Letters in Mathematical Physics **11** (1986) 161–169.

Reduction of Poisson Manifolds

JERROLD E. MARSDEN★
Department of Mathematics, University of California, Berkeley, CA 94720, U.S.A.

and

TUDOR RATIU★★
Department of Mathematics, University of Arizona, Tucson, AZ 85721, U.S.A.

(Received: 15 October 1985)

Abstract. Reduction in the category of Poisson manifolds is defined and some basic properties are derived. The context is chosen to include the usual theorems on reduction of symplectic manifolds, as well as results such as the Dirac bracket and the reduction to the Lie–Poisson bracket.

1. Introduction

Reduction in the symplectic context is a natural outgrowth of the classical theorems of Jacobi and Liouville on the elimination of phase space variables for Hamiltonian systems possessing conserved quantities. Generalizations of these results to a geometric context culminate from work of Smale [34], Kostant [17], Souriau [36], Nehoroshev [32], Meyer [28], and Marsden and Weinstein [24]. Further contributions are due to Marle [21], Kazhdan *et al.* [15], Kummer [19], and others. Expositions of this theory can be found in Abraham and Marsden [1], Arnold [4], Woodhouse [41], Marsden [22], and Guillemin and Sternberg [12].

Reduction has many applications, which we cannot review here, as they are too extensive. We just mention the papers of Cushman and Rod [7], Guillemin and Sternberg [11], Iwai [13, 14], Marsden and Weinstein [25, 26], Marsden *et al.* [27], Deprit [8], Arms *et al.* [2], Marsden *et al.* [23] as representative of just some of the interesting applications. Some of the basic ideas relevant to the general theory of reduction in the Poisson context are already given in Marsden and Weinstein [25, 26], Marsden *et al.* [27], and Montgomery *et al.* [31] (see also Tulczyjew [38]).

One of the main reduction theorems in the symplectic context states that if (P, Ω) is a symplectic manifold, $J\colon P \to \mathfrak{g}^*$ is an Ad^*-equivariant momentum map for a canonical G-action, $\mu \in \mathfrak{g}^*$ is a regular value of J (or a 'clean' value of J)[‡] and if the isotropy group G_μ of μ acts freely and properly on $J^{-1}(\mu)$, then there is a unique symplectic structure Ω_μ on

$$P_\mu = J^{-1}(\mu)/G_\mu$$

★ Research supported by DOE contract DE-AT03-85ER 12097.
★★ Supported by an A. P. Sloan Foundation fellowship.
‡ 'Clean values' are what Marsden and Weinstein [24] call 'weakly regular values'.

such that

$$i_\mu^*\Omega = \pi_\mu^*\Omega_\mu ,$$

where $i_\mu: J^{-1}(\mu) \to P$ is the inclusion and $\pi_\mu: J^{-1}(\mu) \to P_\mu$ is the projection. In this context, the Poisson brackets on P_μ are given as follows. Let f and h be smooth real valued functions on P_μ and let F and H be G-invariant extensions of $f \circ \pi_\mu$ and $h \circ \pi_\mu$. Then

$$\{F, H\}_P \circ i_\mu = \{f, h\}_{P_\mu} \circ \pi_\mu .$$

For this to be valid, F and H must be extensions of $f \circ \pi_\mu$ and $h \circ \pi_\mu$ off $J^{-1}(\mu)$ which are G-invariant; the brackets are not related this way if arbitrary extensions (or G_μ invariant extensions) are used. The purpose of this Letter is to provide a Poisson context general enough to include this example as well as others, such as Dirac brackets (see, e.g., Sniatycki [35] and Flato *et al.* [10]) and the Lie–Poisson bracket (see, e.g., Marsden *et al.* [27]).

