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Reduction technique for matrix nonlinear evolution equations solvable by the spectral transform — Source link

Francesco Calogero, Antonio Degasperis

Institutions: University of Groningen

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Reduction technique for matrix nonlinear evolution equations solvable by the spectral transform

F. Calogero [a)]
Institute for Theoretical Physics, University of Groningen, Groningen, The Netherlands

A. Degasperis
Istituto di Fisica, Università di Roma, 00185 Roma, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Roma, Italy

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The main purpose of this paper is to describe a technique of reduction, whereby from the class of evolution equations for matrices of order N solvable via the spectral transform associated to the (matrix) linear Schrödinger eigenvalue problem, one derives subclasses of nonlinear evolution equations involving less than N^2 fields. To illustrate the method, from the equations for matrices of order 2 two subclasses of equations for 2 fields (rather than 4) are obtained. The first class coincides, or rather includes, that solvable via the spectral transform associated to the generalized Zakharov–Shabat spectral problem; further reduction to nonlinear evolution equations for a single field reproduces a number of well-known equations, but also yields a novel one (highly nonlinear). The second class also yields highly nonlinear equations; some examples are given, including another novel evolution equation for a single field.

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1. INTRODUCTION

Recently we have introduced and discussed a class of matrix nonlinear evolution equations that can be solved via the spectral transform associated with the matrix Schrödinger spectral (or "scattering") problem.[1] These equations involve generally N^2 fields (here and below N is the order of the matrices under consideration); but this number can be reduced by identifying equations (or rather classes of equations) that are satisfied by matrices having some special structure. For instance the requirement that a matrix of order N be Hermitian halves the number of independent fields (from N^2 complex fields to N^2 real fields); the requirement that it be symmetrical reduces the number of independent fields to $\frac{1}{2}N(N+1)$; and so on. Such reductions are, however, rather trivial, and the corresponding restrictions on the class of evolution equations, that are required to guarantee compatibility with the time evolution, are easily established.[1] But other reductions are also possible, that decrease the number of independent fields by inducing nontrivial relations between different matrix elements that are compatible with the time evolution (for appropriately restricted classes of equations). The main purpose of the present paper is to introduce a technique to identify such reductions. The method is then illustrated by applying it to the case of matrices of order 2, thereby obtaining, from the general class of equations involving 4 independent fields, sublcasses of equations involving only two fields, or just a single one. One such class coincides with (or rather includes, since there is one added element of generality) that solvable via the spectral transform associated to the generalized Zakharov-Shabat spectral problem[2]; a result that has been obtained independently by Jaulent and Leon.[3]

For matrices of order 4, the simpler equation of the class solvable via the Schrödinger spectral transform has been analyzed by Bruschi, Levi, and Ragnisco.[4] This equation involves of course 16 fields; reduced versions involving respectively 10, 8, 6, 5, or 4 fields have also been obtained, by identifying the cases in which some of the 16 fields, if vanishing at the initial time, continue to vanish for all time.[4] Thus these reductions are rather simple; although the equations obtained in this manner are certainly far from trivial. All these reductions can be treated by the technique described in this paper, but this technique is actually richer. We plan to present the results obtained by its application to matrices of order 3 and 4 in separate papers.

The plan of this paper, and an outline of its content, can be evinced from the titles of the following sections and subsections. Here we merely report two novel, highly nonlinear evolution equations involving a single field, whose solvability is demonstrated below. The first reads

$$u_t = u_{xxx} - 6u_x\{u^2 - (u + u_{xx} - 2u^3)^2/[a^2 - 4(u^2 + u_x^2 - u^4)]\}; \tag{1.1}$$

$u \equiv u(x,t)$, $u(+\infty,t) = 0$, $u(-\infty,t) = 0$ if $a^2 \neq 1$,
$u(-\infty,t) =$ arbitrary constant if $a^2 = 1$.

The second reads

$$v_t = v_{xxx} - \tfrac{1}{8}v_x^3 + v_x[A\exp(v) + B\exp(-v) + C];$$
$v \equiv v(x,t)$, $v(+\infty,t) = 0$, $v(-\infty,t) = 0$
or
$$v(-\infty,t) = \ln(B/A). \tag{1.2}$$

2. PRELIMINARIES AND NOTATION

The class of matrix nonlinear evolution equations solvable via the spectral transform associated with the matrix Schrödinger spectral problem reads[1]

$$Q_t = \alpha_m(\underline{L})[\sigma_m, Q] + \beta_\mu(\underline{L})G\sigma_\mu. \tag{2.1}$$

[a)] Permanent address: Istituto di Fisica, Università di Roma, 00185 Roma, Italy.

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Here and below $Q \equiv Q(x,t)$ is a matrix of order N vanishing (sufficiently fast[1]) asymptotically,

$$Q(\pm\infty,t)=0. \tag{2.2}$$

Latin subscripts run from 1 to N^2-1, Greek subscripts from 0 to N^2-1 and repeated subscripts are summed upon. The N^2 matrices σ_μ provide a basis for matrices of order N, with $\sigma_0=\mathbf{1}$; the $2N^2-1$ functions $\alpha_m(z)$ and $\beta_\mu(z)$ are ratios of entire functions (in all interesting cases, they are in fact rational functions; in most interesting cases, they are just polynomials of low degree); except for these restrictions, these functions are arbitrary, and it is their choice that characterizes each particular evolution equation of the class (2.1). The possibility to solve the Cauchy problem for (2.1) via the spectral transform technique is maintained even if the functions α_m and β_μ depend explicitly on the time variable t; but we assume, for the sake of simplicity, that they are time independent. Then the evolution Eq. (2.1) is invariant under time translations; the (Cauchy) problem we shall always have in mind is the determination of $Q(x,t)$ for $t>0$ given by

$$Q(x,0)=\bar{Q}(x) \tag{2.3}$$

(of course with $\bar{Q}(\pm\infty)=0$). Finally the integro-differential operators $\underline{L}$ and $\underline{G}$ are defined by the following formulas that detail their action on the generic matrix $F(x)$:

$$\underline{L}F(x)=F_{xx}(x)-2\{Q(x,t),F(x)\}+\underline{G}\int_x^\infty dx'\,F(x'), \tag{2.4}$$

$$\underline{G}F(x)=\{Q_x(x,t),F(x)\}+\left[Q(x,t),\int_x^\infty dx'\,[Q(x',t),F(x')]\right]. \tag{2.5}$$

Here of course, as well as above and below, subscripted variables denote partial differentiation, and the square and curly brackets with a comma inside indicate as usual commutators and anticommutators:

$$[A,B]=AB-BA,\quad \{A,B\}=AB+BA. \tag{2.6}$$

The solvability via spectral transform of (2.1) hinges essentially on the fact that the corresponding evolution equation for the reflection coefficient $R(k,t)$ is linear[1]:

$$R_t(k,t)=[A(-4k^2),R(k,t)] + 2ik\,\{B(-4k^2),R(k,t)\}; \tag{2.7}$$

here and always below

$$A(z)\equiv\alpha_m(z)\sigma_m,\quad B(z)\equiv\beta_\mu(z)\sigma_\mu\,. \tag{2.8}$$

