often even data for medium size decision problems may be impractical to be elicited via pairwise comparisons. The more the comparisons are, the higher is the likelihood that the decision maker will introduce erroneous data. This paper introduces a dual formulation to a given multi-criteria decision making problem, which can significantly alleviate the previous problems. Some theoretical results establish that this is possible when the number of alternatives is greater than the number of decision criteria plus one.

KEY WORDS: Pairwise comparisons, multi-criteria decision making, the Analytic Hierarchy Process (AHP), duality.

1. Some Background Information

This paper deals with a critical problem in decision analysis. It examines the issue of eliciting qualitative information by using a short sequence of pairwise comparisons. Pairwise comparisons have long been proposed as an effective and intuitive way for eliciting qualitative data for multi-criteria decision making (MCDM) problems. Probably, they became best known as part of the Analytic Hierarchy Process (AHP) (Saaty,1980 and 1994). However, pairwise comparisons can be used in conjunction with any decision method that requires to quantify qualitative data. This paper proposes an alternative approach on how such comparisons should be elicited.

In the typical MCDM problem the decision maker is given m alternatives (denoted as A_1, A_2, A_3, ..., A_m) to evaluate in terms of n decision criteria (denoted as C_1, C_2, C_3, ..., C_n). The decision maker's task is to determine the relative performance of the m alternatives. First of all, the decision maker needs to form the entries of a decision matrix $\boldsymbol{A}$ and also determine the vector W with the weights of importance of the n criteria. Usually, the entry a_{ij} of the decision matrix $\boldsymbol{A}$ represents the performance of alternative A_i when it is examined in terms of criterion C_j. Next the decision maker has to use the data in the decision matrix along with the criteria weights and somehow evaluate the performance of each alternative in terms of all the decision criteria considered simultaneously. This problem becomes difficult to solve when the decision criteria are expressed in different units of measure (e.g, dollars, hours, pounds, etc.). How to evaluate alternatives in a multi-dimensional setting, is a central problem in MCDM.

There is a plethora of different schools of thought and proposed methodologies on how such problems can be solved. The focus of this paper is **not** on an evaluation of such methods. For some comparative studies and evaluations of such methods the interested reader may wish to consult with the studies reported by Lootsma (1990), and (Triantaphyllou, and Mann, 1989). Among these methods the Analytic Hierarchy Process (AHP) (Saaty, 1980 and 1994) enjoys a wide acceptance by practitioners (especially by those who use the *Expert Choice* (1990) computer package) and theoreticians. However, it also has its critics (e.g., (Belton and Gear, 1983), (Dyer and Ravinder, 1983), (Dyer and Wendell, 1985), (Dyer, 1990a and 1990b), (Triantaphyllou and Mann, 1994), and (Winkler, 1990)).

A central issue in the AHP is the elicitation of qualitative data via a sequence of pairwise comparisons. It is noticeable that a central role for the commercial acceptance of the AHP is that practitioners find pairwise comparisons to be an effective and intuitive way for eliciting qualitative information. Some theoretical issues on the use of comparisons and scales of measurement can be found in (Saaty, 1980) and (Triantaphyllou *et al.*, 1994).

Suppose that a decision maker wishes to elicit the relative priorities, or weights of importance, of n entities. These n entities could be n decision criteria, or n alternatives to be examined in terms of a single decision criterion. Then, the decision maker must elicit the value of $n(n-1)/2$ pairwise comparisons. If a decision problem involves m alternatives and n decision criteria (multiple hierarchical levels are not considered at this point), then the total number of the required pairwise comparisons is: $n(n-1)/2 + n(m(m-1)/2)$.

The above number can be quite large, even for moderate values of m and n. For instance, for $m = 15$ and $n = 10$, the total number of the required comparisons is equal to 1,095. This number increases quadratically as the values of m and n increase. If the decision maker is inaccurate in the elicitation of some of the comparisons, then the accuracy of the rest of the comparisons can alleviate the burden caused from the inaccurate ones (Saaty, 1980). That is, small inaccuracies in some of the comparisons may not be critical due to the redundancy from the rest of the comparisons. This is the main reason why all the comparisons are needed. However, a high number of comparisons can make the data elicitation process tedious and thus it may compromise the accuracy of the individual judgments. This, in turn, can become the reason to introduce errors in the elicitation process. Therefore, it is important to seek methods for reducing pairwise comparisons, without jeopardizing the benefits of having redundant ones. Some methods (e.g., (Harker, 1987a and 1987b)) have proposed to skip certain comparisons. However, such a measure reduces the redundancy in the judgments and then a single error may have a bigger impact and lead to the wrong ranking of the alternatives.

This paper presents an alternative approach. It slightly changes the way for eliciting pairwise comparisons and it does not skip any comparisons. This is done by formulating a type of a **dual problem**. As a result, the proposed approach can reduce the total number of required pairwise comparisons when the number of alternatives is larger than the number of decision criteria plus one (i.e., when $m > n + 1$).

This paper is organized as follows. The next section briefly overviews the traditional way for eliciting pairwise comparisons. This proposed dual approach is presented in the third section. The fourth section demonstrates the proposed approach in terms of an illustrative example. The fifth section presents some numerical results on how much the number of comparisons can be reduced for problems of different size. Some concluding remarks, and areas of possible extensions, are presented in the last section.

