Refinable maps in the theory of shape

by

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Abstract. A map $r\colon X\to Y$ between metric compacta is said to be *refinable* if for every $\varepsilon>0$ there is an ε-mapping f of X onto Y such that $\sup\{\operatorname{dist}(r(x),f(x))|\ x\in X\}<\varepsilon$. We establish certain properties of refinable maps in the theory of shape. In particular it is shown that if X is a movable continuum with $\operatorname{Fd}(X)\leqslant 1$, every refinable map $r\colon X\to Y$ preserves shape, but there exist 1-dimensional continua X, Y and a refinable map $r\colon X\to Y$ which does not preserve shape.

0. Introduction. The term compactum is used to mean a compact metric space. A connected compactum is a continuum. A map $f\colon X\to Y$ between compacta is said to be an *ε-mapping*, $\varepsilon>0$, if f is surjective and $\operatorname{diam} f^{-1}(y)<\varepsilon$ for each $y\in Y$. If x and y are points of a metric space, $d(x,y)$ denotes the distance from x to y. A map $r\colon X\to Y$ between compacta is *refinable* [9] if for every $\varepsilon>0$ there is an ε-mapping $f\colon X\to Y$ such that $d(r,f)=\sup\{d(r(x),f(x))|\ x\in X\}<\varepsilon$. Such a map f is called an *ε-refinement* of r. For the pointed case, a map $r\colon (X,x)\to(Y,y)$ between pointed compacta is refinable if for every $\varepsilon>0$ there is a map $f\colon (X,x)\to(Y,y)$ which is an ε-refinement of r. By the definitions we know that each refinable map is surjective, each near homeomorphism is refinable and if there is a refinable map from a compactum X to a compactum Y, then X is Y-like. But, any converse assertions of them are not true.

Throughout this paper, by an ANR we mean an ANR for the class of metrizable spaces.

In this paper, we shall investigate shape theoretic properties of refinable maps. In the first section, we prove that a refinable map induces a pseudo-isomorphism in shape category. In the next section, we show that if X is a movable continuum with $\operatorname{Fd}(X)\leqslant 1$, every refinable map $r\colon X\to Y$ preserves shape, but there exist 1-dimensional continua X, Y and a refinable map $r\colon X\to Y$ which does not preserve shape. Moreover we show that under some conditions refinable maps preserve FANR. In the last section, we give more detailed information on the refinable map $r\colon X\to Y$ when Y is a compact ANR.

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1. Refinable maps and pseudo-isomorphisms. By HCW we mean the category of spaces having the homotopy type of CW-complexes and homotopy classes of maps. For a map f $[f]$ denotes the homotopy class determined by f. If K is a category, by $\boldsymbol{K}$ we mean the category of inverse systems in K and system map in K, also by pro-K the homotopy category of $\boldsymbol{K}$ [14].

First, we give the following definitions.

1.1. Definition. Let K be an arbitrary category. A system map [14] $\boldsymbol{f} = \{f, f_\beta, B\}$: $\{X_\alpha, p_{\alpha\alpha'}, A\} \to \{Y_\beta, q_{\beta\beta'}, B\}$ of $\boldsymbol{K}$ is a *pseudo-isomorphism* if for each $\beta \in B$ and each $\alpha \geqslant f(\beta)$ there exist $g(\alpha, \beta) \geqslant \beta$ and a morphism $g_{(\alpha,\beta)}$: $Y_{g(\alpha,\beta)} \to X_\alpha$ such that for every $\beta' \geqslant g(\alpha, \beta)$ there exist $h(\beta') \geqslant \alpha$ and a morphism $h_{\beta'}$: $X_{h(\beta')} \to Y_{\beta'}$ such that

$$f_\beta p_{f(\beta)\alpha} g_{(\alpha,\beta)} = q_{\beta g(\alpha,\beta)} \quad \text{and} \quad g_{(\alpha,\beta)} q_{g(\alpha,\beta)\beta'} h_{\beta'} = p_{\alpha h(\beta')} .$$

A morphism f: $\{X_\alpha, p_{\alpha\alpha'}, A\} \to \{Y_\beta, q_{\beta\beta'}, B\}$ of pro-K is a *pseudo-isomorphism* if it has a pseudo-isomorphism $\boldsymbol{f}$: $\{X_\alpha, p_{\alpha\alpha'}, A\} \to \{Y_\beta, q_{\beta\beta'}, B\}$ of $\boldsymbol{K}$ as the representation, i.e. $f = [\boldsymbol{f}]$.

1.2. Definition. In the shape category [13], a shaping f: $X \to Y$ is a *pseudo-isomorphism* if there is a pseudo-isomorphism $\boldsymbol{f} = \{f, [f_\beta], B\}$: $\{X_\alpha, [p_{\alpha\alpha'}], A\} \to \{Y_\beta, [q_{\beta\beta'}], B\}$ of **HCW** such that

(#) $S[f_\beta p_{f(\beta)}] = S[q_\beta] f$, where $\{X_\alpha, [p_{\alpha\alpha'}], A\}$ and $\{Y_\beta, [q_{\beta\beta'}], B\}$ are associated with X and Y respectively, p_α: $X \to X_\alpha$, $q_\beta: Y \to Y_\beta$ are projections [18, Definition 1.2], and S denotes the shape functor (cf. [13], [18, Theorem 2.3]).