In fact, to solve completely the Cauchy problem via the spectral transform, the time evolution of the appropriate parameters corresponding to the discrete part of the spectrum (if any) must also be given[1]; but we assume for simplicity that these results can all be extracted by analytic continuation in k of R to the poles on the upper imaginary axis[1]; so that in the following we limit our analysis to the time evolution of R. This simplifies considerably our presentation; of course the results are then, strictly speaking, established only for Hermitian matrices Q vanishing asymptotically faster than exponentially; but they clearly have a more general validity, as can be easily demonstrated by looking directly also at the time evolution of the part of the spectral transform associated to discrete eigenvalues.[1]

The basic tool of our treatment obtains from the Wronskian-type formulas[1]

$$2ik\,[F(-4k^2),R(k)] = \int_{-\infty}^{+\infty} dx\bar{\Psi}(x,k)\{f_m(\underline{L})[\sigma_m,Q(x)]\}\Psi(x,k), \tag{2.9a}$$

$$(2ik)^2\{H(-4k^2),R(k)\} = \int_{-\infty}^{+\infty} dx\bar{\Psi}(x,k)\{h_\mu(\underline{L})\underline{G}\sigma_\mu\}\Psi(x,k). \tag{2.9b}$$

Here and below, $f_m(z)$ and $h_\mu(z)$ are arbitrary entire functions (in fact, in all applications below, low-order polynomials);

$$F(z)\equiv f_m(z)\sigma_m,\quad H(z)\equiv h_\mu(z)\sigma_\mu\,; \tag{2.10}$$

$\bar{\Psi}$ and Ψ are appropriate matrix solutions of the Schrödinger equation characterizing the spectral problem[1]; while the remaining symbols have already been defined. We have not indicated explicitly, in these equations, the time dependence (of R, Q, $\bar{\Psi}$, and Ψ); indeed these equations are merely a consequence of the spectral problem, having nothing to do with the time evolution. But they remain of course valid if Q, and therefore also R, $\bar{\Psi}$ and Ψ, depend on time (such dependence is indeed, from the spectral point of view, purely parametric).

3. REDUCTION TECHNIQUE

The task here is to identify matrices Q having a special structure that is maintained as they evolve in time according to (2.1), or rather according to some appropriate subclass of (2.1). The essential requirement characterizing such a special structure is that it induce, at any given time, relations between the different matrix elements of Q, so as to reduce the number of these that can be assigned independently (as functions of x, for any given t and in particular for $t=0$); *these relations need not be algebraic, but can in fact be integro-differential* (see below).

Since the time evolution (2.1) of Q is complicated, while the corresponding time evolution (2.7) of R is simple [indeed this simplicity constitutes the foundation of the spectral transform technique to solve (2.1)], it is clearly easier to find matrices R that have a special structure compatible with the time evolution. On the other hand, since there is a one-to-one correspondence between R and Q (up to the discrete spectrum part of the spectral transform, that, as explained above, is ignored in this analysis), clearly to any reduction in the number of independent elements of R (each being a function of k) there corresponds an analogous reduction in the number of independent elements of Q (each being a function of x).

Thus the main question is to translate a special structure of R into the corresponding special structure of Q; or rather, to identify those special structures of R that make such a translation easy (namely, to identify those restrictions on R such that the corresponding restrictions on Q are easily ascertained). A convenient tool to achieve this goal was reported at the end of Sec. 2, for the results (2.9) imply that, if the matrix $Q(x,t)$ satisfies the (nonlinear integro-differential) equation

$$f_m(\underline{L})[\sigma_m,Q] + h_\mu(\underline{L})\underline{G}\sigma_\mu = 0, \tag{3.1}$$

the corresponding matrix $R(k,t)$ satisfies the linear equation

$$[F(-4k^2),R] + 2ik\{H(-4k^2),R\} = 0, \tag{3.2}$$

where the matrices F and H are of course defined by (2.10). Note that in these equations the $2N^2-1$ functions $f_m(z)$ and $h_\mu(z)$ are arbitrary (they must be entire; in all practical applications they will be low-order polynomials).

The matrix equation (3.2) yields of course, for given F and H, N^2 homogeneous linear equations for the N^2 elements of R; thus, for a generic choice of F and H, it is compatible only with the trivial solution $R=0$. But for appropriate choices of F and H, the restriction (3.2) merely implies a reduction in the number of independent elements of R; and the corresponding relation for Q is then explicitly given by (3.1). Note that this last equation is generally integro-differential and nonlinear [see (2.4) and (2.5)]; however, if the functions $f_m(z)$ and $h_\mu(z)$ are polynomials of very low order (zero, or perhaps one) (3.1) can be explicitly solved; namely the relations between the different matrix elements of Q implied by (3.1) can be rewritten as explicit expressions of some elements in terms of the others (see below).

Of course this process of reduction can be applied more than once, namely it can be required that R satisfy n equations of type (3.2) (with $F(z)=F^{(j)}(z)$, $H(z)=H^{(j)}(z)$, $j=1,2,...,n$), the corresponding Q being then constrained by the n corresponding equations of type (3.1).

Thus, this technique provides the possibility to translate appropriate types of constraint on $R(k)$ into the corresponding constraints on $Q(x)$, and vice versa. Let us emphasize that one is displaying here certain properties of the spectral transform, that have *a priori* nothing to do with the time evolution, and which may indeed also have applications just in the context of the spectral (or "scattering") problem. But of course if Q, and therefore R, evolve in time, the question of compatibility of any condition imposed on these matrices arises: if at the initial time Q resp. R satisfy a certain restriction of type (3.1) resp. (3.2), shall they satisfy it for all subsequent time? We identify below subclasses of the evolution Eq. (2.1) for which this is the case; clearly each evolution equation of these subclasses may be considered to describe the evolution of M fields, with $M<N^2$ (the precise value of M in each case depending on the specific case under consideration, see, for instance, the examples discussed below).

As we have already mentioned, rather than discussing the compatibility of a restriction of type (3.1) with the time evolution (2.1) of Q it is convenient to consider the compatibility of the corresponding restriction of type (3.2) with the time evolution (2.7) of R; the correspondence between R and Q being then a guarantee that one kind of compatibility implies the other.

Let us thus define

$$Z(k,t) = [F(-4k^2),R(k,t)] + 2ik\{H(-4k^2),R(k,t)\}, \tag{3.3}$$

in order to ascertain when $Z(k,t)=0$ is compatible with (2.7). Differentiating with respect to t and using (2.7) one easily obtains

$$Z_t(k,t) = [A(-4k^2),Z(k,t)] + 2ik\{B(-4k^2),Z(k,t)\} + C(k,t) \tag{3.4}$$

with

$$C(k,t) \equiv [R(k,t),([A(-4k^2),F(-4k^2)] - 4k^2[B(-4k^2),H(-4k^2)])] - 2ik\{R(k,t),([B(-4k^2),F(-4k^2)] + [A(-4k^2),H(-4k^2)])\}. \tag{3.5}$$

Thus $Z(k,0)=0$ implies $Z(k,t)=0$ for $t>0$ provided

$$C(k,t)=0; \tag{3.6}$$

this last equation is therefore the compatibility condition.

