

Reduction of superintegrable systems: the anisotropic harmonic oscillator

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We introduce a new $2N$ -parametric family of maximally superintegrable systems in N dimensions, obtained as a reduction of an anisotropic harmonic oscillator in a $2N$ -dimensional configuration space. These systems possess closed bounded orbits and integrals of motion which are rational in the momenta. They generalize known examples of superintegrable models in the Euclidean plane.

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I. INTRODUCTION

The aim of this paper is to introduce a new class of maximally superintegrable systems that are obtained as a symplectic reduction of the anisotropic harmonic oscillator. These systems depend on a set of N real and N integer parameters and possess integrals of motion rational in the momenta. The Hamiltonian defining this family is

$$H_N = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i=1}^N \frac{k_i}{x_i^2} + \frac{\omega^2}{2} \sum_{i=1}^N n_i^2 x_i^2 \quad (1)$$

We recall that in classical mechanics, superintegrable (also known as noncommutatively integrable [20]) systems are characterized by the fact that they possess more than N integrals of motion functionally independent, globally defined in a $2N$ -dimensional phase space. In particular, when the number of integrals is $2N - 1$, the systems are said to be maximally superintegrable. The dynamics of these systems is particularly interesting: all bounded orbits are closed and periodic. This issue, for the spherically symmetric potentials, was first noticed by Bertrand [3]. The phase space topology is also very rich: it has the structure of a symplectic bifoliation, consisting of the usual Liouville–Arnold invariant fibration by Lagrangian tori and of a (coisotropic) polar foliation [21], [9]. Apart from the harmonic oscillator and the Kepler potential, many other potentials turn out to be superintegrable, like the Calogero–Moser potential, the Smorodinsky–Winternitz systems, the Euler top, etc.

A considerable effort has recently been devoted to the search for superintegrable systems as well as to the study of the algebraic and analytic properties of these models. For a recent review of the topic, see [27].

The notion of superintegrability possesses an interesting analog in quantum mechanics. Sommerfeld and Bohr were the first to notice that systems allowing separation

of variables in more than one coordinate system may admit additional integrals of motion. Superintegrable systems show accidental degeneracy of the energy levels, which can be removed by taking into account the quantum numbers associated to the additional integrals of motion. One of the best examples of this phenomenon is provided by the Coulomb atom [10], [2], [16], which is superintegrable in N dimensions [5], [25]. A systematic search for quantum mechanical potentials exhibiting the property of superintegrability was started in [15], [22] and [29]. These models in many cases are also exactly solvable, i.e. they possess a spectrum generating algebra, which allows to compute the whole energy spectrum essentially by algebraic manipulations [26]. In classical mechanics, the multiseparability of the Hamilton–Jacobi equation implies that there should exist at least two different sets of N quadratic integrals of motion in involution. Reduction techniques both in classical and quantum mechanics are well-known (see, for instance, [4]). Essentially, the common idea of several of the existing approaches is to start from a free motion Hamiltonian defined in a suitable higher-dimensional space and to project it down into an appropriate subspace. In this way, one gets a reduced Hamiltonian that is no longer free: an integrable potential appears in the lower-dimensional space [23]. A different point of view, that we adopt here, is to start instead directly from a nontrivial (i.e. not free) dynamical system in a given phase space and to reduce it to a proper subspace, in such a way that the superintegrability of the considered system be inherited by the reduced one.

In this work, we study the reduction of an anisotropic harmonic oscillator, defined in a $2N$ -dimensional classical configuration space. This system is maximally superintegrable. It is described by the Hamiltonian

$$H_{2N} = \frac{1}{2} \sum_{i=1}^{2N} \hat{p}_i^2 + \frac{\omega^2}{2} \sum_{i=1}^{2N} n_i^2 y_i^2. \quad (2)$$