2. Poisson Reduction

Let $(P, \{\,,\}_P)$ be a Poisson manifold★, $M \subset P$ a submanifold and $i: M \to P$ the inclusion. Let $E \subset TP|M$ be a subbundle of the tangent bundle of P restricted to M. We make some assumptions at the outset:

- (A1) $E \cap TM$ is an integrable subbundle of TM, so defines a foliation Φ on M.
- (A2) The foliation Φ is *regular*, so the space of leaves M/Φ is a manifold with projection $\pi: M \to M/\Phi$ a submersion.
- (A3) The bundle E *leaves* $\{\,,\}_P$ *invariant* in the sense that if K, L are smooth functions on P with differentials vanishing on E then $\mathrm{d}\{K, L\}_P$ also vanishes on E.

DEFINITION. We say (P, M, E) is *Poisson reducible* if M/Φ has a Poisson structure $\{\,,\}_{M/\Phi}$ such that for any (locally defined) smooth functions f, h on M/Φ, and (locally defined) smooth extensions F, H of $f \circ \pi, h \circ \pi$ with differentials vanishing on E, we have

$$\{F, H\}_P \circ i = \{f, h\}_{M/\Phi} \circ \pi . \tag{2.1}$$

Under suitable technical hypotheses (as in Chernoff and Marsden [6]), the results of this Letter hold in infinite dimensions, but we work in the finite-dimensional case for simplicity.

POISSON REDUCTION THEOREM. *Let assumptions* (A1)–(A3) *hold and regard the Poisson structure on P as a map $B: T^*P \to TP$. The triple (P, M, E) is Poisson reducible if and only if*

$$B(E^0) \subset TM + E , \tag{2.2}$$

★ A Poisson manifold is a manifold whose ring of C^∞ functions is a Lie algebra whose bracket is also a derivation with respect to the usual ring structure. These key properties are explicitly isolated in Dirac [9], p. 10, but are implicit in the works of Lie. The term 'Poisson manifold' was coined by Lichnerowicz [20] and Bayen *et al.* [5].

where $E_x^0 = \{\alpha_x \in T_x^*P \mid \alpha_x(E_x) = 0\}$ *is the annihilator of* E.

Proof. First assume (P, M, E) is Poisson reducible. Let $\beta_x \in E_x^0$, where $x \in M$ and let F be C^∞ in a neighborhood of x such that $\mathrm{d}F$ vanishes on E and equals β_x at x. Let $\alpha_x \in E_x^0 \cap (T_xM)^0 = (E_x + T_xM)^0$ and choose an extension K of the zero function on M such that $\mathrm{d}K(x) = \alpha_x$ and $\mathrm{d}K$ vanishes on E. Thus,

$$\langle \alpha_x, B_x(\beta_x)\rangle = \{K, F\}_P(x) = \{0, f\}_{M/\Phi}(\pi(x)) = 0\,,$$

where f is the function on M/Φ induced from $F \mid M$. Thus, $B_x(\beta_x) \in E_x + T_xM$.

Conversely, assume $B(E^0) \subset TM + E$. Let $f, h \in C^\infty(M/\Phi)$, the C^∞ functions on M/Φ and let F, H be extensions of $\pi^*f = f \circ \pi$ and $\pi^*h = h \circ \pi$ whose differentials vanish on E. By condition (A3), $\{F, H\}_P$ is constant on the leaves of Φ and so induces a function on M/Φ. We now show that this induced function is independent of the extensions, thereby defining a function $\{f, h\}_{M/\Phi}$ satisfying (2.1). Let H' be another extension of π^*h satisfying $\mathrm{d}H' \mid E = 0$. Thus, $H - H'$ vanishes on M, so its differential vanishes on $E + TM$. Thus, by (2.2),

$$\langle \mathrm{d}(H - H')(x), B_x(\mathrm{d}F(x))\rangle = 0$$

and so

$$\{F, H\}_P(x) = \{F, H'\}_P(x)\,.$$

Thus, $\{F, H\}_P(x)$ is independent of how $h \circ \pi$ is extended, as long as the differential of the extension vanishes along E. By antisymmetry of the bracket, it is also independent of the $f \circ \pi$ extension. Thus, $\{f, h\}_{M/\Phi}$ is well defined and is uniquely determined by (2.1). It remains to show that M/Φ is a Poisson manifold with this bracket. Antisymmetry, bilinearity and the derivation properties of the bracket are directly inherited from these properties on P and uniqueness. For Jacobi's identity, notice that if F and H are extensions of $f \circ \pi$ and $h \circ \pi$ whose differentials vanish along E, then so does $\mathrm{d}\{F, H\}_P$ by (A3), so $\{F, H\}_P$ is an extension of $\{f, h\}_{M/\Phi} \circ \pi$ that we can use in (2.1) to give the identity