The following proposition is easily seen from the definitions.

1.3. Proposition. *If a shaping f: $X \to Y$ is a pseudo-isomorphism, then every system map $\boldsymbol{f}$ which satisfies the condition (#) is a pseudo-isomorphism of* **HCW**.

1.4. Remark. For a shaping f, the following implications are true, but any of converse are not (cf. Examples 2.6 and 2.7)

$$\begin{array}{ccc} f \text{ is a shape equivalence} & \to & f \text{ is a domination} \\ \downarrow & & \downarrow \\ f \text{ is a pseudo-isomorphism} & \to & f \text{ is a weak domination} \end{array}$$

(see [6] for the definition of the weak domination.)

1.5. Theorem. *Let X and Y be compacta. If a map r: $X \to Y$ is refinable, then the shaping $S[r]$ induced by the map r is a pseudo-isomorphism.*

Proof. Let $\{X_n, P_{n,n+1}, N\}$ and $\{Y_n, q_{n,n+1}, N\}$ be inverse sequences of compact polyhedra such that $X = \operatorname{invlim}\{X_n, p_{n,n+1}, N\}$ and $Y = \operatorname{invlim}\{Y_n, q_{n,n+1}, N\}$. Let p_n: $X \to X_n$ and q_n: $Y \to Y_n$ be natural projections respectively. Since each Y_n is a compact ANR, there are positive number ε_n and δ_n such that any two ε_n-near maps to Y_n are homotopic and

(1) if $x, y \in Y$ and $d(x, y) < \delta_n$, then $d(q_n(x), q_n(y)) < \frac{1}{4}\varepsilon_n$.

By S. Mardešić and J. Segal [16], for each n there is a map f_n: $X_n \to Y_n$ such that

(2) $d(f_n p_n, q_n r) < \frac{1}{4}\varepsilon_n$ and $f_n p_{n,n+1} \simeq q_{n,n+1} f_{n+1}$.

Let $\boldsymbol{f} = \{1, [f_n], N\}$. We shall show that $\boldsymbol{f}$ is a pseudo-isomorphism of **HCW**. For each n, choose a positive number η_n such that any two η_n-near maps to X_n are homotopic and

(3) if $x, y \in X_n$ and $d(x, y) < \eta_n$, then $d(f_n(x), f_n(y)) < \frac{1}{4}\varepsilon_n$.

By [15], for each n there exist a map g_n': $Y \to X_n$ and a δ_n-refinement r_n of r such that

(4) $$d(p_n, g_n' r_n) < \eta_n .$$

Then by (4) we have

(5) $$p_n \simeq g_n' r_n .$$

Let us show that $f_n g_n' \simeq q_n$. For each $y \in Y$ there is $x \in X$ such that $r_n(x) = y$. By (1), (2), (3) and (4),

$$\begin{aligned} & d(q_n(y), f_n g_n'(y)) \\ & = d(q_n r_n(x), f_n g_n' r_n(x)) \leqslant d(q_n r_n(x), q_n r(x)) + d(q_n r(x), f_n p_n(x)) + d(f_n p_n(x), f_n g_n' r_n(x)) \\ & \leqslant \tfrac{1}{4}\varepsilon_n + \tfrac{1}{4}\varepsilon_n + \tfrac{1}{4}\varepsilon_n < \varepsilon_n . \end{aligned}$$

Hence we obtain

(6) $$q_n \simeq f_n g_n' .$$

It follows from (2), (5) and (6) that $\boldsymbol{f}$ is a pseudo-isomorphism of **HCW**. Thus the shaping $S[r]$ is a pseudo-isomorphism.

1.6. Remark. For the pointed case, we have the same result as Theorem 1.5. In this paper all results proved for the absolute case are also true for the pointed case.

1.7. Proposition. *If a shaping f: $X \to Y$ is a pseudo-isomorphism, then*

(1) *f is an epimorphism.*

(2) *if X is movable* ([3], [17]), *then Y is movable.*

(3) *X is approximatively n-connected* ([3], [6 p. 50]) *if and only if Y is so.*

(4) $\operatorname{ddim} X = \operatorname{ddim} Y$, *where* $\operatorname{ddim} X$ *implies the deformation dimension of X in the sense of J. Dydak* [6, p. 23].

(5) *X is of trivial shape* ([3], [6, p. 50]) *if and only if Y is so.*

Proof. Let $\boldsymbol{f} = \{f, [f_\beta], B\}$: $\{X_\alpha, [p_{\alpha\alpha'}], A\} \to \{Y_\beta, [q_{\beta\beta'}], B\}$ be a pseudo-isomorphism of **HCW** satisfying the condition (#) of Definition 1.2.

(1) Since f is a weak domination by Remark 1.4, by [6, Proposition 2.9] f is an epimorphism.

(2) Since f is a weak domination and X is movable, by [6, Theorem 2.11] Y is movable.

