Refined asymptotic expansions of solutions to fractional diffusion equations

Kazuhiro Ishige and Tatsuki Kawakami

Abstract

In this paper, as an improvement of the paper [K. Ishige, T. Kawakami and H. Michihisa, SIAM J. Math. Anal. 49 (2017) pp. 2167–2190], we obtain the higher order asymptotic expansions of the large time behavior of the solution to the Cauchy problem for inhomogeneous fractional diffusion equations and nonlinear fractional diffusion equations.

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Addresses:

K. I.: Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
E-mail: `ishige@ms.u-tokyo.ac.jp`

T. K.: Applied Mathematics and Informatics Course,
Faculty of Advanced Science and Technology, Ryukoku University,
1-5 Yokotani, Seta Oe-cho, Otsu, Shiga 520-2194, Japan.
E-mail: `kawakami@math.ryukoku.ac.jp`

1 Introduction

This paper is concerned with the large time behavior of a solution to the Cauchy problem for an inhomogeneous fractional diffusion equation

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = f(x,t) \quad \text{in} \quad \mathbb{R}^N \times (0,\infty), \quad u(x,0) = \varphi(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, $\partial_t := \partial/\partial t$, $0 < \theta < 2$ and

$$\varphi \in L^1_K := L^1(\mathbb{R}^N, (1+|x|)^K \, dx) \quad \text{with} \quad K \geq 0.$$

Here $(-\Delta)^{\theta/2}$ is the fractional power of the Laplace operator. Inhomogeneous fractional diffusion equation (1.1) appears in the study of various nonlinear problems with anomalous diffusion, the Laplace equation with a dynamical boundary condition, and so on. Under suitable integrability conditions on the inhomogeneous term f, the solution u to problem (1.1) behaves like a suitable multiple of the fundamental solution G_θ to the linear fractional diffusion equation

$$\partial_t v + (-\Delta)^{\frac{\theta}{2}} v = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,\infty)$$

as $t \to \infty$. In this paper we obtain the higher order asymptotic expansions (HOAE) of the large time behavior of the solution u. Furthermore, we study the precise description of the large time behavior of solutions to the Cauchy problem for nonlinear fractional diffusion equations such as

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = \lambda |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N \times (0,\infty), \quad u(x,0) = \varphi(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $p > 1$ and $\varphi \in L^1_K$ with $K \geq 0$. This paper is an improvement of [15] and it corresponds a fractional version of the papers [11, 12, 14].

The large time behavior of solutions to nonlinear parabolic equations has been studied extensively in many papers by various methods. Here we just refer to the papers [1, 4, 7–15, 17, 21–26], which are closely related to this paper. Among others, in [11, 12, 14], HOAE of solutions behaving like suitable multiples of the Gauss kernel have already been well established. The property that

$$\bigcup_{t>0} e^{t\Delta} L^1_K \subset L^1_K \quad \text{for} \quad K \geq 0$$

plays an important role in [11, 12, 14] and it follows from the exponential decay of the Gauss kernel at the space infinity. For fractional diffusion equations, if $0 \leq K < \theta$, then

$$\bigcup_{t>0} e^{-t(-\Delta)^{\theta/2}} L^1_K \subset L^1_K \tag{1.3}$$

holds and the arguments in [11, 12, 14] are also applicable to fractional diffusion equations. However, if $K \geq \theta$, then property (1.3) fails. This fact prevents to establish analogous asymptotic expansions of solutions to the case of $\theta = 2$. In [15] the authors of this paper and Michihisa studied a mechanism for property (1.3) to fail in the case of $K \ge \theta$, and obtained HOAE of $e^{-(-\Delta)^{\theta/2}}\varphi$. This argument is applicable to the study of HOAE of solutions to inhomogeneous fractional diffusion equations and nonlinear fractional diffusion equations, however HOAE of [15] to problem (1.1) do not have refined forms.