$$\pi^*\{\{f, h\}_{M/\Phi}, k\}_{M/\Phi} = i^*\{\{F, H\}_P, K\}_P$$

and so Jacobi's identity on M/Φ is inherited from that on P. □

The functorality property of Poisson reduction is given by the following.

COROLLARY. *Let* (P_i, M_i, E_i) *be Poisson reducible* $i = 1, 2$, *and assume that* $\phi\colon P_1 \to P_2$ *is a Poisson map such that* $\phi(M_1) \subset M_2$, $T\phi(E_1) \subset E_2$, *and* ϕ *maps the leaves of* Φ_1 *into leaves of* Φ_2. *Then* ϕ *induces a unique Poisson map* $\hat{\phi}\colon M_1/\Phi_1 \to M_2/\Phi_2$ *satisfying* $\pi_2 \circ \phi = \hat{\phi} \circ \pi_1$ *called the reduction of* ϕ.

Proof. Since ϕ maps leaves into leaves, the map $\hat{\phi}$ exists, is smooth, and is unique with the given property. To show that it is Poisson, let $f, h\colon M_2/\Phi_2 \to \mathbb{R}$ be smooth (local) functions and let $F, H\colon P_2 \to \mathbb{R}$ be smooth (local) extensions of $f \circ \pi_2$ and $h \circ \pi_2$ whose differentials vanish on E_2. Then for any $v \in E_1$ we have

$$\mathrm{d}(F \circ \phi) \circ v = \mathrm{d}F \cdot T\phi(v) = 0\,,$$

since $T\phi(E_1) \subset E_2$. Therefore, $F \circ \phi$ and $H \circ \phi$ are smooth (local) extensions of $f \circ \hat{\phi}$ and $h \circ \hat{\phi}$ whose differentials vanish on E_1. Therefore,

$$\begin{aligned} \pi_1^* \hat{\phi}^* \{f, h\}_{M_2/\Phi_2} &= \{f, h\}_{M_2/\Phi_2} \circ \pi_2 \circ \phi \\ &= \{F, H\}_{P_2} \circ \phi \circ i_1 \\ &= \{F \circ \phi, H \circ \phi\}_{P_1} \circ i_1 \\ &= \{f \circ \hat{\phi}, h \circ \hat{\phi}\}_{M_1/\Phi_1} \circ \pi_1 \\ &= \pi_1^* \{\hat{\phi}^* f, \hat{\phi}^* h\}_{P_1}, \end{aligned}$$

which, since π_1 is onto, shows that $\hat{\phi}$ is a Poisson map. □

Next we study the dynamic counterpart of the Poisson reduction theorem. If $F: P \to \mathbb{R}$ is a smooth Hamiltonian, we say that the submanifold $M \subset P$ is *conserved* for F, if $X_F(x) \in T_x M$ for all $x \in M$. For example, if $B(E^0) \subset TM$, then M is conserved for all functions F whose differentials annihilate E. As we shall see in the next section, in many examples $B(E^0) \subset TM$ holds.

DYNAMIC POISSON REDUCTION THEOREM. *Let (P, M, E) be Poisson reducible and $H: P \to \mathbb{R}$ be a smooth function for which M is conserved. Then the flow ϕ_t of X_H induces Poisson diffeomorphisms $\hat{\phi}_t$ on M/Φ. The vector field on M/Φ whose flow is $\hat{\phi}_t$ equals X_h, where $h: M/\Phi \to \mathbb{R}$ is uniquely determined by H via $h \circ \pi = H \mid M$. In addition, the vector fields $H_H \mid M$ and X_h are π-related.*

Proof. Since M is conserved for H, the flow ϕ_t of X_H leaves M invariant, so by the previous corollary $\hat{\phi}_t$ is a flow of Poisson diffeomorphisms on M/Φ. Therefore, if Y is the vector field on M/Φ whose flow is $\hat{\phi}_t$, X_H, and Y are π-related.