Note that C, as defined by (3.5), depends on the matrices F and H, that characterize the restrictive condition (3.2), and on the matrices A and B, that characterize the evolution equation (2.7); it depends moreover on R itself, that is of course *a priori* unknown except for the requirement that it satisfy the restriction (3.2). Thus (3.6) is required to hold for any R compatible with (3.2). Of course (3.6) is required to hold for all values of k.

There is always at least one evolution equation of the class (2.1) for which the compatibility condition holds, namely the "scalar" case corresponding to

$$\alpha_m(z) = \beta_m(z) = 0, \tag{3.7a}$$

or equivalently

$$A=0, \quad B=\beta_0(-4k^2)\mathbb{1}. \tag{3.7b}$$

Examples in which the reduction process is compatible with a larger subclass of (2.1) than this are given below.

If the compatibility condition (3.6) is satisfied, a matrix Q, that has been reduced by the condition (3.1) to have only $M<N^2$ independent elements (each being a function of x, for given t), may be required to evolve in time according to (2.1). Then this matrix evolution equation, although corresponding formally to N^2 scalar equations, yields in fact only M coupled evolution equations, the remaining N^2-M being automatically satisfied. Thus one is finally left with a system of M coupled evolution equations for M fields; these may be assigned (as functions of x, for $-\infty<x<\infty$) at any given time (and in particular at the initial time $t=0$), their values at all subsequent times being then determined by the requirement that they obey the system of evolution equations.

In conclusion, the process of reduction can be summarized as follows: (i) choose the matrices $F(z)$ and $H(z)$; (ii) ascertain the constraint they imply on R through (3.2); (iii) ascertain the constraint implied on $A(z)$ and $B(z)$ by the requirement that (3.6) hold for any R compatible with (3.2), as determined in step (ii) [of course with the same $F(z)$ and $H(z)$ in (3.6) as in (3.2)]. All these steps are algebraic, and they determine the class of reduced evolution equations. The corresponding structure for the matrix Q is determined by (3.1); this last step need not be purely algebraic. This process of reduction may be performed more than once, with different (judicious!) choices of F and H.

4. APPLICATION TO MATRICES OF ORDER 2

In this section the analysis is restricted to matrices of order 2, in which case the natural choice for the basic matrices σ_μ identifies them with the standard Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad (4.1)$$
$$\{\sigma_m, \sigma_n\} = 2\delta_{mn}, \quad [\sigma_m, \sigma_n] = 2i\epsilon_{mnl}\sigma_l .$$

Here of course δ_{mn} is the (symmetrical) Kronecker symbol ($\delta_{mn} = 1$ if $m = n$, $\delta_{mn} = 0$ if $m \neq n$) and ϵ_{mnl} is the (completely antisymmetrical) Ricci symbol ($\epsilon_{123} = 1$). It will be convenient to represent also the matrices $Q(x,t)$ and $R(k,t)$ in this basis, writing

$$Q \equiv Q_\mu \sigma_\mu = Q_0 + Q_m \sigma_m, \quad R \equiv R_\mu \sigma_\mu = R_0 + R_m \sigma_m . \qquad (4.2)$$

Thus our task here is (i) to analyze the constraint condition (3.2) [for various possible choices of the matrices $F(z)$ and $H(z)$] and to investigate how it reduces the number of independent components of R; (ii) to identify, using the condition (3.6), the subclass of the nonlinear evolution equations (2.1) that are compatible with the constraint; (iii) to extract from the corresponding constraint (3.1) relations determining some of the elements of Q in terms of the others, or equivalently some of the components Q_μ in terms of the remaining ones; (iv) to write explicitly the novel class of nonlinear evolution equations for the reduced number of fields, introducing at this stage if need be an appropriate notation (to make contact with known results) and discussing some specific examples.

We note first of all that, as can be easily shown, there is no choice of the matrices F and H in (3.2) that reduces the number of independent components of R from 4 to 3. There exist instead several possibilities to reduce the independent components to 2; and then the reduction process can be applied once more (sometimes rather trivially, sometimes nontrivially; see below) to reduce to one field only. The more interesting instances are discussed in Sec. 4.1–4.4.

4.1 Simple example: The class of nonlinear evolution equations solvable via the generalized Zakharov–Shabat spectral problem as a subcase of the class of nonlinear evolution equations solvable via the matrix Schrödinger spectral problem

Set

$$F(z) = 0, \quad H(z) = \sigma_3, \qquad (4.1.1)$$

in (3.2). There immediately follows

$$R(k,t) = R_1(k,t)\sigma_1 + R_2(k,t)\sigma_2, \quad (R_0(k,t) = R_3(k,t) = 0). \qquad (4.1.2)$$

It is also easy to obtain the corresponding relations for $Q(x,t)$ that obtain inserting (4.1.1) in (3.1):

$$Q(x,t) = Q_0(x,t) + Q_1(x,t)\sigma_1 + Q_2(x,t)\sigma_2, \quad (Q_3(x,t) = 0), \qquad (4.1.3)$$

$$Q_0(x,t) = \left[\int_x^\infty dx'\, Q_1(x',t)\right]^2 + \left[\int_x^\infty dx'\, Q_2(x',t)\right]^2. \qquad (4.1.4)$$

To obtain the last equation, we have used the boundary condition $Q_0(+\infty,t) = 0$; the condition $Q_0(-\infty,t) = 0$ implies a constraint on Q_1 and Q_2 (see below).

We consider next the compatibility condition (3.6), and using (4.1.2) it is easily seen that it implies

$$A(z) = \alpha_3(z)\sigma_3, \quad (\alpha_1(z) = \alpha_2(z) = 0), \qquad (4.1.5a)$$
$$B(z) = \beta_0(z), \quad (\beta_1(z) = \beta_2(z) = \beta_3(z) = 0). \qquad (4.1.5b)$$

[Actually the compatibility condition does not constrain β_3, but the validity of (3.2) with (4.1.1) implies that the β_3 term does not contribute in the nonlinear evolution equation (2.1); thus by setting $\beta_3 = 0$ no generality is lost.]