We will prove that it can be suitably reduced to the system (1), and that this new system is still maximally superintegrable, as a consequence of the reduction procedure we adopt. This goal is achieved under the assumption $n_1 = n_2, \dots, n_{2N-1} = n_{2N}$. From a geometrical point of view, the approach we adopt reposes on the Marsden–Weinstein symplectic reduction scheme [1], [17], [19]. Given a symplectic manifold (M, Ω) , let K_1, \dots, K_k be k functions in involution:

$$\{K_i, K_j\} = 0 \quad i, j = 1, \dots, k. \quad (3)$$

Assume also that dK_i be independent at each point. Since the flows of the associated Hamiltonian vector fields X_{K_1}, \dots, X_{K_k} do commute, they can be used to define a symplectic action of $G = \mathbb{R}^k$ on the manifold. Let J be the momentum map of this action, and μ be a regular value for J . Then we can conclude that $P_\mu := J^{-1}(\mu)/G$ is still a symplectic manifold, of dimension $4N - 2k$, called the reduced phase space. In our case, $K_i, i = 1, \dots, k$ are components of the angular momentum, $J = K_1 \times \dots \times K_k$ is the momentum map, $G = SO(2) \times SO(2) \times \dots \times SO(2)$ (N times), and the reduced space is $P_\mu = J^{-1}(\mu)/T^k$, where T^k is the k -dimensional torus and $\dim P_\mu = N$. This procedure is a generalization of what in celestial mechanics, since the work of Jacobi, is called "elimination of the nodes" (see [1], chapter IX for details). The reduced Hamiltonian is reminiscent of the structure of the original Hamiltonian, defined in the $4N$ -dimensional phase space, but also possesses a Rosochatius-type term, involving parameters k_i corresponding to the variables that become ignorable, in addition to the harmonic part. Therefore, using the reduction procedure, we obtain the parametric family of Hamiltonian systems (1), defined on a reduced phase P_μ .

The transformations we consider, although very simple, are non-trivial, since the reduced Hamiltonian is not shape-invariant. Nevertheless, since the reduced system turns out to be maximally superintegrable, bounded orbits still remain closed in the reduced space.

When $N = 2$, the maximal superintegrability of the system obtained with $n_1 = n_2 = 1$ was already established, as well as its exact solvability [26].

A new phenomenon emerging from our analysis is that $N - 1$ of the integrals of our model (1) are rational functions rather than polynomials in the momenta, whenever $N \geq 2$. This is quite striking, since for planar systems describing the motion of one particle the existence of superintegrability with rational integrals was never observed before. Instead, systems possessing higher-order polynomial integrals of motion were already known [6], [12], [13], [14], [24].

This paper is directly related to the recent interesting work by Verrier and Evans [28], that performed a similar reducing transformation for the Kepler potential. They found a superintegrable system in three dimensions possessing a quartic integral. They also conjectured that

the system (1) in three space dimensions should be maximally superintegrable, although the explicit expression of the integrals remained to be determined. In the following, we will prove this conjecture, and also we will establish that the system (1) is maximally superintegrable in full generality, i.e. for N arbitrary, providing explicitly the corresponding set of integrals of the motion.

The paper is organized as follows. In Section II, the main properties of the anisotropic oscillator are briefly reviewed. Then its reduction to the planar case is studied in detail. We will show how superintegrability is preserved under a multipolar change of variables and subsequent reduction. In Section III, the same problem is treated and solved in full generality. Some open problems are discussed in the final Section.

II. REDUCTION OF THE ANISOTROPIC OSCILLATOR

The anisotropic oscillator in the two-dimensional case both in classical and quantum mechanics was discussed by Jauch and Hill [16]. The system (2) is also known to be superintegrable in $2N$ dimensions, if the ratios of the frequencies of motions are rational. Hence let us assume

$$\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2} = \dots = \frac{\omega_{2N}}{n_{2N}} = \omega, \quad n_i \in \mathbb{N} \quad (4)$$

Following [16], we define the set of invariants in an auxiliary phase space, with coordinates $z_i, \bar{z}_i, i = 1, \dots, 2N$. Precisely,

$$z_j = \hat{p}_j - in_j \omega_j y_j, \quad \bar{z}_j = \hat{p}_j + in_j \omega_j y_j. \quad (5)$$