2. Eliciting Relative Weights from Pairwise Comparisons

Suppose that an MCDM problem calls for the evaluation of m alternatives in terms of n decision criteria. Then the decision maker needs to extract the relative weights of the n criteria and also the relative (if the criteria are defined in different units of measure) performance values of the m alternatives in terms of each one of the decision criteria. The first task is accomplished by forming an $n \times n$ judgment (also called pairwise comparison) matrix. The (i,j) element of this matrix refers to the evaluation of the importance of the *i-th* criterion when it is compared with the *j-th* criterion. The second task is accomplished by forming n matrices of order $m \times m$ each. In this paper these matrices will be denoted as $\boldsymbol{P}^{\boldsymbol{k}}$. The (i,j) element of the *k-th* matrix (for $k = 1,2,3,...,n$), denoted as p_{ij}^{k}, refers to the evaluation of the importance of the *i-th* alternative when it is compared with the *j-th* alternative in terms of the *k-th* criterion. That is, the decision maker is asked to answer a sequence of questions of the following form:

> "*What is the relative importance p_{ij}^{k} of alternative A_i when it is compared with alternative A_j in terms of the decision criterion C_k?*"

Often the value of p_{ij}^{k} cannot be determined directly. This happens when the decision criterion C_k expresses a qualitative aspect of the alternatives. In such a case the decision maker first assigns a linguistic statement that best expresses his/her opinion of the relative importance of the two alternatives A_i and A_j when they are compared in terms of criterion C_k. Next, this linguistic statement is assigned to some numerical value according to a predetermined scale of measurement (Saaty, 1980), (Triantaphyllou *et al.*, 1994). A judgment matrix $\boldsymbol{P}^{\boldsymbol{k}}$ derived by using the previous pairwise comparisons is a reciprocal one. That is, the property $p_{ij}^{k} = 1/p_{ji}^{k}$ holds for any element (pairwise comparison) p_{ij}^{k}.

The next step is to extract the relative importances (or weights) implied by the previous comparisons. Saaty asserts that to answer this question one has to estimate the **right principal eigenvector** of the previous matrix. Given a judgment matrix with pairwise comparisons, the corresponding right principal eigenvector can be approximated by using the geometric mean of each row (Saaty, 1980). That is, the elements in each row are multiplied with each other and then the *n-th* root is taken (where n is the number of elements in the row). Next these numbers are normalized by dividing them with their sum.

An evaluation of the eigenvalue approach can be found in (Triantaphyllou and Mann, 1990). It should be stated here that there is a number of alternative approaches which do not use the eigenvector concept and can extract relative priorities from pairwise matrices. These include techniques by Fichtner (1986) and Bryson (1995). In (Triantaphyllou *et al.*, 1990) a least squares formulation is presented and it is compared with some other priority extraction approaches. The relevance and usefulness of the proposed duality approach is not restricted to those cases when the right principal eigenvector method is used.

One of the most practical issues in the pairwise comparisons approach is that it allows for slightly inconsistent pairwise comparisons. If all the comparisons are perfectly consistent, then the relation $p_{ij}^{k} = p_{iy}^{k} \times p_{yj}^{k}$ should always be true for any combination of comparisons taken from the judgment matrix.

However, perfect consistency rarely occurs in practice. In the AHP the pairwise comparisons in a judgment matrix are considered to be adequately consistent if the corresponding consistency ratio (*CR*) is less than 10% (Saaty, 1980). The *CR* coefficient is calculated as follows. First the consistency index (*CI*) needs to be estimated. This is done by adding the columns of the judgment matrix and multiplying the resulting vector by the vector of priorities (i.e., the approximated right principal eigenvector) obtained earlier. This yields an approximation of the maximum eigenvalue, denoted by λ_{max}. Then, the *CI* value is calculated by using the formula: $CI = (\lambda_{max} - n)/(n - 1)$.

The concept of the *RCI* (Random Consistency Index) is used next. Given a value of *n* (e.g., the number of items to be compared) the *RCI* value corresponds to the average random consistency index (calculated by using the formula $CI = (\lambda_{max} - n)/(n - 1)$) derived from a sample of size 500 of randomly generated reciprocal matrices with entries from the set {1/9, 1/8, 1/7, ..., 1, 2, ..., 7, 8, 9} (Saaty, 1980). The concept of the *RCI* was introduced by Saaty in order to establish (by means of a statistical test of hypothesis) an upper limit on how much inconsistency may be tolerated in a decision process. Next, the *CR* value of a judgment matrix is obtained by dividing the *CI* value by the corresponding *RCI* value as given in Table 1. If the *CR* value is greater than 0.10, then a re-evaluation of the pairwise comparisons is recommended (because the corresponding consistency ratio is considered as high). This is repeated until a *CR* value of 0.10 or less is achieved. Some alternative consistency indicators have also been proposed (e.g., the ones by (Golden and Wang, 1989) and (Bryson, 1995).

Table 1: *RCI* values for different values of *n* (Saaty, 1980).

n	1	2	3	4	5	6	7	8	9
RCI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45

After the alternatives are compared with each other in terms of each one of the decision criteria and the individual priority vectors are derived, the synthesis step is taken. The priority vectors become the columns of the decision matrix (not to be confused with the judgment matrices with the pairwise comparisons). The weights of importance of the criteria are also determined by using pairwise comparisons.

3. A Duality Approach for Eliciting Comparisons

Recall that when two alternatives are considered in the traditional approach, the decision maker is asked to estimate the value of p_{ij}^{k}, the relative importance of alternative A_i when it is compared with alternative A_j in terms of the decision criterion C_k, by answering the following type of questions:

"*What is the relative importance* p_{ij}^k *of alternative* A_i *when it is compared with alternative* A_j *in terms of the decision criterion* C_k*?*"

We will call this kind of questions the **prime pairwise comparisons** and the corresponding judgment matrices will be called the **prime judgment matrices** and they will be denoted as $\boldsymbol{P}^{\boldsymbol{k}} = (p_{ij}^k)$, for $k = 1, 2, 3, ...,n$.

In the proposed dual approach the previous question takes a different form. Instead of comparing two alternatives at a time, now the relative performance of two decision criteria is examined within a given alternative. That is, now the typical question is of the following form:

"*What is the relative importance* d_{ij}^k *of criterion* C_i *when it is compared with criterion* C_j *in terms of alternative* A_k*?*"

In analogy to the prime pairwise comparisons, we will call this kind of questions the **dual pairwise comparisons** and the corresponding judgment matrices are called the **dual judgment matrices** and they will be denoted as $\boldsymbol{D}^{\boldsymbol{k}} = (d_{ij}^k)$, for $k = 1, 2, 3, ...,m$.

