# REDUCTION OF THE SEMISIMPLE 1:1 RESONANCE 

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The method of "averaging" is often used in Hamiltonian systems of two degrees of freedom to find periodic orbits. Such periodic orbits can be reconstructed from the critical points of an associated "reduced" Hamiltonian on a "reduced space". This paper details the construction of the reduced space and the reduced Hamiltonian for the semisimple $1: 1$ resonance case. The reduced space will be a 2 -sphere in $\mathrm{R}^{3}$, and the reduced differential equations will be Euler's equations restricted to this sphere. The orbit projection from the energy surface in phase space to this sphere will be the Hopf map. The results of the paper are related to problems in physics on "degeneracies" due to symmetries of classical two-dimensional harmonic oscillators and their quantum analogues for the hydrogen atom.

## 1. Introduction

It is the purpose of this paper to detail the construction of the reduced space and the reduced Hamiltonian for the two degree of freedom semisimple 1:1 resonance case (the terminology will be explained below). This is one of the simplest nontrivial examples where reduction can be explicitly carried out with applications to such problems as the HénonHeiles Hamiltonian (see [5], [13, 14], and [21]). A special case of such a system is the twodimensional isotropic harmonic oscillator. Jauch and Hill [12] observed that the quantum mechanical degeneracies of this harmonic oscillator were due to its $\operatorname{SU}(2)$ dynamical symmetry group (see also [6]). Later Dulock and McIntosh

[^0][7] found that for the harmonic oscillator the space of all orbits of a given energy (which is topologically a 3 -sphere in $\mathrm{R}^{4}$ ) could be mapped by the Hopf mapping to the standard 2 -sphere in $\mathbf{R}^{3}$ (see also [9, p. 169] and [13-15]). The Hopf mapping (given in (1.5) below) maps distinct orbits of the oscillator (which are great circles on the 3 -sphere) to distinct points of the 2 sphere which is then the reduced space (orbit space) for this oscillator. Finally it was realized by [2], [20, p. 120], and [17] that the dynamics of the harmonic oscillator induced dynamics on the reduced phase space which are given by Euler's equations for a rigid body (see also [13, p. 58]). For some of the background to [12] and [7] we refer to [8] and [3].

An important use of the reduced space is for finding periodic orbits of real analytic Hamiltonian systems of differential equations with two degrees of freedom. This involves "averaging" the Hamiltonian $H$ out to some
order and truncating to obtain a polynomial Hamiltonian in "normal form" as in (1.2) and (1.3) below. If the truncated Hamiltonian system has periodic orbits that are actually reparametrized orbits of the flow generated by the quadratic terms of $H$, then these will show up as critical points of an associated reduced Hamiltonian on the reduced space [1, p. 306]. The explicit construction of the reduced Hamiltonian for our case turns out to be trivial and is detailed in (2.24)-(2.26) below. Having located such periodic orbits for the truncated Hamiltonian, one can try to carry them over to the original, full Hamiltonian. In the case that the periodic orbits originate as the energy increases from an equilibrium point of the Hamiltonian at energy zero, this can often be done by some perturbation scheme that then gives their existence in the original Hamiltonian flow at low positive energies (see for example [19]). Such a program is carried out in detail in [5], [13, 14], and [21].

Let ( $\mathrm{R}^{4}, \omega=\mathrm{d} x_{1} \wedge \mathrm{~d} y_{1}+\mathrm{d} x_{2} \wedge \mathrm{~d} y_{2}$ ) be a symplectic vector space with global coordinates ( $x_{1}, x_{2}, y_{1}, y_{2}$ ). We define the linear map $\omega^{b}$ by $\omega^{b}\left(\partial / \partial x_{i}\right)=\mathrm{d} y_{i}$ and $\omega^{b}\left(\partial / \partial y_{i}\right)=-\mathrm{d} x_{i}$ for $i=1,2$, and set $\omega^{*}=\left(\omega^{b}\right)^{-1}$. By the semisimple 1:1 resonance case we mean that the quadratic terms $\mathrm{H}_{2}$ of our Hamiltonian have the form
$H_{2}\left(x_{1}, x_{2}, y_{1}, y_{2}\right)=\frac{1}{2}\left(x_{1}^{2}+x_{2}^{2}+y_{1}^{2}+y_{2}^{2}\right)$.
$\mathrm{H}_{2}$ is semisimple because the linear Hamiltonian vector field $X_{2}=\omega^{*}\left(\mathrm{~d} \mathrm{H}_{2}\right)$ is diagonalizable over the complex numbers (see (3.5) and (3.6) below). Moreover, since $i$ is a double eigenvalue of $X_{2}$ at the origin (which is an equilibrium point of $X_{2}$ ), we say $X_{2}$ (or $H_{2}$ itself) is in $1: 1$ resonance.