Thus the subclass of nonlinear evolution equations for the two fields Q_1 and Q_2 reads

$$Q_t(x,t) = 2\beta_0(\underline{L})Q_x(x,t) + \alpha_3(\underline{L})[\sigma_3, Q(x,t)], \qquad (4.1.6)$$

where of course $\underline{L}$ is defined by (2.4) and Q is expressed in terms of Q_1 and Q_2 by (4.1.3) and (4.1.4). The corresponding equation for the reflection coefficient $R(k,t)$ reads of course

$$R_t(k,t) = 4ik\beta_0(-4k^2)R(k,t) + \alpha_3(-4k^2)[\sigma_3, R(k,t)]. \qquad (4.1.7)$$

To show the complete correspondence of these equations to those solvable via the generalized Zakharov–Shabat spectral problem (Ref. 2) we introduce the matrix

$$V(x,t) = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix} = \sigma_1 q_1(x,t) + i\sigma_2 q_2(x,t), \qquad (4.1.8)$$

so that

$$q = q_1 + q_2, \quad r = q_1 - q_2; \quad q_1 = \tfrac{1}{2}(q + r), \quad q_2 = \tfrac{1}{2}(q - r), \qquad (4.1.9)$$

and we relate it to $Q(x,t)$ via the formula

$$Q = V_x + V^2 = \begin{pmatrix} qr & q_x \\ r_x & qr \end{pmatrix}, \qquad (4.1.10)$$

so that

$$\int_x^\infty dx'\, Q_1(x',t) = -q_1(x,t),$$
$$\int_x^\infty dx'\, Q_2(x',t) = -iq_2(x,t),$$
$$Q_1 = q_{1x}, \quad Q_2 = iq_{2x} . \qquad (4.1.11)$$

This last formula provides some motivation for introducing the "matrix Miura transformation"[5] (4.1.10). The corresponding formula for R reads

$$R(k,t) = \begin{pmatrix} 0 & \alpha^{(-)}(-k,t) \\ \alpha^{(+)}(k,t) & 0 \end{pmatrix}. \qquad (4.1.12)$$

With these notations (4.1.6) and (4.1.7) become

$$\sigma_3 v_t(x,t) + \gamma(\underline{L}_{ZS})v(x,t) = 0, \qquad (4.1.13)$$
$$\alpha_t^{(\pm)}(k,t) \pm \gamma(k)\alpha^{(\pm)}(k,t) = 0, \qquad (4.1.14)$$

with

$$v(x,t) \equiv \begin{pmatrix} r(x,t) \\ q(x,t) \end{pmatrix} \qquad (4.1.15)$$

$$\gamma(k) = -4ik\beta_0(-4k^2) + 2\alpha_3(-4k^2), \qquad (4.1.16)$$

the matrix integro-differential operator $\underline{L}_{ZS}$ being defined by the formula

$$L_{ZS}\begin{pmatrix} u^{(1)}(x) \\ u^{(2)}(x) \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} u_x{}^{(1)}(x) \\ -u_x{}^{(2)}(x) \end{pmatrix} + i\begin{pmatrix} r(x,t) \\ q(x,t) \end{pmatrix} \times \int_x^\infty dx'\,[r(x',t)u^{(2)}(x') - q(x',t)u^{(1)}(x')], \quad (4.1.17a)$$

or equivalently

$$L_{ZS}u(x) = (2i)^{-1}\sigma_3 u_x(x) - v(x,t)\int_x^\infty dx'\,(v(x',t),\sigma_2 u(x')). \quad (4.1.17b)$$

The complete equivalence of these equations to those of Calogero and Degasperis[2] is apparent.

Actually the class of nonlinear evolution equations obtained here is more general than that of Calogero and Degasperis,[2] because there one had the condition that the two fields q and r vanish asymptotically ($x \to \pm\infty$) together with all their derivatives, while here one must require that Q vanish asymptotically ($x \to \pm\infty$) with all its derivatives, namely [see (4.1.10)] all the derivatives of the two fields q and r are required to vanish asymptotically, but the two fields themselves need not both vanish as $x \to -\infty$ [that they should vanish as $x \to +\infty$ is implied by (4.1.11) and (4.1.9)]

$$q(x,t) \underset{x\to+\infty}{\longrightarrow} 0, \quad r(x,t) \underset{x\to+\infty}{\longrightarrow} 0, \quad q(x,t)r(x,t) \underset{x\to-\infty}{\longrightarrow} 0. \quad (4.1.18)$$

To display an explicit example, we set

$$\alpha_3(z) = (2i)^{-1}(a + bz), \quad \beta_0(z) = \tfrac{1}{2}(c + dz). \quad (4.1.19)$$

Then the nonlinear evolution equations read

$$r_t = iar + ib\,[r_{xx} - 2(qr)r] + cr_x + d\,[r_{xxx} - 6(qr)r_x], \quad (4.1.20a)$$

$$q_t = -iaq - ib\,[q_{xx} - 2(qr)q] + cq_x + d\,[q_{xxx} - 6(qr)q_x], \quad (4.1.20b)$$

or equivalently [see (4.1.9)]

$$q_{1t} = -iaq_2 - ib\,[q_{2xx} - 2(q_1^2 - q_2^2)q_2] + cq_{1x} + d\,[q_{1xxx} - 6(q_1^2 - q_2^2)q_{1x}], \quad (4.1.21a)$$

$$q_{2t} = -iaq_1 - ib\,[q_{1xx} - 2(q_1^2 - q_2^2)q_1] + cq_{2x} + d\,[q_{2xxx} - 6(q_1^2 - q_2^2)q_{2x}]. \quad (4.1.21b)$$

A reduction of the class of nonlinear evolution equations solvable via the matrix Schrödinger spectral problem to the class solvable via the "generalized Zakharov–Shabat spectral problem" can be performed also in the case of matrices of order N, in close analogy to the treatment given here. We propose, however, to treat this problem in a separate paper, where we shall also provide a more detailed analysis of the connection between the two spectral problems [such an analysis may also serve to better motivate the transformations (4.1.10) and especially (4.1.12), that have been given here without much explanation of their origin].

4.2 Further reduction: Identification of a novel highly nonlinear class of solvable equations for a single field

In Sec. 4.1 we described the reduction of the class of nonlinear evolution equations solvable via the 2×2 matrix Schrödinger spectral problem to that solvable by the generalized Zakharov–Shabat problem Ref. 2. As is well-known, several classical nonlinear evolution equations are contained in this class, including in particular the nonlinear Schrödinger equation, the modified K dV equation and the sine-Gordon equation. These equations (in particular the last two, that are generally written for a single real field) can be obtained by applying once more the reduction technique; but these developments are too trivial and well-known to deserve reporting. In this section we consider instead a less trivial additional reduction of the class of evolution equations (4.1.6) [with (4.1.3) and (4.1.4)], namely that resulting from the choice, in (3.1) and (3.2), of

$$F(z) = \sigma_1 + i(c_0 + c_1 z)\sigma_2, \quad H(z) = 0, \quad (4.2.1)$$

where c_0 and c_1 are constant.

It is then immediately seen that

$$R(k,t) = R_1(k,t)[\sigma_1 + i(c_0 - 4k^2c_1)\sigma_2] = R_1(k,t)F(-4k^2). \quad (4.2.2)$$

The derivation of the corresponding formula for Q, resulting from the constraint (3.1) that now reads

$$[\sigma_1,Q] + i(c_0 + c_1\underline{L})[\sigma_2,Q] = 0, \quad (4.2.3)$$

is less elementary; we outline the main steps in the Appendix. The final result is most conveniently written in terms of the fields q_1 and q_2 of Sec. 4.1 [see in particular (4.1.8–4.1.11)], and it reads

$$q_2 = [c_0q_1 - c_1(2q_1^3 - q_{1xx})]/[1 - 4c_0c_1q_1^2 + 4c_1^2(q_1^4 - q_{1x}^2)]^{1/2}. \quad (4.2.4)$$

Next one considers the compatibility condition (3.6), and it is easily seen that it implies $\alpha_3 = 0$.

