

REDUCTIONS OF POINTS ON ALGEBRAIC GROUPS, II

PETER BRUIN

Mathematisch Instituut, Universiteit Leiden, Postbus 9512, 2300 RA Leiden, The Netherlands
e-mail: P.J.Bruin@math.leidenuniv.nl

ANTONELLA PERUCCA

*Department of Mathematics, University of Luxembourg, 6, Avenue de la Fonte,
4364 Esch-sur-Alzette, Luxembourg*
e-mail: antonella.perucca@uni.lu

(Received 15 October 2019; revised 16 March 2020; accepted 26 June 2020; first published online 28 July 2020)

Abstract. Let A be the product of an abelian variety and a torus over a number field K , and let $m \geq 2$ be a square-free integer. If $\alpha \in A(K)$ is a point of infinite order, we consider the set of primes \mathfrak{p} of K such that the reduction $(\alpha \bmod \mathfrak{p})$ is well defined and has order coprime to m . This set admits a natural density, which we are able to express as a finite sum of products of ℓ -adic integrals, where ℓ varies in the set of prime divisors of m . We deduce that the density is a rational number, whose denominator is bounded (up to powers of m) in a very strong sense. This extends the results of the paper *Reductions of points on algebraic groups* by Davide Lombardo and the second author, where the case m prime is established.

2010 *Mathematics Subject Classification*. Primary: 11F80; Secondary: 14L10, 11G05, 11G10

1. Introduction. This article is the continuation of the paper *Reductions of points on algebraic groups* by Davide Lombardo and the second author [4]. We refer to this other work for the history of the problem, which started in the 1960s with work of Hasse on the multiplicative orders of rational numbers modulo primes.

Let A be the product of an abelian variety and a torus over a number field K , and let $m \geq 2$ be a square-free integer. If $\alpha \in A(K)$ is a point of infinite order, we consider the set of primes \mathfrak{p} of K such that the reduction $(\alpha \bmod \mathfrak{p})$ is well defined and has order coprime to m . This set admits a natural density (see Theorem 7), which we denote by $\text{Dens}_m(\alpha)$.

The main question is whether we can write

$$\text{Dens}_m(\alpha) = \prod_{\ell} \text{Dens}_{\ell}(\alpha), \quad (1.1)$$

where ℓ varies over the prime divisors of m . Let $K(A[m])$ be the m -torsion field of A . We prove that (1.1) holds if $K(A[m]) = K$ (i.e. if $A(K)$ contains all m -torsion points) or, more generally, if the degree $[K(A[\ell]) : K]$ is a power of ℓ for every prime divisor ℓ of m (see Corollary 18). Indeed, (1.1) holds if the torsion fields/Kummer extensions of α related to different prime divisors of m are linearly disjoint over K . In general, (1.1) does not hold: see Section 7.2 for an explicit example.

We are able to express $\text{Dens}_m(\alpha)$ as an integral over the image of the m -adic representation (see Theorem 16) and also as a finite sum of products of ℓ -adic integrals (see

Theorem 19). The latter decomposition allows us to prove that $\text{Dens}_m(\alpha)$ is a rational number whose denominator is uniformly bounded in a very strong sense (see Corollary 20).

Finally, we study Serre curves in detail in Section 6. With the partition given in Section 6.3, one can very easily compute $\text{Dens}_m(\alpha)$ if the m^n -Kummer extensions of α (defined in Section 3) have maximal degree for all n or, more generally, if the degrees of these extensions are known and are the same with respect to the base fields K and $K(A[m])$.

In general, to compute the density $\text{Dens}_m(\alpha)$ for the product of an abelian variety and a torus, we only need information on the Galois group of the m^n -torsion fields/Kummer extensions of α for some sufficiently large n . Thus, a theoretical algorithm to compute the density exists, because the growth in n of the m^n -torsion fields/Kummer extensions of α is eventually maximal (see Proposition 5 and Remark 6 in view of [4, Lemma 11]).

Finally, we point out that since the category of algebraic groups that we consider is stable under products, our results allow us to replace α by a finitely generated subgroup of $A(K)$; see Remark 22.

2. Integration on profinite groups. For every profinite group G , we write μ_G for the normalised Haar measure on G . More generally, if X is a G -torsor, we write μ_X for the normalised Haar measure on X , defined by transporting μ_G along any isomorphism $G \cong X$ of G -torsors.

LEMMA 1. *Let G be a profinite group, and let H be an open subgroup of G . Suppose that we have $G = \prod_{\ell} G_{\ell}$, where ℓ varies in a finite set of prime numbers, and each G_{ℓ} is a profinite group containing a pro- ℓ -group G'_{ℓ} as an open subgroup. Let $G' = \prod_{\ell} G'_{\ell}$ and $H' = H \cap G'$. For each $x \in H/H'$, let $H(x)$ be the fibre over x of the quotient map $H \rightarrow H/H'$.*

- (1) *The subgroup H' is open in H , and for each $x \in H/H'$, the normalised Haar measure on the H' -torsor $H(x)$ is*

$$\mu_{H(x)} = (H : H') \mu_H|_{H(x)}.$$

- (2) *We can write*

$$H' = \prod_{\ell} H'_{\ell},$$

where each H'_{ℓ} is a pro- ℓ -group, and the normalised Haar measures on H' and the H'_{ℓ} are related by

$$\mu_{H'} = \prod_{\ell} \mu_{H'_{\ell}}.$$

- (3) *We can write the H' -torsor $H(x)$ as*

$$H(x) = \prod_{\ell} H_{\ell}(x),$$

where each $H_{\ell}(x)$ is a H'_{ℓ} -torsor, and the normalised Haar measures on $H(x)$ and the $H_{\ell}(x)$ are related by

$$\mu_{H(x)} = \prod_{\ell} \mu_{H_{\ell}(x)}.$$

Proof. The claim that H' is open in H holds because G' is open in G . The measure $\mu_H|_{H(x)}$ is H' -invariant and satisfies $\int_{H(x)} \mu_H = \frac{1}{(H:H')}$; this proves (1). Because G' is a

product of pro- ℓ -groups for pairwise different ℓ , every closed subgroup of G' is similarly a product of pro- ℓ -groups. This shows the existence of the H'_ℓ as in (2); the claim about $\mu_{H'}$ follows because $\prod_\ell \mu_{H'_\ell}$ satisfies the properties of the normalised Haar measure on H' . Finally, (3) is proved in the same way as (2). \square

PROPOSITION 2. *With the notation of Lemma 1, let $f: H \rightarrow \mathbb{C}$ be an integrable function.*

(1) *We have*

$$\int_H f d\mu_H = \frac{1}{(H:H')} \sum_{x \in H/H'} \int_{H(x)} f d\mu_{H(x)}.$$

(2) *Suppose that for each $x \in H/H'$, the restriction of f to $H(x)$ admits a product decomposition*

$$f|_{H(x)} = \prod_\ell f_{x,\ell},$$

where the $f_{x,\ell}: H_\ell(x) \rightarrow \mathbb{C}$ are integrable functions. Then we have

$$\int_H f d\mu_H = \frac{1}{(H:H')} \sum_{x \in H/H'} \prod_\ell \int_{H_\ell(x)} f_{x,\ell} d\mu_{H_\ell(x)}.$$

Proof. Part (1) follows by rewriting $\int_H f d\mu_H$ as $\sum_{x \in H/H'} \int_{H(x)} f d\mu_H$ and applying Lemma 1(1). Part (2) follows from part (1), Lemma 1(3) and the assumption on f . \square

3. The arboreal representation. Let K be a number field, and let \bar{K} be an algebraic closure of K . Let A be a connected commutative algebraic group over K , and let b_A be the first Betti number of A . We fix a square-free integer $m \geq 2$. Below, we let ℓ vary in the set of prime divisors of m . We also fix a point $\alpha \in A(K)$.

We define $T_m A$ as the projective limit of the torsion groups $A[m^n]$ for $n \geq 1$; we can write $T_m A = \prod_\ell T_\ell A$, where the Tate module $T_\ell A$ is a free \mathbb{Z}_ℓ -module of rank b_A .

We define the *torsion fields*

$$K_{m^{-n}} := K(A[m^n]) \quad \text{for } n \geq 1$$

and

$$K_{m^{-\infty}} := \bigcup_{n \geq 1} K_{m^{-n}}.$$

The Galois action on the m -power torsion points of A gives the m -adic representation of A , which maps $\text{Gal}(\bar{K}/K)$ to the automorphism group of $T_m A$. We can also speak of the *mod m^n representation*, which describes the Galois action on $A[m^n]$. Choosing a \mathbb{Z}_ℓ -basis for $T_\ell A$ for every prime divisor ℓ of m , we can identify the image of the m -adic representation with a subgroup of $\prod_\ell \text{GL}_{b_A}(\mathbb{Z}_\ell)$ and the image of the mod m^n representation with a subgroup of $\prod_\ell \text{GL}_{b_A}(\mathbb{Z}/\ell^n \mathbb{Z})$.

For $n \geq 1$, let $m^{-n}\alpha$ be the set of points in $A(\bar{K})$ whose m^n th multiple equals α . We also write

$$m^{-\infty}\alpha = \varprojlim_{n \geq 1} m^{-n}\alpha.$$

This is the set of sequences $\beta = \{\beta_n\}_{n \geq 1}$ such that $m\beta_1 = \alpha$ and $m\beta_{n+1} = \beta_n$ for every $n \geq 1$; it is a torsor under $T_m A$. We note that $m^{-n}0 = A[m^n]$ and $m^{-\infty}0 = T_m A$.