In this paper we improve and refine arguments in [15] by taking into an account of the Taylor expansion of the kernel G_θ with respect to both of the space and the time variables, and obtain HOAE of solutions to inhomogeneous fractional diffusion equations and nonlinear fractional parabolic equations. Our arguments also reveal a mechanism for the solution u to problem (1.1) to break the property that $u(t) \in L^1_K$ for $t > 0$.

We introduce some notations. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $k \ge 0$, let $[k] \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be such that $k - 1 < [k] \le k$. Let $\nabla := (\partial/\partial x_1, \dots, \partial/\partial x_N)$. For any multi-index $\alpha \in \mathbb{M} := \mathbb{N}_0^N$, set

$$|\alpha| := \sum_{i=1}^N \alpha_i, \quad \alpha! := \prod_{i=1}^N \alpha_i!, \quad x^\alpha := \prod_{i=1}^N x_i^{\alpha_i}, \quad \partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

For any $\alpha = (\alpha_1, \dots, \alpha_N)$, $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{M}$, we say $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $i \in \{1, \dots, N\}$. Let $1 \le q \le \infty$ and $K \ge 0$. Let $\|\cdot\|_q$ be the usual norm of $L^q := L^q(\mathbb{R}^N)$. Set

$$|||f|||_{q,K} := \|f_K\|_q \quad \text{with} \quad f_K(x) := |x|^K f(x).$$

Let

$$f \in L^q_K := \left\{ f \in L^q \,:\, \|f\|_{L^q_K} < \infty \right\}, \quad \text{where} \quad \|f\|_{L^q_K} := \|f\|_q + |||f|||_{q,K}.$$

For any $f \in L^1_K$ and $\alpha \in \mathbb{M}$ with $|\alpha| \le K$, set

$$M_\alpha(f) := \int_{\mathbb{R}^N} x^\alpha f(x)\, dx.$$

We are ready to state our main results on the asymptotic expansions of solutions to inhomogeneous fractional diffusion equations. In what follows, set $K_\theta := [K/\theta]$. Furthermore, set

$$g_{\alpha,m}(x,t) := \frac{(-1)^{|\alpha|+m}}{\alpha! m!} (\partial_t^m \partial_x^\alpha G_\theta)(x, t+1)$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$, where $\alpha \in \mathbb{M}$ and $m \in \mathbb{N}_0$.

Theorem 1.1 *Let $N \ge 1$, $0 < \theta < 2$, $0 \le \ell \le K$, and $1 \le q \le \infty$. Let $\varphi \in L^1_K$ and f be a measurable function in $\mathbb{R}^N \times (0,\infty)$ such that*

$$E_{K,q}[f] \in L^1_{\mathrm{loc}}(0,\infty), \tag{1.4}$$

where

$$\begin{aligned} E_{K,q}[f](t) := (t+1)^{\frac{K}{\theta}} &\left[t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)} \|f(t)\|_q + \|f(t)\|_1 \right] \\ &+ t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)} |||f(t)|||_{q,K} + |||f(t)|||_{1,K} \quad \textit{for} \quad t > 0. \end{aligned} \tag{1.5}$$

Let $u \in C(\mathbb{R}^N \times (0,\infty))$ be a solution to problem (1.1), *that is, u satisfies*

$$u(x,t) = \int_{\mathbb{R}^N} G_\theta(x-y,t)\varphi(y)\,dy + \int_0^t \int_{\mathbb{R}^N} G_\theta(x-y,t-s)f(y,s)\,dy\,ds$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Then

$$\sup_{0<t<\tau} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)} |||u(t)-w(t)|||_{q,\ell} < \infty \quad \text{for} \quad \tau > 0, \tag{1.6}$$

where

$$w(x,t) := \sum_{m=0}^{K_\theta} \sum_{|\alpha|\le K} \left\{ M_\alpha(\varphi) + \int_0^t (s+1)^m M_\alpha(f(s))\,ds \right\} g_{\alpha,m}(x,t). \tag{1.7}$$

