Reeb Vector Fields and Open Book Decompositions

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Joint work with Vincent Colin

Contact structures and Reeb vector fields

Let M be a closed oriented 3-manifold, and let ξ be a cooriented contact structure on M, i.e., $\xi = \ker \alpha$, where α is a global 1-form on M and $\alpha \wedge d\alpha > 0$.

To any contact 1-form α we can assign its *Reeb vector field* $R = R_\alpha$ (a contact vector field $\pitchfork \xi$) defined by:

$$R \lrcorner\, d\alpha = 0,$$

$$R \lrcorner\, \alpha = 1.$$

Dichotomy of contact structures

Contact structures, in dimension three, come in two flavors: *tight* or *overtwisted* (thanks to the work of Bennequin, clarified by Eliashberg). ξ is *overtwisted* if it admits an overtwisted disk, i.e., an embedded disk D such that $\xi_x = T_x D$ for all $x \in \partial D$. ξ is *tight* if it does not admit an overtwisted disk.

Basic Facts:

1. (Pfaff-Darboux) Every contact 3-manifold (M, ξ) is locally diffeomorphic to $(\mathbf{R}^3, dz - ydx = 0)$.
2. (Bennequin) $(\mathbf{R}^3, dz - ydx = 0)$ is tight.

The Weinstein conjecture

Weinstein conjecture: For any (M, ξ) and any Reeb vector field R of ξ, there is a closed orbit of R.

Progress in dimension 3:

- Hofer (1992-3): True if (M, ξ) is OT, $\pi_2(M) \neq 0$, or covered by (S^3, ξ_{std}).
- Taubes (2006): True in general.

The work of Hofer (together with the Floer package) started the subject of *contact homology*.

Contact homology

Contact homology is one of the simplest specializations of *Symplectic Field Theory*, proposed by Eliashberg, Givental, and Hofer and being developed by many people.

It is the homology of a chain complex whose chain groups are generated by closed orbits of R and whose boundary maps count holomorphic cylinders in the symplectization.

The *symplectization* is the 4-manifold $\mathbf{R} \times M$ with the symplectic form $d(e^t \alpha)$, where $t \in \mathbf{R}$. We choose an *adapted* almost complex structure J which maps ξ to itself and $\frac{\partial}{\partial t} \mapsto R$, $R \mapsto -\frac{\partial}{\partial t}$.

Contact homology

Assume all periodic orbits are *nondegenerate* (the Poincaré return map does not have 1 as an eigenvalue). Then let $\mathcal{P}$ be the set of "good" periodic orbits of R. The chain group is $\mathbf{Q}\langle\mathcal{P}\rangle$.

Define:

$$\partial\gamma = \sum_{\mu(\gamma,\gamma')=1} \#(\text{holomorphic cylinders from } \gamma \text{ to } \gamma') \cdot \gamma'.$$

Here $\mu(\gamma, \gamma')$ is the Conley-Zehnder index of the pair γ, γ'.

$\partial^2 = 0$

The above definition often fails because $\partial^2 \neq 0$.

Recall that the usual way of proving $\partial^2 = 0$ is to glue a holomorphic cylinder from γ to γ' and a holomorphic cylinder from γ' to γ''. The broken holomorphic cylinder is the end of a 1-dimensional family of holomorphic cylinders from γ to γ''. If there are no other types of degenerations/bubbling, then the other end of the 1-manifold would also be a broken holomorphic cylinder consisting of a holomorphic cylinder from γ to γ''', followed by another cylinder from γ''' to γ'', and (at least with $\mathbf{Z}/2\mathbf{Z}$-coefficients), we see that $\partial^2 = 0$.

Cylindrical contact homology

However, it is possible that other types of bubbling occur. In that case, we would need to start counting punctured holomorphic spheres with one positive end and n negative ends.

One way of preventing this type of bubbling from happening is to show that there are no periodic orbits with $\mu(\gamma) = 2$ which bound (holomorphic) finite energy planes in $\mathbf{R} \times M$. Then the *cylindrical contact homology*, i.e., the version that only counts holomorphic cylinders, exists.

Let $HC(\alpha)$ be the cylindrical contact homology group. It is an invariant of the contact structure ξ.