(3) Suppose that X is approximatively n-connected. For each $\beta \in B$ there is $\alpha \geqslant f(\beta)$ such that for any map t of the n-sphere S^n to X_α, $p_{f(\beta)\alpha} t$ is null-homotopic. Then there exist $g(\alpha, \beta) \geqslant \beta$ and a map $g_{(\alpha,\beta)}$: $Y_{g(\alpha,\beta)} \to X_\alpha$ such that $[f_\beta][p_{f(\beta)\alpha}][g_{(\alpha,\beta)}] = [q_{\beta g(\alpha,\beta)}]$. Let s: $S^n \to Y_{g(\alpha,\beta)}$ be an arbitrary map. Then we have

$$q_{\beta g(\alpha,\beta)} s \simeq f_\beta p_{f(\beta)\alpha} g_{(\alpha,\beta)} s .$$

Hence, $q_{\beta g(\alpha,\beta)} s$ is null-homotopic, which implies that Y is approximatively n-connected. The converse is similar.

(4) Since f is a weak domination, by [6, Theorem 4.2] we have $\operatorname{ddim} X \geqslant \operatorname{ddim} Y$. The converse is similar.

(5) The proof is the same as (3).

1.8. THEOREM. *If a map r: $X \to Y$ between compacta is refinable, then*

(1) $S[r]$ *is an epimorphism.*

(2) *if X is movable, then Y is movable.*

(3) *X is approximatively n-connected if and only if Y is so.*

(4) $\mathrm{Fd}(X) = \mathrm{Fd}(Y)$ [3, p. 253] *and* $\dim X = \dim Y$.

(5) *X is an* FAR [3] *if and only if Y is so.*

Proof. All of the proofs except $\dim X = \dim Y$ follow from Theorem 1.5 and Proposition 1.7. Let us show $\dim X = \dim Y$. Since r is refinable, for any family β of ANR's X is β-like by [9, Corollary 3.1] if and only if Y is β-like. This implies $\dim X = \dim Y$.

2. Refinable maps preserving shapes. In this section we show that under some conditions refinable maps preserve shapes. Also we give some examples in which refinable maps do not preserve shapes.

The author thanks the referee for some remarks concerning this section.

2.1. PROPOSITION. *For a category K, if f: $X = \{X_\alpha, p_{\alpha\alpha'}, A\} \to Y = \{Y_\beta, q_{\beta\beta'}, B\}$ is a pseudo-isomorphism of* pro-K *and Y is dominated in* pro-K *by an object of K, then f is an isomorphism.*

Proof. Let $f = \{f, f_\beta, B\}$: $X \to Y$ be a pseudo-isomorphism of K such that $f = [f]$. Then for each $\beta \in B$ there exist $\beta' \geqslant \beta$ such that for every $\beta'' \in B$ there exist a morphism $k_{\beta''\beta'}$: $Y_{\beta'} \to Y_{\beta''}$ and $\beta''' \geqslant \beta', \beta''$ such that $k_{\beta''\beta'} q_{\beta'\beta'''} = q_{\beta''\beta'''}$, because Y is dominated in pro-K by an object of K. For each $\alpha \geqslant f(\beta)$, choose $\alpha' \geqslant \alpha, f(\beta')$ such that $q_{\beta\beta'} f_{\beta'} p_{f(\beta')\alpha'} = f_\beta p_{f(\beta)\alpha'}$. Since f is a pseudo-isomorphism of K, there exist $g(\alpha', \beta') \geqslant \beta'$ and a morphism $g_{(\alpha'\beta')}$: $Y_{g(\alpha',\beta')} \to X_{\alpha'}$ satisfying the condition of Definition 1.1. Then for every $\beta'' \geqslant g(\alpha', \beta')$ there exist a morphism $k_{\beta''\beta'}$: $Y_{\beta'} \to Y_{\beta''}$ and $\beta''' \geqslant \beta''$ such that $k_{\beta''\beta'} q_{\beta'\beta'''} = q_{\beta''\beta'''}$. Also, there exist $h(\beta''') \geqslant \alpha', f(\beta''')$ and a morphism $h_{\beta'''}$: $X_{h(\beta''')} \to Y_{\beta'''}$ such that

$$g_{(\alpha',\beta')} q_{g(\alpha',\beta')\beta'''} h_{\beta'''} = p_{\alpha' h(\beta''')} \quad \text{and} \quad f_{\beta'} p_{f(\beta')h(\beta''')} = q_{\beta'\beta'''} f_{\beta'''} p_{f(\beta''')h(\beta''')} .$$

Then we have

$$\begin{aligned} q_{\beta''\beta'''} f_{\beta'''} p_{f(\beta''')h(\beta''')} &= k_{\beta''\beta'} q_{\beta'\beta'''} f_{\beta'''} p_{f(\beta''')h(\beta''')} = k_{\beta''\beta'} f_{\beta'} p_{f(\beta')h(\beta''')} \\ &= k_{\beta''\beta'} f_{\beta'} p_{f(\beta')\alpha'} g_{(\alpha',\beta')} q_{g(\alpha',\beta')\beta'''} h_{\beta'''} \\ &= k_{\beta''\beta'} q_{\beta' g(\alpha',\beta')} q_{g(\alpha'\beta')\beta'''} h_{\beta'''} \\ &= q_{\beta''\beta'''} h_{\beta'''} . \end{aligned}$$

Hence, we obtain

$$g_{(\alpha',\beta')} q_{g(\alpha',\beta')\beta'''} f_{\beta'''} p_{f(\beta''')h(\beta''')} = g_{(\alpha',\beta')} q_{g(\alpha',\beta')\beta''} q_{\beta''\beta'''} h_{\beta'''} = p_{\alpha' h(\beta''')} ,$$

and

$$f_\beta p_{f(\beta)\alpha'} g_{(\alpha',\beta')} = q_{\beta g(\alpha',\beta')} .$$

This implies that f is an isomorphism.