Furthermore, there exists $C>0$ such that, for any $\varepsilon > 0$ and $T > 0$,

$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t)-w(t)|||_{q,\ell} \le \varepsilon t^{-\frac{K}{\theta}} + Ct^{-\frac{K}{\theta}} \int_T^t E_{K,q}[f](s)\,ds \tag{1.8}$$

holds for large enough $t > 0$. In particular, if

$$\int_0^\infty E_{K,q}[f](s)\,ds < \infty,$$

then

$$\lim_{t\to\infty} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{K-\ell}{\theta}} |||u(t)-w(t)|||_{q,\ell} = 0. \tag{1.9}$$

Theorem 1.1 corresponds to [14, Theorems 1.1, 1.2] for $\theta = 2$ and it is an improvement of [15, Theorem 3.1 (ii)]. Our asymptotic profile w has a pretty simpler form than that of [15]. (See Remarks 3.1 and 5.2.) We also remark that, under condition (1.4), both of $u(\cdot,t)$ and $w(\cdot,t)$ do not necessarily belong to L^q_ℓ, while $u(t)-w(t) \in L^q_\ell$. In other words, the function w may break the property that $u(t) \in L^q_\ell$ for $t > 0$. Furthermore, we have:

Corollary 1.1 *Assume the same conditions as in Theorem* 1.1. *Let $u \in C(\mathbb{R}^N \times (0,\infty))$ be a solution to problem* (1.1). *Then there exists $R > 0$ such that*

$$u(t) \in \left\{ h + \sum_{(\alpha,m)\in\Lambda^q_K} a_{\alpha,m} g_{\alpha,m}(x,t) \,:\, h \in L^q_K \text{ with } \|h\|_{L^q_K} \le R,\, \{a_{\alpha,m}\} \subset [-R,R] \right\}$$

for $t > 0$, where $\Lambda^q_K := \{(\alpha,m) \in \mathbb{M} \times \mathbb{N}_0 \,:\, g_{\alpha,m}(\cdot,0) \notin L^q_K\}$.

We explain the idea of the proof of Theorem 1.1. We improve and refine arguments in the previous papers [11,12,14,15] to obtain HOEA of the solution u to problem (1.1), in particular, the integral term

$$\int_0^t \int_{\mathbb{R}^N} G_\theta(x-y,t-s)f(y,s)\,dy\,ds.$$

In [15], following the arguments in [11, 12, 14], the authors of this paper and Michihisa expanded the integral kernel $G_\theta(x-y, t-s)$ by the Taylor expansions with respect to the space derivatives of $G_\theta(x, t-s)$. Then the slow decay of $G_\theta(x,t)$ makes difficult to obtain refined HOAE of the solution u to problem (1.1). In this paper we expand the integral kernel $G_\theta(x-y, t-s)$ by the Taylor expansions with respect to both of space and time variable derivatives of $G_\theta(x,t)$. (This is the same sprit as in [9].) Indeed, we introduce the following integral kernels by the use of the Taylor expansions of G_θ:

$$
\begin{aligned}
\mathcal{S}_\ell^m(x,y,t) &:= (\partial_t^m G_\theta)(x-y,t) - \sum_{|\alpha|\le \ell} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_t^m \partial_x^\alpha G_\theta)(x,t) y^\alpha \\
&= \frac{1}{[\ell]!} \int_0^1 (1-\tau)^{[\ell]} \frac{\partial^{[\ell]+1}}{\partial \tau^{[\ell]+1}} (\partial_t^m G_\theta)(x-\tau y, t)\, d\tau, \\
\mathcal{T}(x,y,t,s) &:= G_\theta(x-y,t-s) - \sum_{m=0}^{K_\theta} \frac{(-1)^m}{m!} (\partial_t^m G_\theta)(x-y,t) s^m \\
&= \frac{1}{K_\theta!} \int_0^1 (1-\tau)^{K_\theta} \frac{\partial^{K_\theta+1}}{\partial \tau^{K_\theta+1}} G_\theta(x-y, t-\tau s)\, d\tau,
\end{aligned}
\tag{1.10}
$$