Open book decompositions

Let K be a link in a closed oriented 3-manifold M. Then an *open book decomposition* with binding K is a homeomorphism between $(S \times [0,1]/\sim, (\partial S \times [0,1])/\sim)$ and (M, K). Here:

1. S is a compact oriented surface with $\partial S \neq \emptyset$;
2. $h : S \xrightarrow{\sim} S$ is a diffeomorphism which is the identity on ∂S;
3. $\sim$ is given by $(x,1) \sim (h(x),0)$ for all $x \in S$ and $(y,t) \sim (y,t')$ for all $y \in \partial S,\ t,t' \in [0,1]$.

Thurston-Winkelnkemper

According to the work of Thurston-Winkelnkemper from the 1970's, given an open book (S, h), we can construct a contact structure ξ which is *adapted* to (S, h):

1. ξ is close to TS away from the binding;
2. ξ is positively transverse to K. Here K is oriented as ∂S.
3. There exist a Reeb vector field of ξ which is transverse to the pages S in the interior of S and tangent to (and directing) the binding ∂S.

Correspondence between open books and contact structures

Theorem (Giroux)

There is a 1-1 correspondence between isotopy classes of contact structures (M, ξ) and open book decompositions modulo "positive stabilization".

Hence, the study of contact structures "reduces" to the study of monodromy maps $h \in Aut(S, \partial S)$.

Role of Stein fillability and tightness

Theorem (Akbulut-Ozbagci, Giroux, inspired by Loi-Piergallini)

ξ is Stein fillable iff there is an adapted open book decomposition (S, h) such that h is a product of positive Dehn twists.

Theorem (H.-Kazez-Matić)

ξ is tight iff all adapted open book decompositions (S, h) are right-veering.

Roughly speaking, "right-veering" means it sends all properly embedded arcs on S to the right.

Nielsen-Thurston classification of surface homeomorphisms

By the Nielsen-Thurston classification of surface homeomorphisms, any h is freely homotopic to ψ, which is one of the following:

1. Periodic, i.e., $\psi^n = id$.
2. Pseudo-Anosov, i.e., there exist stable and unstable geodesic laminations Λ^s and Λ^u so that $\psi(\Lambda^s) = \Lambda^s$ and $\psi(\Lambda^u) = \Lambda^u$.
3. Reducible, i.e., ψ fixes a multicurve (embedded 1-manifold).

The pseudo-Anosov case

Today we focus on the *pseudo-Anosov case*. For simplicity, assume ∂S is connected.

To each h we can associate a pseudo-Anosov ψ as well as a *fractional Dehn twist coefficient* c, which is the amount of rotation about the boundary. More precisely, let $F : S \times [0,1] \to S$ be the free homotopy from $h(x) = F(x,0)$ to $\psi(x) = F(x,1)$. Then the trace of F on $\partial S \times [0,1]$ remembers the amount of winding about the boundary of S.

The pseudo-Anosov case

The connected component of $S - \Lambda^s$ containing ∂S is a semi-open annulus A whose metric completion has geodesic boundary $\partial S \sqcup \lambda_1 \sqcup \cdots \sqcup \lambda_n$, where λ_i are infinite geodesics. The region between λ_i and λ_{i+1} is called a "prong". ψ cyclically permutes the prongs.

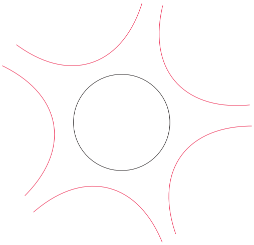

Figure: There are $n = 5$ prongs.

The pseudo-Anosov case

Notice that $c = \frac{k}{n}$, where n is the number of prongs and $k \in \mathbf{Z}$.

Theorem (H.-Kazez-Matić)

If $c \leq 0$, then the corresponding (M, ξ) is overtwisted.

Therefore, we only need to consider $c > 0$, if we are interested in *tight* contact structures.

Working Conjecture

Conjecture

Suppose ∂S is connected. If (S, h) is pseudo-Anosov and $c > \frac{1}{n}$, then the cylindrical contact homology is well-defined and nontrivial. Moreover, the cylindrical contact homology is infinitely generated, and the number of generators $[\gamma]$ with action $\int_\gamma \alpha \leq L$ (with respect to a contact 1-form α) is exponentially growing with respect to L. Here n is the number of prongs.

This result echoes results of Gabai-Oertel on essential laminations. In particular, M has universal cover $\mathbf{R}^3$ (and is hyperbolic) when $c > \frac{1}{n}$.