The following theorem follows from Theorem 1.5 and Proposition 2.1.

2.2. THEOREM. *Let X and Y be compacta and a map r: $X \to Y$ be refinable. If Y is an* FANR, *then* $S[r]$ *is a shape equivalence.*

2.3. COROLLARY [9, Corollary 3.4]. *Let X and Y be compact* ANR*'s and a map r: $X \to Y$ be refinable. Then r is a homotopy equivalence.*

2.4. THEOREM. *Let X and Y be compacta and a map r: $X \to Y$ be refinable. If X is a movable continuum with* $\mathrm{Fd}(X) \leqslant 1$, *then* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. *Moreover if either X or Y is an* FANR, $S[r]$ *is a shape equivalence.*

Proof. Since X is a movable continuum with $\mathrm{Fd}(X) \leqslant 1$, by Theorem 1.8 Y is a movable continuum with $\mathrm{Fd}(Y) \leqslant 1$. Then there exist continua X_1 and Y_1 such that $\mathrm{Sh}(X) = \mathrm{Sh}(X_1)$, $\mathrm{Sh}(Y) = \mathrm{Sh}(Y_1)$, $\dim X_1 \leqslant 1$ and $\dim Y_1 \leqslant 1$. Further, by A. Trybulec [21] there exist plane continua X_2 and Y_2 such that $\mathrm{Sh}(X_1) = \mathrm{Sh}(X_2)$ and $\mathrm{Sh}(Y_1) = \mathrm{Sh}(Y_2)$. By K. Borsuk [3, p. 221, 267], it is enough to consider the following two cases.

Case (1). If Y_2 is an FANR, then Y would be an FANR. It follows from Theorem 2.2 that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

Case (2). If Y_2 is not an FANR, we must show that X_2 is not an FANR. Suppose that X_2 is an FANR. By [3, p. 267] we have,

(i) the projection p_1: $\check{H}_1(X) \to \text{pro-}H_1(X)$ is an isomorphism where H_1 denotes a 1-dimentional homology with integer coefficients, and

(ii) pro-$H_1(Y)$ is not dominated in pro-groups (e.g. see [14]) by a group.

Then, by Theorem 1.5 the composition pro-$H_1(r) p_1$: $\check{H}_1(X) \to \text{pro-}H_1(Y)$ is a weak domination of pro-groups. Since $\check{H}_1(X)$ is a finitely generated abelian group, by [6, Lemma 2.20] pro-$H_1(Y)$ is dominated in pro-groups by a group. This implies a contradiction. Thus we conclude that X_2 is not an FANR. By [3, p. 221], $\mathrm{Sh}(X_2) = \mathrm{Sh}(Y_2)$ hence $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

The second part of Theorem follows from the first part and Theorem 2.2. This completes the proof.

2.5. COROLLARY. *Let X and Y be continua in the plane R^2. If there exists a refinable map of X to Y, then two continua X, $Y \subset R^2$ decompose the plane into the same number of components.*

2.6. EXAMPLE. Let k be a natural number and $\{X_n^k, p_{n,n+1}^k, N\}$ be an inverse sequence of the k-sphere $X_n^k = S^k$, where the bonding map $p_{n,n+1}^k$ is a map of S^k onto S^k of $\deg(p_{n,n+1}^k) > 1$ for each n. Consider the following sets:

$$Z(k) = \operatorname{invlim}\{X_n^k, p_{n,n+1}^k, N\},$$

$$Y_n^k = \{(x_1, x_2, \ldots, x_{k+1}) \in R^{k+1} \mid (x_1 - 1/n)^2 + x_2^2 + \ldots + x_{k+1}^2 = 1/n^2\},$$

$$Y(k) = \bigcup_{n=1}^{\infty} Y_n^k, \quad y_0 = (0, 0, \ldots, 0) \in R^{k+1}, \text{ where } R \text{ is reals}.$$

By identifying a point $z_0 \in Z(k)$ and $y_0 \in Y(k)$ we obtain a continuum

$$(X(k), *) = (Z(k), z_0) \vee (Y(k), y_0).$$

Now, define a map r_k: $X(k) \to Y(k)$ by

$$r_k(x) = \begin{cases} x & \text{if } x \in Y(k), \\ y_0 & \text{if } x \in Z(k). \end{cases}$$