It is easily checked that the expressions

$$c_{ij} = z_j^{n_k} \bar{z}_k^{n_j} \quad (6)$$

provide integrals of motion. They can be also arranged in a real-valued form, as the combinations $(1/2)(c_{ij} + \bar{c}_{ij})$ and $(1/2i)(c_{ij} - \bar{c}_{ij})$. In particular, among these integrals we have the angular momenta

$$L_{ik} = y_i \hat{p}_k - y_k \hat{p}_i \quad (7)$$

(when $n_i = n_k$) and

$$T_{ik} = \frac{1}{2} \hat{p}_i \hat{p}_k + n_i n_k \omega^2 y_i y_k \quad (8)$$

We will now study reductions of the anisotropic oscillator (2) and establish the superintegrability of the corresponding dynamical systems.

Hamiltonian and first integrals: the planar case

We recall the definition of a momentum map. For further details, see for instance [1]. Let (M, Ω) be a $2n$ dimensional symplectic manifold. Suppose that a Lie group

G acts on M and leaves Ω invariant. Let \mathfrak{g} be the Lie algebra of G , \mathfrak{g}^* its dual, and \langle, \rangle the natural pairing between the two spaces.

A momentum map for the G -action on (M, Ω) is a map $J : M \rightarrow \mathfrak{g}^*$ such that, for all $X \in \mathfrak{g}$,

$$d(\langle J, X \rangle) = i_X \Omega$$

In particular, if the manifold is exact, i.e. $\Omega = d\theta$, we have

$$J_X = i_X \theta$$

We will also assume that the map is equivariant with respect to the coadjoint action Ad^* of G on \mathfrak{g}^* , i.e.:

$$\langle \text{Ad}_g^* \xi, X \rangle = \langle \xi, \text{Ad}_{g^{-1}} X \rangle,$$

for all $g \in G, \xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

Let us first consider a simple case, when the anisotropic oscillator is defined in a symplectic manifold with $n = 4$. So, $\Omega = \sum_{i=1}^4 dy^i \wedge d\hat{p}_i$. In order to make the reduction possible, we will select frequencies to be equal in pairs, so that we have only two independent frequencies. Hence the system (2) takes the special form

$$H_4 = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2 + \hat{p}_4^2) + \frac{n_1^2 \omega^2}{2}(y_1^2 + y_2^2) + \frac{n_2^2 \omega^2}{2}(y_3^2 + y_4^2). \quad (9)$$

In the auxiliary coordinates $z_1, \bar{z}_1, \dots, z_4, \bar{z}_4$, we have explicitly

$$\begin{aligned} z_1 &= \hat{p}_1 - i n_1 \omega y_1, & z_2 &= \hat{p}_2 - i n_1 \omega y_2, \\ z_3 &= \hat{p}_3 - i n_2 \omega y_3, & z_4 &= \hat{p}_4 - i n_2 \omega y_4. \end{aligned}$$

Consequently, the Hamiltonian reads

$$H_4 = \frac{1}{2} \sum_{i=1}^4 |z_i|^2. \quad (10)$$

Put in a matrix form, the set of invariants (6) can be written as

$$Z = \begin{pmatrix} z_1 \bar{z}_1 & z_1 \bar{z}_2 & z_1^{n_2} \bar{z}_3^{n_1} & z_1^{n_2} \bar{z}_4^{n_1} \\ z_2 \bar{z}_1 & z_2 \bar{z}_2 & z_2^{n_2} \bar{z}_3^{n_1} & z_2^{n_2} \bar{z}_4^{n_1} \\ z_3^{n_1} \bar{z}_1^{n_2} & z_3^{n_1} \bar{z}_2^{n_2} & z_3 \bar{z}_3 & z_3 \bar{z}_4 \\ z_4^{n_1} \bar{z}_1^{n_2} & z_4^{n_1} \bar{z}_2^{n_2} & z_4 \bar{z}_3 & z_4 \bar{z}_4 \end{pmatrix}. \quad (11)$$