We define the fields

$$K_{m^{-n}\alpha} := K(m^{-n}\alpha) \quad \text{for } n \geq 1$$

and

$$K_{m^{-\infty}\alpha} := \bigcup_{n \geq 1} K_{m^{-n}\alpha}.$$

We call the field extension $K_{m^{-n}\alpha}/K_{m^{-n}}$ the m^n -Kummer extension defined by the point α . We view the m -adic representation as a representation of $\text{Gal}(K_{m^{-\infty}\alpha}/K)$.

We fix an element $\beta \in m^{-\infty}\alpha$ and define the *arboreal representation*

$$\begin{aligned} \omega_{\alpha, m^\infty} : \text{Gal}(K_{m^{-\infty}\alpha}/K) &\longrightarrow T_m A \rtimes \text{Aut}(T_m A) \\ \sigma &\longmapsto (t, M), \end{aligned}$$

where M is the image of σ under the m -adic representation and $t = \sigma(\beta) - \beta$. Then, $\omega_{\alpha, m^\infty}$ is an injective homomorphism of profinite groups identifying $\text{Gal}(K_{m^{-\infty}\alpha}/K)$ with a subgroup of

$$T_m A \rtimes \text{Aut}(T_m A) \cong \prod_{\ell} \mathbb{Z}_{\ell}^{b_A} \rtimes \prod_{\ell} \text{GL}_{b_A}(\mathbb{Z}_{\ell}) \cong \prod_{\ell} (\mathbb{Z}_{\ell}^{b_A} \rtimes \text{GL}_{b_A}(\mathbb{Z}_{\ell})).$$

Likewise, for each $n \geq 1$, the choice of β defines a homomorphism

$$\begin{aligned} \omega_{\alpha, m^n} : \text{Gal}(K_{m^{-n}\alpha}/K) &\longrightarrow A[m^n] \rtimes \text{Aut}(A[m^n]) \\ \sigma &\longmapsto (t, M), \end{aligned}$$

where t and M are defined in a similar way as above. This identifies $\text{Gal}(K_{m^{-n}\alpha}/K)$ with a subgroup of

$$A[m^n] \rtimes \text{Aut}(A[m^n]) \cong \prod_{\ell} ((\mathbb{Z}/\ell^n \mathbb{Z})^{b_A} \rtimes \text{GL}_{b_A}(\mathbb{Z}/\ell^n \mathbb{Z})).$$

We denote by $\mathcal{G}(\ell^\infty)$ the image of the ℓ -adic representation in $\text{Aut}(T_\ell A) \cong \text{GL}_{b_A}(\mathbb{Z}_\ell)$ and by $\mathcal{G}(\ell^n)$ the image of the mod ℓ^n representation in $\text{Aut}(A[\ell^n]) \cong \text{GL}_{b_A}(\mathbb{Z}/\ell^n \mathbb{Z})$. Similarly, we denote by $\mathcal{G}(m^\infty)$ the image of the m -adic representation in $\text{Aut}(T_m A) \cong \prod_{\ell} \text{GL}_{b_A}(\mathbb{Z}_{\ell})$ and by $\mathcal{G}(m^n)$ the image of the mod m^n representation in $\text{Aut}(A[m^n]) \cong \prod_{\ell} \text{GL}_{b_A}(\mathbb{Z}/\ell^n \mathbb{Z})$.

We write $d_{A, \ell}$ for the dimension of the Zariski closure of $\mathcal{G}(\ell^\infty)$ in $\text{GL}_{b_A, \mathbb{Q}_\ell}$, and we put

$$D_{A, m} = \prod_{\ell | m} \ell^{d_{A, \ell}}.$$

We note that the $d_{A, \ell}$ and $D_{A, m}$ do not change when replacing K by a finite extension. Moreover, assuming the Mumford–Tate conjecture, all $d_{A, \ell}$ are equal to d_A , the dimension of the Mumford–Tate group, implying $D_{A, m} = m^{d_A}$. This is known, for example, when A is an elliptic curve; in this case, d_A equals 2 if A has complex multiplication, and 4 otherwise.

DEFINITION 3. We say that $(A/K, m)$ satisfies *eventual maximal growth of the torsion fields* if there exists a positive integer n_0 such that for all $N \geq n \geq n_0$ we have

$$[K_{m^{-N}} : K_{m^{-n}}] = D_{A, m}^{N-n}.$$

We say that $(A/K, m, \alpha)$ satisfies *eventual maximal growth of the Kummer extensions* if there exists a positive integer n_0 such that for all $N \geq n \geq n_0$ we have

$$[K_{m^{-N}\alpha} : K_{m^{-n}\alpha}] = (m^{b_A} D_{A,m})^{N-n}. \quad (3.1)$$

REMARK 4. Condition (3.1) means that there is eventual maximal growth of the torsion fields, that $K_{m^{-n}\alpha}$ and $K_{m^{-N}\alpha}$ are linearly disjoint over $K_{m^{-n}\alpha}$ and that we have

$$[K_{m^{-N}\alpha} : K_{m^{-N}}(m^{-n}\alpha)] = m^{b_A(N-n)}.$$

If there is eventual maximal growth of the Kummer extensions, the rational number

$$C_m := m^{b_A n} / [K_{m^{-n}\alpha} : K_{m^{-n}}] \quad (3.2)$$

is independent of n for $n \geq n_0$. In fact, C_m is an integer because ω_{α, m^n} maps $\text{Gal}(K_{m^{-n}\alpha}/K_{m^{-n}})$ injectively into $A[m^n] \cong (\mathbb{Z}/m\mathbb{Z})^{b_A}$.

PROPOSITION 5. *If A is a semiabelian variety, then $(A/K, m)$ satisfies eventual maximal growth of the torsion fields. If A is the product of an abelian variety and a torus and $\mathbb{Z}\alpha$ is Zariski dense in A , then $(A/K, m, \alpha)$ satisfies eventual maximal growth of the Kummer extensions.*

Proof. By [4, Lemma 12], if A is a semiabelian variety and ℓ is a prime divisor of m , then $(A/K, \ell, \alpha)$ satisfies eventual maximal growth of the torsion fields. We also know that the degree $[K_{\ell^{-n}} : K_{\ell^{-1}}]$ is a power of ℓ for each n . Therefore, the extensions $K_{m^{-1}}K_{\ell^{-n}}$ for $\ell \mid m$ are linearly disjoint over $K_{m^{-1}}$ and the first assertion follows. By [4, Remark 9], the second assertion holds for $(A/K, \ell, \alpha)$, where ℓ is any prime divisor of m . We conclude because the degrees of these Kummer extensions are powers of ℓ . \square

4. Relating the density and the arboreal representation.

4.1. The existence of the density. Let $(A/K, m, \alpha)$ be as in Section 3. From now on, we assume that $(A/K, m, \alpha)$ satisfies eventual maximal growth of the Kummer extensions.

REMARK 6. This is not a restriction if A is the product of an abelian variety and a torus by Proposition 5. Indeed, consider the number of connected components of the Zariski closure of $\mathbb{Z}\alpha$. If this number is not coprime to m , then the density $\text{Dens}_m(\alpha)$ is zero by [5, Main Theorem] while if it is coprime to m we may replace α by a multiple to reduce to the case where the Zariski closure of $\mathbb{Z}\alpha$ is connected. Finally, we may replace A by the Zariski closure of $\mathbb{Z}\alpha$ and reduce to the case where $\mathbb{Z}\alpha$ is Zariski dense. Also notice that if A is simple (i.e. has exactly two connected algebraic subgroups), then eventual maximal growth of the Kummer extensions is satisfied as soon as α has infinite order.

The $T_m A$ -torsor $m^{-\infty}\alpha$ from Section 3 defines a Galois cohomology class

$$C_\alpha \in H^1(\text{Gal}(K_{m^{-\infty}\alpha}/K), T_m A).$$

For any choice of $\beta \in m^{-\infty}\alpha$, this is the class of the cocycle

$$\begin{aligned} c_\beta : \text{Gal}(K_{m^{-\infty}\alpha}/K) &\longrightarrow T_m A \\ \sigma &\longmapsto \sigma(\beta) - \beta. \end{aligned}$$

We also consider the restriction map with respect to the cyclic subgroup generated by some element $\sigma \in \text{Gal}(K_{m^{-\infty}\alpha}/K)$:

$$\text{Res}_\sigma: H^1(\text{Gal}(K_{m^{-\infty}\alpha}/K), T_m A) \longrightarrow H^1(\langle \sigma \rangle, T_m A).$$

THEOREM 7. *If $(A/K, m, \alpha)$ satisfies eventual maximal growth of the Kummer extensions, then the density $\text{Dens}_m(\alpha)$ exists and equals the normalised Haar measure in $\text{Gal}(K_{m^{-\infty}\alpha}/K)$ of the subset*

$$\begin{aligned} S_\alpha &:= \{\sigma \in \text{Gal}(K_{m^{-\infty}\alpha}/K) \mid C_\alpha \in \ker(\text{Res}_\sigma)\} \\ &= \{\sigma \in \text{Gal}(K_{m^{-\infty}\alpha}/K) \mid \sigma(\beta) = \beta \text{ for some } \beta \in m^{-\infty}\alpha\}. \end{aligned}$$

Proof. The generalisations of [2, Theorem 3.2] and [4, Theorem 7] to the composite case are straightforward. \square

Similarly to [4, Remark 21], we may equivalently consider S_α as a subset of either $\text{Gal}(\overline{K}/K)$ or $\text{Gal}(K_{m^{-\infty}\alpha}/K)$ with their respective normalised Haar measures.