In the conventional AHP it is assumed that the criteria are independent of the alternatives. The relative weights of the criteria are extracted by comparing the criteria among themselves. Also the alternatives are compared among themselves in terms of each criterion. If the criteria are assumed to depend on the alternatives, then Saaty (1994, Chapter 8) proposes the elicitation of questions that compare criteria among themselves in terms of each of the alternatives as is the case with the dual comparisons.

However, one can argue that seldomly a decision maker examines a set of criteria without having in mind the alternatives to be evaluated and vice-versa. In other words, since criteria and alternatives are the key entities in a given MCDM problem, one has to simultaneously keep them on focus all the time during the decision making process.

For instance, if one considers the two criteria *"cost"* and *"functionality"* in purchasing a product, then these two criteria have different relative importance if the problem is to purchase a TV set or a new house. In the first case a 20% price change may not be so critical, while in the case of purchasing a new house a 20% price change may be more detrimental. Therefore, it is also natural to accept the premise that perfect independence of the criteria and the alternatives seldomly exists. This is a highly debatable issue in the decision analysis community and different authors have expressed different positions (Saaty, 1994, Chapter 8). Thus, the use of the proposed question format is not totally new. At this point we state the first key assumption that governs the developments described in this paper.

Assumption #1:

In a given MCDM problem the criteria ***influence the perception*** *of the alternatives and vice-versa.*

When the above assumption is not acceptable, Saaty (1994) proposes to use the so-called supermatrix technique. When the decision maker elicits all possible dual comparisons of the criteria in terms of a given alternative (say alternative A_i, for i = 1,2,3,...,m), then the normalized weight vector of this judgment matrix $\boldsymbol{D^i}$ corresponds to a normalized **row** (actually the i-th row) of the decision matrix. Therefore, by using the previous type of dual comparisons the decision maker can determine all the rows of the decision matrix in a normalized manner (i.e., the elements in each row sum up to one or are divided by the maximum entry of that row). We call this decision matrix $\boldsymbol{H}$, since each row has been normalized **horizontally**.

Let h_{ij} (for i=1,2,3,...,m and j=1,2,3,...,n) denote the (i,j) element of the decision matrix normalized in terms of each row via a sequence of m (dual) judgment matrices with dual comparisons. It can be noticed that now there are m judgment matrices of size $n \times n$ each, while in the traditional (prime) approach there are n such matrices of size $m \times m$ each. Moreover, suppose that the decision maker also forms the pairwise comparisons of a single judgment matrix in the traditional (prime) fashion. That is, the decision maker forms a single $m \times m$ judgment matrix.

To help fix ideas, suppose that the judgment matrix formed in the traditional (prime) fashion examines all the m alternatives in terms of the first decision criterion. According to our previous notation, this matrix is denoted as $\boldsymbol{P^1}$, since it involves the (prime) comparisons of the alternatives in terms of the first criterion. Let v_{i1} (for i = 1, 2, 3, ..., m) be the elements of the corresponding weight (column) vector derived from this prime judgment matrix. Note that the elements v_{i1} are normalized **vertically**.

Given the values of the m rows of the decision matrix normalized in terms of each row (e.g., the h_{ij} values) and the values of the single normalized column (e.g., the v_{i1} values), then it is algebraically straightforward to derive the elements of any column in the decision matrix normalized in terms of each column (i.e., normalized vertically). We call this matrix $\boldsymbol{V}$, since it represents the decision matrix with its elements normalized **vertically**. It can be easily verified that this can be achieved by employing the following formula (3.1):

$$h_{ij}^{/} = \left(\frac{h_{i1}}{v_{i1}}\right) v_{ij} \tag{3.1}$$

It is important to note here that the sum of all the $h_{ij}^{/}$ elements in a given column (except the first one) does not necessarily add up to one.

In general (if the k-th column is selected for normalization via an additional prime judgment matrix derived according to the traditional approach), the previous expression becomes:

$$h_{ij}^{/} = \left(\frac{h_{ik}}{v_{ik}}\right) v_{ij} \tag{3.2}$$

When the previously derived elements $h_{ij}^{/}$ are normalized by dividing each element by the sum of the entries of its column, the last expression yields:

$$h_{ij} = \frac{\left(\frac{h_{ik}}{v_{ik}}\right) v_{ij}}{\sum_{y=1}^{m}\left(\left(\frac{h_{yk}}{v_{yk}}\right) v_{yj}\right)}, \quad for\ \ i = 1,2,3,...,m,\ and \quad j = 1,2,3,...,n\,. \tag{3.3}$$

Therefore, it is possible for one to derive the decision matrix normalized in any desirable way by using dual pairwise comparisons. At this point we state the second key assumption that governs these developments:

Assumption #2:

*Given matrix **V** and one row of matrix **H**, then matrix **H** can be derived according to relation (3.3), assuming **invariance of proportions**.*

It should be mentioned at this point that the above considerations hold true for consistent or inconsistent judgment matrices (assuming that the inconsistences are less than 10% (Saaty, 1980)). The presence of high inconsistency can cause the pairwise comparisons of a (prime or dual) judgment matrix to be re-elicited by the decision maker(s). The present developments refer to the number of such comparisons. Once an acceptable (that is, with the value of the *CR* coefficient less or equal than the 10% limit) judgment matrix has been derived, the next step is to extract the pertinent weights or relative importances.

When the dual comparisons are used, a different sequence of judgment matrices is formed of dimensions different than those formed when comparisons are elicited in the traditional approach. It should be emphasized here that v_{ij} represent the entries of the decision matrix normalized vertically (i.e., the entries of each column add up to one). On the other hand, h_{ij} represent the entries normalized horizontally (i.e., the entries of each row add up to one). Next, the question which is naturally raised at this point is under what conditions the number of comparisons in the dual approach is smaller. This is the subject of the following theorem and corollaries.