In conclusion, a class of nonlinear evolution equations for the single field

$$u(x,t) \equiv q_1(x,t), \quad (4.2.5)$$

solvable by the spectral transform technique obtains setting $\alpha_3 = 0$ in (4.1.6), letting $\beta_0(z)$ be an arbitrary entire function (or more generally, the ratio of two entire functions), and expressing the matrix Q in terms of the single field u as implied by (4.1.8)–(4.1.11), and (4.2.4–4.2.5). Equivalently but more simply, the same class of nonlinear evolution equations obtains from (4.1.13), with $\gamma(z)$ odd in z and entire (or, more generally, the ratio of two entire functions), the fields r and q being given in terms of u by (4.1.9), (4.2.4), and (4.2.5).

The asymptotic boundary conditions that must supplement this class of equations, so as to assure consistency, via (4.2.5), (4.1.11), (4.1.10), and (4.1.9), with the assumed asymptotic vanishing of Q and its derivatives, require u to vanish with its derivatives as $x \to +\infty$,

$$0 = u(+\infty,t) = u_x(+\infty,t) = u_{xx}(+\infty,t) = \cdots, \quad (4.2.6a)$$

and moreover that all the derivatives of u vanish as $x \to -\infty$,

$$0 = u_x(-\infty,t) = u_{xx}(-\infty,t) = \cdots; \quad (4.2.6b)$$

but the value of u itself as $x \to -\infty$ is required to vanish only if $c_0^2 \neq 1$ (elementary algebra shows that, if $c_0^2 = 1$, any asymptotic value of u is consistent with the requirement that the matrix Q vanish asymptotically):

$$u(-\infty,t) = 0 \quad \text{if} \quad c_0^2 \neq 1,$$

$$u(-\infty,t) = \text{arbitrary constant}, \quad \text{if} \quad c_0^2 = 1. \quad (4.2.6c)$$

In fact it is easily seen that, provided $c_0^2 = 1$ and $c_1 \neq 0$, u might even diverge as $x \to -\infty$ [but the derivatives of u must vanish, see (4.2.6b)].

Of course another requirement on u is that q_2, as given by (4.2.4) and (4.2.5), be finite for $-\infty < x < \infty$; a condition sufficient to guarantee this is the requirement that u itself be regular and that the denominator on the right-hand side of (4.2.4) not vanish for real x. It is clearly sufficient that all these conditions hold at the initial time, since they are then automatically guaranteed to hold throughout the time evolution.

A simple example of nonlinear evolution equation of this class obtains inserting (4.2.4)–(4.2.5) in (4.1.21a) (of course with $a = b = 0$, as required by the consistency condition that forces $\alpha_3(z)$ to vanish; see above). It reads

$$u_t = cu_x + d\{u_{xxx} - 6u_x[u^2 - (c_0u - 2c_1u^3 + c_1u_{xx})^2/ (1 - 4c_0c_1u^2 + 4c_1^2u^4 - 4c_1^2u_x^2)]\}. \quad (4.2.7)$$

The change of dependent and independent variables

$$u(x,t) = (c_0/c_1)^{1/2}u'(x',t'), \quad x = (c_0/c_1)^{1/2}(x + ct), \quad t' = d(c_0/c_1)^{3/2}t, \quad (4.2.8)$$

yields for $u'(x',t')$ the neater equation

$$u_t = u_{xxx} - 6u_x\{u^2 - (u - 2u^3 + u_{xx})^2/ [a^2 - 4(u^2 - u^4 + u_x^2)]\}, \quad (4.2.9)$$

that we have written omitting all primes (for notational convenience; and we persevere below), and setting $c_0 = 1/a$. The boundary conditions for this equation are

$$0 = u(+\infty,t) = u_x(+\infty,t) = u_{xx}(+\infty,t) = \cdots; \quad (4.2.10a)$$

$$0 = u_x(-\infty,t) = u_{xx}(-\infty,t) = \cdots; \quad (4.2.10b)$$

$$u(-\infty,t) = 0 \quad \text{if} \quad a^2 \neq 1,$$

$$u(-\infty,t) = \text{arbitrary constant} \quad \text{if} \quad a^2 = 1. \quad (4.2.10c)$$

Let us emphasize once more that the technique to solve this equation is through the equivalence of (4.2.7) to (2.1) with $\alpha_m = 0, \beta_m = 0, \beta_0(z) = \frac{1}{2}(c + dz)$ and Q given in terms of u by (4.1.8–4.1.11) and (4.2.4–4.2.5). This implies of course not only the possibility of solving the Cauchy problem[1] [given $u(x,0)$ one can clearly compute $Q(x,0)$; and given $Q(x,t)$ one can recover $u(x,t)$ with just one quadrature, as implied by (4.1.11)], but also to obtain all the results associated with the solvability of (2.1) by the spectral transform technique: An infinite number of conserved quantities, Bäcklund transformation, all the soliton phenomenology.[1] Here we merely report the single soliton solution of (4.2.9), that reads

$$u(x,t) = 2pa[(1 + 4p^2)^2 - a^2]^{-1/2}/\cosh\{2p[x - \xi(t)]\}, \quad (4.2.11)$$

with

$$\xi(t) = \xi_0 - 4p^2t. \quad (4.2.12)$$

In writing this equation we assume that p is a positive constant such that the square root on the right-hand side of (4.2.11) is real, or equivalently, such that the quantity ν defined setting

$$\coth\nu = (1 + 4p^2)/a, \quad (4.2.13)$$

is real. Note that the soliton of this equation moves with constant speed; this is in contrast with the generic behavior of the solitons associated with matrix nonlinear evolution equations,[1] but it agrees with the generic behavior of the solitons of the nonlinear evolution equations of the Zakharov–Shabat class (4.1.13),[2] of which after all (4.2.9) is merely a subcase (although one that would not have been easy to discover without the technique given above). We also report the spectral transform[1] of the matrix Q corresponding to (4.2.11); it has of course $R = 0$, and a *two-fold degenerate* discrete eigenvalue $-p^2$, so that the matrix C associated with it[1] has the structure

$$C = C_1 + C_2 \quad (4.2.14)$$

$$C_j = 2p\exp(2p\,\xi_j)P_j, \quad j = 1,2, \quad (4.2.15)$$

$$\xi_1 = \xi - (2p)^{-1}\ln\sinh\mu, \quad \xi_2 = \xi_1 - i\pi/(2p), \quad (4.2.16)$$

$$P_j = \tfrac{1}{2}(\mathbb{1} + \hat{n}_m^{(j)}\sigma_m), \quad j = 1,2, \quad (4.2.17)$$

$$\hat{n}_1^{(j)} = \cosh[\nu - (-)^j\mu], \quad \hat{n}_2^{(j)} = i\sinh[\nu - (-)^j\mu], \quad \hat{n}_3^{(j)} = 0, \quad j = 1,2. \quad (4.2.18)$$

Note that the constant μ is in fact not present in C,

$$C = 2p\exp(2p\xi)(\sigma_1\sinh\nu + i\sigma_2\cosh\nu), \quad (4.2.19)$$

and accordingly does not appear in (4.2.11). Let us emphasize that, for equations obtained by reduction, the fact that the solitons may correspond to degenerate discrete eigenvalues appears not to be exceptional.[4]

Let us finally discuss some limiting properties of the solutions $u(x,t;a)$ of Eq. (4.2.9).