PROPOSITION 8. *If L/K is any Galois extension that is linearly disjoint from $K_{m^{-\infty}\alpha}$ over K , then we have $\text{Dens}_L(\alpha) = \text{Dens}_K(\alpha)$.*

Proof. The generalisation of [4, Proposition 22] to the composite case is straightforward. \square

4.2. Counting elements in the image of the arboreal representation. By Theorem 7, computing $\text{Dens}_m(\alpha)$ comes down to computing the Haar measure of S_α in $\text{Gal}(K_{m^{-\infty}\alpha}/K)$. This is why we now investigate the Galois groups $\text{Gal}(K_{m^{-n}\alpha}/K)$ for positive integers n .

For $M \in \mathcal{G}(m^n)$ we define

$$\mathcal{W}_{m^n}(M) := \{t \in A[m^n] \mid (t, M) \in \text{Gal}(K_{m^{-n}\alpha}/K)\} \quad (4.1)$$

and

$$w_{m^n}(M) := \frac{\#(\text{Im}(M - I) \cap \mathcal{W}_{m^n}(M))}{\# \text{Im}(M - I)} \in \mathbb{Q}. \quad (4.2)$$

We note that $\mathcal{W}_{m^n}(M)$ is a $\text{Gal}(K_{m^{-n}\alpha}/K_{m^{-n}})$ -torsor and in particular satisfies

$$\#\mathcal{W}_{m^n}(M) = [K_{m^{-n}\alpha} : K_{m^{-n}}].$$

For every prime divisor ℓ of m and every $n \geq 1$, we consider the Galois group of the compositum $K_{\ell^{-n}\alpha}K_{m^{-1}}$ over K and the inclusion

$$\iota_{\alpha, \ell^n}: \text{Gal}(K_{\ell^{-n}\alpha}K_{m^{-1}}/K) \hookrightarrow (A[\ell^n] \rtimes \mathcal{G}(\ell^n)) \times \mathcal{G}(m).$$

For all $x \in \mathcal{G}(m)$ and $V \in \mathcal{G}(\ell^n)$, we define

$$\mathcal{W}_{x, \ell^n}(V) := \{\tau \in A[\ell^n] \mid (\tau, V, x) \in \text{Im } \iota_{\alpha, \ell^n}\} \quad (4.3)$$

and

$$w_{x, \ell^n}(V) := \frac{\#(\text{Im}(V - I) \cap \mathcal{W}_{x, \ell^n}(V))}{\# \text{Im}(V - I)} \in \mathbb{Z}[1/\ell]. \quad (4.4)$$

We denote by π_* the projection onto $\mathcal{G}(*)$.

PROPOSITION 9. *If $x \in \mathcal{G}(m)$ and $M \in \mathcal{G}(m^n)$ are such that $\pi_m M = x$, then we have*

$$\mathcal{W}_{m^n}(M) = \prod_{\ell} \mathcal{W}_{x, \ell^n}(\pi_{\ell^n} M)$$

and

$$w_{m^n}(M) = \prod_{\ell} w_{x, \ell^n}(\pi_{\ell^n} M).$$

Proof. Since the extensions $K_{\ell^{-n}\alpha}K_{m^{-1}}/K_{m^{-1}}$ have pairwise coprime degrees and hence are linearly disjoint, giving an element of $\text{Gal}(K_{m^{-n}\alpha}/K)$ mapping to $x \in \mathcal{G}(m)$ is equivalent to giving, for each prime $\ell \mid n$, an element of $\text{Gal}(K_{\ell^{-n}\alpha}K_{m^{-1}}/K)$ mapping to x . Hence, given an element $t = \sum_{\ell} t_{\ell}$ in $A[m^n] = \bigoplus_{\ell} A[\ell^n]$, we have $t \in \mathcal{W}_{m^n}(M)$ if and only if for every ℓ we have $t_{\ell} \in \mathcal{W}_{x, \ell^n}(\pi_{\ell^n} M)$. Therefore (t, M) is in $\text{Gal}(K_{m^{-n}\alpha}/K)$ if and only if $(t_{\ell}, \pi_{\ell^n} M, x)$ is in the image of ι_{α, ℓ^n} for all ℓ . This implies the first claim. The second claim follows because we have $\text{Im}(M - I) = \bigoplus_{\ell} \text{Im}(\pi_{\ell^n} M - I)$. \square

LEMMA 10. *For all $x \in \mathcal{G}(m)$ and $V \in \mathcal{G}(\ell^\infty)$, the value $w_{x, \ell^n}(V)$ is constant for n sufficiently large.*

Proof. This is proved as in [4, Lemma 25]. \square

By Lemma 10, we can define

$$w_{x, \ell^\infty}(V) = \lim_{n \rightarrow \infty} w_{x, \ell^n}(V) \in \mathbb{Z}[1/\ell]. \quad (4.5)$$

From Proposition 9 we deduce that for all $M \in \mathcal{G}(m^\infty)$, the value $w_{m^n}(M)$ is also constant for n sufficiently large, so we can analogously define

$$w_{m^\infty}(M) = \lim_{n \rightarrow \infty} w_{m^n}(M) \in \mathbb{Q}. \quad (4.6)$$

PROPOSITION 11. *If $M \in \mathcal{G}(m^\infty)$ is such that $\pi_m M = x$, then we have*

$$w_{m^\infty}(M) = \prod_{\ell} w_{x, \ell^\infty}(\pi_{\ell^\infty} M).$$

Proof. Taking the limit as $n \rightarrow \infty$ in Proposition 9 yields the claim. \square

The following lemma gives sufficient conditions for the sets $\mathcal{W}_{m^n}(M)$ and the functions $w_{m^n}(M)$ and $w_{m^\infty}(M)$ to admit product decompositions without a dependence on the element $x \in \mathcal{G}(m)$. It will not be used in the remainder of this article.

LEMMA 12. *For all primes $\ell \mid m$ and all $n \geq 1$, the following conditions are equivalent:*

- (1) *The intersection of the fields $K_{m^{-1}}$ and $K_{\ell^{-n}\alpha}$ is contained in $K_{\ell^{-n}}$.*
- (2) *The intersection of the fields $K_{m^{-1}}K_{\ell^{-n}}$ and $K_{\ell^{-n}\alpha}$ equals $K_{\ell^{-n}}$.*
- (3) *The fields $K_{m^{-1}}K_{\ell^{-n}}$ and $K_{\ell^{-n}\alpha}$ are linearly disjoint over $K_{\ell^{-n}}$.*
- (4) *We have $[K_{m^{-1}}K_{\ell^{-n}\alpha} : K_{m^{-1}}K_{\ell^{-n}}] = [K_{\ell^{-n}\alpha} : K_{\ell^{-n}}]$.*
- (5) *We have $[K_{m^{-n}}K_{\ell^{-n}\alpha} : K_{m^{-n}}] = [K_{\ell^{-n}\alpha} : K_{\ell^{-n}}]$.*

If these conditions are satisfied for all primes $\ell \mid m$ and all $n \geq 1$, then the following statements hold:

- (6) *We have $C_m = \prod_{\ell} C_{\ell}$.*
- (7) *For all $n \geq 1$ and all $M \in \mathcal{G}(m^n)$ we have $\mathcal{W}_{m^n}(M) = \prod_{\ell} \mathcal{W}_{\ell^n}(\pi_{\ell^n} M)$.*

(8) For all $n \geq 1$ and all $M \in \mathcal{G}(m^n)$ we have $w_{m^n}(M) = \prod_{\ell} w_{\ell^n}(\pi_{\ell^n} M)$.

(9) For all $M \in \mathcal{G}(m^\infty)$ we have $w_{m^\infty}(M) = \prod_{\ell} w_{\ell^\infty}(\pi_{\ell^\infty} M)$.

Proof. The equivalence of the conditions (1)–(4) follows from Galois theory, using the fact that all the fields involved are Galois extensions of K . The conditions (4) and (5) are equivalent because $[K_{m^{-1}} K_{\ell^{-n}\alpha} : K_{m^{-1}} K_{\ell^{-n}}]$ is a power of ℓ and $[K_{m^{-n}} : K_{m^{-1}} K_{\ell^{-n}}]$ is prime to ℓ . If condition (5) holds for a given $n \geq 1$ and all primes $\ell \mid m$, then we have

$$\begin{aligned} [K_{m^{-n}\alpha} : K_{m^{-n}}] &= \prod_{\ell} [K_{m^{-n}} K_{\ell^{-n}\alpha} : K_{m^{-n}}] \\ &= \prod_{\ell} [K_{\ell^{-n}\alpha} : K_{\ell^{-n}}]. \end{aligned}$$

This implies that if (5) is true for all primes $\ell \mid m$ and all $n \geq 1$, then (6) and (7) hold. Finally, it is clear that (7) implies (8) and (9). \square

4.3. Partitioning the image of the m -adic representation. We view elements of $\mathcal{G}(m^\infty)$ as automorphisms of $A[m^\infty] = \bigcup_{n \geq 1} A[m^n]$. We then classify elements $M \in \mathcal{G}(m^\infty)$ according to the group structure of $\ker(M - I)$ and according to the projection $\pi_m(M) \in \mathcal{G}(m)$. Note that if $\ker(M - I)$ is finite, then it is a product over the primes $\ell \mid m$ of finite abelian ℓ -groups that have at most b_A cyclic components.

Let F be a group of the form $\prod_{\ell \mid m} F_\ell$, where F_ℓ is a finite abelian ℓ -group with at most b_A cyclic components. We define the set

$$\mathcal{M}_F := \{M \in \mathcal{G}(m^\infty) \mid \ker(M - I : A[m^\infty] \rightarrow A[m^\infty]) \cong F\}, \quad (4.7)$$

and for every $x \in \mathcal{G}(m)$ we define the set

$$\mathcal{M}_{x,F} := \{M \in \mathcal{G}(m^\infty) \mid \ker(M - I : A[m^\infty] \rightarrow A[m^\infty]) \cong F, \pi_m(M) = x\}.$$

We denote by $\mathcal{M}_F(*)$ and $\mathcal{M}_{x,F}(*)$, respectively, the images of these sets under the reduction map $\mathcal{G}(m^\infty) \rightarrow \mathcal{G}(*)$. We also write

$$\mathcal{M} := \bigcup_F \mathcal{M}_F = \bigcup_{x,F} \mathcal{M}_{x,F}, \quad (4.8)$$

the union being taken over all $x \in \mathcal{G}(m)$ and over all groups $F = \prod_{\ell} F_\ell$ as above, up to isomorphism.