Now let $f: M/\Phi \to \mathbb{R}$ be an arbitrary smooth map and let $F: P \to \mathbb{R}$ be an extension of $f \circ \pi$ to P such that $\mathrm{d}F \in E^0$. Then for any $x \in M$ we have

$$\begin{aligned} \mathrm{d}f(\pi(x)) \cdot X_h(\pi(x)) &= \{f, h\}_{M/\Phi}(\pi(x)) \\ &= \{F, H\}_P(x) = \mathrm{d}F(x) \cdot X_H(x) \\ &= \mathrm{d}f(\pi(x)) \cdot T\pi(x) \cdot X_H(x) \end{aligned}$$

and so $X_h(\pi(x)) = T\pi(x) \cdot X_H(x)$, i.e., $X_H \mid M$ and X_h are also π-related, which in turn implies $X_h = Y$. □

Thus, if $B(E^0) \subset TM$, *the dynamics of X_H on M projects to that of X_h on M/Φ*. If the leaves of Φ are orbits of a Poisson group action, then the dynamics of X_H on M can be reconstructed from the reduced dynamics of X_h and the group action, as in the symplectic case (Abraham and Marsden [1], p. 305).

3. Examples and Remarks

A. *Let $P = T^*G$ where G is a Lie group, $M = P$ and E be the tangent space to the left G-orbits. Then (P, M, E) is Poisson reducible and $M/\Phi = \mathfrak{g}^*$ with the $(-)$ Lie–Poisson* *structure*:

$$\{F, G\}(\mu) = -\left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right]\right\rangle$$

for $\mu \in \mathfrak{g}^*$ *and* $\delta F/\delta\mu \in \mathfrak{g}$ *the functional derivative*. (For the right G-orbits we get the (+) Lie–Poisson structure.) See, for example, Marsden *et al.* [27] for the proof. This is the Poisson reduction (implicitly) used by Arnold [3] in passing from material to spatial coordinates in fluid dynamics, and by Marsden and Weinstein [25] for the Vlasov equation. More generally, if G acts on P canonically, then P/G is the Poisson reduction of $M = P$ with E the tangent bundle to the G-orbits.

B. *Let* P *be a Poisson manifold and* $J: P \to \mathfrak{g}^*$ *an* Ad**-equivariant momentum map for the canonical action of a Lie group* G. *Let* μ *be a regular (or clean) value of* J *and* $M = J^{-1}(\mu)$. *Let* E *be the tangent spaces to the* G*-orbits. Then* (P, M, E) *is Poisson reducible*. Indeed, to check (A1) we observe that $E \cap TM$ consists of the tangent spaces to the G_μ-orbits by equivariance of J. Thus, (A1) is satisfied and (A2) holds if G_μ acts freely and properly, so $M/\Phi = J^{-1}(\mu)/G_\mu$ is a manifold. (A3) holds since the G-action is canonical and $B(E^0) \subset TM$ (so (2.2) holds) since J is conserved for G-invariant functions.

Thus, $J^{-1}(\mu)/G_\mu$ *inherits a unique Poisson structure satisfying* (2.1). If P is symplectic, the Poisson structure on $J^{-1}(\mu)/G_\mu$ is that of the reduced symplectic structure Ω_μ. This follows from the fact that from either point of view, G-invariant functions produce Hamiltonian vector fields π-related to their reductions (proved above in the Poisson case and in Marsden and Weinstein [24] in the symplectic case).