Theorem 1:

The percent (%) of change of the number of comparisons between the prime and the dual problem is given by the following formula:

$$\frac{m(n-1)(m-n-1)}{n(n-1+m(m-1))} \, x \, 100 \,. \tag{3.4}$$

Proof: Since the problem has m alternatives and n decision criteria, one $n\times n$ judgment matrix is required to derive the criteria weights and n judgement matrices of size $m\times m$ each are required to derive (under the traditional approach) the relative weights of the m alternatives in terms of each one of the n decision criteria. Thus, the total number of required pairwise comparisons according to the traditional (primal) approach is equal to:

$$\frac{n(n-1)}{2} + n\frac{m(m-1)}{2} \,. \tag{3.5}$$

Similarly, for the dual problem the decision maker must form one matrix of size $n\times n$ for the weights of the decision criteria, plus m judgement matrices of size $n\times n$ (one for each of the m rows of the decision matrix), plus one matrix of size $m\times m$ (in order to normalize any one of the n columns of the decision matrix). Therefore, the total number of pairwise comparisons under the dual approach is:

$$\frac{n(n-1)}{2} + m\frac{n(n-1)}{2} + \frac{m(m-1)}{2} \,. \tag{3.6}$$

Then, the net decrease on the number of comparisons can be found as the difference of expression (3.6) from expression

(3.5), given as (3.7), below (after some elementary algebraic simplifications take place):

$$\frac{1}{2} m(n-1)(m-n-1). \tag{3.7}$$

Therefore, the percent (%) change of the number of comparisons between the prime and the dual problem is given as expression (3.4). ■

Corollary 1:

The dual problem requires less pairwise comparisons if the number of alternatives in the problem is greater than the number of decision criteria plus one.

Proof: This follows directly from the fact that expression (3.7) must be greater than zero. Thus, $m - n - 1 > 0$, or $m > n + 1$. This is true because the value of n is always greater than 1 (and also $m > 0$). ■

Therefore, if a problem has more alternatives than decision criteria (plus one), then the decision making process can explicitly benefit from the smaller number of comparisons needed by the proposed dual approach. It should be stated here that in many real-life problems there are more decision criteria than alternatives. However, in some real-life problems the number of alternatives may be dramatically high. For instance, in ranking a number of employees for possible pay raises, the number of alternatives (i.e., the individual employees) is often very large, when compared to the decision criteria (which describe their job performance). The rate of reduction on the number of required comparisons is given by expression (3.4) in Theorem 1. The next corollary states that these reduction rates converge to a fixed quantity when the number of criteria is kept constant and the number of alternatives approaches to infinity.

Corollary 2:

The percent (%) of change of the number of comparisons between the prime and the dual problem, for a given number of criteria N, approaches the value (N - 1)/N when the number of alternatives approaches infinity.

Proof: This follows directly from expression (3.4), in Theorem 1, if one sets $n = N$ and then takes the limit when m approaches to infinity. Also, the function in (3.4) is continuous and increases monotonically. Therefore, this limit can also serve as **an upper bound** on the reduction rate which can be achieved by using the proposed duality approach. ■

A related issue is to examine what happens if a problem is defined in a multi-level hierarchy. The previous considerations can easily be extended to this general case. The proposed duality approach can directly be applied on each individual level of the hierarchy. In particular, the duality approach will be beneficial if the number of sub-criteria in one level, is greater than the number of criteria in the previous level plus one.

4. A Numerical Example

The previous analyses are next demonstrated in terms of an illustrative example. Suppose that a single-level hierarchy multi-criteria decision making problem involves the five alternatives A_1, A_2, A_3, A_4, and A_5, which have to be evaluated in terms of the three decision criteria C_1, C_2, and C_3. These three criteria are assumed to have weights of importance equal to: $W = (5/8, 1/8, 2/8)$. Let the **actual values** of these alternatives in terms of the three decision criteria be as in the following decision matrix:

	C_1 5/8	C_2 1/8	C_3 2/8
A_1	5	3	4
A_2	2	5	3
A_3	4	2	5
A_4	2	4	1
A_5	3	3	2

In reality the decision maker does not know the above actual data. However, as it will be shown next, the decision maker can extract their relative values by using pairwise comparisons.

4.1. The Primal Approach

Under the traditional approach, the decision maker compares the alternatives with each other in terms of one decision criterion at a time. This is done by means of three judgment matrices of size 5×5 each. Next, we assume for simplicity in the calculations that the decision maker is always perfectly consistent. Then, the three judgment matrices with the pairwise comparisons for each one of the decision criteria are as follows:

For Criterion C_1

$$Matrix\ P^1 = \begin{bmatrix} 1 & 5/2 & 5/4 & 5/2 & 5/3 \\ 2/5 & 1 & 2/4 & 2/2 & 2/3 \\ 4/5 & 4/2 & 1 & 4/2 & 4/3 \\ 2/5 & 2/2 & 2/4 & 1 & 2/3 \\ 3/5 & 3/2 & 3/4 & 3/2 & 1 \end{bmatrix}$$

For Criterion C_2

$$Matrix\ P^2 = \begin{bmatrix} 1 & 3/5 & 3/2 & 3/4 & 3/3 \\ 5/3 & 1 & 5/2 & 5/4 & 5/3 \\ 2/3 & 2/5 & 1 & 2/4 & 2/3 \\ 4/3 & 4/5 & 4/2 & 1 & 4/3 \\ 3/3 & 3/5 & 3/2 & 3/4 & 1 \end{bmatrix}$$