Clearly $u(x,t;\infty)$ satisfies the mK dV equation

$$u_t = u_{xxx} - 6\eta u^2u_x, \quad \eta = \pm 1 \quad (4.2.20)$$

(with $\eta = +1$); and indeed in this limit (4.2.11) yields the single-soliton solution of the mK dV equation (this solution is imaginary; indeed it is (4.2.20) with $\eta = -1$ that has real solitons).

Another limiting case obtains setting

$$u(\epsilon x,\epsilon^3t;2/\epsilon) = \int_x^\infty dx'\sin\left[2\int_{x'}^\infty dx''\,\bar{u}(x'',t)\right], \quad (4.2.21a)$$

$$\bar{u}(x,t) = \frac{1}{2}\frac{d}{dx}\arcsin\left[\frac{d}{dx}u(\epsilon x,\epsilon^3t;2/\epsilon)\right], \quad (4.2.21b)$$

with $\epsilon \to 0$. It is then easily seen that $\bar{u}(x,t)$ satisfies again the mK dV equation (4.2.20) (with $\eta = -1$).

A third limiting case obtains setting instead

$$u(\epsilon x,\epsilon^3t;1) = -\int_x^\infty dx'\exp\left[-2\int_{x'}^\infty dx''\tilde{u}(x'',t)\right], \quad (4.2.22a)$$

$$\tilde{u}(x,t) = \frac{1}{2}\frac{d}{dx}\ln\left[\frac{d}{dx}u(\epsilon x,\epsilon^3t;1)\right], \quad (4.2.22b)$$

again with $\epsilon \to 0$. It is easily seen that also $\tilde{u}(x,t)$ satisfies the mK dV equation (4.2.20) with $(\eta = +1)$.

It is remarkable that these three limiting procedures all yield solutions of the mK dV equation. It should however be noted that while the first two prescriptions produce solutions of mK dV that vanish asymptotically, the last one appears to yield solutions that diverge asymptotically; the way is thereby opened to the study of the Cauchy problem for the mK dV equation with diverging asymptotic behavior, that however does not appear to be anywhere as interesting and important as the analogous problem for the K dV equation.[6]

4.3. Another class of nonlinear evolution equations involving two fields

In this section we consider another reduction of the class of evolution equations (2.1) for matrices of order 2, that again decreases the number of independent fields from 4 to 2, but in a different fashion than in Sec. 4.1. It obtains setting in (3.2)

$$F(z) = -i(\gamma_0 + \gamma_1 z)\sigma_1, \quad H(z) = \gamma\sigma_3. \tag{4.3.1}$$

There immediately follows

$$R(k,t) = -[(\gamma_0 - 4k^2\gamma_1)/(2i\gamma k)]R_2(k,t) + R_1(k,t)\sigma_1 + R_2(k,t)\sigma_2, \quad (R_3(k,t) = 0), \tag{4.3.2a}$$

or equivalently

$$R(k,t) = R_0(k,t) + R_1(k,t)\sigma_1 - [2i\gamma k/(\gamma_0 - 4k^2\gamma_1)]R_0(k,t)\sigma_2, \quad (R_3(k,t) = 0). \tag{4.3.2b}$$

These two expressions display the fact that R contains now only 2 independent components; while their equivalence is quite obvious, the first is to be preferred in the special case $\gamma_0 = \gamma_1 = 0$, the second in the special case $\gamma = 0$ (see below).

The corresponding expression for Q obtains inserting (4.3.1) in (3.1). After some labor, that we consider sufficiently straightforward not to warrant any reporting, there obtains the result

$$Q(x,t) = Q_0(x,t) + Q_1(x,t)\sigma_1 + Q_2(x,t)\sigma_2, \quad (Q_3(x,t) = 0), \tag{4.3.3}$$

$$Q_0(x,t) = (\gamma + 2\gamma_1 W_2)^{-2}[\gamma_1(\gamma + 2\gamma_1 W_2)W_{2xx} - \gamma_1^2 W_{2x}^2 + W_2^2(\gamma + \gamma_1 W_2)^2 + \gamma_0 W_2(\gamma + \gamma_1 W_2) + \gamma^2 W_1^2 + 4\gamma_1^2 U^2 - 4\gamma_1\gamma U W_1], \tag{4.3.4}$$

$$W_j \equiv W_j(x,t) = \int_x^\infty dx'\, Q_j(x',t),$$

$$Q_j(x,t) = -W_{jx}(x,t), \quad j = 1,2, \tag{4.3.5}$$

$$U \equiv U(x,t) = -\int_x^\infty dx'\, Q_1(x',t) W_2(x',t), \tag{4.3.6a}$$

$$U(x,t) = -W_1(x,t)W_2(x,t) + \int_x^\infty dx'\, Q_2(x',t)W_1(x',t). \tag{4.3.6b}$$

Note the similarity of this definition of W_j to the definition (4.1.11) of the fields q_j; the differences are caused by the need, in Sec. 4.1, to reproduce the notation of Calogero and Degasperis.[2]

We consider next the compatibility condition (3.6), and using (4.3.2) it is easily seen that it implies

$$\alpha_1(z) = i[(\gamma_0 + \gamma_1 z)/\gamma]\beta_3(z), \tag{4.3.7a}$$

$$\alpha_2(z) = \beta_2(z) = 0, \tag{4.3.7b}$$

$$\alpha_3(z) = i[\gamma z/(\gamma_0 + \gamma_1 z)]\beta_1(z), \tag{4.3.7c}$$

with $\beta_0(z)$, $\beta_1(z)$, and $\beta_2(z)$ arbitrary entire functions, or rather ratios of entire functions. Note moreover that the constraint condition (3.2) with (4.3.1), together with (4.3.7a), implies that the α_1 and β_3 terms in (2.1) cancel each other, so that one can set, without loss of generality,

$$\alpha_1(z) = \beta_3(z) = 0. \tag{4.3.7d}$$

In conclusion the class of nonlinear evolution equations that we have now obtained corresponds to (2.1) with the functions $\alpha_m(z)$ and $\beta_\mu(z)$ restricted by the conditions (4.3.7) and with the matrix Q given by (4.3.3–4.3.6). This class, for any choice of the functions α_m and β_μ [compatible with (4.3.7)], yields two coupled evolution equations for the two fields $Q_1(x,t)$ and $Q_2(x,t)$, or equivalently for the fields $W_1(x,t)$ and $W_2(x,t)$ of (4.3.5) (indeed the evolution equations have generally a neater appearance when written in terms of the dependent variables W_j rather than Q_j; see below). The boundary conditions to be required are clearly

$$0 = W_j(+\infty,t) = W_{jx}(+\infty,t) = W_{jxx}(+\infty,t) = \cdots, \quad j = 1,2, \tag{4.3.8a}$$

and

$$0 = W_{jx}(-\infty,t) = W_{jxx}(-\infty,t) = \cdots, \quad j = 1,2. \tag{4.3.8b}$$