PROPOSITION 13. *The following holds:*

- (1) The sets $\mathcal{M}_{x,F}$ are measurable in $\mathcal{G}(m^\infty)$, and the set \mathcal{M} of (4.8) is measurable in $\mathcal{G}(m^\infty)$.
- (2) If $n > v_\ell(\exp F)$ for all $\ell \mid m$, then we have

$$\mu_{\mathcal{G}(m^\infty)}(\mathcal{M}_{x,F}) = \mu_{\mathcal{G}(m^n)}(\mathcal{M}_{x,F}(m^n)).$$

- (3) We have $\mu_{\mathcal{G}(m^\infty)}(\mathcal{M}_{x,F}) = 0$ if and only if $\mathcal{M}_{x,F} = \emptyset$.
- (4) If $(A/K, m)$ satisfies eventual maximal growth of the torsion fields, then we have

$$\mu_{\mathcal{G}(m^\infty)}(\mathcal{M}) = 1.$$

Proof. This is proved as in [4, Lemma 23]. \square

5. The density as an integral. Suppose that $(A/K, m, \alpha)$ satisfies eventual maximal growth of the Kummer extensions. Recall from Remark 6 that this is not a restriction if A is the product of an abelian variety and a torus. By Theorem 7, computing $\text{Dens}_m(\alpha)$ comes down to computing the Haar measure of S_α in $\text{Gal}(K_{m^{-\infty}\alpha}/K)$. The generalisation of [4, Remark 19] to the composite case gives

$$S_\alpha = \{(t, M) \in \text{Gal}(K_{m^{-\infty}\alpha}/K) \mid M \in \mathcal{G}(m^\infty) \text{ and } t \in \text{Im}(M - I)\}.$$

In view of (4.8), we consider the sets

$$S_{x,F} := \{(t, M) \in \text{Gal}(K_{m^{-\infty}\alpha}/K) \mid M \in \mathcal{M}_{x,F} \text{ and } t \in \text{Im}(M - I)\}.$$

By assertion (4) of Proposition 13 and our assumption that $(A/K, m, \alpha)$ satisfies eventual maximal growth of the torsion fields, the set S_α is the disjoint union of the sets $S_{x,F}$ up to a set of measure 0. To see that the Haar measure of $S_{x,F}$ is well defined and to compute it, we define for every $n \geq 1$ the set

$$S_{x,F,m^n} = \{(t, M) \in \text{Gal}(K_{m^{-n}\alpha}/K) \mid M \in \mathcal{M}_{x,F}(m^n) \text{ and } t \in \text{Im}(M - I)\}.$$

PROPOSITION 14. *Suppose $n > n_0$ and $n > \max_\ell \{v_\ell(\exp F)\}$ for every ℓ , where n_0 is as in Definition 3. Then the set S_{x,F,m^n} is the image of $S_{x,F}$ under the projection to $\text{Gal}(K_{m^{-n}\alpha}/K)$.*

Proof. The set S_{x,F,m^n} clearly contains the reduction modulo m^n of $S_{x,F}$. To prove the other inclusion, consider $(t_{m^n}, M_{m^n}) \in S_{x,F,m^n}$ and a lift $(t, M) \in \text{Gal}(K_{m^{-\infty}\alpha}/K)$. Since n is sufficiently large with respect to F , we have $\ker(M - I) \cong F$. Clearly, M_{m^n} and M have the same projection $x \in \mathcal{G}(m)$. To conclude, it suffices to ensure $t \in \text{Im}(M - I)$. Take $\tau_{m^n} \in A[m^n]$ satisfying $(M_{m^n} - I)(\tau_{m^n}) = t_{m^n}$, and some lift τ of τ_{m^n} to $T_m(A)$: we may replace t by $(M - I)\tau$ because the difference is in $m^n T_m(A)$ and since $n > n_0$ we know that $\text{Gal}(K_{m^{-\infty}\alpha}/K)$ contains $m^n T_m(A) \times \{I\}$. \square

THEOREM 15. *We have*

$$\mu(S_{x,F}) = \frac{C_m}{\#F} \int_{\mathcal{M}_{x,F}} w_{m^\infty}(M) d\mu_{\mathcal{G}(m^\infty)}(M),$$

where C_m is the constant of (3.2) and w_{m^∞} is as in (4.6).

Proof. Choose n large enough so that $n > n_0$ and $n > \max_\ell \{v_\ell(\exp F)\}$ for every ℓ , where n_0 is as in Definition 3. By definition (see (4.1)) we can write

$$\#S_{x,F,m^n} = \sum_{M \in \mathcal{M}_{x,F}(m^n)} \#(\text{Im}(M - I) \cap \mathcal{W}_{m^n}(M)).$$

By definition (see (4.2)), we can express the summand as

$$\#\text{Im}(M - I) \cdot w_{m^n}(M) = \frac{w_{m^n}(M) \cdot m^{bn}}{\#F},$$

so from (3.2), we deduce

$$\frac{\#S_{x,F,m^n}}{\#\text{Gal}(K_{m^{-n}\alpha}/K)} = \frac{1}{\#\mathcal{G}(m^n)} \sum_{M \in \mathcal{M}_{x,F}(m^n)} \frac{C_m}{\#F} \cdot w_{m^n}(M).$$

By (3.1), the left-hand side is a non-increasing function of n , and therefore, it admits a limit for $n \rightarrow \infty$, which is $\mu(S_{x,F})$. The right-hand side is an integral over $\mathcal{M}_{x,F}(m^n)$ with

respect to the normalised counting measure of $\mathcal{G}(m^n)$, and the matrices in $\mathcal{M}_{x,F}$ are exactly the matrices in $\mathcal{G}(m^\infty)$ whose reduction modulo m^n lies in $\mathcal{M}_{x,F}(m^n)$. Taking the limit over n , we thus find the formula in the statement. \square

THEOREM 16. *We have*

$$\begin{aligned} \text{Dens}_m(\alpha) &= C_m \sum_F \frac{1}{\#F} \int_{\mathcal{M}_F} w_{m^\infty}(M) d\mu_{\mathcal{G}(m^\infty)}(M) \\ &= C_m \int_{\mathcal{G}(m^\infty)} \frac{w_{m^\infty}(M)}{\#\ker(M-I)} d\mu_{\mathcal{G}(m^\infty)}(M), \end{aligned} \quad (5.1)$$

where the function w_{m^∞} is as in (4.6), the constant C_m is as in (3.2), and F varies over the products over the primes $\ell \mid m$ of finite abelian ℓ -groups with at most b_A cyclic components.

Proof. To prove the first equality, note that \mathcal{M}_F is the disjoint union of the $\mathcal{M}_{x,F}$ for $x \in \mathcal{G}(m)$. By Theorem 7, we may write $\text{Dens}_m(\alpha) = \mu(S_\alpha) = \sum_{x,F} \mu(S_{x,F})$ and then it suffices to apply Theorem 15. The second equality follows because the union of the sets \mathcal{M}_F from (4.7) has measure 1 in $\mathcal{G}(m^\infty)$ by Proposition 13. \square

COROLLARY 17 ([4, Theorem 1 and Remark 27]). *In the special case $m = \ell$, we have*

$$\begin{aligned} \text{Dens}_\ell(\alpha) &= C_\ell \sum_F \frac{1}{\#F} \int_{\mathcal{M}_F} w_{\ell^\infty}(M) d\mu_{\mathcal{G}(\ell^\infty)}(M) \\ &= C_\ell \int_{\mathcal{G}(\ell^\infty)} \frac{w_{\ell^\infty}(M)}{\#\ker(M-I)} d\mu_{\mathcal{G}(\ell^\infty)}(M), \end{aligned} \quad (5.2)$$

where F varies among the finite abelian ℓ -groups with at most b_A cyclic components.

Notice that we have $\#\ker(M-I) = \ell^{v_\ell(\det(M-I))}$ for every $M \in \mathcal{G}(\ell^\infty)$; this shows the equivalence with [4, Theorem 1].

COROLLARY 18. *Let ℓ vary among the prime divisors of m . If the fields $K_{\ell^{-\infty}\alpha}$ are linearly disjoint over K , then we have*

$$\text{Dens}_m(\alpha) = \prod_\ell \text{Dens}_\ell(\alpha).$$

Proof. Note that we have $C_m = \prod_\ell C_\ell$. By assumption, we also have $\mathcal{G}(m^\infty) = \prod_\ell \mathcal{G}(\ell^\infty)$, which implies $\mu_{\mathcal{G}(m^\infty)} = \prod_\ell \mu_{\mathcal{G}(\ell^\infty)}$, and $w_{m^\infty}(M) = \prod_\ell w_{\ell^\infty}(\pi_{\ell^\infty} M)$. We conclude that (5.1) is the product of the expressions (5.2) for $\ell \mid m$. \square

The conditions of Corollary 18 are satisfied, for example, if $K_{m^{-1}} = K$, or more generally if the degree $[K_{\ell^{-1}} : K]$ is a power of ℓ for each ℓ . Under weaker conditions, $\text{Dens}_m(\alpha)$ is not in general the product of the $\text{Dens}_\ell(\alpha)$, but we can still express it as a sum of products of ℓ -adic integrals, as the following result shows.