For Criterion C_3

$$Matrix\ P^3 = \begin{bmatrix} 1 & 4/3 & 4/5 & 4/1 & 4/2 \\ 3/4 & 1 & 3/5 & 3/1 & 3/2 \\ 5/4 & 5/3 & 1 & 5/1 & 5/2 \\ 1/4 & 1/3 & 1/5 & 1 & 1/2 \\ 2/4 & 2/3 & 2/5 & 2/1 & 1 \end{bmatrix}$$

For instance, element (2,3) (= 5/2) in the second matrix denotes that when alternative A_2 is compared with alternative A_3 in terms of criterion C_2, then the decision maker feels that their relative importance is best represented by the ratio 5/2. A similar interpretation holds for the rest of the entries. The decision maker also forms a single judgment matrix of size 3×3 for the three decision criteria. This judgment matrix, denoted as ***C***, is as follows:

$$Matrix\ \mathrm{C} = \begin{bmatrix} 1 & 5/1 & 5/2 \\ 1/5 & 1 & 1/2 \\ 2/5 & 2/1 & 1 \end{bmatrix}$$

As it was mentioned in section 3, although alternatives are not explicitly mentioned in deriving the relative weights of the three decision criteria, the decision maker always keeps them as reference during the elicitation process. Furthermore, the primal and dual judgment matrices can be slightly inconsistent. This is not the case in this example for simplification reasons.

Next, the relative priorities are extracted from each one of the previous four judgment matrices. Since these matrices are perfectly consistent, it is easy to verify that the extracted priorities are the same regardless of which method (i.e., the eigenvector approach or a least squares approach) is used. These vectors are as follows:

$$\textit{From } \boldsymbol{P^1} \quad v_1 = (\, v_{i,1} \,) = \begin{bmatrix} 5/16 \\ 2/16 \\ 4/16 \\ 2/16 \\ 3/16 \end{bmatrix} \qquad \textit{From } \boldsymbol{P^2} \quad v_2 = (\, v_{2,i} \,) = \begin{bmatrix} 3/17 \\ 5/17 \\ 2/17 \\ 4/17 \\ 3/17 \end{bmatrix} \qquad \textit{From } \boldsymbol{P^3} \quad v_3 = (\, v_{3,i} \,) = \begin{bmatrix} 4/15 \\ 3/15 \\ 5/15 \\ 1/15 \\ 2/15 \end{bmatrix} \qquad \textit{From } \boldsymbol{C} \quad W = \begin{bmatrix} 5/8 \\ 1/8 \\ 2/8 \end{bmatrix}$$

These vectors form the normalized columns and the criteria weights of the decision matrix as follows:

	C_1	C_2	C_3
	5/8	1/8	2/8
A_1	5/16	3/17	4/15
A_2	2/16	5/17	3/15
A_3	4/16	2/17	5/15
A_4	2/16	4/17	1/15
A_5	3/16	3/17	2/15

From the above considerations it follows that for this illustrative example, and under the traditional (primal) approach, the decision maker needs to form 4 (= 3 + 1) judgment matrices with a total of **33** (= 3(3-1)/2 + 3[5(5-1)/2]) pairwise comparisons.

4.2. The Dual Approach

Under the dual approach the decision maker is first required to derive the decision matrix normalized in terms of the rows. This is accomplished by forming five judgment matrices which compare the criteria among themselves in terms of each one of the five alternatives. These matrices are as follows (again, perfect consistency is assumed in the elicitation of the pairwise comparisons):

$$\textit{Matrix } \mathrm{D}^1 = \begin{bmatrix} 1 & 5/3 & 5/4 \\ 3/5 & 1 & 3/4 \\ 4/5 & 4/3 & 1 \end{bmatrix} \qquad \textit{Matrix } \mathrm{D}^2 = \begin{bmatrix} 1 & 2/5 & 2/3 \\ 5/2 & 1 & 5/3 \\ 3/2 & 3/5 & 1 \end{bmatrix}$$

$$\textit{Matrix } \mathrm{P}^3 = \begin{bmatrix} 1 & 4/2 & 4/5 \\ 2/4 & 1 & 2/5 \\ 5/4 & 5/2 & 1 \end{bmatrix} \qquad \textit{Matrix } \mathrm{D}^4 = \begin{bmatrix} 1 & 2/4 & 2/1 \\ 4/2 & 1 & 4/1 \\ 1/2 & 1/4 & 1 \end{bmatrix}$$

$$\textit{Matrix } \mathrm{D}^5 = \begin{bmatrix} 1 & 3/3 & 3/2 \\ 3/3 & 1 & 3/2 \\ 2/3 & 2/3 & 1 \end{bmatrix}$$

For the case of the three criteria, the decision maker forms a judgment matrix which is identical to matrix ***C*** formed during the primal approach. Working as in the previous sub-section, the priority vectors are derived from each one of the previous matrices. These vectors are as follows:

From D^1

$$h^1 = (\ h_{1,i}\ ) = \begin{bmatrix} 5/12 \\ 3/12 \\ 4/12 \end{bmatrix}$$

From D^2

$$h^2 = (\ h_{2,i}\ ) = \begin{bmatrix} 2/10 \\ 5/10 \\ 3/10 \end{bmatrix}$$

From D^3

$$h^3 = (\ h_{3,i}\ ) = \begin{bmatrix} 4/11 \\ 2/11 \\ 5/11 \end{bmatrix}$$

From D^4

$$h^4 = (\ h_{4,i}\ ) = \begin{bmatrix} 4/11 \\ 2/11 \\ 5/11 \end{bmatrix}$$

From D^5

$$h^5 = (\ h_{5,i}\ ) = \begin{bmatrix} 3/8 \\ 3/8 \\ 2/8 \end{bmatrix}$$

From C

$$W = \begin{bmatrix} 5/8 \\ 1/8 \\ 2/8 \end{bmatrix}$$

These vectors form the normalized rows and the criteria weights of the decision matrix. Thus, the normalized decision matrix is as follows:

	C_1	C_2	C_3
	5/8	1/8	2/8
A_1	5/12	3/12	4/12
A_2	2/10	5/10	3/10
A_3	4/11	2/11	5/11
A_4	2/7	4/7	1/7
A_5	3/8	3/8	2/8

The previous decision matrix can be normalized in terms of the columns, by using one more judgment matrix of size 5×5. This extra matrix represents the pairwise comparisons derived when the five alternatives are compared in terms of any one of the three decision criteria. For easy demonstration, suppose that the decision maker chooses to compare the five alternatives in terms of the first decision criterion (the case of using the second or the third decision criterion can be developed in an identical manner). The corresponding judgment matrix was provided in the previous sub-section as matrix $\boldsymbol{P}^{\mathbf{1}}$. Therefore, the normalized column is: $(5/16, 2/16, 4/16, 2/16, 3/16)^T$.

