

## REDUCTIVE COMPACTIFICATIONS OF SEMITOPOLOGICAL SEMIGROUPS

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We consider the enveloping semigroup of a flow generated by the action of a semitopological semigroup on any of its semigroup compactifications and explore the possibility of its being one of the known semigroup compactifications again. In this way, we introduce the notion of  $E$ -algebra, and show that this notion is closely related to the reductivity of the semigroup compactification involved. Moreover, the structure of the universal  $E\mathcal{F}$ -compactification is also given.

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**1. Introduction.** A semigroup  $S$  is called *right reductive* if  $a = b$  for each  $a, b \in S$ , since  $at = bt$  for every  $t \in S$ . For example, all right cancellative semigroups and semigroups with a right identity are right reductive.

From now on,  $S$  will be a semitopological semigroup, unless otherwise is stipulated. By a *semigroup compactification* of  $S$  we mean a pair  $(\psi, X)$ , where  $X$  is a compact Hausdorff right topological semigroup, and  $\psi : S \rightarrow X$  is a continuous homomorphism with dense image such that, for each  $s \in S$ , the mapping  $x \rightarrow \psi(s)x : X \rightarrow X$  is continuous. The  $C^*$ -algebra of all bounded complex-valued continuous functions on  $S$  will be denoted by  $\mathcal{C}(S)$ . For  $\mathcal{C}(S)$ , the left and right translations,  $L_s$  and  $R_t$ , are defined for each  $s, t \in S$  by  $(L_s f)(t) = f(st) = (R_t f)(s)$ ,  $f \in \mathcal{C}(S)$ . The subset  $\mathcal{F}$  of  $\mathcal{C}(S)$  is said to be left translation invariant if for all  $s \in S$ ,  $L_s \mathcal{F} \subseteq \mathcal{F}$ . A left translation invariant unital  $C^*$ -subalgebra  $\mathcal{F}$  of  $\mathcal{C}(S)$  is called  *$m$ -admissible* if the function  $s \rightarrow T_\mu f(s) = \mu(L_s f)$  is in  $\mathcal{F}$  for all  $f \in \mathcal{F}$  and  $\mu \in S^\mathcal{F}$  (where  $S^\mathcal{F}$  is the spectrum of  $\mathcal{F}$ ). Then the product of  $\mu, \nu \in S^\mathcal{F}$  can be defined by  $\mu\nu = \mu \circ T_\nu$  and the Gelfand topology on  $S^\mathcal{F}$  makes  $(\epsilon, S^\mathcal{F})$  a semigroup compactification (called the  $\mathcal{F}$ -compactification) of  $S$ , where  $\epsilon : S \rightarrow S^\mathcal{F}$  is the evaluation mapping.

Some  $m$ -admissible subalgebras of  $\mathcal{C}(S)$ , that we will need, are left multiplicatively continuous functions  $\mathcal{LMC}$ , distal functions  $\mathcal{D}$ , minimal distal functions  $\mathcal{MD}$ , and strongly distal functions  $\mathcal{SD}$ . We also write  $\mathcal{GD}$  for  $\mathcal{MD} \cap \mathcal{SD}$ ; and we define  $\mathcal{LX} := \{f \in \mathcal{C}(S); f(st) = f(s) \text{ for all } s, t \in S\}$ . For a discussion of the universal property of the corresponding compactifications of these function algebras see [1, 2].

**2. Reductive compactifications and  $E$ -algebras.** Let  $(\psi, X)$  be a compactification of  $S$ , then the mapping  $\sigma : S \times X \rightarrow X$ , defined by  $\sigma(s, x) = \psi(s)x$ , is separately continuous and so  $(S, X, \sigma)$  is a flow. If  $\Sigma_X$  denotes the enveloping semigroup of the flow  $(S, X, \sigma)$  (i.e., the pointwise closure of semigroup  $\{\sigma(s, \cdot) : s \in S\}$  in  $X^X$ ) and the mapping  $\sigma_X : S \rightarrow \Sigma_X$  is defined by  $\sigma_X(s) = \sigma(s, \cdot)$  for all  $s \in S$ , then  $(\sigma_X, \Sigma_X)$  is a compactification of  $S$  (see [1, Proposition 1.6.5]).