THEOREM 19. *Denote by $H(x) = \prod_\ell H_\ell(x)$ the set of matrices in $\mathcal{G}(m^\infty) \subseteq \prod_\ell \mathcal{G}(\ell^\infty)$ mapping to x in $\mathcal{G}(m)$. We then have*

$$\text{Dens}_m(\alpha) = \frac{C_m}{\#\mathcal{G}(m)} \sum_{x \in \mathcal{G}(m)} \prod_\ell \int_{H_\ell(x)} \frac{w_{x,\ell^\infty}(M)}{\#\ker(M-I)} d\mu_{H_\ell(x)}(M), \quad (5.3)$$

where w_{x,ℓ^∞} is as in (4.5).

Proof. Write $S_x = \bigcup_F S_{x,F}$ and recall from Proposition 13 that the set of matrices M for which $\ker(M - I)$ is infinite has measure zero in $\mathcal{G}(m^\infty)$. By Theorem 15, we have

$$\mu(S_x) = \sum_F \mu(S_{x,F}) = C_m \int_{H(x)} \frac{w_{m^\infty}(M)}{\#\ker(M - I)} d\mu_{\mathcal{G}(m^\infty)}(M).$$

The assertion follows from Propositions 2 and 11. \square

COROLLARY 20. *The density $\text{Dens}_m(\alpha)$ is a rational number. Moreover, for every positive integer b , there exists a non-zero polynomial $p_b(t) \in \mathbb{Z}[t]$ with the following property: whenever K is a number field and A is the product of an abelian variety and a torus such that the first Betti number of A equals b , then for all $\alpha \in A(K)$ and all square-free integers $m \geq 2$ such that $(A/K, m, \alpha)$ satisfies eventual maximal growth of the Kummer extensions, we have*

$$\text{Dens}_m(\alpha) \cdot \prod_{\ell} p_b(\ell) \in \mathbb{Z}[1/m],$$

where ℓ varies over the prime divisors of m .

Proof. Recall that C_m is an integer. In view of Lemma 10, we can consider each ℓ -adic integral in (5.3) and proceed as in the proof of [4, Theorem 36]. \square

REMARK 21. For elliptic curves, it is also possible to bound the minimal denominator of $\text{Dens}_m(\alpha)$. Indeed, let us consider (5.3), recalling that C_m is an integer. Each of the finitely many functions w_{x,ℓ^∞} takes only finitely many values: these are rational numbers whose minimal denominator divides ℓ^{2n_0} , where n_0 is large enough so that condition (3.1) holds for all $N \geq n \geq n_0$. If $M \in \mathcal{M}_\ell(a, b)$ (see Section 6.2), then $\#\ker(M - I) = \ell^{2a+b}$. The crucial fact is the independence of the number of lifts [3, Theorem 28]; the case distinction for the normaliser of a Cartan subgroup does not matter because we separately count the matrices in the Cartan subgroup and those in its complement. This means that the measure of $\mathcal{M}_\ell(a, b) \cap H_\ell(x)$ is a fraction of that of $\mathcal{M}_\ell(a, b)$: this ratio can take only finitely many values and can be understood by working modulo ℓ^{n_0} . We may then need to multiply the denominator in the measure of $\mathcal{M}_\ell(a, b)$ by an integer which is at most $\#\text{GL}_2(\ell^{n_0})$. Essentially we need to evaluate finitely many geometric series because of the eventual maximal growth of the torsion fields (the degrees $[K(E[\ell^n]) : K]$ for n sufficiently large form a geometric progression) and we may reason as in [4, Theorems 5 and 6].

REMARK 22. We may replace the point α by a finitely generated subgroup G of $A(K)$. Indeed, let $\alpha_1, \dots, \alpha_r$ be generators for G . We may then consider the point $\beta = (\alpha_1, \dots, \alpha_r)$ in the product $A^r(K)$. Then the density $\text{Dens}_m(\beta)$ for the single point β is exactly the density of primes \mathfrak{p} of K such that the order of $(G \bmod \mathfrak{p})$ is coprime to m .

6. Serre curves.

6.1. Definition of Serre curves. Let E be an elliptic curve over a number field K . We choose a Weierstrass equation for E of the form

$$E: y^2 = (x - x_1)(x - x_2)(x - x_3), \quad (6.1)$$

where $x_1, x_2, x_3 \in K(E[2])$ are the x -coordinates of the points of order 2. The discriminant of the right-hand side of (6.1) is $\Delta = \sqrt{\Delta}^2$, where

$$\sqrt{\Delta} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1).$$

We thus have $K(\sqrt{\Delta}) \subseteq K(E[2])$, and we define a character

$$\begin{aligned}\psi_E: \operatorname{Gal}(K(E[2])/K) &\longrightarrow \{\pm 1\} \\ \sigma &\longmapsto \sigma(\sqrt{\Delta})/\sqrt{\Delta}.\end{aligned}$$

For any choice of basis of the 2-torsion of E , we have the 2-torsion representation

$$\rho_{E,2}: \operatorname{Gal}(K(E[2])/K) \longrightarrow \operatorname{GL}_2(\mathbb{Z}/2\mathbb{Z}).$$

Let ψ be the unique non-trivial character $\operatorname{GL}_2(\mathbb{Z}/2\mathbb{Z}) \rightarrow \{\pm 1\}$; this corresponds to the sign character under any isomorphism of $\operatorname{GL}_2(\mathbb{Z}/2\mathbb{Z})$ with S_3 . The character ψ_E factors as

$$\psi_E = \psi \circ \rho_{E,2}.$$

From now on, we take $K = \mathbb{Q}$. All number fields that we will consider will be subfields of a fixed algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} .

Let d be an element of \mathbb{Q}^\times . Let m_d be the conductor of $\mathbb{Q}(\sqrt{d})$; this is the smallest positive integer such that \sqrt{d} lies in the cyclotomic field $\mathbb{Q}(\zeta_{m_d})$. Let d_{sf} be the square-free part of d . We have

$$m_d = \begin{cases} |d_{\text{sf}}| & \text{if } d_{\text{sf}} \equiv 1 \pmod{4}, \\ 4|d_{\text{sf}}| & \text{otherwise.} \end{cases}$$

We define a character

$$\begin{aligned}\varepsilon_d: \operatorname{Gal}(\mathbb{Q}(\zeta_{m_d})/\mathbb{Q}) &\longrightarrow \{\pm 1\} \\ \sigma &\longmapsto \sigma(\sqrt{d})/\sqrt{d}.\end{aligned}$$

If σ is the automorphism of $\mathbb{Q}(\zeta_{m_d})$ defined by $\sigma(\zeta_{m_d}) = \zeta_{m_d}^a$ with $a \in (\mathbb{Z}/m_d\mathbb{Z})^\times$, then $\varepsilon_d(\sigma)$ equals the Jacobi symbol $\left(\frac{d_{\text{sf}}}{a}\right)$. We view ε_d as a character of $\operatorname{GL}_2(\mathbb{Z}/m_d\mathbb{Z})$ by composing with the determinant.

For all $n \geq 1$, we have a canonical projection

$$\pi_n: \operatorname{GL}_2(\widehat{\mathbb{Z}}) \rightarrow \operatorname{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$

Fixing a $\widehat{\mathbb{Z}}$ -basis for the projective limit of the torsion groups $E[n](\overline{\mathbb{Q}})$, we have a torsion representation

$$\rho_E: \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \operatorname{GL}_2(\widehat{\mathbb{Z}}).$$

The image of ρ_E is contained in the subgroup

$$H_\Delta = \{M \in \operatorname{GL}_2(\widehat{\mathbb{Z}}) \mid \psi(\pi_2(M)) = \varepsilon_\Delta(\pi_{m_\Delta}(M))\}$$

of index 2 in $\operatorname{GL}_2(\widehat{\mathbb{Z}})$. This expresses the fact that $\sqrt{\Delta}$ is contained in both $\mathbb{Q}(E[2])$ and $\mathbb{Q}(E[m_\Delta])$. An elliptic curve is said to be a *Serre curve* if the image of ρ_E is equal to H_Δ . As proven by N. Jones [1], almost all elliptic curves over \mathbb{Q} are Serre curves.

6.2. Counting matrices. Let ℓ be a prime number. For all integers $a, b \geq 0$, we write $\mathcal{M}_\ell(a, b)$ for the set of matrices $M \in \operatorname{GL}_2(\mathbb{Z}_\ell)$ such that the kernel of $M - I$ as an endomorphism of $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^2$ is isomorphic to $\mathbb{Z}/\ell^a\mathbb{Z} \times \mathbb{Z}/\ell^{a+b}\mathbb{Z}$.

If \mathcal{N} is a non-empty subset of $\mathcal{M}_\ell(a, b)$ that is the preimage in $\mathcal{M}_\ell(a, b)$ of its reduction modulo ℓ^n (which means that \mathcal{N} contains the intersection of $\mathcal{M}_\ell(a, b)$ with the set of preimages of $(\mathcal{N} \bmod \ell^n)$ in $\text{GL}_2(\mathbb{Z}_\ell)$), then we have

$$\frac{\mu_{\text{GL}_2(\mathbb{Z}_\ell)}(\mathcal{N})}{\mu_{\text{GL}_2(\mathbb{Z}_\ell)}(\mathcal{M}_\ell(a, b))} = \frac{\mu_{\text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})}(\mathcal{N} \bmod \ell^n)}{\mu_{\text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})}(\mathcal{M}_\ell(a, b) \bmod \ell^n)} \quad (6.2)$$

by [3, Theorem 27] (where the number of lifts is independent of the matrix). Notice that if $a \geq n$, then $(\mathcal{N} \bmod \ell^n)$ consists of the identity.