The next step is to use the previous five normalized rows and the normalized column to derive the normalized columns of the decision matrix. It can be observed that only the last two columns of the decision matrix need to be calculated in this example. For instance, when formula (3.3) is applied to calculate the value of $v_{1,2}$, it turns out that its value is equal to 3/17. This is true because from relation (3.3) the value of $v_{1,2}$ should be equal to:

$$v_{12} \ = \ \frac{\left(\frac{v_{11}}{h_{11}}\right)h_{12}}{\sum_{y=1}^{5}\left(\left(v_y\frac{1}{h_y}1\right)h_y 2\right)} \ =$$

$$= \frac{\frac{5/16}{5/12} x\, 3/12}{\left(\frac{5/16}{5/12} x\, 3/12\right)+\left(\frac{2/16}{2/10} x\, 5/10\right)+\left(\frac{4/16}{4/11} x\, 2/11\right)+\left(\frac{2/16}{2/7} x\, 4/7\right)+\left(\frac{3/16}{3/8} x\, 3/8\right)} =$$

$$= \frac{3/16}{3/16 + 5/16 + 2/16 + 4/16 + 3/16} = 3/17\,.$$

In a similar manner, it can be shown that the second column (after normalization) is equal to:

$(3/17, 5/17, 2/17, 4/17, 3/17)^T$.

Similarly, the third column (after normalization) is equal to:

$(4/15, 3/15, 5/15, 1/15, 2/15)^T$.

Therefore, the normalized decision matrix is as follows:

	C_1 5/8	C_2 1/8	C_3 2/8
A_1	5/16	3/17	4/15
A_2	2/16	5/17	3/15
A_3	4/16	2/17	5/15
A_4	2/16	4/17	1/15
A_5	3/16	3/17	2/15

The above matrix is identical to the matrix derived with the primal approach. However, in the dual approach the decision maker needed to form 5 (= 1 + 3 + 1) judgment matrices with a total of **28** (= 3(3-1)/2 + 5[3(3-1)/2] + 5(5-1)/2) pairwise comparisons. This represents a reduction of 15.14% from the total number of pairwise comparisons required under the primal approach. Although this reduction may not seem to be very significant, when the number of alternatives is much higher than the number of decision criteria, the benefits of using the dual approach increase dramatically. This is further investigated in the computational results presented in the next section.

As a final note, it can be observed that the dual matrices are of smaller rank that the prime ones (3 versus 5). As a result, they are easier to be consistent the first place. However, the lack of the redundancy that occurs in larger judgement matrices, makes the impact of an individual error to be larger than otherwise. The opposite is true with the larger prime matrices. The main issue is that with the dual matrices fewer comparisons are required, thus the problems associated with the elicitation of numerous comparisons are easier to be controlled.

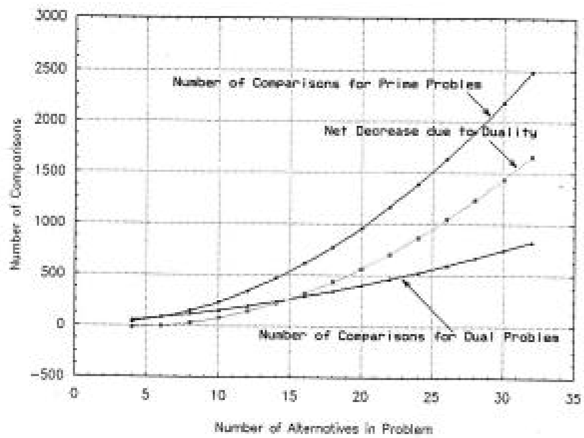

Figure 1. Total number of comparisons and reduction achieved when the dual approach is used. The number of criteria $n = 5$.

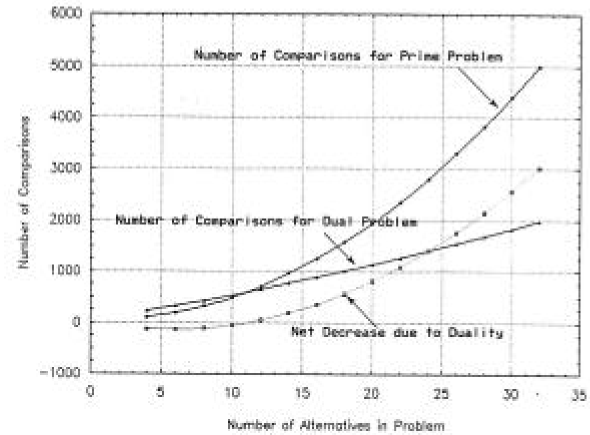

Figure 2. Total number of comparisons and reduction achieved when the dual approach is used. The number of criteria $n = 10$.