for x, $y \in \mathbb{R}^N$ and $0 \le s < t$, where $0 \le \ell \le K$ and $m \in \mathbb{N}_0$. Then

$$
\begin{aligned}
& G_\theta(x-y,t-s) \\
&= \sum_{m=0}^{K_\theta} \frac{(-1)^m}{m!} (\partial_t^m G_\theta)(x-y,t) s^m + \mathcal{T}(x,y,t,s) \\
&= \sum_{m=0}^{K_\theta} \sum_{|\alpha|\le \ell} \frac{(-1)^{|\alpha|+m}}{\alpha! m!} (\partial_t^m \partial_x^\alpha G_\theta)(x,t) y^\alpha s^m + \sum_{m=0}^{K_\theta} \frac{(-1)^m}{m!} \mathcal{S}_\ell^m(x,y,t) s^m + \mathcal{T}(x,y,t,s).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mathcal{R}(x,y,t,s) &:= \mathcal{T}(x,y,t,s) + \sum_{m=0}^{K_\theta} \frac{(-1)^m}{m!} \mathcal{S}_K^m(x,y,t) s^m \\
&= G_\theta(x-y,t-s) - \sum_{m=0}^{K_\theta} \sum_{|\alpha|\le K} \frac{(-1)^{|\alpha|+m}}{\alpha! m!} (\partial_t^m \partial_x^\alpha G_\theta)(x,t) y^\alpha s^m.
\end{aligned}
\tag{1.11}
$$

Then it follows from (1.7) that

$$
\begin{aligned}
u(x,t) - w(x,t) &= \int_{\mathbb{R}^N} \mathcal{R}(x,y,t+1,1) \varphi(y)\, dy \\
&\quad + \int_0^t \int_{\mathbb{R}^N} \mathcal{R}(x,y,t+1,s+1) f(y,s)\, dy\, ds.
\end{aligned}
\tag{1.12}
$$

Thanks to the decay of the derivatives of G_θ and (1.10), we see that

$$\mathcal{T}(x,y,t,s) = O(|x-y|^{-N-K-\varepsilon}) \quad \text{as} \quad |x-y| \to \infty,$$
$$\mathcal{S}_K^m(x,y,t) = O(|x|^{-N-K-\varepsilon}) \quad \text{as} \quad |x| \to \infty,$$

for some $\varepsilon > 0$. These decay of the integral kernels at the space infinity enables us to establish HOAE of solutions to problem (1.1) and to obtain Theorem 1.1. These arguments require delicate integral estimates on the integral kernels $\mathcal{S}_\ell^m$ and $\mathcal{T}$.

Theorem 1.1 is applicable to problem (1.2) and it gives asymptotic profiles of solutions to problem (1.2) as a linear combination of the derivatives of G_θ (see Theorem 5.1). Furthermore, taking a suitable approximation of the nonlinear term in problem (1.2), we obtain refined asymptotic expansions of the solution to problem (1.1) (see Theorem 5.2). Here we state the following result, which is a variation of Theorem 5.2.

Theorem 1.2 *Let $N \geq 1$, $0 < \theta < 2$, $\lambda \in \mathbb{R}$, and $\varphi \in L^1_K \cap L^\infty$ with $K \geq 0$. Let $u \in C(\mathbb{R}^N \times (0,\infty))$ be a solution to problem (1.2) with $p > 1 + \theta/N$ and satisfy*

$$\sup_{t>0}\, (t+1)^{\frac{N}{\theta}} \|u(t)\|_\infty < \infty. \tag{1.13}$$

Then there exists $M_* \in \mathbb{R}$ such that

$$M_* := \lim_{t\to\infty} \int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} \varphi(x)\,dx + \int_0^\infty \int_{\mathbb{R}^N} F(u(x,t))\,dx\,dt,$$

where $F(u(x,t)) := \lambda |u(x,t)|^{p-1} u(x,t)$.