As for the values of the fields W_j as $x \to -\infty$, the relevant condition must be read from (4.3.4), corresponding to the requirement

$$Q_0(-\infty,t) = 0. \tag{4.3.8}$$

The first example we consider corresponds to the choice

$$\gamma_1 = 0, \quad \beta_0(z) = \tfrac{1}{2}(c + dz), \quad \beta_1(z) = -\tfrac{1}{2} b\gamma_0/\gamma. \tag{4.3.9}$$

Then one obtains for the two fields

$$u(x,t) \equiv W_1(x,t), \quad v(x,t) \equiv W_2(x,t) + \tfrac{1}{2}\gamma_0/\gamma \tag{4.3.10}$$

the evolution equations

$$u_t = -b[v_{xx} - 2v(u^2 + v^2 - C^2)] + cu_x + d[u_{xxx} - 6u_x(u^2 + v^2 - C^2)], \tag{4.3.11a}$$

$$v_t = b[u_{xx} - 2u(u^2 + v^2 - C^2)] + cv_x + d[v_{xxx} - 6v_x(u^2 + v^2 - C^2)], \tag{4.3.11b}$$

where we have introduce the constant

$$C = \tfrac{1}{2}\gamma_0/\gamma. \tag{4.3.12}$$

Assuming the constants b, c, d, and C^2, as well as the fields u^2 and v^2, to be real, one can introduce the complex field $\phi(x,t)$ setting

$$\phi(x,t) = u(x,t) + iv(x,t). \tag{4.3.13}$$

Then the two evolution equations (4.3.11) combine into the single equation

$$\phi_t = ib[\phi_{xx} - 2\eta(|\phi|^2 - |C|^2)\phi] + c\phi_x + d[\phi_{xxx} - 6\eta(|\phi|^2 - |C|^2)\phi_x], \quad \eta = \pm 1 \tag{4.3.14}$$

(for $\eta = +1$, u, v, and C are real; for $\eta = -1$, they are imaginary). Moreover the boundary conditions for the field $\phi(x,t)$, besides requiring the asymptotic vanishing of all its derivatives, read

$$\phi(+\infty,t) = iC, \quad |\phi(-\infty,t)| = |C|. \tag{4.3.15}$$

Thus the field

$$\psi(x,t) = \exp(-iat + i\mu)\phi(x,t), \tag{4.3.16}$$

where a and μ are real constants, satisfies the "generalized Hirota equation"[7]

$$\psi_t = -ia\psi + ib\,[\psi_{xx} - 2\eta(|\psi|^2 - |C|^2)\psi] + c\psi_x + d\,[\psi_{xxx} - 6\eta(|\psi|^2 - |C|^2)\psi_x], \tag{4.3.17}$$

with boundary conditions

$$|\psi(+\infty,t)| = |\psi(-\infty,t)| = |C|. \tag{4.3.18}$$

[Note that the last equation need not imply $\psi(+\infty,t) = \psi(-\infty,t)$.]

Of course subcases of this equation are the (generalized) versions of the nonlinear Schrödinger equation and of the mK dV equation, that obtain respectively for $a = c = d = 0$, $b = 1$, reading

$$i\psi_t = -\psi_{xx} + 2\eta(|\psi|^2 - |C|^2)\psi,$$
$$\eta = \pm 1, \quad |\psi(\pm\infty,t)| = |C|, \tag{4.3.19}$$

and for $a = b = c = 0$, $d = 1$, $\psi(x,t) = \psi^*(x,t) = u(x,t)$, reading

$$u_t = u_{xxx} - 6\eta(u^2 - |C|^2)u,$$
$$\eta = \pm 1, \quad u^2(\pm\infty,t) = |C|^2. \tag{4.3.20}$$

The second example we consider corresponds to the choice

$$\gamma_1 \neq 0, \quad \beta_0(z) = \tfrac{1}{2}(c + dz), \quad \beta_1(z) = 0. \tag{4.3.21}$$

One obtains then the two nonlinear evolution equations

$$W_{jt}(x,t) = cW_{jx}(x,t) + d\,[W_{jxxx}(x,t) - 6Q_0(x,t)\,W_{jx}(x,t)], \quad j = 1,2, \tag{4.3.22}$$

with Q_0 given in terms of W_1 and W_2 by (4.3.4)–(4.3.6). These equations are rather complicated; but they yield a simpler equation if a further reduction is performed. This is discussed in Sec. 4.4.

4.4 Further reduction: Novel solvable nonlinear evolution equation for a single field

The further reduction that we apply here is directly suggested by the structure of (4.3.22), that is clearly compatible with the position

$$W_2(x,t) \equiv u(x,t), \quad W_1(x,t) = \rho u(x,t), \tag{4.4.1}$$

ρ being a constant. This implies [see (4.3.5)–(4.3.6)]

$$U(x,t) = -\tfrac{1}{2}\rho u^2(x,t) \tag{4.4.2}$$

and [see (4.3.4)]

$$Q_0(x,t) = (\gamma + 2\gamma_1 u)^{-2}\{\gamma_1(\gamma + 2\gamma_1 u)u_{xx} - \gamma_1^2 u_x^2 + u(\gamma + \gamma_1 u)[\gamma_0 + (1+\rho^2)(\gamma + \gamma_1 u)u]\}. \tag{4.4.3}$$

Thus one obtains now for the single field $u(x,t)$, or rather for the field

$$u'(x',t') = 2(\gamma_1/\gamma)u(x,t), \quad x' = \lambda x + \mu t, \quad t' = d\lambda^3 t, \tag{4.4.4}$$

the nonlinear evolution equation (hereafter all equations are written for the primed variables, but dropping all primes for notational convenience)

$$u_t = \frac{\partial}{\partial x}\,[u_{xx} - \tfrac{3}{2}u_x^2/(1+u) + \tfrac{1}{3}A(1+u)^3 - B/(1+u) + Cu], \tag{4.4.5}$$

where

$$A = -\tfrac{3}{8}(1+\rho^2)\gamma^2/(\gamma_1^2\lambda^2), \tag{4.4.6a}$$
$$B = A + \tfrac{3}{2}\gamma_0/(\gamma_1\lambda^2), \tag{4.4.6b}$$
$$C = -A - B + (c\lambda - \mu)/(d\lambda^3). \tag{4.4.6c}$$

Of course some of these constants can be eliminated or set to unity by appropriate choices of the constants λ and μ.