We will consider polynomial Hamiltonian functions of the form
$H=\sum_{i=1}^{n} H_{2 i}$,
where the $H_{2 i}$ are homogeneous polynomials of
degree ( $2 i$ ) in the variables ( $x_{1}, x_{2}, y_{1}, y_{2}$ ). We will assume that $H$ is in normal form w.r.t. $H_{2}$; that is, the Lie derivative of $H$ w.r.t. the Hamiltonian vector field $X_{2}$ vanishes. In standard notation [1, p. 79 and p. 192] we have
$L_{X_{2}}(H)=X_{2}(H)=\left\{H, H_{2}\right\}=0$,
where $\{$,$\} denotes Poisson brackets defined$ w.r.t. the symplectic form $\omega$ above. We show in section 3 that (1.3) is equivalent to
$H=H\left(W_{1}, W_{2}, W_{3}, W_{4}\right)$,
where the $W_{j}, j=1,2,3,4$, are quadratic polynomials in the variables ( $x_{1}, x_{2}, y_{1}, y_{2}$ ) given in (2.10) below. The $W_{j}$ are referred to as "Hopf variables" and the map
$\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \rightarrow\left(W_{1}, W_{2}, W_{3}\right)$
is the standard "Hopf map" when restricted to the 3 -sphere in $\mathrm{R}^{4}$. Thus $H$ as in (1.2) being in normal form w.r.t. $H_{2}$ means that $H$ can be expressed as a polynomial (1.4) in these Hopf variables.

We mention two standard ways of computing the normal form w.r.t. $H_{2}$ of a given Hamiltonian up to some order, so that on truncating the higher order terms we have (1.2) with (1.3). The original scheme for computing the Birkhoff normal form of a Hamiltonian was modified by Gustavson in [10] to apply to resonance cases such as (1.1). An account of this method can also be found in [18, pp. 10-13]. An alternate technique uses the method of Lie series (see [1, $\mathrm{pp} .500-502]$ and the references therein). This latter method was applied to the Hénon-Heiles Hamiltonian in [5].
We can now summarize the results of this paper. Let $S_{r}^{n}$ denote the standard $n$-sphere in $\mathbf{R}^{n+1}$ centered at the origin with radius $r$. In section 2 we will carry out the reduction process on the Hamiltonian vector field $X_{H}=\omega^{*}(\mathrm{~d} H)$ restricted to $M=S_{(2 h)^{1 / 2}}^{3}=H_{2}^{-1}(h) \subset \mathrm{R}^{4}$ (see [1, p.

299 and p. 304]). We will find that the reduced space $M_{R}=M_{R}(h)$ can be identified as the sphere $S_{(2 h)}^{2} \subset R^{3}$ by factoring the canonical projection $\pi: M \rightarrow M_{\mathrm{R}}$ through the following commutative diagram:

$M_{\mathrm{R}} \approx \mathrm{S}_{(2 h)}^{2} \stackrel{\lambda^{-1}}{\longleftarrow} \mathrm{su}(2)$
Here $J$ is the Ad*-equivariant momentum map $^{*}$ associated to a linear $\mathrm{SU}(2)$ action on $\mathrm{R}^{4}, \gamma$ is the standard Killing form on the Lie algebra $\operatorname{su}(2)$ with induced linear map $\gamma^{b}: \operatorname{su}(2) \rightarrow \operatorname{su}(2)^{*}$ and $\gamma^{*}=\left(\gamma^{b}\right)^{-1}$, and $(2 \lambda): \mathrm{R}^{3} \rightarrow \operatorname{su}(2)$ is a natural identification map. The orbit map $\pi$ sends distinct periodic orbits of the flow of $X_{2}$ in $M$ to distinct points in $M_{R}$. The factor of 2 in the identification map (2 $\lambda$ ) has been put in so that $\pi$ will be precisely the standard Hopf mapping [11, p. 654], and $\left(\gamma^{b} \circ \lambda\right)\left(M_{R}\right)$ will be an orbit in $\mathrm{su}(2)^{*}$ of the coadjoint action of $\mathrm{SU}(2)$ on the dual of its Lie algebra.

The factoring of $\pi$ in (1.6) allows us to calculate explicitly the induced symplectic form $\omega_{R}$ on $M_{R}$ satisfying the pullback relation $\pi^{*} \omega_{\mathrm{R}}=$ $i^{*} \omega$, where $i: M \rightarrow \mathrm{R}^{4}$ is inclusion. We can then find the reduced Hamiltonian $K$ that is associated to $H$ by $K \circ \pi=H \circ i$, and the reduced vector field $X_{K}=\left(\omega_{\mathrm{R}}\right)^{*}(\mathrm{~d} K)$ on $M_{\mathrm{R}}$. In (2.29) we show that this reduced vector field is just Euler's equations restricted to the sphere $M_{R}$ (see [13, p. 58]). That the orbit projection $\pi$ is the Hopf map was expressed in [7].

We will use the notation of [1] throughout (see also [16, Lectures 3 and 4]).