C. *In Example B, if we let* $M = J^{-1}(\mathcal{O})$, *where* $\mathcal{O}$ *is a coadjoint orbit and* E *be the tangent bundle to the* G*-orbits, then* (P, M, E) *is also Poisson reducible*. Here, $M/\Phi = J^{-1}(\mathcal{O})/G$. For a description of the symplectic structure in the symplectic case, see Marle [21], Kazhdan *et al.* [15], and Marsden [22]. Both structures coincide in this case and $J^{-1}(\mathcal{O})/G$ is canonically diffeomorphic to $J^{-1}(\mu)/G_\mu$ for $\mu \in \mathcal{O}$. (This requires some proof, but is routine to supply.)

D. *Let* P *be a Poisson manifold,* $M \subset P$ *a submanifold and let* $E = B((TM)^0)$. *Then* (P, M, E) *is Poisson reducible (assuming* $E \cap TM$ *is a subbundle).*

First of all, the *characteristic distribution* $E \cap TM$ is integrable. Indeed, by Frobenius' theorem and the fact that Hamiltonian vector fields X_F with $\mathrm{d}F \mid TM = 0$ span $B((TM)^0)$ pointwise, it is enough to show that $[X_F, X_H] \in E \cap TM$ for two such X_F, X_H. Clearly $[X_F, X_H] \in TM$. To show it lies in E just notice that by Jacobi's identity,

$$[X_F, X_H] = -X_{\{F,H\}_P} \tag{3.1}$$

so if we can show $\mathrm{d}\{F, H\}_P$ vanishes on TM, we will be done with (A1). However, $\{F, H\}_P(x) = \mathrm{d}F(x) \cdot X_H(x)$ vanishes at points of M since $\mathrm{d}F$ annihilates TM and X_H is tangent to M. We assume (A2) holds and note that (A3) follows from (3.1). Finally, $B(E^0) \subset TM$ follows from the general fact that for a subspace $F_x \subset T_xP$,

$$B_x((B_x(F_x^0))^0) \subset F_x,$$

as is readily verified. (In the symplectic case, $B_x(F_x^0)$ is the symplectic orthogonal complement to F_x.)

If P is symplectic and $M \subset P$ is a symplectic submanifold, then $E = B((TM))^0$ is the symplectic orthogonal complement of TM and thus $E \cap TM = \{0\}$, so the Poisson structure induced on $M/\Phi = M$ is the same as the one determined by the symplectic structure on M.

If P is symplectic and $M \subset P$ is Lagrangian, then $E \cap TM = TM$, so the reduced Poisson manifold M/Φ is trivial.

E. *Let (P, Ω) be symplectic and $i: M \hookrightarrow P$ a submanifold. Let E be the characteristic distribution of $i^*\Omega$. Then (P, M, E) is Poisson reducible if and only if M is coisotropic.* (The 'if' part is given in Guillemin and Sternberg [12], p. 177.) It is well-known that M/Φ is also symplectic in this case (Abraham and Marsden [1], p. 298). The Poisson structure on M/Φ is that of the symplectic structure when M is coisotropic. For P a Poisson manifold and $E = B((TM)^0) \cap TM$, (P, M, E) is also Poisson reducible if and only if M is coisotropic (in the Poisson sense). All these statements are routinely verified.

F. *Reduction in Symplectic Leaves.* Let (P, M, E) be Poisson reducible and let $S \subset P$ be a symplectic leaf (see Kirillov [16]), with $S \cap M \neq \phi$. Assume that the leaves of Φ intersect S cleanly (Guillemin and Sternberg [12], p. 180). Then $S \cap \Phi$ defines a foliation on S and the symplectic leaves in M/Φ are given by the connected components of $S \cap M/S \cap \Phi$. For example, if P is a symplectic manifold and the Lie group G acts freely and properly on P with Ad*-equivariant momentum map J, the symplectic leaves of the Poisson manifold P/G are the reduced manifolds $J^{-1}(\mathcal{O})/G$, for $\mathcal{O}$ a coadjoint orbit. See Marsden *et al.* [23] for further examples involving semidirect products.