Assume $N(p+\theta) > N + K$ and $\varphi \in L^\infty_k$ with $k = \min\{N+\theta, K\}$. Let $1 \leq q \leq \infty$. Then

$$\sup_{t>0}\, (t+1)^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t)|||_{q,\ell} < \infty,$$

where $0 \leq \ell \leq K$ with $0 \leq \ell < \theta + N(1-1/q)$. Furthermore, for any $\sigma > 0$

$$\sup_{t>0}\, t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t)-v(t)|||_{q,\ell} < \infty,$$
$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t)-v(t)|||_{q,\ell} = o\left(t^{-\frac{K}{\theta}}\right) + O\left(t^{-\frac{K}{\theta}} \int_1^t s^{\frac{K}{\theta}-A_p} h_\sigma(s)\,ds\right) \quad \text{as} \quad t \to \infty,$$

where $0 \leq \ell \leq K$. Here

$$v(x,t) := \sum_{m=0}^{K_\theta} \sum_{|\alpha|\leq K} c_{\alpha,m}(t) g_{\alpha,m}(x,t) + \int_0^t e^{-(t-s)(-\Delta)^{\frac{\theta}{2}}} F_\infty(s)\,ds,$$
$$c_{\alpha,m}(t) := M_\alpha(\varphi) + \int_0^t (s+1)^m M_\alpha(F(u(s)) - F_\infty(s))\,ds,$$
$$F_\infty(x,t) := F\left(M_* G_\theta(x,t+1)\right), \quad h_\sigma(t) := t^{-(A_p-1)+\sigma} + t^{-1} + t^{-\frac{1}{\theta}}.$$

Theorem 1.2 corresponds to [12, Corollary 1.1] for $\theta = 2$. See Remark 5.1 for condition (1.13).

The rest of this paper is organized as follows. In Section 2 we collect some properties of the fundamental solution G_θ. In Section 3 we obtain some estimates on the integral kernels $\mathcal{S}_\ell^m(x,y,t)$ and $\mathcal{T}(x,y,t,s)$, and prove Theorem 1.1 and Corollary 1.1. In Section 4 we apply Theorem 1.1 to obtain HOAE of solutions to the Cauchy problem for a convection type inhomogeneous fractional diffusion equation. In Section 5 we apply Theorem 1.1 to study HOAE of solutions to the Cauchy problem for nonlinear fractional diffusion equations. Furthermore, we prove Theorem 1.2.

2 Preliminaries

We recall some properties of the fundamental solution $G_\theta = G_\theta(x,t)$, In what follows, by the letter C we denote generic positive constants (independent of x and t) and they may have different values also within the same line.

Let $0 < \theta < 2$. The fundamental solution $G_\theta = G_\theta(x,t)$ is represented by

$$G_\theta(x,t) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{ix\cdot\xi} e^{-t|\xi|^\theta}\, d\xi, \quad (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Then we have:

(**G**) $G_\theta = G_\theta(x,t)$ is a positive smooth function in $\mathbb{R}^N \times (0,\infty)$ with the following properties:

- (i) $G_\theta(x,t) = t^{-\frac{N}{\theta}} G_\theta(t^{-\frac{1}{\theta}}x, 1)$ for $x \in \mathbb{R}^N$ and $t > 0$;
- (ii) $\sup_{x\in\mathbb{R}^N} (1+|x|)^{N+\theta+|\alpha|} |(\partial_x^\alpha G_\theta)(x,1)| < \infty$ for $\alpha \in \mathbb{M}$;
- (iii) $G_\theta(\cdot,1)$ is radially symmetric and decreasing with respect to $r := |x|$. Furthermore,
$$\liminf_{|x|\to+\infty} (1+|x|)^{N+\theta+j} (\partial_r^j G_\theta)(x,1) > 0, \quad j \in \mathbb{N}_0;$$
- (iv) $G_\theta(x,t) = \displaystyle\int_{\mathbb{R}^N} G_\theta(x-y,t-s)\, G_\theta(y,s)\, dy$ for $x \in \mathbb{R}^N$ and $t > s > 0$;
- (v) $\displaystyle\int_{\mathbb{R}^N} G_\theta(x,t)\, dx = 1$ for $t > 0$.

See [3,4]. (See also [13,14,19].)

Let $\alpha \in \mathbb{M}$ and $m \in \mathbb{N}_0$. Let

$$1 \le q \le \infty, \quad 0 \le \ell < \theta m' + |\alpha| + N\left(1 - \frac{1}{q}\right) \quad \text{with} \quad m' := \max\{m, 1\}.$$

It follows from (**G**)-(i), (ii) and [15, Lemma 2.1] that

$$|(\partial_t^m \partial_x^\alpha G_\theta)(x,t)| \le C t^{-\frac{N+|\alpha|}{\theta} - m} \big(1 + t^{-\frac{1}{\theta}}|x|\big)^{-(N+\theta m'+|\alpha|)} \tag{2.1}$$

for $x \in \mathbb{R}^N$ and $t > 0$. This implies that

$$\sup_{t>0} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{|\alpha|-\ell}{\theta}+m} |||(\partial_t^m \partial_x^\alpha G_\theta)(t)|||_{q,\ell} < \infty. \tag{2.2}$$

Lemma 2.1 *Let $1 \le q \le r \le \infty$, $\alpha \in \mathbb{M}$, and $m \in \mathbb{N}_0$. Let*

$$0 \le \ell < \theta m' + |\alpha| + N\left(\frac{1}{q} - \frac{1}{r}\right). \tag{2.3}$$

Then there exists $C > 0$ such that

$$t^{\frac{N}{\theta}\left(\frac{1}{q}-\frac{1}{r}\right)+\frac{|\alpha|}{\theta}+m} \left|\left|\left| \partial_t^m \partial_x^\alpha e^{-t(-\Delta)^{\theta/2}} \varphi \right|\right|\right|_{r,\ell} \le C t^{\frac{\ell}{\theta}} \|\varphi\|_q + C|||\varphi|||_{q,\ell}$$

for $\varphi \in L^q_\ell$ and $t > 0$. Here

$$\left[e^{-t(-\Delta)^{\theta/2}} \varphi\right](x) := \int_{\mathbb{R}^N} G_\theta(x-y,t)\varphi(y)\,dy, \quad (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Proof. Assume (2.3). It follows that

$$|x|^\ell \left[\partial_t^m \partial_x^\alpha e^{-t(-\Delta)^{\theta/2}} \varphi\right](x) \le C \int_{\mathbb{R}^N} \left[|x-y|^\ell + |y|^\ell\right] |(\partial_t^m \partial_x^\alpha G_\theta)(x-y,t)||\varphi(y)|\,dy$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$. The Young inequality together with (2.2) implies that

$$\begin{aligned}
&\left|\left|\left| \partial_t^m \partial_x^\alpha e^{-t(-\Delta)^{\theta/2}} \varphi \right|\right|\right|_{r,\ell} \\
&\le C|||\partial_t^m \partial_x^\alpha G_\theta(t)|||_{p,\ell} \|\varphi\|_q + C\|\partial_t^m \partial_x^\alpha G_\theta(t)\|_p |||\varphi|||_{q,\ell} \\
&\le C t^{-\frac{N}{\theta}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{|\alpha|}{\theta}-m+\frac{\ell}{\theta}} \|\varphi\|_q + C t^{-\frac{N}{\theta}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{|\alpha|}{\theta}-m} |||\varphi|||_{q,\ell}
\end{aligned}$$

for $t > 0$, where $p \in [1,\infty]$ with $1/r = 1/p + 1/q - 1$. Then we obtain the desired inequality, and the proof is complete. $\Box$

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first prepare the following lemma.

