REDUCTIVE COMPACTIFICATIONS OF SEMITOPOLOGICAL SEMIGROUPS

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We consider the enveloping semigroup of a flow generated by the action of a semitopological semigroup on any of its semigroup compactifications and explore the possibility of its being one of the known semigroup compactifications again. In this way, we introduce the notion of E-algebra, and show that this notion is closely related to the reductivity of the semigroup compactification involved. Moreover, the structure of the universal $E\mathcal{F}$ -compactification is also given.

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1. Introduction. A semigroup S is called *right reductive* if a = b for each $a, b \in S$, since at = bt for every $t \in S$. For example, all right cancellative semigroups and semigroups with a right identity are right reductive.

From now on, S will be a semitopological semigroup, unless otherwise is stipulated. By a *semigroup compactification* of S we mean a pair (ψ, X) , where X is a compact Hausdorff right topological semigroup, and $\psi: S \to X$ is a continuous homomorphism with dense image such that, for each $s \in S$, the mapping $x \to \psi(s)x: X \to X$ is continuous. The C^* -algebra of all bounded complex-valued continuous functions on S will be denoted by $\mathfrak{C}(S)$. For $\mathfrak{C}(S)$, the left and right translations, L_S and R_t , are defined for each $s,t \in S$ by $(L_sf)(t) = f(st) = (R_tf)(s)$, $f \in \mathfrak{C}(S)$. The subset \mathcal{F} of $\mathfrak{C}(S)$ is said to be left translation invariant if for all $s \in S$, $L_s\mathcal{F} \subseteq \mathcal{F}$. A left translation invariant unital C^* -subalgebra \mathcal{F} of $\mathfrak{C}(S)$ is called m-admissible if the function $s \to T_\mu f(s) = \mu(L_s f)$ is in \mathcal{F} for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$ (where $S^{\mathcal{F}}$ is the spectrum of \mathcal{F}). Then the product of $\mu, \nu \in S^{\mathcal{F}}$ can be defined by $\mu \nu = \mu \circ T_\nu$ and the Gelfand topology on $S^{\mathcal{F}}$ makes $(\varepsilon, S^{\mathcal{F}})$ a semigroup compactification (called the \mathcal{F} -compactification) of S, where $\varepsilon: S \to S^{\mathcal{F}}$ is the evaluation mapping.

Some m-admissible subalgebras of $\mathscr{C}(S)$, that we will need, are left multiplicatively continuous functions \mathscr{LMC} , distal functions \mathscr{D} , minimal distal functions \mathscr{MD} , and strongly distal functions \mathscr{SD} . We also write \mathscr{SD} for $\mathscr{MD} \cap \mathscr{SD}$; and we define $\mathscr{LH} := \{f \in \mathscr{C}(S); \ f(st) = f(s) \ \text{for all} \ s,t \in S\}$. For a discussion of the universal property of the corresponding compactifications of these function algebras see [1,2].

2. Reductive compactifications and *E***-algebras.** Let (ψ, X) be a compactification of S, then the mapping $\sigma: S \times X \to X$, defined by $\sigma(s, x) = \psi(s)x$, is separately continuous and so (S, X, σ) is a flow. If Σ_X denotes the enveloping semigroup of the flow (S, X, σ) (i.e., the pointwise closure of semigroup $\{\sigma(s, \cdot): s \in S\}$ in X^X) and the mapping $\sigma_X: S \to \Sigma_X$ is defined by $\sigma_X(s) = \sigma(s, \cdot)$ for all $s \in S$, then (σ_X, Σ_X) is a compactification of S (see [1, Proposition 1.6.5]).

One can easily verify that $\Sigma_X = \{\lambda_X : x \in X\}$, where $\lambda_X(y) = xy$ for each $y \in X$. If we define the mapping $\theta : X \to \Sigma_X$ by $\theta(x) = \lambda_X$, then θ is a continuous homomorphism with the property that $\theta \circ \psi = \sigma_X$. So (σ_X, Σ_X) is a factor of (ψ, X) , that is $(\psi, X) \ge (\sigma_X, \Sigma_X)$. By definition, θ is one-to-one if and only if X is right reductive. So we get the next proposition, which is an extension of the Lawson's result [3, Lemma 2.4(ii)].

PROPOSITION 2.1. Let (ψ, X) be a compactification of S. Then $(\sigma_X, \Sigma_X) \cong (\psi, X)$ if and only if X is right reductive.