PROPOSITION 23. *If \mathcal{N} is a subset of $\mathcal{M}_\ell(a, b)$ that is the preimage in $\mathcal{M}_\ell(a, b)$ of its reduction modulo ℓ , then we have*

$$\mu_{\text{GL}_2(\mathbb{Z}_\ell)}(\mathcal{N}) = \mu_{\text{GL}_2(\mathbb{Z}/\ell\mathbb{Z})}(\mathcal{N} \bmod \ell) \cdot \begin{cases} 1 & \text{if } a = b = 0 \\ \ell^{-b}(\ell - 1) & \text{if } a = 0, b \geq 1 \\ \ell^{-4a} \cdot \ell(\ell - 1)^2(\ell + 1) & \text{if } a \geq 1, b = 0 \\ \ell^{-4a-b} \cdot (\ell - 1)^2(\ell + 1)^2 & \text{if } a \geq 1, b \geq 1. \end{cases}$$

Proof. We are working with $\text{GL}_2(\mathbb{Z}_\ell)$, so we can apply [3, Proposition 33] (see also [3, Definition 19]). This gives the assertion for the set $\mathcal{M}_\ell(a, b)$; we can conclude because of (6.2). \square

We now collect some results in the case $\ell = 2$. From [3, Theorem 2], we know

$$\mu_{\text{GL}_2(\mathbb{Z}_2)}(\mathcal{M}_2(a, b)) = \begin{cases} 1/3 & \text{if } a = b = 0 \\ 1/2 \cdot 2^{-b} & \text{if } a = 0, b \geq 1 \\ 2^{-4a} & \text{if } a \geq 1, b = 0 \\ 3/2 \cdot 2^{-4a-b} & \text{if } a \geq 1, b \geq 1. \end{cases}$$

We consider the action of $\text{GL}_2(\mathbb{Z}/2^3\mathbb{Z})$ on $\mathbb{Q}(\zeta_{2^3})$ defined by $M\zeta_{2^3} = \zeta_{2^3}^{\det M}$. The matrices $M \in \text{GL}_2(\mathbb{Z}/2^3\mathbb{Z})$ that fix $\sqrt{-1}$ are those with $\det(M) = 1, 5$. The ones that fix $\sqrt{2}$ are those with $\det(M) = 1, 7$. The ones that fix $\sqrt{-2}$ are those with $\det(M) = 1, 3$.

For $a, b \in \{0, 1, 2, 3\}$ and $z \in \{-1, 2, -2\}$, we write $\mathcal{N}_2(a, b, z)$ for the set of matrices in $\mathcal{M}_2(a, b)$ that fix \sqrt{z} .

LEMMA 24. *We have*

$$\frac{\mu_{\text{GL}_2(\mathbb{Z}_2)}(\mathcal{N}_2(a, b; -1))}{\mu_{\text{GL}_2(\mathbb{Z}_2)}(\mathcal{M}_2(a, b))} = \begin{cases} 1/2 & \text{for } a = 0, b \geq 0 \\ 2/3 & \text{for } a = 1, b = 0 \\ 1/3 & \text{for } a = 1, b \geq 1 \\ 1 & \text{for } a \geq 2, b \geq 0 \end{cases}$$

and

$$\frac{\mu_{\mathrm{GL}_2(\mathbb{Z}_2)}(\mathcal{N}_2(a, b; \pm 2))}{\mu_{\mathrm{GL}_2(\mathbb{Z}_2)}(\mathcal{M}_2(a, b))} = \begin{cases} 1/2 & \text{for } a \leq 1, b \geq 0 \\ 2/3 & \text{for } a = 2, b = 0 \\ 1/3 & \text{for } a = 2, b \geq 1 \\ 1 & \text{for } a \geq 3, b \geq 0. \end{cases}$$

Proof. For $a, b \in \{0, 1, 2, 3\}$ and $d \in (\mathbb{Z}/2^3\mathbb{Z})^\times$, let $h(a, b, d)$ be the number of matrices $M \in \mathrm{GL}_2(\mathbb{Z}/2^3\mathbb{Z})$ such that $\det(M) = d$ and $\ker(M - I) \cong \mathbb{Z}/2^a\mathbb{Z} \times \mathbb{Z}/2^{a+b}\mathbb{Z}$. Using [9] one can easily count these matrices:

- $h(0, 0, d) = 128$, $h(0, 1, d) = 96$ and $h(0, 2, d) = h(0, 3, d) = 48$ for all d ;
- $h(1, 0, d) = 32$ for $d = 1, 5$ and $h(1, 0, d) = 16$ for $d = 3, 7$;
- for $b = 1, 2$ we have $h(1, b, d) = 12$ for $d = 1, 5$ and $h(1, b, d) = 24$ for $d = 3, 7$;
- $h(2, 0, 1) = 4$, $h(2, 0, 5) = 2$ and $h(2, 0, d) = 0$ for $d = 3, 7$;
- $h(2, 1, 1) = 3$, $h(2, 1, 5) = 6$ and $h(2, 1, d) = 0$ for $d = 3, 7$;
- $h(3, 0, 1) = 1$ (the identity matrix) and $h(3, 0, d) = 0$ for $d = 3, 5, 7$.

This classification and (6.2) lead to the measures in the statement. \square

LEMMA 25. For all $a, b \geq 0$ and all $M \in \mathcal{M}_2(a, b)$, we have

$$\psi(M) = \begin{cases} -1 & \text{if } a = 0 \text{ and } b \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Consider matrices $M \in \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$. The matrices

$$M \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

satisfy $\psi(M) = 1$ and $\dim_{\mathbb{F}_2} \ker(M - I) \in \{0, 2\}$. The matrices

$$M \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

satisfy $\psi(M) = -1$ and $\dim_{\mathbb{F}_2} \ker(M - I) = 1$. This implies the claim. \square

Now let ℓ be an odd prime number. We write

$$\ell^* = (-1)^{(\ell-1)/2} \ell,$$

so ε_{ℓ^*} is a character of $(\mathbb{Z}/\ell\mathbb{Z})^\times$ and also of $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ via the determinant.

LEMMA 26. Let M vary in $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \setminus \{I\}$, where ℓ is an odd prime number.

- (1) There are $\frac{1}{2}(\ell + 1)^2(\ell - 2)$ matrices M satisfying $\varepsilon_{\ell^*}(M) = 1$ and $\ell \mid \det(M - I)$.
- (2) There are $\frac{1}{2}\ell(\ell^3 - 2\ell^2 - \ell + 4)$ matrices M satisfying $\varepsilon_{\ell^*}(M) = 1$ and $\ell \nmid \det(M - I)$.
- (3) There are $\frac{1}{2}\ell(\ell^2 - 1)$ matrices M satisfying $\varepsilon_{\ell^*}(M) = -1$ and $\ell \mid \det(M - I)$.
- (4) There are $\frac{1}{2}\ell(\ell^2 - 1)(\ell - 2)$ matrices M satisfying $\varepsilon_{\ell^*}(M) = -1$ and $\ell \nmid \det(M - I)$.

Proof. (1) Write $\chi(M)$ for the characteristic polynomial of M . The condition $\varepsilon_{\ell^*}(M) = 1$ is equivalent to $\det(M) = \chi(0)$ being a square in $(\mathbb{Z}/\ell\mathbb{Z})^\times$, and the condition $\ell \mid \det(M - I)$ is equivalent to $\chi(1) = 0$ in $\mathbb{Z}/\ell\mathbb{Z}$. Thus, the matrices M satisfying both conditions are those for which there exists $s \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ with

$$\chi(M) = (x - 1)(x - s^2).$$

The matrices with $\chi(0) \neq 1$ (giving $\frac{\ell-1}{2} - 1$ possibilities for χ) are diagonalisable, and we only have to choose the two distinct eigenspaces; this gives $(\ell + 1)\ell$ matrices for every such χ . The matrices with $\chi(0) = 1$ are the identity (which we are excluding) and the $\ell^2 - 1$ matrices conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Note that (1) can also be obtained from [7, Table 1].

- (2) There are $\frac{1}{2}\#\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ matrices satisfying $\varepsilon_{\ell^*} = 1$, and we only need to subtract the identity and the matrices from (1).
- (3) There are $\ell^3 - 2\ell$ matrices in $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ having 1 as an eigenvalue (see for example [3, Proof of Theorem 2]), and we only need to subtract the identity and the matrices from (1).
- (4) There are $\frac{1}{2}\#\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ matrices satisfying $\varepsilon_{\ell^*} = -1$, and we only need to subtract the matrices from (3). Alternatively, there are $\#\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) - (\ell^3 - 2\ell)$ matrices that do not have 1 as eigenvalue, and we only need to subtract the matrices from (2). \square

6.3. Partitioning the image of the m -adic representation. Let E be a Serre curve over \mathbb{Q} . Let Δ be the minimal discriminant of E , and let Δ_{sf} be its square-free part. We write $\Delta_{\mathrm{sf}} = zu$, where $z \in \{1, -1, 2, -2\}$ and where u is an odd fundamental discriminant. Then $|u|$ is the odd part of m_Δ , and we have $\varepsilon_\Delta = \varepsilon_z \cdot \varepsilon_u$ as characters of $(\mathbb{Z}/m_\Delta\mathbb{Z})^\times$.