Then, r_k is refinable. In fact, for a given $\varepsilon > 0$ choose a natural number n_0 such that $4/n_0 < \varepsilon$. Since $Z(k)$ is S^k-like, there is an $\frac{1}{2}\varepsilon$-mapping f: $(Z(k), z_0) \to (Y_{n_0+1}^k, y_0)$. Define a map g: $X(k) \to Y(k)$ by

$$g(x) = \begin{cases} x & \text{if } x \in \bigcup_{n=1}^{n_0} Y_n^k \cup \bigcup_{n=n_0+2}^{\infty} Y_n^k, \\ y_0 & \text{if } x \in Y_{n_0+1}^k, \\ f(x) & \text{if } x \in Z(k). \end{cases}$$

Clearly we have $\operatorname{diam} g^{-1}(y) < \varepsilon$ and $d(r_k, g) < \varepsilon$, hence r_k is refinable. On the other hand, $Y(k)$ is movable but $X(k)$ is not movable [17], which implies $\operatorname{Sh}(X(k)) \neq \operatorname{Sh}(Y(k))$ for each k. Note that $X(1)$ and $Y(1)$ are contained in R^3 and $\dim X(1) = \dim Y(1) = 1$. Hence this example shows that the movability of X in Theorem 2.4 can not be removed.

2.7. EXAMPLE. Let A be an arbitrary continuum. Consider the following sets in $A \times [0, 3]$.

$$A_t = A \times \{t\} \subset A \times [0, 3] \quad \text{for} \quad 0 \leqslant t \leqslant 3,$$

$$X(A) = A_0 \cup \bigcup_{n=1}^{\infty} A_{1/n} \cup A_2 \cup \bigcup_{n=1}^{\infty} A_{2+1/n},$$

$$Y(A) = A_0 \cup \bigcup_{n=1}^{\infty} A_{1/n}.$$

Define a map r_A: $X(A) \to Y(A)$ by

$$r_A(a, t) = \begin{cases} (a, t) & \text{if } (a, t) \in A_0 \cup \bigcup_{n=1}^{\infty} A_{1/n}, \\ (a, 0) & \text{if } (a, t) \in A_2 \cup \bigcup_{n=1}^{\infty} A_{2+1/n}. \end{cases}$$

Then the map r_A is refinable. On the other hand, $\operatorname{Sh}(X(A)) \neq \operatorname{Sh}(Y(A))$, because the decomposition space $\square(X(A))$ of $X(A)$ can not be embedded in the decomposition space $\square(Y(A))$ of $Y(A)$ [3, p. 214]. If A is a one point set, $X(A)$ and $Y(A)$ are movable compacta of dimension 0. Hence this example shows that the connectivity of X in Theorem 2.4 can not be removed.

2.8. EXAMPLE. Let Y_n^k, $Y(k)$ and y_0 be the same in Example 2.6. Since each Y_n^k is the k-sphere, there is a homeomorphism h_n: $Y_{2n-1}^k \to Y_n^k$ for each n such that $h_n(y_0) = y_0$. Define a map h: $Y(k) \to Y(k)$ by

$$h(y) = \begin{cases} h_n(y) & \text{if } y \in Y_{2n-1}^k \quad (n = 1, 2, \ldots), \\ y_0 & \text{if } y \in Y_{2n}^k \quad (n = 1, 2, \ldots). \end{cases}$$

Then, by the same way as in Example 2.6 we know that h is a near homeomorphism. Note that $Y(1) \subset R^2$. Clearly, h does not induce a shape equivalence. Hence this example shows that if X is not an FANR in Theorem 2.4, every refinable map r: $X \to Y$ does not necessarily induce a shape equivalence.

2.9. Remark. (1) For an arbitrary inverse sequence $\{X_n, p_{n,n+1}, N\}$ of compacta with bonding maps $p_{n,n+1}$ onto, the method of the construction of the refinable map in Example 2.6 can be generalized. Therefore for any nonmovable continuum we can construct a refinable map which does not preserve shape.

(2) T. Watanabe showed that a refinable map does not necessarily preserve shape for compacta in R^3 by using the example of K. Borsuk [2].

Applying Theorem 2.4, we get the following

2.10. COROLLARY. *Let a map r: $X \to Y$ between compacta be refinable and Y be a plane compactum. If X is an* ANR *(resp.* AR), *then Y is an* ANR *(resp.* AR).

Proof. Without loss of generality, we may assume that X and Y are continua. Since X is locally connected, Y is locally connected. By Theorem 1.8 we have $\operatorname{Fd}(X) = \operatorname{Fd}(Y) \leqslant 1$. Further, by Theorem 2.4 Y is an FANR, hence by K. Borsuk [4, Theorem 14.1] we conclude that Y is an ANR. If X is an AR, then by Theorem 1.8 Y is an FAR, hence we conclude that Y is an AR.

2.11. COROLLARY [19]. *Let a map r: $X \to Y$ between compacta be refinable. If X is a 1-dimensional* ANR, *then Y is a 1-dimensional* ANR.

Proof. We may assume that X and Y are continua. By Theorem 1.8 Y is a 1-dimensional locally connected continuum. Further, by Theorem 2.4 Y is an FANR, hence by [4, Corollary 13.6] we conclude that Y is an ANR.

Now, the following problem is raised; if X is an FANR, does every refinable map $r\colon X\to Y$ induce a shape equivalence? If $\mathrm{Fd}(X)\leqslant 1$, we have an affirmative answer from Theorem 2.4. Moreover, we obtain the following partial answer.