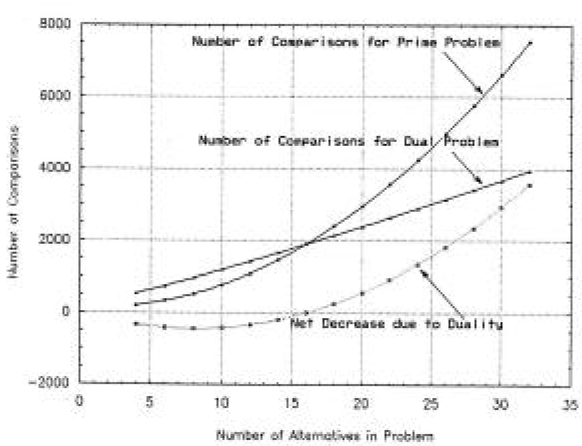

Figure 3. Total number of comparisons and reduction achieved when the dual approach is used. The number of criteria $n = 15$.

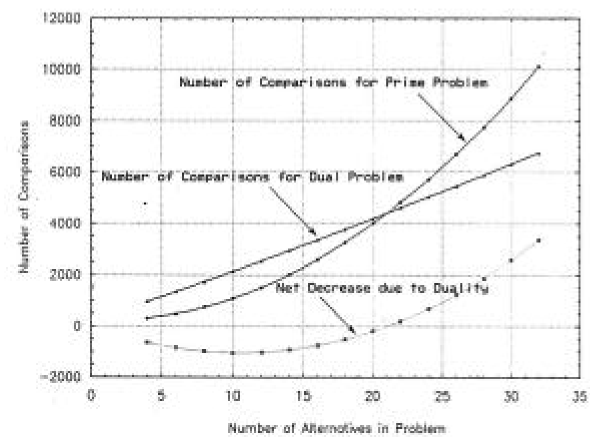

Figure 4. Total number of comparisons and reduction achieved when the dual approach is used. The number of criteria $n = 20$.

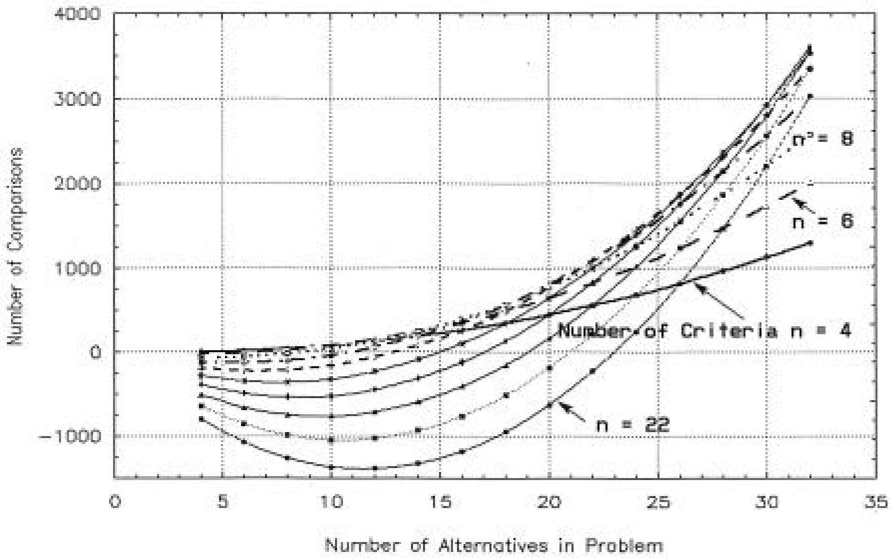

Figure 5. Net reduction on the number of comparisons when the dual approach is used. Results for problems of various sizes.

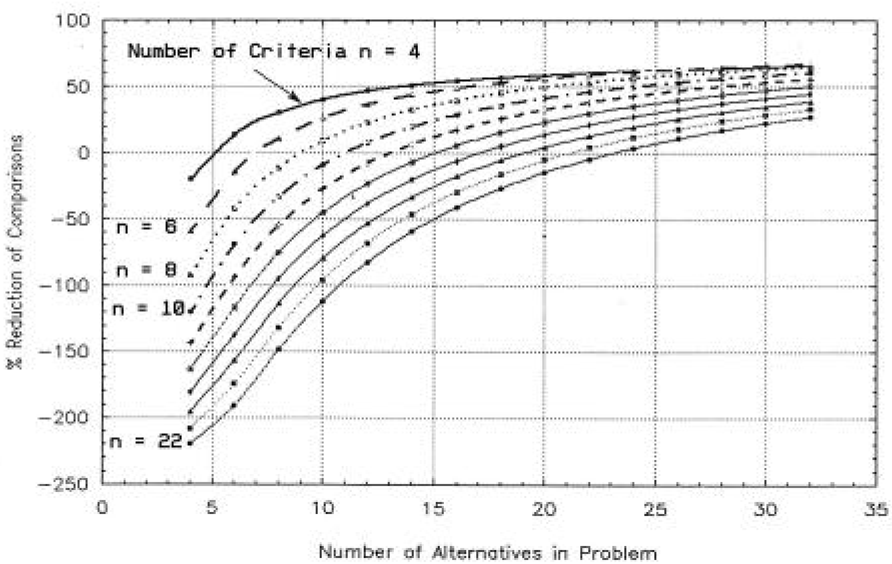

Figure 6. Percent (%) reduction on the number of comparisons when the dual approach is used. Results for problems of various sizes.

5. Numerical Results for Problems of Different Sizes

Consider expressions (3.5), (3.6), and (3.7) which provide the total numbers of comparisons required under the primal and dual approaches, as well as their net difference. Figures 1, 2, 3, and 4, depict these values when the number of decision criteria n is equal to 10, 15, 20, and 25, respectively. As it is also shown in these expressions, the values of these functions increase quadratically with the value of m (number of alternatives).

Moreover, when the condition of corollary 1 is satisfied (i.e., when $m > n + 1$), then the net decrease due to duality is positive. It is also interesting to observe that in all these four representative plots, the number of comparisons for the dual problem seems to increase almost linearly with the number of alternatives. In reality the rate of increase is still quadratic with the number of alternatives m, but with the number of criteria chosen in these figures this increase appears to be almost linear. This, of course, is a nice characteristic of the number of comparisons required by using the dual approach. The previous observation holds true for the ranges of the m and n parameters examined in this study. Clearly, if the number of alternatives increases more than 35, then the number of comparisons under the dual approach will assume a more quadratic rate of increase. However, one may argue that many real-life problems involve less than 35 alternatives (which is the upper limit in the previous plots).