## 2. Reduction of $X_{H}$

We begin the reduction of the Hamiltonian vector field $X_{H}$ by showing that the hypotheses of the reduction theorem [1, p. 299 and p. 304] are satisfied for the $S^{1}$-action on $R^{4}$ generated by
$\mathrm{X}_{2}$. Since $H$ is in normal form, $0=L_{X_{2}}(H)=$ $-L_{X_{H}}\left(H_{2}\right)$, implying that $H_{2}$ is an integral of $X_{H}$. Thus for $h>0$ the flow of $X_{H}$ leaves the 3sphere $M=H_{2}^{-1}(h)$ invariant, and $X_{H}$ restricts to $M$. The flow of the linear Hamiltonian vector field $X_{2}$ defines an $S^{1}$-action $\Phi:[0,2 \pi) \times R^{4} \rightarrow R^{4}$ by
$\Phi(t, x)=\Phi_{t}(x)=\left(\begin{array}{cc}(\cos t) I_{2} & (\sin t) I_{2} \\ -(\sin t) I_{2} & (\cos t) I_{2}\end{array}\right) x$,
where $I_{2}=\left(\begin{array}{ll}1 & 9 \\ 0 & 1\end{array}\right)$ and $x=\left(x_{1}, x_{2}, y_{1}, y_{2}\right) . \Phi_{t}$ is a linear symplectic mapping of ( $\mathrm{R}^{4}, \omega$ ), and the $\mathrm{S}^{1}$-momentum mapping associated to $\Phi$ is just $\mathrm{H}_{2}$ itself which is trivially $\mathrm{Ad}^{*}$-equivariant. Since $\Phi$ restricts to a free and proper $S^{1}$-action on $M$, the reduction process in [1, p. 299], applied to the momentum map $H_{2}$, yields a smooth orbit manifold $M_{R}=M / S^{1}$. The smooth surjective orbit map $\pi: M \rightarrow M_{\mathrm{R}}$ together with $\omega$ determines a unique symplectic form $\omega_{\mathrm{R}}$ on $\mathrm{M}_{\mathrm{R}}$ by the pullback relation $\pi^{*} \omega_{\mathrm{R}}=i^{*} \omega$, where $i: M \rightarrow R^{4}$ is inclusion. Now (1.3) implies that $H$ is invariant under the action of $\Phi_{i}$, hence by [1, p. 304] there is an induced Hamiltonian $K$ on $M_{\mathrm{R}}$ such that $K \circ \pi=H \circ i$ and
$\pi_{*} X_{H}(x)=X_{K}(\pi(x)), \quad$ for all $x \in M$,
where $\pi_{*}$ is the tangent map of $\pi$.
We now determine a model for ( $M_{R}, \omega_{R}, \pi$ ) by factoring $\pi$ through a momentum mapping associated to a linear $\mathrm{SU}(2)$ action on ( $\mathrm{R}^{4}, \omega$ ). One can easily show that any two models for the reduced space are symplectomorphic.

With $A, B \in \operatorname{gl}(2, R)$ we let

$$
\begin{align*}
\mathrm{U}(2) & =\operatorname{Sp}(4, \mathrm{R}) \cap \mathrm{SO}(4, \mathrm{R}) \\
& =\left\{\left.\left(\begin{array}{cc}
A-B \\
B & A
\end{array}\right) \right\rvert\,\left(A A^{\mathrm{t}}+B B^{\mathrm{t}}\right)=I_{2}, A B^{\mathrm{t}}=B A^{\mathrm{t}}\right\} \tag{2.3}
\end{align*}
$$

be the unitary group with Lie algebra

$$
\begin{align*}
\mathbf{u}(2) & =\mathrm{sp}(4, \mathrm{R}) \cap \mathrm{so}(4, \mathrm{R}) \\
& =\left\{\left.\left(\begin{array}{cc}
a-b \\
b & a
\end{array}\right) \right\rvert\, a=-a^{\mathrm{t}}, b=b^{\mathrm{t}}\right\} . \tag{2.4}
\end{align*}
$$

(We remark that in many books on Lie groups $\mathrm{Sp}(4, R)$ is denoted by $\mathrm{Sp}(2, \mathrm{R})$, etc.) Then
$\operatorname{SU}(2)=\left\{\left.\left(\begin{array}{lr}A-B \\ B & A\end{array}\right) \in \mathrm{U}(2) \right\rvert\, \operatorname{det}(A+\mathrm{i} B)=1\right\}$
is the special unitary group with Lie algebra
$\operatorname{su}(2)=\left\{\left.\left(\begin{array}{cc}a-b \\ b & a\end{array}\right) \in u(2) \right\rvert\, \operatorname{Tr}(b)=0\right\}$.
Let
$e_{1}=\left(\begin{array}{rr|rr}0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0\end{array}\right), \quad e_{2}=\left(\begin{array}{rr|rr}0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0\end{array}\right)$,
$e_{3}=\left(\begin{array}{rr|rr}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{array}\right), \quad e_{4}=\left(\begin{array}{rr|rr}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline-1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0\end{array}\right)$.