2.12. THEOREM. *Let a map $r\colon (X,x)\to(Y,y)$ between pointed compacta be refinable. If X is a connected pointed FANR and the first shape group $\check{\pi}_1(X,x)$ is trivial, then $S[r]$ is a shape equivalence.*

Proof. Since the Wall obstruction $\omega(\check{\pi}_1(X,x))\in\tilde{K}^0(\check{\pi}_1(X,x))$ vanishes, by [7] there is a compact polyhedron (P,p) such that $\mathrm{Sh}(X,x)=\mathrm{Sh}(P,p)$. Further, since $\pi_1(P,p)=\check{\pi}_1(X,x)=0$, by [20, p. 509] $\pi_i(P,p)=\check{\pi}_i(X,x)$ is a finitely generated abelian group for each i. Then, by [10] the projection $p_i\colon \check{\pi}_i(X,x)\to\to \text{pro-}\pi_i(X,x)$ is an isomorphism of pro-groups (e.g., see [14]), because (X,x) is movable and $\check{\pi}_i(X,x)$ is a countable group. Then, by Theorem 1.5 we conclude that the composition $\text{pro-}\pi_i(r)p_i\colon \check{\pi}_i(X,x)\to\text{pro-}\pi_i(Y,y)$ is a weak domination of pro-groups. Since by [6, Lemma 2.20] $\text{pro-}\pi_i(Y,y)$ is stable and by Theorem 1.8 $\mathrm{Fd}(Y)=\mathrm{Fd}(X)<\infty$, by ([8], [6, p. 46]) Y is a pointed FANR. By Theorem 2.2 we conclude that $S[r]$ is a shape equivalence.

3. Refinable maps onto ANR's.

In this section, we give detailed information on the refinable map $r\colon X\to Y$ when Y is a compact ANR.

3.1. THEOREM. *Let $r\colon X\to A$ be a map between compacta and A be an ANR. Then the map r is refinable if and only if there exist an inverse sequence $\{A_i, p_{i,i+1}, N\}$ such that for each i, $A_i=A$ and $p_{i,i+1}$ is an onto map which is homotopic to 1_A, and a homeomorphism $h\colon X\to \operatorname{invlim}\{A_i,p_{i,i+1},N\}$ such that $\lim_{i\to\infty}(p_ih)=r$, where for each i $p_i\colon \operatorname{invlim}\{A_i,p_{i,i+1},N\}\to A_i$ is a natural projection.*

Proof. It is enough to give the proof of necessity. It is essentially due to S. Mardešić and J. Segal ([12], [15]).

Inductively, we can find for each i, maps $p_{i,i+1}\colon A_{i+1}\to A_i$ onto, positive number ε_i with $\lim\varepsilon_i=0$, ε_i-refinement f_i of r, and positive number $\eta_i<\varepsilon_i$ having the following properties.

(1) For any set $N_j\subset A_j$ with $\mathrm{diam}(N_j)\leqslant\eta_j$, we have $\mathrm{diam}\,p_{ij}(N_j)<\eta_i/2^{j-i}$.

(2) If $d(x,y)\geqslant 2\varepsilon_i$ for $x,y\in X$, then $d(f_i(x),f_i(y))>2\eta_i$.

(3) $d(f_i,p_{i,i+1}f_{i+1})\leqslant\frac{1}{2}\eta_i$ for each i.

Then the sequence $\{p_{ij}f_j\}_{j=1,2,\dots}$ is a Cauchy sequence for each i. Put $h_i=\lim_{j\to\infty}(p_{ij}f_j)$. Then we have $h_i=p_{i,i+1}h_{i+1}$ and $d(f_i,h_i)\leqslant\eta_i$. Therefore there is a map $h\colon X\to\operatorname{invlim}\{A_i,p_{i,i+1},N\}$ such that $p_ih=h_i$ for each i. Then h is a homeomorphism [15]. On the other hand, we have

$$\begin{aligned} d(r(x),p_ih(x)) &= d(r(x),h_i(x))\\ &\leqslant d(r(x),f_i(x))+d(f_i(x),h_i(x))\\ &\leqslant \varepsilon_i+\eta_i<2\varepsilon_i \quad\text{for each } x\in X.\end{aligned}$$

Hence, $\lim_{i\to\infty}(p_ih)=r$. Since A is a compact ANR, $r\simeq p_ih$ for allmost all i. Then, $r\simeq h_i=p_{i,i+1}h_{i+1}\simeq p_{i,i+1}r$ for allmost all i. Since $S[r]$ is an epimorphism, we obtain $p_{i,i+1}\simeq 1_A$ for allmost all i. This completes the proof.

A map $f\colon X\to Y$ is said to be monotone if f is surjective and for each $y\in Y$ $f^{-1}(y)$ is connected.

3.2. THEOREM. *A map r of a $(S_1\vee S_2\vee\dots\vee S_n)$-like continuum X to $S_1\vee S_2\dots\vee S_n$ is refinable if and only if it is monotone, where $S_1\vee S_2\vee\dots\vee S_n$ $(n\geqslant 1)$ denotes a one point union of n circles.*

Proof. We shall show only the case $n=2$. The case $n\neq 2$ is similarly proved. If r is refinable, r is monotone [9, Corollary 1.2] because $S_1\vee S_2$ is locally connected.