Now let m be a square-free positive integer. If $m = 2$, or if m is odd, or if u does not divide m , then we have

$$\mathcal{G}(m^\infty) = \prod_{\ell} \mathcal{G}(\ell^\infty).$$

If $m \neq 2$ is even and u divides m , then $\mathcal{G}(m^\infty)$ has index 2 in $\prod_{\ell} \mathcal{G}(\ell^\infty)$. The defining condition for the image of the m -adic representation is then $\psi = \varepsilon_\Delta$, or equivalently

$$\psi \cdot \varepsilon_z = \varepsilon_u.$$

We may then partition $\mathcal{G}(m^\infty) \subseteq \prod_{\ell|m} \mathcal{G}(\ell^\infty)$ into two sets that are products, namely

$$(\mathcal{G}(2^\infty) \cap \{\psi \cdot \varepsilon_z = 1\}) \times (\mathcal{G}(|u|^\infty) \cap \{\varepsilon_u = 1\}) \times \mathcal{G}\left(\left|\frac{m}{2u}\right|^\infty\right)$$

and

$$(\mathcal{G}(2^\infty) \cap \{\psi \cdot \varepsilon_z = -1\}) \times (\mathcal{G}(|u|^\infty) \cap \{\varepsilon_u = -1\}) \times \mathcal{G}\left(\left|\frac{m}{2u}\right|^\infty\right).$$

The set $\mathcal{G}(|u|^\infty) \cap \{\varepsilon_u = 1\}$ is the disjoint union of sets of the form $\prod_{\ell|u} (\mathcal{G}(\ell^\infty) \cap \{\varepsilon_{\ell^*} = \pm 1\})$, choosing an even number of minus signs; for the set $\mathcal{G}(|u|^\infty) \cap \{\varepsilon_u = -1\}$ we have to choose an odd number of minus signs. Since each $\ell \mid u$ is odd, the two sets $\mathcal{G}(\ell^\infty) \cap \{\varepsilon_{\ell^*} = \pm 1\}$ can be investigated with the help of Lemma 26. Finally, the two sets $\mathcal{G}(2^\infty) \cap \{\psi \cdot \varepsilon_z = \pm 1\}$ can be investigated using Lemmas 24 and 25.

7. Examples.

7.1. Example (non-surjective mod 3 representation). Consider the non-CM elliptic curve

$$E: y^2 + y = x^3 + 6x + 27$$

of discriminant $-3^{19} \cdot 17$ and conductor $153 = 3^2 \cdot 17$ over \mathbb{Q} [8, label 153.b2]. The group $E(\mathbb{Q})$ is infinite cyclic and is generated by the point

$$\alpha = (5, 13).$$

We will compute the following values (by testing the primes up to 10^6 , we have computed an approximation to $\text{Dens}_6(\alpha)$ using [9]):

Point	Dens_2	Dens_3	Dens_6	primes $< 10^6$
$\alpha = (5, 13)$	11/21	23/104	$253/2184 = 11.584 \dots \%$	11.624%
$2\alpha = (-1, 4)$	16/21	23/104	$46/273 = 16.849 \dots \%$	16.885%
$3\alpha = (-7/4, -31/8)$	11/21	77/104	$121/312 = 38.782 \dots \%$	38.730%
$6\alpha = (137/16, 1669/64)$	16/21	77/104	$22/39 = 56.410 \dots \%$	56.373%
$4\alpha = (3, -9)$	37/42	23/104	$851/4368 = 19.482 \dots \%$	19.479%
$9\alpha = (\frac{19649}{12100}, -\frac{9216643}{1331000})$	11/21	95/104	$1045/2184 = 47.847 \dots \%$	47.791%

The image of the 3-adic representation is the inverse image of its reduction modulo 3, the image of the mod 3 representation is isomorphic to the symmetric group of order 6 and the 3-adic Kummer map is surjective [4, Example 6.4]. The image of the mod 3 representation has a unique subgroup of index 2, so the field $\mathbb{Q}(E[3])$ contains as its only quadratic subextension the cyclotomic field $\mathbb{Q}(\sqrt{-3})$.

The image of the 2-adic representation is $\text{GL}_2(\mathbb{Z}_2)$; see [8]. By [2, Theorem 5.2], the 2-adic Kummer map is surjective: the assumptions of that result are satisfied because the prime $p = 941$ splits completely in $E[4]$, but the point $(\alpha \bmod p)$ is not 2-divisible over \mathbb{F}_p . Since the image of the mod 2 representation has a unique subgroup of index 2, the field $\mathbb{Q}(E[2])$ contains as its only quadratic subextension the field $\mathbb{Q}(\sqrt{-51})$ (the square-free part of the discriminant of E is -51).

We have $\mathbb{Q}(E[2]) \cap \mathbb{Q}(E[9]) = \mathbb{Q}$ because the residual degree modulo 22699 of the extension $\mathbb{Q}(E[2], E[9])/\mathbb{Q}(E[9])$ is divisible by 3 and the degree of this extension is even because $\mathbb{Q}(\sqrt{-51})$ is not contained in $\mathbb{Q}(E[3])$. We deduce $\mathbb{Q}(E[2]) \cap \mathbb{Q}(E[3^\infty]) = \mathbb{Q}$ by applying [4, Theorem 14 (i)] (where $K = \mathbb{Q}(E[2])$).

Moreover, we have $\mathbb{Q}(E[3]) \cap \mathbb{Q}(E[4]) = \mathbb{Q}$ because $\mathbb{Q}(\sqrt{-3})$ is not contained in $\mathbb{Q}(E[4])$: the prime 941 is not congruent to 1 modulo 3 and splits completely in $\mathbb{Q}(E[4])$. By [4, Theorem 14 (i)], we conclude that $\mathbb{Q}(E[3]) \cap \mathbb{Q}(E[2^\infty]) = \mathbb{Q}$.

The 2-adic Kummer extensions of α have maximal degree also over $\mathbb{Q}(E[3])$, in view of the maximality of the 2-Kummer extension, because the prime 4349 splits completely in $\mathbb{Q}(2^{-2}\alpha)$ but not in $\mathbb{Q}(\sqrt{-3})$; see [4, Theorem 14 (ii)] (where $K = \mathbb{Q}(\sqrt{-3})$).

The 3-adic Kummer extensions of α have maximal degree also over $\mathbb{Q}(E[2])$ because the prime 217981 splits completely in $\mathbb{Q}(3^{-2}\alpha)$ but 3 divides the residual degree of $\mathbb{Q}(E[2])$; see [4, Theorem 14 (ii)] (where $K = \mathbb{Q}(E[2])$).

We thus have $\mathcal{G}(6^\infty) = \mathcal{G}(2^\infty) \times \mathcal{G}(3^\infty)$, the 2^∞ Kummer extensions are independent from $\mathbb{Q}(E[3])$, and the 3^∞ Kummer extensions are independent from $\mathbb{Q}(E[2])$. We are thus in the situation that the fields $\mathbb{Q}(2^{-\infty}\alpha)$ and $\mathbb{Q}(3^{-\infty}\alpha)$ are linearly disjoint over \mathbb{Q} . We deduce from Corollary 18 that the equality

$$\text{Dens}_6(\alpha) = \text{Dens}_2(\alpha) \cdot \text{Dens}_3(\alpha)$$

holds for α and for its multiples. The 2-densities can be evaluated by [4, Theorem 35], for the 3-densities see [4, Example 6.4].

7.2. The Serre curve $y^2 + y = x^3 + x^2$. The elliptic curve

$$E: y^2 + y = x^3 + x^2$$

of discriminant -43 and conductor 43 over \mathbb{Q} [8, label 43.a1] is a Serre curve [6, Example 5.5.7]. The group $E(\mathbb{Q})$ is infinite cyclic and is generated by the point

$$\alpha = (0, 0).$$

The point α satisfies

$$\text{Dens}_2(\alpha) \cdot \text{Dens}_{43}(\alpha) \neq \text{Dens}_{2 \cdot 43}(\alpha)$$

because, as we will show below, we have

$$\begin{aligned} \text{Dens}_2(\alpha) &= \frac{11}{21}, \quad \text{Dens}_{43}(\alpha) = \frac{143510179}{146927088}, \\ \text{Dens}_2(\alpha) \cdot \text{Dens}_{43}(\alpha) &= \frac{143510179}{280497168} \sim 51.16279\%, \\ \text{Dens}_{2 \cdot 43}(\alpha) &= \frac{526206455}{1028489616} \sim 51.16303\%. \end{aligned}$$

We will also compute the following values (by testing the primes up to 10^6 , we have computed an approximation to $\text{Dens}_{2 \cdot 43}(\alpha)$ using [9]):

Point	$\text{Dens}_{2 \cdot 43}$	primes $< 10^6$
$\alpha = (0, 0)$	$526206455/1028489616 = 51.163 \dots \%$	51.136%
$2\alpha = (-1, -1)$	$42521603/57138312 = 74.418 \dots \%$	74.397%
$4\alpha = (2, 3)$	$1769960107/2056979232 = 86.046 \dots \%$	86.072%

By looking at the reduction modulo 293, we see that α is not divisible by 2 over the 4-torsion field of E . Therefore, by [2, Theorem 5.2], for every prime number ℓ and for every $n \geq 1$, the degree of the ℓ^n -Kummer extension is maximal, i.e.

$$[\mathbb{Q}_{\ell^{-n}\alpha} : \mathbb{Q}_{\ell^{-n}}] = \ell^{2n}.$$

The 43-adic Kummer extensions have maximal degree also over $\mathbb{Q}(E[2])$, i.e.

$$[\mathbb{Q}_{43^{-n}\alpha}(E[2]) : \mathbb{Q}_{43^{-n}}(E[2])] = 43^{2n},$$

because the degree $[\mathbb{Q}(E[2]) : \mathbb{Q}] = 6$ is coprime to 43.

The extensions $\mathbb{Q}(2^{-1}\alpha)$ and $\mathbb{Q}(E[2 \cdot 43])$ are linearly disjoint over $\mathbb{Q}(E[2])$, as can be seen by investigating the residual degree for the reduction modulo the prime 29327, which splits completely in $\mathbb{Q}(E[2])$. Indeed, the residual degree of the extension $\mathbb{Q}(2^{-1}\alpha)$ equals 4, while the residual degree of the extension $\mathbb{Q}(E[2 \cdot 43])$ is odd because the prime is congruent to 1 modulo 43, and there are points of order 43 in the reductions (the subgroup of the upper unitriangular matrices in $\mathrm{GL}_2(\mathbb{Z}/43\mathbb{Z})$ has order 43).