Then $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ is a basis for $\mathrm{u}(2)$, and $\left\{e_{1}, e_{2}, e_{3}\right\}$ is a basis for su(2).

For $V \in S U(2)$ and $x \in \mathrm{R}^{4}$ the linear symplectic action $(V, x) \rightarrow V \cdot x$ of $\mathrm{SU}(2)$ on $\left(\mathrm{R}^{4}, \omega\right)$ has an associated momentum mapping $J: \mathrm{R}^{4} \rightarrow$ su(2)* given by (see [1, pp. 287-288(iv)])
$J(x) \cdot e=\frac{1}{2} \omega(e x, x)$,
where $e=\left(a_{1} e_{1}+a_{2} e_{2}+a_{3} e_{3}\right) \in \operatorname{su}(2)$ and the $a_{j}$, $j=1,2,3$, are real coefficients.

With (,) as the standard inner product on $\mathrm{R}^{4}$, we have

$$
\begin{align*}
J(x) \cdot e & =(1 / 2)\left\langle e x, e_{4} x\right\rangle \\
& =(1 / 2)\left[a_{1} W_{1}+a_{2} W_{2}+a_{3} W_{3}\right], \tag{2.9}
\end{align*}
$$

where the $W_{i}$ are the Hopf variables (see [11, p. 654])
$W_{1}=2\left(x_{1} x_{2}+y_{1} y_{2}\right)$,
$W_{2}=2\left(x_{2} y_{1}-x_{1} y_{2}\right)$,
$W_{3}=\left(x_{1}^{2}+y_{1}^{2}-x_{2}^{2}-y_{2}^{2}\right)$,
$W_{4}=\left(x_{1}^{2}+x_{2}^{2}+y_{1}^{2}+y_{2}^{2}\right)=2 H_{2}$.
The $W_{j}(x)=\left\langle e_{j} x, e_{4} x\right\rangle$ for $j=1,2,3,4$, satisfy the identity
$\left(W_{1}^{2}+W_{2}^{2}+W_{3}^{2}\right)=W_{4}^{2}$.
The mapping $J$ can then be written
$J(x)=\frac{1}{2}\left[W_{1} e_{1}^{*}+W_{2} e_{2}^{*}+W_{3} e_{3}^{*}\right]$,
where $\left\{e_{i}^{*}\right\}$ is the basis of $\operatorname{su}(2)^{*}$ dual to $\left\{e_{i}\right\}$.
We now show that $J(M)$ is a coadjoint orbit in $\operatorname{su}(2)^{*}$. For $V \in \operatorname{SU}(2)$ and $e \in \operatorname{su}(2)$ we have for every $x \in \mathrm{R}^{4}$

$$
\begin{align*}
J(V x) \cdot e & =\frac{1}{2} \omega(e V x, V x)=\frac{1}{2} \omega\left(V^{-1} e V x, x\right) \\
& =J(x) \cdot V^{-1} e V, \tag{2.13}
\end{align*}
$$

or
$J(V x)=\operatorname{Ad}_{V^{-1} \circ}^{*} J(x)$.
Thus, $J$ is an Ad*-equivariant momentum map- $^{*}$ ping for the linear symplectic action of $\mathrm{SU}(2)$ on $\left(R^{4}, \omega\right)$. Since $\operatorname{SU}(2)$ gives a transitive action on the 3 -sphere $M$, we have for $x_{0}=\left((2 h)^{1 / 2}, 0,0,0\right)$ that

$$
\begin{align*}
J(M) & =J\left(\operatorname{SU}(2) \cdot x_{0}\right) \\
& =\left\{\operatorname{Ad}_{V^{-1}}^{*} \circ J\left(x_{0}\right) \mid V \in \operatorname{SU}(2)\right\} \subset \operatorname{su}(2)^{*}, \tag{2.15}
\end{align*}
$$

where $J\left(x_{0}\right)=h e_{3}^{*}$. Thus, in particular, the coadjoint orbit $J(M)$ is even dimensional and has a standard symplectic form given by the Kirillov-Kostant-Souriau theorem [1, p. 302]. We now determine a symplectic form $\Omega$ on this coajoint orbit so that $J^{*} \Omega=i^{*} \omega$ where $i: M \rightarrow$ $R^{4}$ is inclusion.
Let $x \in \mathrm{R}^{4}=P$ and $J(x)=\mu \in \operatorname{su}(2)^{*}=Q$. Let
$\xi_{P}(x)$ denote the infinitesimal generator at $x$ corresponding to $\xi \in \operatorname{su}(2)$ of the above linear action of $\operatorname{SU}(2)$ on $P$, and $\xi_{Q}(\mu)$ the corresponding entity for the $\mathrm{Ad}^{*}$-action of $\mathrm{SU}(2)$ on $Q$ (see [1, p. 267] for definitions). Letting $J_{*}$ denote the tangent map of $J$, the equivariance relation (2.14) implies $J_{*} \xi_{P}(x)=\xi_{Q}(\mu)$ (see [1, p. 270]). Let $\{$,$\} denote the Poisson brackets given$ by the standard symplectic form $\omega$ on $\mathrm{R}^{4}$ (see [1, p. 192]). On infinitesimalizing the $\mathrm{Ad}^{*}$-equivariance of $J$ in formula (2.14), we obtain for $\xi, \eta \in$ su(2) on setting $J^{*} \Omega=i^{*} \omega$ (see [1, p. 276 and p. 281] for notation):