Conversely, suppose that r is monotone. Let ε be an arbitrary positive number. Now, let $A_0,A_1,\dots,A_m$ and $B_0,B_1,\dots,B_m$ denote circular chains of non-overlapping closed intervals of S_1 and S_2 respectively such that $s_0\in\mathrm{Int}(A_0\cup B_0)$, where s_0 is a common point of S_1 and S_2, and $\mathrm{diam}(A_0\cup B_0)<\frac{1}{8}\varepsilon$, $\mathrm{diam}\,A_i<\frac{1}{8}\varepsilon$, $\mathrm{diam}\,B_i<\frac{1}{8}\varepsilon$ for each $1\leqslant i\leqslant m$. We may assume that m is a sufficiently large integer. Put $C_i=r^{-1}(A_i)$ and $D_i=r^{-1}(B_i)$ for $0\leqslant i\leqslant m$ and choose points $x_j\in C_j$ and $y_j\in D_j$ for $1\leqslant j\leqslant m$ such that

$$(1)\qquad r(x_j)\in\mathrm{Int}A_j\quad\text{and}\quad r(y_j)\in\mathrm{Int}B_j.$$

Then the sequences $C_0,C_1,\dots,C_m$ and $D_0,D_1,\dots,D_m$ are circular chains of subcontinua of X such that $X=\bigcup_{i=0}^m C_i\cup\bigcup_{i=0}^m D_i$, because r is monotone [11, p. 131]. Then by (1) there is a positive number $\eta<\varepsilon$ such that

$$(2)\qquad d(x_i,\bigcup_{j\in J_i}C_j\cup\bigcup_{j=0}^m D_j)>\eta\quad\text{for}\quad 1\leqslant i\leqslant m,$$

$$d(y_i,\bigcup_{j=0}^m C_j\cup\bigcup_{j\in J_i}D_j)>\eta\quad\text{for}\quad 1\leqslant i\leqslant m,$$

where $J_i=\{0,1,\dots,i-1,i+1,\dots,m\}$ $(1\leqslant i\leqslant m)$, and

(3) if $d(C_i,C_j)<\eta$ for $0\leqslant i,j\leqslant m$, then $|i-j|\leqslant 1$ or $|i-j|=m$,
if $d(D_i,D_j)<\eta$ for $0\leqslant i,j\leqslant m$, then $|i-j|\leqslant 1$ or $|i-j|=m$, and
$d(C_i,D_j)>\eta$ for $1\leqslant i,j\leqslant m$.

Since X is $(S_1\vee S_2)$-like, there is an η-mapping $f\colon X\to(S_1\vee S_2)$. Then, by (2) and (3) we can prove that $s_0\in f(C_0\cup D_0)$ and the sequences $f(C_2),f(C_3),\dots$ $\dots,f(C_{m-1})$ and $f(D_2),f(D_3),\dots,f(D_{m-1})$ are chains of closed intervals in S_1 or S_2. Clearly, there is a homeomorphism $h\colon S_1\vee S_2\to S_1\vee S_2$ such that

$$(4)\qquad hf(x_i)=r(x_i)\quad\text{and}\quad hf(y_i)=r(y_i)\quad\text{for}\quad 2\leqslant i\leqslant m-1.$$

Now, hf is an η-mapping, hence an ε-mapping. Also, by (1), (2), (3) and (4) we have

(5) $hf(C_i)\subset[r(x_{i-1}), r(x_{i+1})]\subset A_{i-1}\cup A_i\cup A_{i+1}$ for $3\leqslant i\leqslant m-2$,

(6) $hf(D_i)\subset[r(y_{i-1}), r(y_{i+1})]\subset B_{i-1}\cup B_i\cup B_{i+1}$ for $3\leqslant i\leqslant m-2$,

and

(7) $$hf(\bigcup_{i=0}^{2} C_i\cup\bigcup_{j=m-1}^{m} C_j\cup\bigcup_{i=0}^{2} D_i\cup\bigcup_{j=m-1}^{m} D_j)\subset[r(x_3), r(x_{m-2})]\cup[r(y_3), r(y_{m-2})]$$
$$\subset\bigcup_{i=0}^{3} A_i\cup\bigcup_{j=m-2}^{m} A_j\cup\bigcup_{i=0}^{3} B_i\cup\bigcup_{j=m-2}^{m} B_j,$$

where $[x, y]$ denotes the smallest interval in a circle S^1 from x to y if $d(x, y)<\operatorname{diam} S^1$ for $x, y\in S^1$. Then it follows from (5), (6) and (7) that $d(r, hf)<\varepsilon$, which implies that r is refinable. This completes the proof.

Combining Theorems 3.1 and 3.2 we have

3.3. COROLLARY. *If there is a monotone map r of a $(S_1\vee S_2\vee\ldots\vee S_n)$-like continuum X to $S_1\vee S_2\vee\ldots\vee S_n$, then $S[r]$ is a shape equivalence. Moreover, X is homeomorphic to the limit of the inverse sequence of $S_1\vee S_2\vee\ldots\vee S_n$ satisfying the conditions of Theorem* 3.1.