G. Let P be a Poisson manifold and $M = J^{-1}(\mathcal{O})$, as in Example C. Now let $E = B((TM)^0)$, so $E \cap TM$ is the characteristic distribution of M. Thus, M/Φ is Poisson, by Example D. As in the symplectic case (Marle [21], Kazhdan *et al.* [15]), one can show M/Φ is Poisson diffeomorphic to $[J^{-1}(\mathcal{O})/G] \times \mathcal{O}$.

H. An example arising in coupled systems is as follows (Krishnaprasad and Marsden [11]). Let P be a Poisson manifold and G a group acting canonically on P (on the left). Then G also acts canonically on the Poisson manifold $(T^*G \times P)/G$. The map $\phi: T^*G \times P \to \mathfrak{g}^* \times P$; $\phi(\alpha_g, x) = ((TL_g)^*\alpha_g, g^{-1} \cdot x)$ identifies $(T^*G \times P)/G$ with $\mathfrak{g}^* \times P$. Thus $\mathfrak{g}^* \times P$ is a Poisson manifold. One computes the inherited bracket to be

$$\{F, H\}(\mu, x) = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right]\right\rangle + \{F, H\}_P - \mathrm{d}_x F \cdot \left(\frac{\delta H}{\delta \mu}\right)_P + \mathrm{d}_x H \cdot \left(\frac{\delta F}{\delta \mu}\right)_P, \tag{3.2}$$

where the first term is the $(-)$ Lie–Poisson bracket, $\delta F/\delta\mu \in \mathfrak{g}$ is the functional derivative, and for $\xi \in \mathfrak{g}$, ξ_P is its infinitesimal generator on P. If $P = \mathfrak{h}^*$, the dual of another Lie algebra $\mathfrak{h}$, (3.2) is the Lie–Poisson bracket of the dual of the semidirect product of $\mathfrak{g}$ with $\mathfrak{h}$. If the action of G on P has an Ad*-equivariant momentum map

$J: P \to \mathfrak{g}^*$, the map $\alpha: \mathfrak{g}^* \times P \to \mathfrak{g}^* \times P$ defined by

$$\alpha(\mu, x) = (\mu + J(x), x)$$

transforms the bracket (3.2) into the decoupled bracket

$$\{F, H\} = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right]\right\rangle + \{F, H\}_P. \tag{3.3}$$

Of course, this example is closely related to the Hamiltonian structures used for the description of a particle in a Yang–Mills field. See Sternberg [37], Weinstein [40], and Montgomery [30]. For an account of this from the Poisson reduction point of view, see Montgomery *et al.* [31].

I. The previous example gives an easy proof of the *Adler–Kostant–Symes Theorem*: *If G is a Lie group, H, K are subgroups such that $G = HK$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ as a vector space direct sum, then Casimir functions of $\mathfrak{g}^*$ are in involution on the product Poisson manifold $\mathfrak{h}^*_- \times \mathfrak{k}^*_+$.* Indeed, K acts on the right on G and, hence, by lift on T^*G, so that by (3.3) the reduced Poisson manifold $(T^*G)/K \simeq (T^*H \times T^*K)/K \simeq T^*H \times \mathfrak{k}^*_+$ has the sum Poisson bracket. Now H acts on this manifold trivially on $\mathfrak{k}^*_+$ and by lift of left translation, on T^*H. The reduced Poisson manifold $H \backslash (T^*H \times \mathfrak{k}^*_+)$ equals, therefore, the product $\mathfrak{h}^*_- \times \mathfrak{k}^*_+$. If F is a Casimir on $\mathfrak{g}^*$, then it is Ad*-invariant and, therefore, its extension to T^*G by left or right translations is both left and right invariant. Therefore, F induces by right invariance a function on $T^*H \times \mathfrak{k}^*_+$ and by left invariance a function on $\mathfrak{h}^*_- \times \mathfrak{k}^*_+$. Since reduction preserves the involutivity of functions, the stated result follows. By taking the infinitesimal version of this proof, it is easy to see that only $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ enter and not the Lie groups G, H, K.