The 2-adic Kummer extensions have maximal degree also over $\mathbb{Q}(E[43])$, i.e.

$$[\mathbb{Q}_{2^{-n}\alpha}(E[43]) : \mathbb{Q}_{2^{-n}}(E[43])] = 2^{2n}.$$

To see this, we consider the intersection L of $\mathbb{Q}_{2^{-n}\alpha}$ and $\mathbb{Q}(E[43])$. This is a Galois extension of \mathbb{Q} , and the group $G = \mathrm{Gal}(L/\mathbb{Q})$ is a quotient of both $(\mathbb{Z}/2^n\mathbb{Z})^2 \rtimes \mathrm{GL}_2(\mathbb{Z}/2^n\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z}/43\mathbb{Z})$. Because $\mathrm{SL}_2(\mathbb{Z}/43\mathbb{Z})$ has no non-trivial quotient that can be embedded into a quotient of $(\mathbb{Z}/2^n\mathbb{Z})^2 \rtimes \mathrm{GL}_2(\mathbb{Z}/2^n\mathbb{Z})$, the quotient map $\mathrm{GL}_2(\mathbb{Z}/43\mathbb{Z}) \rightarrow G$ factors as

$$\mathrm{GL}_2(\mathbb{Z}/43\mathbb{Z}) \xrightarrow{\det} (\mathbb{Z}/43\mathbb{Z})^\times \longrightarrow G$$

This implies that L is a subfield of $\mathbb{Q}(\zeta_{43})$. Furthermore, L contains $\mathbb{Q}(\sqrt{-43})$. Because $(\mathbb{Z}/2^n\mathbb{Z})^2 \rtimes \mathrm{GL}_2(\mathbb{Z}/2^n\mathbb{Z})$ does not have any quotient group of odd order, the maximal subfield of $\mathbb{Q}(\zeta_{43})$ that can be embedded into $\mathbb{Q}_{2^{-n}\alpha}$ is $\mathbb{Q}(\sqrt{-43})$, and we conclude that L equals $\mathbb{Q}(\sqrt{-43})$.

It follows that for $m = 2 \cdot 43$, we have the maximal degree $[\mathbb{Q}_{m^{-n}\alpha} : \mathbb{Q}_{m^{-n}}] = m^{2n}$ and, more generally, that for every multiple P of α we have $[\mathbb{Q}_{m^{-n}P} : \mathbb{Q}_{m^{-n}}] = [\mathbb{Q}_{2^{-n}P} : \mathbb{Q}_{2^{-n}}] \cdot [\mathbb{Q}_{43^{-n}P} : \mathbb{Q}_{43^{-n}}]$. We may then apply [4, Example 28] and various results in this paper to compute the exact densities in the above table, and we use [9] to numerically verify them for the primes up to 10^6 .

We conclude by sketching the computations for the point α . The 43-adic representation is surjective, and the 43-Kummer extensions have maximal degree. By parts (3) and (4) of Lemma 26, we find that $\frac{1}{2 \cdot 42}$ (respectively, $\frac{41}{2 \cdot 42}$) is the counting measure in $\mathrm{GL}_2(\mathbb{Z}/43\mathbb{Z})$ of the matrices such that $\varepsilon_{-43} = -1$ and that are in $(\mathcal{M}_{43}(0, b) \bmod \ell)$ for some $b > 0$ (respectively, for $b = 0$). By multiplying this quantity by $43^{-b} \cdot 42$, we obtain by Proposition 23 that $\mu_{\mathrm{GL}_2(\mathbb{Z}_{43})}(\mathcal{M}_{43}(0, b)) = \frac{1}{2} 43^{-b}$ for $b > 0$. By [4, Example 28], the contribution to Dens_{43} coming from the matrices in $\mathcal{G}(43^\infty)$ such that $\varepsilon_{-43} = -1$ is then

$$\mathrm{Dens}_{43}(\varepsilon_{-43} = -1) = \frac{41}{2 \cdot 42} + \sum_{b>0} \frac{1}{2} \cdot 43^{-2b} = \frac{1805}{2 \cdot 42 \cdot 44}.$$

From [4, Theorem 35] we know that $\mathrm{Dens}_{43}(\alpha) = 143510179/146927088$, and hence the contribution to $\mathrm{Dens}_{43}(\alpha)$ coming from the matrices in $\mathcal{G}(43^\infty)$ such that $\varepsilon_{-43} = +1$ equals

$$\mathrm{Dens}_{43}(\varepsilon_{-43} = 1) = \frac{3261637}{6678504}.$$

Now we work with the 2-adic representation, which is surjective and restrict to counting the contribution to $\mathrm{Dens}_2(\alpha)$ coming from the matrices satisfying $\psi = -1$. In view of Lemma 25 and Proposition 23, we find $\mu_{\mathrm{GL}_2(\mathbb{Z}_2)}(\mathcal{M}_2(0, b)) = 1/2 \cdot 2^{-b}$ for $b > 0$. By [4, Example 28], the contribution to $\mathrm{Dens}_2(\alpha)$ coming from the matrices in $\mathcal{G}(2^\infty)$ such that $\psi = -1$ is therefore

$$\mathrm{Dens}_2(\psi = -1) = \sum_{b>0} 1/2 \cdot 2^{-2b} = 1/6. \quad (7.1)$$

From [4, Theorem 35] we know that $\text{Dens}_2(\alpha) = 11/21$, and hence the contribution to Dens_2 coming from the matrices in $\mathcal{G}(2^\infty)$ such that $\psi = 1$ is

$$\text{Dens}_2(\psi = 1) = 5/14.$$

Finally, by the partition in Section 6.3, we can compute the requested density as the following combination of the above quantities:

$$\begin{aligned} \text{Dens}_{2,43}(\alpha) &= 2(\text{Dens}_2(\psi = 1) \cdot \text{Dens}_{43}(\varepsilon_{-43} = 1) \\ &\quad + \text{Dens}_2(\psi = -1) \cdot \text{Dens}_{43}(\varepsilon_{-43} = -1)). \end{aligned} \quad (7.2)$$

Indeed, let us consider Theorem 19, recalling that $C_m = 1$. Let us call H_+ the subset of $\mathcal{G}(m^\infty)$ consisting of elements whose image in $\mathcal{G}(2)$ satisfies $\psi = 1$ and whose image in $\mathcal{G}(43)$ satisfies $\varepsilon_{-43} = 1$ and define analogously H_- with $\psi = -1$ and $\varepsilon_{-43} = -1$. Write $H_+ = H_{2,+} \times H_{43,+}$, where $H_{2,+} \subseteq \mathcal{G}(2^\infty)$ and $H_{43,+} \subseteq \mathcal{G}(43^\infty)$. Similarly, write $H_- = H_{2,-} \times H_{43,-}$. The formula of Theorem 19, considering the two contributions for $\text{Dens}_{2,43}(\alpha)$ coming from H_+ and H_- , gives

$$\text{Dens}^+ = \frac{\#\mathcal{G}(2)\#\mathcal{G}(43)}{\#\mathcal{G}(2 \cdot 43)} \int_{H_{2,+}} \frac{w_{2^\infty}(M)}{\#\ker(M - I)} d\mu_{\mathcal{G}_{2^\infty}}(M) \cdot \int_{H_{43,+}} \frac{w_{43^\infty}(M)}{\#\ker(M - I)} d\mu_{\mathcal{G}_{43^\infty}}(M),$$

and similarly for Dens^- . This yields formula (7.2).

For the point 2α , by [4, Example 28], we only need to scale (7.1) by a factor 2, giving $1/3$ and $3/7$ as the two contributions to $\text{Dens}_2(2\alpha)$ by [4, Theorem 35]. For the point 4α , we adapt (7.1) as $2 \cdot 1/2 \cdot 2^{-2} + \sum_{b>1} 4 \cdot 1/2 \cdot 2^{-2b}$ and obtain $5/12$ and $13/28$ as the two contributions to $\text{Dens}_2(4\alpha)$.

ACKNOWLEDGMENTS. The authors would like to thank Davide Lombardo and the referee for their valuable comments on the paper.

REFERENCES

1. N. Jones, Almost all elliptic curves are Serre curves, *Trans. Amer. Math. Soc.* **362**(3) (2010), 1547–1570.
2. R. Jones and J. Rouse, Galois theory of iterated endomorphisms, *Proc. Lond. Math. Soc.* (3) **100**(3) (2010), 763–794. Appendix A by Jeffrey D. Achter.
3. D. Lombardo and A. Perucca, The 1-eigenspace for matrices in $\text{GL}_2(\mathbb{Z}_\ell)$, *New York J. Math.* **23** (2017), 897–925.
4. D. Lombardo and A. Perucca, Reductions of points on algebraic groups, *J. Inst. Math. Jussieu* (2020), 1–33. doi:[10.1017/S1474748019000598](https://doi.org/10.1017/S1474748019000598).
5. A. Perucca, Prescribing valuations of the order of a point in the reductions of abelian varieties and tori, *J. Number Theory* **129** (2009), 469–476.
6. J.-P. Serre, Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, *Invent. Math.* **15**(4) (1972), 259–331.
7. A. V. Sutherland, Computing images of Galois representations attached to elliptic curves, *Forum Math. Sigma* **4**(e4) (2016), 79.
8. The LMFDB Collaboration, The L-functions and modular forms database (2016). <http://www.lmfdb.org>.
9. The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 7.3)* (2016). <http://www.sagemath.org>.