The map of the Warsaw Circle onto a circle obtained by shrinking the limiting interval to a point is monotone, hence refinable. Generally we have the following

3.4. COROLLARY. *If X is a hereditarily decomposable circle-like continuum, then there is a refinable map of X to a circle S^1.*

Proof. Since X is a hereditarily decomposable circle-like continuum, by [5, Theorem 3] we conclude that X is not arc-like. Also, since X is decomposable, there exist proper subcontinua M and N such that $X = M\cup N$. Then, by [5, Theorem 4] no proper subcontinuum of X separates X, hence we may assume that

$$M=\overline{X-N}\quad\text{and}\quad N=\overline{X-M}.$$

Then $M\cap N$ is not connected, because X is not arc-like. By [5, Theorem 5] $M\cap N = H\cup K$, where H and K are disjoint subcontinua of X. Now, we define an equivalence relation $\sim$ on X by setting $x\sim y$ if and only if $x = y$ and $x\in X-(H\cup K)$ or $x, y\in H$ or $x, y\in K$. Let $X_1 = X/\sim$ be the quotient space and $p\colon X\to X_1$ the quotient map. Note that $p\colon X\to X_1$, $p|M\colon M\to M_1 = p(M)$ and $p|N\colon N\to N_1 = p(N)$ are monotone maps. Since M, N are irreducible between H and K respectively, M_1, N_1 are irreducible between $p(H)$ and $p(K)$ respectively [11, p. 192]. Further, M_1 and N_1 are atriodic, hereditarily unicoherent and hereditarily decomposable by [11, p. 171]. Then for some intervals $I_i = [a_i, b_i]$ ($i = 1, 2$ and $a_1<b_1<a_2<b_2$), by [1, Theorem 8] there are monotone maps $f_1\colon M_1\to I_1$ and $f_2\colon N_1\to I_2$. Since $p(H)$ and $p(K)$ are one point sets respectively, we may assume that $f_1p(H) = a_1$, $f_1p(K) = b_1$, $f_2p(H) = a_2$ and $f_2p(K) = b_2$.

Clearly, there is an onto map $k\colon I_1\cup I_2\to S^1$ such that $k(a_1) = k(a_2)\neq k(b_1) = k(b_2)$ and $k|I_i$ ($i = 1, 2$) is injective. Define a map $r\colon X\to S^1$ by

$$r(x)=\begin{cases}kf_1p(x) & \text{if } x\in M,\\ kf_2p(x) & \text{if } x\in N.\end{cases}$$

Then we can easily see that r is monotone, hence by Theorem 3.2 r is refinable. This completes the proof.

PROBLEM. Does each refinable map preserve FANR?

References

[1] R. H. Bing, *Snake-like continua*, Duke Math. J. 18 (1951), pp. 653–663.
[2] K. Borsuk, *A note on shape of quasi-homeomorphic compacta*, Comment. Math. Prac. Math. 14 (1970), pp. 25–34.
[3] — *Theory of Shape*, Warszawa 1975.
[4] — *Theory of Retracts*, Warszawa 1967.
[5] C. E. Burgess, *Chainable continua and indecomposability*, Pacific J. Math. 9 (1959), pp. 653–660.
[6] J. Dydak, *The Whitehead and Smale theorems in shape theory*, Dissertationes Math. 156 (1979).
[7] D. A. Edwards and R. Geoghegan, *Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction*, Ann. of Math. 101 (1975), pp. 521–535.
[8] — — *Stability theorems in shape and pro-homotopy*, Trans. Amer. Math. Soc. 222 (1976), pp. 389–403.
[9] J. Ford and J. W. Rogers. Jr, *Refinable maps*, preprint.
[10] J. Keesling, *On the Whitehead theorem in shape theory*, Fund. Math. 92 (1976), pp. 247–253.
[11] K. Kuratowski, *Topology*, Vol. 2, New York–London–Warszawa 1968.
[12] I. Lončar and S. Mardešić, *A note on inverse sequenses of* ANR*'s*, Grasnik Math. Tom 3 (23), No. 1 (1968), pp. 41–48.
[13] S. Mardešić, *Shapes for topological spaces*, General Top. and Appl. 3 (1973), pp. 265–282.
[14] — *On the Whitehead theorem in shape theory I*, Fund. Math. 91 (1976), pp. 51–64.
[15] S. Mardešić and J. Segal, *ε-mappings onto polyhedra*, Trans. Amer. Math. Soc. 109 (1963), pp. 146–164.
[16] — — *Shapes of compacta and* ANR*-systems*, Fund. Math. 72 (1971), pp. 41–59.
[17] — — *Movable compacta and* ANR*-systems*, Bull. Acad. Polon. Sci. 18 (1970), pp. 649–654.
[18] K. Morita, *On shapes of topological spaces*, Fund. Math. 86 (1975), pp. 251–259.
[19] P. Raymond Patten, *The refinable image of 1-dimensional* ANR, preprint.
[20] E. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.
[21] A. Trybulec, *On shapes of movable curves*, Bull. Acad. Polon. Sci. 21 (1973), pp. 727–733.

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