Let us consider now the following change of coordinates:

$$\begin{cases} y_1 = x_1 \cos x_3, & y_2 = x_1 \sin x_3 \\ y_3 = x_2 \cos x_4, & y_4 = x_2 \sin x_4. \end{cases} \quad (12)$$

The corresponding momenta read

$$\begin{aligned} \hat{p}_1 &= -p_3 \frac{\sin x_3}{x_1} + p_1 \cos x_3, & \hat{p}_2 &= p_3 \frac{\cos x_3}{x_1} + p_1 \sin x_3 \\ \hat{p}_3 &= -p_4 \frac{\sin x_4}{x_2} + p_2 \cos x_4, & \hat{p}_4 &= p_4 \frac{\cos x_4}{x_2} + p_2 \sin x_4. \end{aligned}$$

Let us now consider the group T_2 , which is the group $SO(2) \times SO(2)$ in the old coordinates, acting on \mathbb{R}^4

$$\begin{aligned} x'_1 &= x_1 \\ x'_2 &= x_2 \\ x'_3 &= x_3 + a_1 \\ x'_4 &= x_4 + a_2. \end{aligned}$$

This group leaves Ω invariant. The fundamental vector fields on $T^*\mathbb{R}^4$ corresponding to this action are:

$$X_1 = \partial_{x_3}, \quad X_2 = \partial_{x_4} \quad (13)$$

and, if $X = \lambda_1 X_1 + \lambda_2 X_2$, the momentum map J satisfies:

$$J_{(a_1, a_2)} = \theta(\lambda_1 \partial_{x_3} + \lambda_2 \partial_{x_4}) = \lambda_1 p_3 + \lambda_2 p_4 \quad (14)$$

Let us choose a regular point in t_2^* (the dual of the Lie algebra of T_2), for instance $p_3 = \sqrt{2k_1}$, $p_4 = \sqrt{2k_2}$. The inverse image under J is

$$J^{-1}(\sqrt{2k_1}, \sqrt{2k_2}) = (p_1, p_2, \sqrt{2k_1}, \sqrt{2k_2}, x_1, x_2, x_3, x_4) \quad (15)$$

The stabilizer of this point in t_2^* under the coadjoint action of T_2 is the whole group, because its action is trivial on the p coordinates.

The reduced phase space is therefore

$$J^{-1}(\sqrt{2k_1}, \sqrt{2k_2})/T_2 \approx \{(p_1, p_2, x_1, x_2) \in \mathbb{R}^4\} \quad (16)$$

and the reduced Hamiltonian is

$$H_2 = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{k_1}{x_1^2} + \frac{k_2}{x_2^2} + \frac{n_1^2}{2} x_1^2 + \frac{n_2^2}{2} x_2^2 \quad (17)$$

Let F be a first integral of the Hamiltonian $H_4(\hat{p}, y)$, i.e. $\{H_4, F\} = 0$.

We show now how the original ring of integrals can be reduced in the low-dimensional phase space. First, we consider the restriction \hat{F} of the function F to the manifold $J^{-1}(\sqrt{2k_1}, \sqrt{2k_2})$.

Observe that \hat{F} can be defined on the quotient manifold $J^{-1}(\sqrt{2k_1}, \sqrt{2k_2})/T_2$, when it is constant on the equivalence classes, that is, \hat{F} is independent on x_3, x_4 . In this case \hat{F} can be factored out in the following way:

$$\begin{array}{ccc} J^{-1}(\sqrt{2k_1}, \sqrt{2k_2}) & \xrightarrow{\hat{F}} & \mathbb{R} \\ \pi \downarrow & \nearrow F_r & \\ J^{-1}(\sqrt{2k_1}, \sqrt{2k_2})/T_2 & & \end{array}$$

where π is the canonical projection and

$$F_r \circ \pi = \hat{F}$$

Then,

$$\{H_2, F_r\} = 0$$

Let us first study the particular case when the frequencies are $n_1 = n_2 = 1$, $n_3 = n_4 = 2$, $\omega = 1$ and construct explicitly the integrals in a direct way.