$$
\begin{align*}
\Omega\left(\xi_{Q}, \eta_{Q}\right)(\mu) & =\Omega\left(J_{*} \xi_{P}, J_{*} \eta_{P}\right)(J x) \\
& =(J * \Omega)\left(\xi_{P}, \eta_{P}\right)(x) \\
& =\omega\left(\xi_{P}, \eta_{P}\right)(x) \\
& =\omega\left(X_{f(\xi)}, X_{(\xi \eta)}\right)(x) \\
& =\{\hat{J}(\xi), \hat{J}(\eta)\}(x) \\
& =\hat{J}[\xi, \eta](x) \\
& =J(x) \cdot[\xi, \eta] \\
& =\mu \cdot[\xi, \eta] . \tag{2.16}
\end{align*}
$$

Thus $\Omega$ is the Kirillov-Konstant-Souriau form on the coadjoint orbit $J(M)$ (modulo signum; see [1, p. 303]).

We will eventually realize the reduced space $M_{R}$ in $R^{3}$, and for this purpose we must first pass from $\operatorname{su}(2)^{*}$ to $\operatorname{su}(2)$. Let $e=\sum_{i=1}^{3} a_{i} e_{i}$ and $e^{\prime}=$ $\Sigma_{i=1}^{3} a_{i}^{\prime} e_{i}$ be in su(2). We define a positive definite inner product $\gamma$ on $\operatorname{su}(2)$ by

$$
\begin{equation*}
\gamma\left(e, e^{\prime}\right)=-\frac{1}{4} \operatorname{Tr}\left(e \cdot e^{\prime}\right)=\left[a_{1} a_{1}^{\prime}+a_{2} a_{2}^{\prime}+a_{3} a_{3}^{\prime}\right] . \tag{2.17}
\end{equation*}
$$

Then $\gamma$ is a Killing form on su(2) that induces a linear map $\gamma^{b}: \operatorname{su}(2) \rightarrow \operatorname{su}(2)^{*}$ by $\gamma^{b}(e) \cdot e^{\prime}=$ $\gamma\left(e, e^{\prime}\right)$. We set $\gamma^{*}=\left(\gamma^{b}\right)^{-1}$. A direct computation then shows that the following diagram is commutative:


It follows that $\left(\gamma^{*} \circ J\right)(M)$ is an orbit of the adjoint action of $\mathrm{SU}(2)$ on its Lie algebra. Infinitesimalizing the equivariance relation (2.18) gives for $\xi \in \operatorname{su}(2)=T$ and $\operatorname{su}(2)^{*}=Q$ (see [1, p.270])

$$
\begin{equation*}
\gamma^{b}\left(\xi_{T}\right)=\xi_{Q} \circ \gamma^{b}, \tag{2.19}
\end{equation*}
$$

since $\left(\gamma^{b}\right)_{*}=\gamma^{b}$ as $\gamma^{b}$ is linear. A computation similar to (2.16) then shows that the symplectic form $\tilde{\Omega}=\left(\gamma^{b}\right) * \Omega$ on this adjoint orbit satisfies
$\tilde{\Omega}\left(\xi_{T}, \eta_{T}\right)(\nu)=\gamma(\nu,[\xi, \eta])$
for $\xi, \eta, \nu \in \operatorname{su}(2)$ where $\xi_{T}(\nu)=[\xi, \nu]$ (see $[1$, pp. 267-268(b)]).

The last stage in realizing the reduced space $M_{R}$ as a subset of $\mathrm{R}^{3}$ requires an identification of $\operatorname{su}(2)$ with $\mathrm{R}^{3}$. Let $\left\{f_{1}, f_{2}, f_{3}\right\}$ be the standard orthonormal basis of $\mathrm{R}^{3}$. Define a linear map $\lambda: \mathrm{R}^{3} \rightarrow \operatorname{su}(2)$ by $\lambda\left(f_{i}\right)=\frac{1}{2} e_{i}$ for $i=1,2,3$. With $\Gamma$ as the standard inner product on $\mathrm{R}^{3}$, note that $(2 \lambda)$ is an isometry of ( $\mathrm{R}^{3}, \Gamma$ ) with ( $\left.\mathrm{su}(2), \gamma\right)$; that is, $(2 \lambda)^{*} \gamma=\Gamma$.