One can easily verify that  $\Sigma_X = \{\lambda_x : x \in X\}$ , where  $\lambda_x(y) = xy$  for each  $y \in X$ . If we define the mapping  $\theta : X \rightarrow \Sigma_X$  by  $\theta(x) = \lambda_x$ , then  $\theta$  is a continuous homomorphism with the property that  $\theta \circ \psi = \sigma_X$ . So  $(\sigma_X, \Sigma_X)$  is a factor of  $(\psi, X)$ , that is  $(\psi, X) \geq (\sigma_X, \Sigma_X)$ . By definition,  $\theta$  is one-to-one if and only if  $X$  is right reductive. So we get the next proposition, which is an extension of the Lawson's result [3, Lemma 2.4(ii)].

**PROPOSITION 2.1.** *Let  $(\psi, X)$  be a compactification of  $S$ . Then  $(\sigma_X, \Sigma_X) \cong (\psi, X)$  if and only if  $X$  is right reductive.*

A compactification  $(\psi, X)$  is called *reductive* if  $X$  is right reductive. For example, the  $\mathcal{MD}$ -,  $\mathcal{GP}$ -, and  $\mathcal{LE}$ -compactifications are reductive.

An  $m$ -admissible subalgebra  $\mathcal{F}$  of  $\mathcal{C}(S)$  is called an  *$E$ -algebra* if there is a compactification  $(\psi, X)$  such that  $(\sigma_X, \Sigma_X) \cong (\epsilon, S^{\mathcal{F}})$ . In this setting  $(\psi, X)$  is called an  *$E\mathcal{F}$ -compactification* of  $S$ . Trivially for every reductive compactification  $(\psi, X)$ ,  $\psi^*(\mathcal{C}(X))$  is an  $E$ -algebra. But the converse is not, in general, true. For instance, for any compactification  $(\psi, X)$ ,  $\sigma_X^*(\mathcal{C}(\Sigma_X))$  is an  $E$ -algebra; however, it is possible that  $\Sigma_X$  would be nonreductive, as the next example shows.

**EXAMPLE 2.2.** Let  $S = \{a, b, c, d\}$  be the semigroup with the following multiplication table:

	$a$	$b$	$c$	$d$
$a$	$a$	$a$	$a$	$a$
$b$	$a$	$a$	$a$	$c$
$c$	$a$	$a$	$a$	$a$
$d$	$a$	$c$	$a$	$b$

Then for the identity compactification  $(i, X)$  of  $S$ ,  $\Sigma_X$  is not right reductive; in fact,  $\lambda_a \neq \lambda_b$ , however,  $\lambda_{at} = \lambda_{bt}$  for every  $t \in S$ .

**LEMMA 2.3.** *If  $(\psi, X)$  is a compactification satisfying  $X^2 = X$ , then the compactification  $(\sigma_X, \Sigma_X)$  is reductive.*

**PROOF.** Since  $X^2 = X$ , for each  $x_1, x_2 \in X$ , from  $\lambda_{x_1}\lambda_y = \lambda_{x_2}\lambda_y$  for every  $\lambda_y \in \Sigma_X$ , it follows that  $\lambda_{x_1} = \lambda_{x_2}$ . So  $\Sigma_X$  is right reductive.  $\square$

**COROLLARY 2.4.** *Let  $sS$  (or  $Ss$ ) be dense in  $S$ , for some  $s \in S$ , then for every compactification  $(\psi, X)$  of  $S$ , it follows that  $X^2 = X$  and so  $(\sigma_X, \Sigma_X)$  is reductive.*

Now, we are going to construct the universal  $E\mathcal{F}$ -compactification of  $S$ . For this end we need the following lemma.