Current Theorem

The current version of the theorem is:

Theorem

The conjecture holds when $c > \frac{1}{n}$ and the stable lamination is oriented.

Growth Rate of Contact Homology

Example: Standard contact structure on S^3. Modulo taking direct limits, there are two simple periodic orbits. Since multiple covers of simple orbits are also counted, the growth rate of generators of $HC(S^3, \xi_{std})$ is linear with respect to the action.

Example: Tight contact structure (T^3, ξ_1). The closed orbits are in 2-1 correspondence with $\mathbf{Z}^2 - \{(0,0)\}$. Hence HC grows quadratically with respect to the action.

Conjecture

Suppose M is irreducible and atoroidal. Then $HC(M, \xi)$ is cylindrical and grows exponentially with respect to the action if M is hyperbolic and ξ is universally tight.

Sketch of Proof

Suppose h is freely isotopic to id. The general case is only more complicated. I will explain why the cylindrical theory is well-defined.

Step 1. Suppose there is a finite energy plane $\tilde{u} : \mathbf{C} \to \mathbf{R} \times M$. The projection to M is basically a disk $u : D \to M$ with $u(\partial D) = \gamma$. u has finitely many complex branch points and u is positively transverse to R (by the holomorphic condition). Hence all the intersections of D with the binding K are *positive*. Now consider the cut-open disk $D_0 = D - u^{-1}(S \times \{0\} - N(K))$.

Goal: Show that the cut-open disk D_0 cannot possibly exist. (It is useful to project it to S.)

The cut-open disk D_0

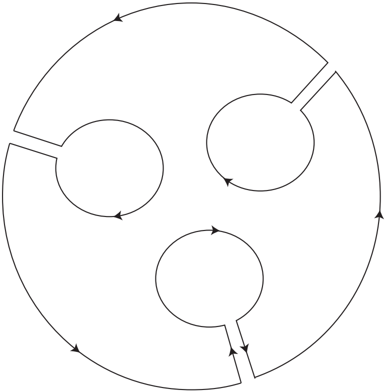

Figure: The cut-open disk D_0.

The Rademacher function

Step 2. Define the *Rademacher function* of a path in the Farey tessellation as follows:

$$\Phi(\delta) = \#(\text{right turns}) - \#(\text{left turns}),$$

if δ begins at the edge $0 \to \infty$ and ends on another edge.

The Rademacher function has the following convenient properties:

1. $\Phi(\delta\delta') = \Phi(\delta) + \Phi(\delta') + 0, 3, \text{or } -3;$
2. $\Phi(\delta^{-1}) = -\Phi(\delta).$

The Rademacher function

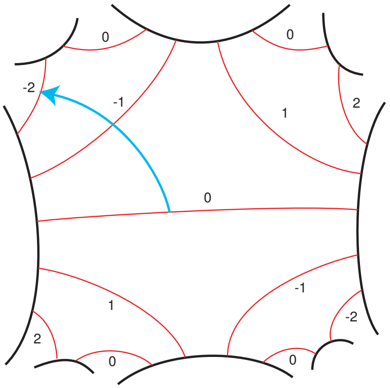

Figure: The tessellation of the universal cover $\widetilde{S}$ of S and values of the Rademacher function on the tessellation.

Sketch of Proof, Continued

We now calculate the Rademacher function of the boundary of the cut-open disk D_0.

For example, if S is the once-punctured torus, triangulate it using arcs. Then pass to the universal cover $\widetilde{S}$. If α is an arc of D from ∂D to an intersection with K, then there is a path $\alpha(h(\alpha))^{-1}$ which is a subarc of the boundary. Now, $\Phi(\alpha(h(\alpha))^{-1})$ is very positive if c is very positive. Hence, if $c \gg 0$, then Φ of the boundary could never be zero!

Sketch of Proof, Continued

A similar technique gives restrictions on holomorphic cylinders which are asymptotic to γ' at the positive end and asymptotic to γ'' at the negative end.

Final and extremely important ingredient: Dynamics of pseudo-Anosov diffeomorphisms, due to Thurston. If $\psi : S \to S$ is pseudo-Anosov, then the number of periodic points with period n grows exponentially with n. The count of such periodic points is essentially the E_1 term of a spectral sequence. The above restrictions on holomorphic cylinders prevents the higher E_m terms from collapsing too much!