We compute from (2.7) the su(2) Lie bracket $\left[e_{1}, e_{2}\right]=\left(e_{1}, e_{2}-e_{2} e_{1}\right)=-2 e_{3}$. Hence $\left[\lambda f_{1}, \lambda f_{2}\right]=-\lambda\left(f_{1} \times f_{2}\right)$ where $\times$ is the standard cross product on $\mathrm{R}^{3}$. There are corresponding identities on cyclic permutation of the indices. Thus $\lambda: \mathrm{R}^{3} \rightarrow \mathrm{su}(2)$ is a Lie algebra isomorphism relative to $(-1) \times$ as the bracket operation on $R^{3}$.

Letting $\tilde{J}=\left(\lambda^{-1} \circ \gamma^{*} \circ J\right)$, we derive from (2.12) that
$\tilde{J}(x)=\left[W_{1} f_{1}+W_{2} f_{2}+W_{3} f_{3}\right]=\left(W_{1}, W_{2}, W_{3}\right) \in \mathrm{R}^{3}$.

Restricting $x \in M$, the identity (2.11) implies

$$
\begin{equation*}
\tilde{J}(M)=S_{(2 h)}^{2} \subset \mathrm{R}^{3} \tag{2.22}
\end{equation*}
$$

Thus the coadjoint orbit in su(2)* has now been mapped onto a sphere in $R^{3}$. We can now compute the symplectic form $\omega_{\mathrm{R}}$ on this sphere satisfying $\omega_{\mathrm{R}}=\lambda^{*} \tilde{\Omega}$. Let $\tilde{\xi} \times \tilde{v}$ and $\tilde{\eta} \times \tilde{\nu}$ be
tangent vectors at $\tilde{\nu} \in \mathrm{S}_{(2 h)}^{2}$. Let $\lambda(\tilde{\xi})=\xi, \lambda(\tilde{\eta})=$ $\eta, \lambda(\tilde{\nu})=\nu$ be in $\operatorname{su}(2)$, and recall that $\lambda(\tilde{\xi} \times \tilde{\nu})=$ $-[\xi, \nu]$, etc. Then using (2.20) we have

$$
\begin{align*}
\omega_{\mathrm{R}}(\tilde{\xi} \times \tilde{\nu}, \tilde{\eta} \times \tilde{\nu})(\tilde{\nu}) & =\left(\lambda^{*} \tilde{\Omega}\right)(\tilde{\xi} \times \tilde{\nu}, \tilde{\eta} \times \tilde{\nu})(\tilde{\nu}) \\
& =\tilde{\Omega}(\lambda(\tilde{\xi} \times \tilde{\nu}), \lambda(\tilde{\eta} \times \tilde{\nu}))(\nu) \\
& =\tilde{\Omega}([\xi, \nu],[\eta, \nu])(\nu) \\
& =\gamma(\nu,[\xi, \eta]) \\
& =-\frac{1}{4} \gamma((2 \lambda)(\tilde{\nu}),(2 \lambda)(\tilde{\xi} \times \tilde{\eta})) \\
& =-\frac{1}{4}\left((2 \lambda)^{*} \gamma\right](\tilde{\nu}, \tilde{\xi} \times \tilde{\eta}) \\
& =-\frac{1}{4} \Gamma(\tilde{\nu}, \tilde{\xi} \times \tilde{\eta}) \tag{2.23}
\end{align*}
$$

since $(2 \lambda)$ is an isometry. Note that the result is independent of the representation of the tangent vectors $\tilde{\xi} \times \tilde{\nu}=(\tilde{\xi}+r \tilde{\nu}) \times \tilde{\nu}$ where $r \in \mathbf{R}^{\prime}$.

For $h=\frac{1}{2}$ formulas (2.21) and (2.22) show that the map $\tilde{J}$ is the Hopf map of $S^{3}$ onto $S^{2}$ (see [11, p. 654]). The Hopf map is a fibration of $S^{3}$ over $S^{2}$ with fiber $S^{1}$ and Hopf invariant +1 . One easily calculates from (2.10) that $L_{X_{2}}\left(W_{i}\right)=0$ for $i=1,2,3,4$, and hence the components of $\tilde{J}$ are integrals of $X_{2}$. Thus each fiber $(\tilde{J})^{-1}(w)$ for $w \in S_{(2 h)}^{2}$ is an integral manifold of $X_{2}$ that is diffeomorphic to $S^{1}$ and hence is a unique $\Phi$ orbit in $M$. We choose $\left(M_{R}, \omega_{\mathrm{R}}, \pi\right)=\left(\mathrm{S}_{(2 h)}^{2}, \omega_{\mathrm{R}}, \tilde{J}\right)$ as a model for the reduced space and note that by our construction $\tilde{J}=\pi: M \rightarrow M_{R}$ satisfies $\pi^{*} \omega_{\mathrm{R}}=i^{*} \omega$ where $\omega_{\mathrm{R}}$ is given by (2.23). We can now compute the induced differential equations on $M_{R}$.