**LEMMA 2.5.** *Let  $\mathcal{F}$  be an  $m$ -admissible subalgebra of  $\mathcal{C}(S)$ . Then  $T_\nu f \in \sigma_{S^\mathcal{F}}^*(\mathcal{C}(\Sigma_{S^\mathcal{F}}))$  for all  $f \in \mathcal{F}$  and  $\nu \in S^{\mathcal{LM}\mathcal{C}}$ .*

**PROOF.** Since  $\Sigma_{S^\mathcal{F}} = \{\lambda_\mu : \mu \in S^\mathcal{F}\}$ , we can define  $g : \Sigma_{S^\mathcal{F}} \rightarrow \mathbb{C}$  by  $g(\lambda_\mu) = \mu(T_\nu f)$ , where  $\mathbb{C}$  denotes the complex numbers. Since the mapping  $\lambda_\mu \rightarrow \mu\nu : \Sigma_{S^\mathcal{F}} \rightarrow S^\mathcal{F}$  is  $p$ -weak\* continuous,  $g$  is a bounded continuous function and it is easy to see that  $\sigma_{S^\mathcal{F}}^*(g) = T_\nu(f)$ . Therefore,  $T_\nu f \in \sigma_{S^\mathcal{F}}^*(\mathcal{C}(\Sigma_{S^\mathcal{F}}))$  for all  $\nu \in S^\mathcal{F}$ . If  $\tilde{\nu} \in S^{\mathcal{LM}\mathcal{C}}$  and  $\nu$  is the restriction of  $\tilde{\nu}$  to  $\mathcal{F}$ , then  $T_{\tilde{\nu}}f = T_\nu f$  for all  $f \in \mathcal{F}$ . So the conclusion follows.  $\square$

**PROPOSITION 2.6.** *Let  $\mathcal{F}$  be an  $E$ -algebra. Then*

$$G_\mathcal{F} := \{f \in \mathcal{LM}\mathcal{C} : T_\nu f \in \mathcal{F} \ \forall \nu \in S^{\mathcal{LM}\mathcal{C}}\} \quad (2.1)$$

*is an  $m$ -admissible subalgebra of  $\mathcal{C}(S)$  and  $(\epsilon, S^{G_\mathcal{F}})$  is the universal  $E\mathcal{F}$ -compactification of  $S$ .*

**PROOF.** It is easy to verify that  $G_\mathcal{F}$  is an  $m$ -admissible subalgebra of  $\mathcal{C}(S)$  containing  $\mathcal{F}$ . By definition of  $G_\mathcal{F}$  we can define the mapping  $\theta : S^\mathcal{F} \rightarrow \Sigma_{S^{G_\mathcal{F}}}$  by  $\theta(\mu) = \lambda_{\tilde{\mu}}$ , where  $\tilde{\mu}$  is an extension of  $\mu$  to  $S^{G_\mathcal{F}}$ . Clearly,  $\theta$  is continuous and  $\theta \circ \epsilon = \sigma_{S^{G_\mathcal{F}}}$ . Thus  $(\epsilon, S^\mathcal{F}) \geq (\sigma_{S^{G_\mathcal{F}}}, \Sigma_{S^{G_\mathcal{F}}})$ . On the other hand, since  $\mathcal{F}$  is an  $E$ -algebra, there exists a compactification  $(\phi, Y)$  of  $S$  such that  $(\sigma_Y, \Sigma_Y) \cong (\epsilon, S^\mathcal{F})$  and  $\mathcal{F} = \sigma_Y^*(\mathcal{C}(\Sigma_Y))$ . By Lemma 2.5, we have  $T_\nu f \in \sigma_Y^*(\mathcal{C}(\Sigma_Y))$ , for each  $\nu \in S^{\mathcal{LM}\mathcal{C}}$  and each  $f \in \phi^*(\mathcal{C}(Y))$ . This means that  $\phi^*(\mathcal{C}(Y)) \subset G_\mathcal{F}$  and so, by [1, Proposition 1.6.7],  $(\sigma_Y, \Sigma_Y) \leq (\sigma_{S^{G_\mathcal{F}}}, \Sigma_{S^{G_\mathcal{F}}})$ . Therefore,  $(\epsilon, S^\mathcal{F}) \cong (\sigma_{S^{G_\mathcal{F}}}, \Sigma_{S^{G_\mathcal{F}}})$  and  $(\epsilon, S^{G_\mathcal{F}})$  is an  $E\mathcal{F}$ -compactification of  $S$ . Finally, if  $(\psi, X)$  is an  $E\mathcal{F}$ -compactification of  $S$  and  $f \in \psi^*(\mathcal{C}(X))$ , then by Lemma 2.5,  $T_\mu f \in \sigma_X^*(\mathcal{C}(\Sigma_X)) = \mathcal{F}$  for all  $\mu \in S^{\mathcal{LM}\mathcal{C}}$ . So  $\psi^*(\mathcal{C}(X)) \subset G_\mathcal{F}$  and  $(\psi, X) \leq (\epsilon, S^{G_\mathcal{F}})$ .  $\square$