We remark in passing that the Adler–Kostant–Symes theorem applied to the central extension of a Lie algebra $\bar{\mathfrak{g}}$ by an element $\varepsilon \in \mathfrak{g}^*$, i.e., $\bar{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}$ with the bracket $[(\xi, a), (\eta, b)] = ([\xi, \eta], \langle \varepsilon, [\xi, \eta] \rangle)$, yields the involution theorem of Mishchenko and Fomenko [29].

J. *Dirac Brackets.* Let P be a Poisson manifold and M a nondegenerate submanifold of P, i.e., $B((TM)^0) \cap TM = \{0\}$. Then M is a symplectic manifold. If $E = B((TM)^0)$, example D insures that M/Φ is a Poisson manifold. But since $E \cap TM = \{0\}$, the leaves of Φ are points, i.e., $M/\Phi = M$. If $M = \psi^{-1}(\mathbf{0})$ *for* $\mathbf{0} \in \mathbb{R}^{2l}$, $\psi = (\psi_1, \ldots, \psi_{2l})$, and the matrix $C = (C_{ij} = \{\psi_i, \psi_j\})$ is nondegenerate, a direct computation shows that the bracket on M is given by Dirac's formula:

$$\{F, H\}_M(x) = \{\tilde{F}, \tilde{H}\}_P(x) - \sum_{i,j=1}^{2l} \{\tilde{F}, \psi_i\}(x) C^{ij}(x) \{\psi_j, \tilde{H}\}(x) \tag{3.4}$$

where $C^{-1} = (C^{ij})$ and $\tilde{F}$, $\tilde{H}$ are arbitrarily smooth extensions of F and H to P. If P is symplectic and N is any submanifold of P, Sniatycki [35] has shown that there exists a symplectic submanifold M in which N is embedded coisotropically. Then the Dirac bracket on M induces a Poisson structure on the reduced manifold N by example E.

In Dirac's language M_i are second-class constraints and N is the final constraint manifold (see also Flato *et al.* [10]).

Finally, we remark that if M is a transverse manifold to the symplectic leaf S through x_0 with $S \cap M = \{x_0\}$ and $M = \psi^{-1}(0)$ and if the matrix C is nondegenerate, then the above formula for the Poisson bracket on M still holds. This follows from the fact that every symplectic leaf of M is the intersection of a symplectic leaf in P with M. This formula applied to the case of $P = \mathfrak{g}^*$ shows that the transverse structure is linear if $\mathfrak{g}_\mu$ defines a reductive splitting of $\mathfrak{g}$ (Weinstein and Molino) and is at most quadratic if the isotropy $\mathfrak{g}_\mu$ has a complement in $\mathfrak{g}$ which is a Lie subalgebra (this result is due to Oh).

Acknowledgement

We thank Ted Courant, Richard Montgomery, Jedrzej Sniatycki, and Alan Weinstein for helpful discussions.