This choice of the frequencies is relevant, since the reduced Hamiltonian is now

$$H_2 = \frac{1}{2} (p_1^2 + p_2^2) + \frac{k_1}{x_1^2} + \frac{1}{2} x_1^2 + \frac{k_2}{x_2^2} + 2x_2^2. \quad (18)$$

and represents a generalization of the system

$$H_2 = \frac{1}{2} (p_1^2 + p_2^2) + \frac{\alpha}{x_1^2} + \frac{1}{2} x_1^2 + 2x_2^2. \quad (19)$$

discovered in [15]. This system is superintegrable both in classical and quantum mechanics, and possesses integrals which are second order in the momenta.

We will prove that the Hamiltonian (18) instead possesses also an integral rational in the momenta.

Let us write some of the invariants of the Hamiltonian (10) which will be useful in the sequel:

$$T_i = |z_i|^2, \quad i = 1, \dots, 4$$

$$L_{12} = \frac{i}{2} (c_{12} - c_{21}) = \frac{i}{2} (z_1 \bar{z}_2 - z_2 \bar{z}_1) = y_1 \hat{p}_2 - y_2 \hat{p}_1$$

$$L_{34} = \frac{i}{4} (c_{34}^{\frac{1}{2}} - c_{43}^{\frac{1}{2}}) = \frac{i}{4} (z_3 \bar{z}_4 - z_4 \bar{z}_3) = y_3 \hat{p}_4 - y_4 \hat{p}_3$$

and

$$T_{12} = \hat{p}_1 \hat{p}_2 + y_1 y_2$$

$$T_{34} = \hat{p}_3 \hat{p}_4 + 4y_3 y_4$$

Notice that they satisfy the Poisson commutation relations

$$\{H_4, L_{ij}\} = 0, \quad \{H_4, T_i\} = 0, \quad \{H_4, T_{ij}\} = 0$$

$$\{L_{12}, T_1 + T_2\} = 0, \quad \{L_{34}, T_1 + T_2\} = 0,$$

$$\{L_{12}, T_3 + T_4\} = 0, \quad \{L_{34}, T_3 + T_4\} = 0$$

The Poisson bracket can be written in terms of the z_i variables:

$$\begin{aligned} & \{f(z_i, \bar{z}_i), g(z_i, \bar{z}_i)\} \\ &= \sum_{i=1}^2 \left(\frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \bar{z}_i} - \frac{\partial f}{\partial \bar{z}_i} \frac{\partial g}{\partial z_i} \right) + 2 \sum_{i=3}^4 \left(\frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \bar{z}_i} - \frac{\partial f}{\partial \bar{z}_i} \frac{\partial g}{\partial z_i} \right) \end{aligned}$$

Let us look for some more invariant quantities commuting with L_{12} and L_{34} . Impose that a function of $z_1, \bar{z}_1, z_2, \bar{z}_2$ commutes with L_{12} :

$$z_2 \partial_{z_1} f - z_1 \partial_{z_2} f + \bar{z}_2 \partial_{\bar{z}_1} f - \bar{z}_1 \partial_{\bar{z}_2} f = 0$$

The invariants of this differential equation, that is the quantities commuting with L_{12} , are:

$$z_1^2 + z_2^2, \quad z_1 \bar{z}_2 - \bar{z}_1 z_2, \quad z_1 \bar{z}_1 + z_2 \bar{z}_2$$