**EXAMPLES 2.7.** (a) We have  $G_{\mathcal{M}\mathcal{D}} = \mathcal{D}$ . To see this, if  $f \in G_{\mathcal{M}\mathcal{D}}$ , then for all  $\mu, \nu, \eta \in S^{\mathcal{LM}\mathcal{C}}$  with  $\eta^2 = \eta$ , we have  $\mu\eta\nu(f) = \mu\eta(T_\nu f) = \mu(T_\nu f) = \mu\nu(f)$ . So  $f \in \mathcal{D}$ . Also if  $f \in \mathcal{D}$ , then for all  $\mu, \nu, \eta \in S^{\mathcal{LM}\mathcal{C}}$  with  $\eta^2 = \eta$ , we have  $\mu\eta(T_\nu f) = \mu\eta\nu(f) = \mu\nu(f) = \mu(T_\nu f)$ . That is,  $T_\nu f \in \mathcal{M}\mathcal{D}$  for all  $\nu \in S^{\mathcal{LM}\mathcal{C}}$  and so  $f \in G_{\mathcal{M}\mathcal{D}}$  (see also [4, Lemma 2.2]).

(b) By a similar proof, we can show that  $G_{\mathcal{g}\mathcal{D}} = \mathcal{F}\mathcal{D}$  (see [4, Lemma 2.2 and Theorem 2.6]).

(c) Let  $\mathcal{R} := \{f \in \mathcal{LM}\mathcal{C}(S) : f(rst) = f(rt) \text{ for } r, s, t \in S\}$ . Clearly,  $\mathcal{R}$  is an  $m$ -admissible subalgebra of  $\mathcal{C}(S)$ . If  $f \in \mathcal{R}$  and  $\nu \in S^{\mathcal{LM}\mathcal{C}}$ , then for each  $r, s, t \in S$  we have  $L_{rt}f(s) = f(rts) = f(rs) = L_rf(s)$ . So  $T_\nu f(rt) = \nu(L_{rt}f) = \nu(L_rf) = T_\nu f(r)$ . That is,  $T_\nu f \in \mathcal{L}\mathcal{E}$ . On the other hand, if  $f \in G_{\mathcal{F}\mathcal{L}}$ , then  $f(rst) = (T_{\epsilon(t)}f)(rs) = (T_{\epsilon(t)}f)(r) = f(rt)$  and so  $f \in \mathcal{R}$ . Therefore,  $G_{\mathcal{F}\mathcal{L}} = \mathcal{R}$ .

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