The boundary condition to be associated with (4.4.5) requires all derivatives of u to vanish asymptotically, and moreover u itself to vanish as $x \to +\infty$ (we are assuming $\lambda > 0$):

$$0 = u(+\infty,t) = u_x(+\infty,t) = u_{xx}(+\infty,t) = \cdots, \tag{4.4.7a}$$
$$0 = u_x(-\infty,t) = u_{xx}(-\infty,t) = \cdots. \tag{4.4.7b}$$

As for the value of u as $x \to -\infty$, the following four possibilities are all compatible with the condition $Q_0(-\infty,t) = 0$:

$$u(-\infty,t) = -1 \pm 1, \quad u(-\infty,t) = -1 \pm (B/A)^{1/2}; \tag{4.4.7c}$$

of course the last one can be contemplated, for real u, only if the ratio B/A is positive (this we assume below).

Another interesting version of the nonlinear evolution equation (4.4.5) obtains setting

$$u(x,t) = \exp[\tfrac{1}{2}v(x,t)] - 1, \tag{4.4.8}$$

since v obeys then the nonlinear equation

$$v_t = v_{xxx} - \tfrac{1}{8}v_x^3 + v_x\,[A\exp(v) + B\exp(-v) + C\,], \tag{4.4.9}$$

while the boundary conditions read

$$0 = v(+\infty,t) = v_x(+\infty,t) = v_{xx}(+\infty,t) = \cdots, \tag{4.4.10a}$$
$$0 = v_x(-\infty,t) = v_{xx}(-\infty,t) = \cdots, \tag{4.4.10b}$$
$$v(-\infty,t) = 0 \quad \text{or} \quad v(-\infty,t) = \ln(B/A). \tag{4.4.10c}$$

Let us note that the expression of the (matrix) reflection coefficient corresponding to the matrix Q of (4.3.3)–(4.3.5) and (4.4.1)–(4.4.3) reads

$$R(k,t) = R_0(k,t)[1 - 2ik\gamma(\gamma_0 - 4k^2\gamma_1)^{-1}(\rho\sigma_1 + \sigma_2)], \tag{4.4.11}$$

and evolves according to the simple equation

$$R_{0t}(k,t) = 2ik\,(c - 4k^2 d)R_0(k,t) \tag{4.4.12}$$

[here we are again using the unprimed t variable; see (4.4.4)].

Finally let us note the limiting cases that can be obtained from (4.4.9) [or equivalently (4.4.5)], setting

$$v(x,t) = \epsilon\psi(x,t), \tag{4.4.13}$$

$$A = A_0 + \tfrac{1}{2}A_1\epsilon^{-1} + A_2\epsilon^{-2}, \tag{4.4.14a}$$

$$B = -\tfrac{1}{2}A_1\epsilon^{-1} + A_2\epsilon^{-2}, \tag{4.4.14b}$$

$$C = -2A_2\epsilon^{-2}, \tag{4.4.14c}$$

and considering the limit $\epsilon \to 0$. Then (4.4.9) becomes

$$\psi_t = \psi_{xxx} + \psi_x(A_0 + A_1\psi + A_2\psi^2) \tag{4.4.15}$$

with the boundary conditions

$$0 = \psi(\pm\infty,t) = \psi_x(\pm\infty,t) = \psi_{xx}(\pm\infty,t) = \cdots . \tag{4.4.16}$$

This equation is, however, already contained in the class considered in Sec. 4.3 [see (4.3.17)].

5. CONCLUDING REMARKS

The main purpose of this paper has been to present the method of reduction. Since the worth of any pie is apparent only in the eating, we have also applied it, but in the simplest context, namely to matrices of order 2. This has not only displayed the connection between the class of nonlinear evolution equations solvable by the spectral transform associated to the Zakharov–Shabat spectral problem[2] and those solvable by the matrix Schrödinger problem,[1,3] but has in fact provided some generalization of the Zakharov–Shabat class (by allowing a less restrictive asymptotic behavior of the solutions). Moreover novel classes of nonlinear evolution equations involving two fields, or a single field only, have been obtained; we have displayed some of these, that provide therefore novel additions to the stock of nonlinear partial differential equations of evolution type solvable by the spectral transform technique. All these equations possess of course all the properties characteristic of the "soliton" equations; and it is straightforward to display such properties using the formalism given in this paper and elsewhere.[1]

A number of additional applications are naturally suggested by the results of this paper; in particular we shall report separately the findings yielded by the application of this approach to matrices of order higher than two.

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APPENDIX

In this Appendix we indicate how the nonlinear integro-differential equation (4.2.3) [with (4.1.8)–(4.1.11) and of course (2.4)–(2.5)] can be solved to yield (4.2.4).

Trivial algebra yields first of all

$$q_{2x} + ic_0q_{1x} + ic_1\Big[q_{1xxx} - 6q_{1x}q_1^2 + 4q_{1x}q_2^2 + 4q_{2x}q_1q_2 + 4q_{2x}\int_x^\infty dx'\, q_{2x}(x')q_1(x')\Big] = 0. \tag{A1}$$

It is then convenient to set

$$w(x) = \int_x^\infty dx' q_{1x}(x')q_2(x'), \tag{A2a}$$

$$q_2(x) = -w_x(x)/q_{1x}(x), \tag{A2b}$$

and to note that the left-hand side of (A1) is a perfect differential, so that integration from x to ∞ yields

$$q_2 + ic_0q_1 + ic_1(q_{1xxx} - 2q_1^3 + 4q_2w) = 0. \tag{A3}$$

Multiply this equation by q_{1x}, and use (A2b) to eliminate q_2. One obtains again in this manner a perfect differential, whose integration from x to ∞ yields the equation

$$2iw + c_0q_1^2 + c_1(q_{1x}^2 - q_1^4 + 4w^2) = 0. \tag{A4}$$

This is immediately solved for w (to identify the correct solution out of the two possible ones note that w must vanish when q_{1x} and q_1 vanish, since this is what happens in the limit $x \to +\infty$), and subsequent insertion in (A2b) yields (4.2.4).

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[6] The typical diverging solution of the K dV equation behaves linearly in x as x diverges; removal of the divergent part by subtraction leaves for the nondivergent part an interesting equation, related to the so-called "cylindrical K dV equation" [see for instance, F. Calogero and A. Degasperis, Lett. Nuovo Cimento **23**, 150 (1978); L.A. Bordag and M.B. Matveev, Leipz. Prepr. (to be published); and the literature referred to in these papers.] The typical diverging solution of the mK dV equation behaves asymptotically proportionally to the square root of x; the equation one obtains after removal by substraction of the divergent part does not appear particularly interesting. This is connected with the fact that, while $q(x,t) = \frac{1}{6}(x/t)$ is an (asymptotically diverging) exact solution of the K dV equation $q_t - q_{xxx} + 6q_xq = 0$, no comparably simple solution of the mK dV equation exists; indeed the solution related to that written above via the Miura transformation (Ref. 5) involves Airy functions.

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