Doing the same calculus with L_{34} , we finally find the following set of invariants under L_{12} and L_{34} :

$$z_1^2 + z_2^2, \quad z_3^2 + z_4^2, \quad |z_1|^2 + |z_2|^2, \quad |z_3|^2 + |z_4|^2 \quad (20)$$

where we have not included L_{12} and L_{34} . These quantities are not necessarily invariant under H_4 . It is easy to check that:

$$\{H_4, |z_1|^2 + |z_2|^2\} = 0, \quad \{H_4, |z_3|^2 + |z_4|^2\} = 0$$

(a trivial result if one writes these expressions in terms of y_i and \hat{p}_i), but H_4 does not commute with the first two expressions in (20). We look for a function of these two expressions satisfying:

$$\{H_4, f(z_1^2 + z_2^2, z_3^2 + z_4^2)\} = 0$$

This equation defines one invariant which can be written as

$$R = \frac{(z_1^2 + z_2^2)^2}{z_3^2 + z_4^2} \quad (21)$$

It is evident that all the invariants can be chosen real and so,

$$\bar{R} = \frac{(\bar{z}_1^2 + \bar{z}_2^2)^2}{\bar{z}_3^2 + \bar{z}_4^2} \quad (22)$$

is also an invariant. Then, R and \bar{R} are invariants under H_4 , L_{12} and L_{34} . Notice that R and \bar{R} are not polynomials, but they can be written in terms of the invariants c_{ij} . For instance, R reads

$$R = \frac{(z_1^2 + z_2^2)^2}{z_3^2 + z_4^2} = \frac{(c_{13} + c_{23})^2 + (c_{14} + c_{24})^2}{c_{33}^2 + c_{44}^2 + c_{34}^2 + c_{43}^2} \quad (23)$$

Reduction of the first integrals

As we said, the integrals as well are restricted by the transformation (12) in the reduced phase space P_μ :

$$\tilde{L} = L|_{\text{reduced}}, \quad \tilde{T} = T|_{\text{reduced}}$$

$$\tilde{L}_{12} = \sqrt{2k_1}, \quad \tilde{L}_{34} = \sqrt{2k_2}$$

$$\tilde{E} = H_2$$

$$\tilde{E}_1 = \frac{1}{2} (T_1 + T_2) = \frac{1}{2} p_1^2 + \frac{k_1^2}{x_1^2} + \frac{1}{2} x_1^2$$

$$\tilde{E}_2 = \frac{1}{2} (T_3 + T_4) = \frac{1}{2} p_2^2 + \frac{k_2^2}{x_2^2} + 2x_2^2$$

$$\{H_2, \tilde{E}_i\} = 0, \quad i = 1, 2, \quad \{\tilde{E}_1, \tilde{E}_2\} = 0,$$

The quantities \tilde{E}_i are the reductions of the invariants

$$|z_1|^2 + |z_2|^2, \quad |z_3|^2 + |z_4|^2$$

The other invariant, R (or its real part) is also reduced to an invariant of the reduced Hamiltonian. For instance, its real part I reads

$$I = \frac{1}{I_2^2 + \frac{k_2}{x_2^2} M_2} \left(K_1^2 - I_1^2 + 2 \frac{k_2}{x_2^2} M_1 \right) \quad (24)$$

where

$$\begin{aligned} I_2 &= \tilde{E}_2 |_{k_2=0} = \frac{1}{2} p_2^2 + 2x_2^2, \\ M_2 &= p_2^2 + \frac{k_2}{x_2^2} - 4x_2^2, \\ I_1 &= 2 \left(\frac{1}{2} p_1^2 + \frac{k_1}{x_1^2} - \frac{1}{2} x_1^2 \right) x_2 - p_1 p_2 x_1, \\ M_1 &= \left(\frac{1}{2} p_1^2 + \frac{k_1}{x_1^2} - \frac{3}{2} x_1^2 \right)^2 - 2 \left(x_1^2 - \frac{k_1}{x_1^2} \right) x_1^2, \\ K_1 &= \left(\frac{1}{2} p_1^2 + \frac{k_1}{x_1^2} - \frac{1}{2} x_1^2 \right) p_2 + 2p_1 x_1 x_2. \end{aligned} \quad (25)$$

Observe that R is a smooth function, whose denominator never vanishes. The quantity I commutes with H_2 and is functionally independent of H_2 and \tilde{E}_1 (or \tilde{E}_2). When $k_2 = 0$, the integral I collapses into one of the integrals of the system (19), namely the one responsible for the separation of variables in parabolic coordinates.