A compactification (ψ, X) is called *reductive* if X is right reductive. For example, the $M\mathfrak{D}$ -, \mathfrak{GP} -, and \mathfrak{L} -compactifications are reductive.

An m-admissible subalgebra \mathscr{F} of $\mathscr{C}(S)$ is called an E-algebra if there is a compactification (ψ,X) such that $(\sigma_X,\Sigma_X)\cong (\epsilon,S^{\mathscr{F}})$. In this setting (ψ,X) is called an $E\mathscr{F}$ -compactification of S. Trivially for every reductive compactification (ψ,X) , $\psi^*(\mathscr{C}(X))$ is an E-algebra. But the converse is not, in general, true. For instance, for any compactification (ψ,X) , $\sigma_X^*(\mathscr{C}(\Sigma_X))$ is an E-algebra; however, it is possible that Σ_X would be nonreductive, as the next example shows.

EXAMPLE 2.2. Let $S = \{a, b, c, d\}$ be the semigroup with the following multiplication table:

	а	b	С	d
a b	а	а	а	а
b	а	а	а	С
С	а	а	а	а
d	а	С	а	b

Then for the identity compactification (i,X) of S, Σ_X is not right reductive; in fact, $\lambda_a \neq \lambda_b$, however, $\lambda_{at} = \lambda_{bt}$ for every $t \in S$.

LEMMA 2.3. If (ψ, X) is a compactification satisfying $X^2 = X$, then the compactification (σ_X, Σ_X) is reductive.

PROOF. Since $X^2 = X$, for each $x_1, x_2 \in X$, from $\lambda_{x_1} \lambda_y = \lambda_{x_2} \lambda_y$ for every $\lambda_y \in \Sigma_X$, it follows that $\lambda_{x_1} = \lambda_{x_2}$. So Σ_X is right reductive.

COROLLARY 2.4. Let sS (or Ss) be dense in S, for some $s \in S$, then for every compactification (ψ, X) of S, it follows that $X^2 = X$ and so (σ_X, Σ_X) is reductive.

Now, we are going to construct the universal $E\mathcal{F}$ -compactification of S. For this end we need the following lemma.

LEMMA 2.5. Let \mathscr{F} be an m-admissible subalgebra of $\mathscr{C}(S)$. Then $T_{\mathcal{V}}f \in \sigma_{S^{\mathscr{F}}}^*(\mathscr{C}(\Sigma_{S^{\mathscr{F}}}))$ for all $f \in \mathscr{F}$ and $v \in S^{\mathscr{LM}\mathscr{C}}$.

PROOF. Since $\Sigma_{S^{\mathfrak{F}}}=\{\lambda_{\mu}:\mu\in S^{\mathfrak{F}}\}$, we can define $g:\Sigma_{S^{\mathfrak{F}}}\mathbb{C}$ by $g(\lambda_{\mu})=\mu(T_{\nu}f)$, where \mathbb{C} denotes the complex numbers. Since the mapping $\lambda_{\mu}\to\mu\nu:\Sigma_{S^{\mathfrak{F}}}\to S^{\mathfrak{F}}$ is p-weak* continuous, g is a bounded continuous function and it is easy to see that $\sigma_{S^{\mathfrak{F}}}^*(g)=T_{\nu}(f)$. Therefore, $T_{\nu}f\in\sigma_{S^{\mathfrak{F}}}^*(\mathfrak{C}(\Sigma_{S^{\mathfrak{F}}}))$ for all $\nu\in S^{\mathfrak{F}}$. If $\tilde{\nu}\in S^{\mathfrak{LMC}}$ and ν is the restriction of $\tilde{\nu}$ to \mathfrak{F} , then $T_{\tilde{\nu}}f=T_{\nu}f$ for all $f\in \mathfrak{F}$. So the conclusion follows.

PROPOSITION 2.6. Let F be an E-algebra. Then

$$G_{\mathcal{F}} := \{ f \in \mathcal{LMC} : T_{\nu} f \in \mathcal{F} \ \forall \nu \in S^{\mathcal{LMC}} \}$$
 (2.1)

is an m-admissible subalgebra of $\mathscr{C}(S)$ and $(\epsilon, S^{G_{\mathscr{F}}})$ is the universal $E\mathscr{F}$ -compactification of S.