References

1. Abraham, R. and Marsden, J., *Foundations of Mechanics*, 2nd edn., Addison-Wesley, 1978.
2. Arms, J., Marsden, J., and Moncrief, V., *Commun. Math. Phys.* **78**, 455–478 (1981).
3. Arnold, V., *Ann. Inst. Fourier, Grenoble* **16**, 319–361 (1966).
4. Arnold, V., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.
5. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D., *Ann. Phys.* **111**, 61, 1978.
6. Chernoff, P. and Marsden, J., *Properties of Infinite Dimensional Hamiltonian Systems*, Springer Lecture Notes in Mathematics **425**, Springer-Verlag, New York, 1974.
7. Cushman, R. and Rod, D. L., *Physica D* **6**, 105–112 (1982).
8. Deprit, A., *Celestial Mech.* **30**, 181–195 (1983).
9. Dirac, P. A. M., *Lectures on Quantum Mechanics*, Belfer Grad. School of Sci., Monograph Series, Vol. 2, Yeshiva University (1964).
10. Flato, M., Lichernowicz, A., and Sternheimer, D., *J. Math. Phys.* **17**, 17–54 (1976).
11. Guillemin, V. and Sternberg, S., *Invent. Math.* **67**, 515–538 (1982).
12. Guillemin, V. and Sternberg, S., *Symplectic Techniques in Physics*, Cambridge University Press, 1984.
13. Iwai, T., *J. Math. Phys.* **23**, 1088–1092 (1982).
14. Iwai, T., *J. Math. Phys.* **26**, 885–893 (1985).
15. Kazhdan, D., Kostant, B., and Sternberg, S., *Commun. Pure Appl. Math.* **31**, 481–508 (1978).
16. Kirillov, A. A., *Russian Math. Surveys* **31**, 55–75 (1976).
17. Kostant, B., *Quantization and Unitary Representations*, Lecture Notes in Mathematics **170**, Springer-Verlag, New York, 1970.
18. Krishnaprasad, P. S. and Marsden, J., Hamiltonian Structures and Stability for Rigid Bodies With Flexible Attachments, *Arch. Rat. Mech. An.* (to appear).
19. Kummer, M., *Indiana Univ. Math. J.* **30**, 281–291 (1981).
20. Lichnerowicz, A., *J. Diff. Geom.* **12**, 253–300 (1977).
21. Marle, G. M., 'Symplectic Manifolds, Dynamical Groups and Hamiltonian Mechanics', in M. Cahen and M. Flato (eds.), *Differential Geometry and Relativity*, D. Reidel, Dordrecht, 1976.
22. Marsden, J., *Lectures on Geometric Methods in Mathematical Physics*, CBMS-NSF Regional Conference Series #37, SIAM, 1981.
23. Marsden, J., Ratiu, R., and Weinstein, A., *Trans. Am. Math. Soc.* **281**, 141–177 (1984).
24. Marsden, J. and Weinstein, A., *Rep. Math. Phys.* **5**, 121–130 (1974).
25. Marsden, J. and Weinstein, A., *Physica D* **4**, 394–405 (1982).
26. Marsden, J. and Weinstein, A., *Physica D* **7**, 305–323 (1983).

27. Marsden, J., Weinstein, A., Ratiu, T., Schmid, R., and Spencer, R. G., Proc. IUTAM-ISIMM Symposium on 'Modern Developments in Analytical Mechanical', Torino, June 7–11, 1982, *Atti della Academia della Scienze di Torino* **117**, 289–340 (1983).
28. Meyer, K. R., 'Symmetries and Integrals in Mechanics', in *Dynamical Systems*, Academic Press, 1973, pp. 259–273.
29. Mishchenko, A. S. and Fomenko, A. T., *Funct. Anal. Appl.* **12**, 113–131 (1978) and *Math. USSR Izvestia* **12**, 371–389 (1978).
30. Montgomery, R., *Lett. Math. Phys.* **8**, 59–67 (1984).
31. Montgomery, R., Marsden, J., and Ratiu, T., *Cont. Math. AMS* **28**, 101–114 (1984).
32. Nehoroshev, N., *Trud. Mosk. Math.* **26**, 1972.
33. Ratiu, T., Involution Theorems in *Springer Lecture Notes in Mathematics* **775**, Springer-Verlag, New York, 1980.
34. Smale, S., *Inv. Math.* **10**, 205–331; **11**, 45–64 (1970).
35. Sniatycki, J., *Ann. Inst. H. Poincaré* **20**, 365–372 (1974).
36. Souriau, J. M., *Structures des Systèmes Dynamiques*, Dunod, Paris, 1970.
37. Sternberg, S., *Proc. Nat. Acad. Sci.* **74**, 5253–5254 (1977).
38. Tulczyjew, W. M., 'Poisson Reductions', in J-P. Dufour (ed.), *Singularités feuilletages et mécanique hamiltonienne*, Hermann, Paris, 1985.
39. Weinstein, A., *Adv. Math.* **6**, 329–346 (1971).
40. Weinstein, A., *Lett. Math. Phys.* **2**, 417–420 (1978).
41. Woodhouse, N., *Geometric Quantization*, Oxford University Press, 1980.