In section 3 below we show that (1.3) implies that $H$ as in (1.2) satisfies
$H_{2 i}\left(x_{1}, x_{2}, y_{1}, y_{2}\right)=K_{i}\left(W_{1}, W_{2}, W_{3}, W_{4}\right)$,
where $K_{i}$ is a homogeneous polynomial of degree $i$ in the Hopf variables (2.10). Letting $w=\left(w_{1}, w_{2}, w_{3}\right)$ be global coordinates on $\mathrm{R}^{3}$, we define
$K^{e}(w)=\sum_{i=1}^{n} K_{i}\left(w_{1}, w_{2}, w_{3}, 2 h\right)$,
and set
$K=K^{e} \mid M_{\mathrm{R}}$.

Clearly $K \circ \pi=H \circ i$, so that $K$ is the reduced Hamiltonian associated to $H$ (see [1, p. 304]). By (2.23) the Hamiltonian vector field $X_{K}(w)=$ $A \times w$ satisfies for an arbitrary tangent vector $B \times w$ at $w \in M_{\mathrm{R}}$ :

$$
\begin{align*}
\Gamma\left(w \times \nabla K^{e}(w), B\right) & =\Gamma\left(\nabla K^{e}(w), B \times w\right) \\
& =\mathrm{d} K(w) \cdot(B \times w) \\
& =\omega_{\mathrm{R}}\left(X_{K}(w), B \times w\right)(w) \\
& =\omega_{\mathrm{R}}(A \times w, B \times w)(w) \\
& =-\frac{1}{4} \Gamma(w, A \times B) \\
& =\frac{1}{4} \Gamma\left(X_{K}(w), B\right) . \tag{2.27}
\end{align*}
$$

Again, the result is independent of the representation of the tangent vector $B \times w=$ $(B+r w) \times w$ where $r \in \mathrm{R}^{1}$. Since $B$ is arbitrary, the reduced vector field $X_{K}$ is just Euler's equations (modulo the factor 4) restricted to $M_{\mathrm{R}}$ :
$\dot{w}=X_{K}(w)=4 w \times \nabla K^{e}(w)$,
or
$\left(X_{K} \circ \pi\right)(x)=4 \pi(x) \times\left[\left(\nabla K^{e}\right) \circ \pi(x)\right]$.
Note that by (2.29) the critical points of $K$ on $M_{\mathrm{R}}$ are just the points where $\nabla K^{e}(w)$ is normal to the sphere $M_{R}$ at $w$. If $X_{K}(\pi(p))=0$, then $p$ is called a relative equilibrium for $H$. There is then an orbit of $X_{H}$ through $p$ that is a reparametrized orbit of $X_{H_{2}}$ (see [1, p. 306]). Under suitable conditions this will be a periodic orbit of $X_{H}$. This approach was used in [5] and [13, 14] to find periodic orbits at low positive energies in the Hénon-Heiles and related Hamiltonians. For computational purposes involving stability questions about the periodic orbit, it is often convenient to rotate the critical point to one of the poles $(0,0, \pm 2 h)$. This can be done by a rotation of $M_{R}$ that is induced by a linear symplectic change of coordinates in $\mathrm{R}^{4}$ given by a suitable $V \in S U(2)$. This construction and the applications to stability analysis of the periodic orbits are given in [5] and [13, 14] (see also [4, pp. 36-38]).

## 3. Normal form of the semisimple 1:1 resonance

We now verify the relation given in formula (2.24). For an alternate proof see [5, lemma 4.1]. Recall that the Hopf variables of (2.10) satisfy $L_{X_{2}}\left(W_{i}\right)=0$ for $i=1,2,3,4$, and that $\left(W_{1}^{2}+\right.$ $\left.W_{2}^{2}+W_{3}^{2}\right)=W_{4}^{2}$. Let $\quad P_{n}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \quad$ (resp. $\left.Q_{m}\left(W_{1}, W_{2}, W_{3}, W_{4}\right)\right)$ be the vector space of homogeneous polynomials with real coefficients in the specified variables of degree $n$ (resp. $m$ ). Because $L_{X_{2}}$ is a derivation we have

$$
\begin{align*}
& Q_{m}\left(W_{1}, W_{2}, W_{3}, W_{4}\right) \\
& \quad \subseteq \operatorname{Ker}\left(L_{X_{2}} \mid P_{2 m}\left(x_{1}, x_{2}, y_{1}, y_{2}\right)\right) \tag{3.1}
\end{align*}
$$