On the other hand, the number of comparisons required under the traditional (prime) approach noticeably increases quadratically in these figures. The previous observations are a compelling reason why the duality approach can be significantly beneficial to most real-life decision problems which use pairwise comparisons to elicit qualitative data. As it was mentioned in section 3, these developments are identical when dealing with perfectly consistent and slightly inconsistent (i.e., with *CR* value less or equal than 10%) judgment matrices.

Figure 5 depicts the net reduction on the number of comparisons (given as expression (3.7)) achieved when the dual approach is applied, for different size problems. Figure 6 presents the percent (%) reductions achieved when the dual approach is used on problems of various sizes. As before, these increases follow quadratic patterns on the size of the decision problems. It is noticeable that in Figure 6 the reduction rates seem to converge to a constant value when the number of alternatives increases. Obviously, this is in direct agreement with corollary 2.

5. Concluding Remarks

The previous analyses demonstrate that the number of comparisons required to solve a decision problem which has m alternatives and n criteria can significantly be reduced when the number of alternatives is greater than the number of criteria plus one. This is achieved in terms of a duality approach. In this approach the decision maker compares how well the criteria perform within a single alternative at a time. It should be noticed that in the traditional way of implementing pairwise comparisons, the decision maker is asked to compare the alternatives in terms of a single criterion at a time or to compare a set of criteria (when their relative weights of importance are extracted).

The achieved reductions on the required total number of pairwise comparisons are given through some analytical formulas. The main findings of this study are also depicted in a number of figures. It is remarkable to emphasize here that these reductions become more dramatic when the size of the problem increases. Thus, the proposed duality approach becomes more practical for large size decision problems.

Although the concept of duality is an old one in decision sciences (e.g., in linear programming), the proposed duality approach is a novel development in multi-criteria decision making. The proposed method can be applied to the AHP, or its variants, as well as to any other method which uses pairwise comparisons to elicit qualitative or fuzzy data from the decision maker(s). More research in this intriguing area may reveal more benefits of using information extracted from the dual formulation of a given multi-criteria decision problem.

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References

Belton, V., and Gear, T., "On a short-coming of Saaty's method of analytic hierarchies," *Omega*, (1983), 228-230.

Bryson, N., "A Goal Programming Method for Generating Priority Vectors," *J. of the Operational Research Society*, 46/5, (1995), 641-648.

Dyer, J.S., and Ravinder, H.V., "Irrelevant alternatives and the Analytic Hierarchy Process," *Technical Report*, Department of Management, The University of Texas at Austin, Austin, TX, (1983).

Dyer, J.S. and Wendell, R.E., "A critique of the Analytic Hierarchy Process," *Technical Report* 84/85-4-24, Department of Management, The University of Texas at Austin, Austin, TX, (1985).

Dyer, J.S., "Remarks on the Analytic Hierarchy Process," *Management Science*, **36/3** (1990a), 249-258.

Dyer, J.S., "A clarification of 'Remarks on the Analytic Hierarchy Process'," *Management Science*, **36/3** (1990b), 274-275.

Harker, P. T., "Derivatives of the Perron Root of a Positive Reciprocal Matrix: With Application to the Analytic Hierarchy Process," *Applied Mathematics and Computation*, **22**, (1987a), 217-232.

Harker, P. T., "Incomplete Pairwise Comparisons in the Analytic Hierarchy Process," *Mathematical Modeling*, Vol. **9/11**, (1987b), 837-848.

Harker, P.T., and Vargas, L.G., "Reply to 'Remarks on the Analytic Hierarchy Process'," *Management Science*, **36/3** (1990), 269-273.

Expert Choice Software. *Expert Choice Inc.* 4922 Ellsworth Ave., Pittsburgh, PA, (1990).

Fichtner, J., "On Deriving Priority Vectors from Matrices of pairwise Comparisons," *Socio-Economic Planning Sciences*, 20/6, (1986), 341-345.

Golden, B.L., and Q. Wang, "An Alternate Measure of Consistency," in: B.L. Golden, E.A. Wassil, and P.T. Harker (eds.), *The Analytic Hierarchy Process*, Springer-Verlag, New York, NY, (1989), 68-81.

Lootsma, F.A., "The French and the American school in multi-criteria decision analysis," *Recherche Operationnele / Operations Research*, **24/3**, (1990), 263-285.

Saaty, T.L., *The Analytic Hierarchy Process*, McGraw-Hill International, New York, NY, (1980).

Saaty, T.L., *Fundamentals of Decision Making and Priority Theory with the AHP*, RWS Publications, Pittsburgh, PA, (1994).

Triantaphyllou, E. and Mann, S.H., "An examination of the effectiveness of four multi-dimensional decision-making methods: A decision-making paradox," *International Journal of Decision Support Systems*, 5, (1989), 303-312.

Triantaphyllou, E., Pardalos, P. M., Mann, S. H., "A Minimization approach to membership evaluation in fuzzy sets and error analysis," *Journal of Optimization Theory and Applications*, **66/2**, (1990), 275-287.

Triantaphyllou, E., and Mann, S.H., "Some critical issues in making decisions with pairwise comparisons," *Proceedings of the Third International Symposium on the AHP*, George Washington University, Washington, DC, (1994), 225-235.

Triantaphyllou, E., Lootsma, F.A., Pardalos, P.M., and Mann, S.M., "On the evaluation and application of different scales for quantifying pairwise comparisons in fuzzy sets," *Multi-Criteria Decision Analysis*, Vol. **3,** (1994), 1-23.

Winkler, R.L., "Decision modeling and rational choice: AHP and utility theory," *Management Science*, **36/3** (1990), 247-248.