III. THE GENERAL CASE

Within the same approach, it is easy to extend the previous picture to the general situation of a reduction from a $2N$ to a N -dimensional configuration space:

$$H_{2N} = \frac{1}{2} \sum_{i=1}^{2N} \hat{p}_i^2 + \frac{\omega^2}{2} \sum_{j=1}^N n_j^2 (y_{2j-1}^2 + y_{2j}^2). \quad (26)$$

Indeed, let us introduce the affine variables

$$z_k = \hat{p}_k - i n_k \omega y_k, \quad k = 1, \dots, 2N$$

so that the Hamiltonian reads

$$H_{2N} = \frac{1}{2} \sum_{k=1}^{2N} |z_k|^2.$$

The Poisson bracket is defined by

$$\{f(z_i, \bar{z}_i), g(z_i, \bar{z}_i)\} = \sum_{j=1}^N \sum_{k=2j-1}^{2j} n_j \left(\frac{\partial f}{\partial z_k} \frac{\partial g}{\partial \bar{z}_k} - \frac{\partial f}{\partial \bar{z}_k} \frac{\partial g}{\partial z_k} \right)$$

The invariants under the group action generated by $L_{12}, \dots, L_{2N-1, 2N}$ are

$$z_1^2 + z_2^2, \dots, z_{2N-1}^2 + z_{2N}^2, \quad (27)$$

apart from the quantities $L_{12}, \dots, L_{2N-1, 2N}$ and the “2-plane energies” which commute with the Hamiltonian H_4

$$|z_1|^2 + |z_2|^2, \dots, |z_{2N-1}|^2 + |z_{2N}|^2 \quad (28)$$

Imposing

$$\{H_{2N}, f(z_1^2 + z_2^2, \dots, z_{2N-1}^2 + z_{2N}^2)\} = 0$$

we get the differential equation

$$n_1(z_1^2 + z_2^2) \partial_1 f + \dots + n_N(z_{2N-1}^2 + z_{2N}^2) \partial_N f = 0.$$

Its general solution depends on $N - 1$ invariants:

$$\frac{(z_1^2 + z_2^2)^{n_N}}{(z_{2N-1}^2 + z_{2N}^2)^{n_1}}, \dots, \frac{(z_{2N-3}^2 + z_{2N-2}^2)^{n_N}}{(z_{2N-1}^2 + z_{2N}^2)^{n_{N-1}}} \quad (29)$$

Then, using the transformation (12) we now reduce the original Hamiltonian to the following one:

$$H_N = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i=1}^N \frac{k_i}{x_i^2} + \frac{\omega^2}{2} \sum_{i=1}^N n_i^2 x_i^2 \quad (30)$$

The corresponding reduced invariants are:

$$|z_1|^2 + |z_2|^2, \dots, |z_{2N-1}|^2 + |z_{2N}|^2, \quad (31)$$

$$\frac{(z_1^2 + z_2^2)^{n_N}}{(z_{2N-1}^2 + z_{2N}^2)^{n_1}}, \dots, \frac{(z_{2N-3}^2 + z_{2N-2}^2)^{n_N}}{(z_{2N-1}^2 + z_{2N}^2)^{n_{N-1}}} \quad (32)$$

There are $2N - 1$ functionally independent integrals and consequently the system is maximally superintegrable, proving the conjecture of [28].

We have thus added a new maximally superintegrable system in N dimensions to previously known ones (see, e.g., [7], [8], [11], [27]).

IV. OPEN PROBLEMS

From the previous considerations, it emerges that it would be desirable to construct systematically transformations mapping a superintegrable system into another system, that is also superintegrable, and defined in a reduced phase space. It seems natural to associate such transformations to the rich symmetry structure possessed by superintegrable systems. For instance, changes of variables of the type (12) are clearly related to invariance properties under rotation. From this point of view, the role of higher order groups of transformations generated by the flow associated to integrals that are polynomials in the momenta remains to be fully investigated. A quantum mechanical version of this reduction procedure is also to be understood.

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