We wish to show that equality holds in (3.1). This will be done by computing the dimensions of the vector spaces on both sides of (3.1). Using the above relation between the Hopf variables we see that as vector spaces we have

$$
\begin{align*}
& Q_{m}\left(W_{1}, W_{2}, W_{3}, W_{4}\right)=Q_{m}\left(W_{1}, W_{2}, W_{3}\right) \\
& \quad+W_{4} \cdot Q_{m-1}\left(W_{1}, W_{2}, W_{3}\right) . \tag{3.2}
\end{align*}
$$

Now the dimension of the right-hand side of (3.2) is
$\frac{1}{2}(m+2)(m+1)+\frac{1}{2}(m+1)(m)=(m+1)^{2}$.
To compute the dimension of $\operatorname{Ker}\left(L_{X_{2}} \mid P_{n}\right)$ we introduce complex conjugate variables $(k=1,2)$

$$
\begin{array}{ll}
z_{k}=x_{k}+\mathrm{i} y_{k}, & \partial / \partial z_{k}=\frac{1}{2}\left[\partial / \partial x_{k}-\mathrm{i} \partial / \partial y_{k}\right]  \tag{3.4}\\
\bar{z}_{k}=x_{k}-\mathrm{i} y_{k}, & \partial / \partial \bar{z}_{k}=\frac{1}{2}\left[\partial / \partial x_{k}+\mathrm{i} \partial / \partial y_{k}\right]
\end{array}
$$

This coordinate change diagonalizes

$$
\begin{equation*}
X_{2}=\left[y_{1}\left(\partial / \partial x_{1}\right)+y_{2}\left(\partial / \partial x_{2}\right)-x_{1}\left(\partial / \partial y_{1}\right)-x_{2}\left(\partial / \partial y_{2}\right)\right] \tag{3.5}
\end{equation*}
$$

to

$$
\begin{align*}
\tilde{X}_{2}= & -\mathrm{i}\left[z_{1}\left(\partial / \partial z_{1}\right)+z_{2}\left(\partial / \partial z_{2}\right)\right. \\
& \left.-\bar{z}_{1}\left(\partial / \partial \bar{z}_{1}\right)-\bar{z}_{2}\left(\partial / \partial \bar{z}_{2}\right)\right] \tag{3.6}
\end{align*}
$$

and transforms $P_{n}\left(x_{1}, x_{2}, y_{1}, y_{2}\right)$ to

$$
\begin{aligned}
& H P_{n}\left(z_{1}, z_{2}, \bar{z}_{1}, \bar{z}_{2}\right) \\
& =\left\{\sum c\left(i_{1}, i_{2}, j_{1}, j_{2}\right) z_{1}^{i_{1}} z_{2}^{i_{2}} \bar{z}_{1}^{j_{1}} \bar{z}_{2}^{j_{2}} \mid\left(i_{1}+i_{2}+j_{1}+j_{2}\right)=n,\right.
\end{aligned}
$$

and
$\left.c\left(i_{1}, i_{2}, j_{1}, j_{2}\right)=\bar{c}\left(j_{1}, j_{2}, i_{1}, i_{2}\right)\right\}$.
Since any polynomial in $H P_{n}$ lying in $\operatorname{Ker}\left(L_{\hat{X}_{2}} \mid H P_{n}\right)$ corresponds to a unique polynomial in $P_{n}$ lying in $\operatorname{Ker}\left(L_{X_{2}} \mid P_{n}\right)$ under the above coordinate change (3.4), we have
$\operatorname{dim} \operatorname{Ker}\left(L_{X_{2}} \mid P_{n}\right)=\operatorname{dim} \operatorname{Ker}\left(L_{\tilde{X}_{2}} \mid H P_{n}\right)$.

Now,

$$
\begin{equation*}
L_{\tilde{X}_{2}}\left(z_{1}^{i_{1}} z_{2}^{i_{2}} \bar{z}_{1}^{j_{1}} \bar{z}_{2}^{j_{2}}\right)=-\mathrm{i}\left(i_{1}+i_{2}-j_{1}-j_{2}\right)\left(z_{1}^{i_{1}} z_{2}^{i_{2}} \bar{z}_{1}^{j_{1}} \bar{z}_{2}^{j_{2}}\right) \tag{3.9}
\end{equation*}
$$

implies that $\operatorname{dim} \operatorname{Ker}\left(L_{\dot{X}_{2}} \mid H P_{n}\right)$ is the number of non-negative integer solutions of
$\left(i_{1}+i_{2}-j_{1}-j_{2}\right)=0$,
$\left(i_{1}+i_{2}+j_{1}+j_{2}\right)=n$,
which is equivalent to
$2\left(i_{1}+i_{2}\right)=n=2\left(j_{1}+j_{2}\right)$.

Thus $n=2 m$ for some $m$ in order that $\operatorname{Ker}\left(L_{X_{2}} \mid P_{n}\right)$ be non-zero. The number of nonnegative integer solutions of (3.11) when $n=2 m$ is $(m+1)^{2}$, and hence equality holds in (3.1).

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