PROOF. It is easy to verify that $G_{\mathcal{F}}$ is an m-admissible subalgebra of $\mathscr{C}(S)$ containing \mathcal{F} . By definition of $G_{\mathcal{F}}$ we can define the mapping $\theta: S^{\mathcal{F}} \to \Sigma_S G_{\mathcal{F}}$ by $\theta(\mu) = \lambda_{\tilde{\mu}}$, where $\tilde{\mu}$ is an extension of μ to $S^{G_{\mathcal{F}}}$. Clearly, θ is continuous and $\theta \circ \epsilon = \sigma_S G_{\mathcal{F}}$. Thus $(\epsilon, S^{\mathcal{F}}) \geq (\sigma_S G_{\mathcal{F}}, \Sigma_S G_{\mathcal{F}})$. On the other hand, since \mathcal{F} is an E-algebra, there exists a compactification (ϕ, Y) of S such that $(\sigma_Y, \Sigma_Y) \cong (\epsilon, S^{\mathcal{F}})$ and $\mathcal{F} = \sigma_Y^*(\mathscr{C}(\Sigma_Y))$. By Lemma 2.5, we have $T_V f \in \sigma_Y^*(\mathscr{C}(\Sigma_Y))$, for each $V \in S^{\mathcal{L}M\mathcal{C}}$ and each $f \in \phi^*(\mathscr{C}(Y))$. This means that $\phi^*(\mathscr{C}(Y)) \subset G_{\mathcal{F}}$ and so, by [1, Proposition 1.6.7], $(\sigma_Y, \Sigma_Y) \leq (\sigma_S G_{\mathcal{F}}, \Sigma_S G_{\mathcal{F}})$. Therefore, $(\epsilon, S^{\mathcal{F}}) \cong (\sigma_S G_{\mathcal{F}}, \Sigma_S G_{\mathcal{F}})$ and $(\epsilon, S^{G_{\mathcal{F}}})$ is an $E\mathcal{F}$ -compactification of S. Finally, if (ψ, X) is an $E\mathcal{F}$ -compactification of S and S

EXAMPLES 2.7. (a) We have $G_{\mathcal{M}\mathfrak{D}} = \mathfrak{D}$. To see this, if $f \in G_{\mathcal{M}\mathfrak{D}}$, then for all $\mu, \nu, \eta \in S^{\mathcal{L}\mathcal{M}\mathfrak{C}}$ with $\eta^2 = \eta$, we have $\mu \eta \nu(f) = \mu \eta(T_\nu f) = \mu(T_\nu f) = \mu \nu(f)$. So $f \in \mathfrak{D}$. Also if $f \in \mathfrak{D}$, then for all $\mu, \nu, \eta \in S^{\mathcal{L}\mathcal{M}\mathfrak{C}}$ with $\eta^2 = \eta$, we have $\mu \eta(T_\nu f) = \mu \eta \nu(f) = \mu \nu(f) = \mu(T_\nu f)$. That is, $T_\nu f \in \mathcal{M}\mathfrak{D}$ for all $\nu \in S^{\mathcal{L}\mathcal{M}\mathfrak{C}}$ and so $f \in G_{\mathcal{M}\mathfrak{D}}$ (see also [4, Lemma 2.2]).

- (b) By a similar proof, we can show that $G_{\text{GP}} = \mathcal{SD}$ (see [4, Lemma 2.2 and Theorem 2.6]).
- (c) Let $\Re := \{f \in \mathcal{LMC}(S) : f(rst) = f(rt) \text{ for } r, s, t \in S\}$. Clearly, \Re is an m-admissible subalgebra of $\mathcal{C}(S)$. If $f \in \Re$ and $v \in S^{\mathcal{LMC}}$, then for each $r, s, t \in S$ we have $L_{rt}f(s) = f(rts) = f(rs) = L_rf(s)$. So $T_vf(rt) = v(L_{rt}f) = v(L_rf) = T_vf(r)$. That is, $T_vf \in \mathcal{L}$. On the other hand, if $f \in G_{\mathcal{L}}$, then $f(rst) = (T_{\epsilon(t)}f)(rs) = (T_{\epsilon(t)}f)(r) = f(rt)$ and so $f \in \Re$. Therefore, $G_{\mathcal{L}} = \Re$.

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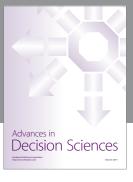
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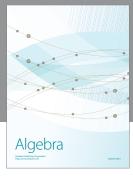
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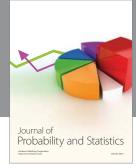
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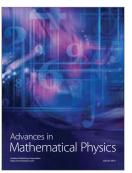


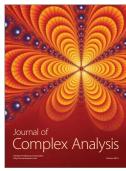


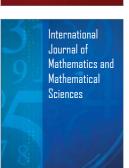


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