Lemma 3.1 *Assume the same conditions as in Theorem 1.1. Then*

$$t^{\frac{N}{\theta}\left(1-\frac{1}{r}\right)} (t+1)^{\frac{K-\ell}{\theta}} |||f(t)|||_{r,\ell} \le E_{K,q}[f](t), \quad t > 0,$$

where $0 \le \ell \le K$ and $1 \le r \le q$.

Proof. Let $0 \le \ell \le K$ and $1 \le q \le \infty$. It follows that

$$(t+1)^{-\frac{\ell}{\theta}}|x|^\ell \le C + C(t+1)^{-\frac{K}{\theta}}|x|^K, \quad (x,t) \in \mathbb{R}^N \times (0,\infty).$$

This together with (1.5) implies that

$$\begin{aligned}(t+1)^{-\frac{\ell}{\theta}}|||f(t)|||_{r,\ell} &\le C\|f(t)\|_r + C(t+1)^{-\frac{K}{\theta}}|||f(t)|||_{r,K} \\ &\le C\|f(t)\|_1^\lambda \|f(t)\|_q^{1-\lambda} + C(t+1)^{-\frac{K}{\theta}}|||f(t)|||_{1,K}^\lambda |||f(t)|||_{q,K}^{1-\lambda} \\ &\le Ct^{-\frac{N}{\theta}(1-\frac{1}{r})}(t+1)^{-\frac{K}{\theta}}E_{K,q}[f](t), \quad t>0,\end{aligned}$$

where $1/r = \lambda + (1-\lambda)/q$. Thus Lemma 3.1 follows. $\square$

Next we prove a lemma on the integral kernel $\mathcal{S}_\ell^m(x,y,t)$.

Lemma 3.2 *Let $m \in \mathbb{N}_0$, $0 \le \ell \le K$, $1 \le q \le \infty$, and $j = 0,1$.*

(a) *There exists $C_1 > 0$ such that*

$$|||\nabla^j \mathcal{S}_\ell^m(\cdot,y,t)|||_{q,\ell} \le C_1 t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-m-\frac{j}{\theta}}|y|^\ell, \quad (y,t) \in \mathbb{R}^N \times (0,\infty).$$

(b) *There exists $C_2 > 0$ such that*

$$\left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{S}_K^m(\cdot,y,t)\varphi(y)\,dy \right|\right|\right|_{q,\ell} \le C_2 t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-m-\frac{K+j-\ell}{\theta}}|||\varphi|||_{1,K}, \quad t>0,$$

for $\varphi \in L^1_K$.

(c) *Let $\varphi \in L^1_K$. Then*

$$\lim_{t\to\infty} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+m+\frac{K+j-\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{S}_K^m(\cdot,y,t)\varphi(y)\,dy \right|\right|\right|_{q,\ell} = 0.$$

Proof. Let $0 \le \ell \le K$, $1 \le q \le \infty$, and $j = 0,1$. We prove assertion (a). Let x, $y \in \mathbb{R}^N$ and $t > 0$. It follows that

$$|x-\tau y| \ge |x| - |y| \ge |x|/2 \quad \text{if} \quad |x| \ge 2|y| \quad \text{and} \quad 0 \le \tau \le 1.$$

Then, by (1.10) and (2.1) we have

$$\begin{aligned}&|x|^\ell |\nabla^j \mathcal{S}_\ell^m(x,y,t)| \\ &\le C\int_0^1 |x|^\ell \left|(\partial_t^m \nabla^{[\ell]+j+1}G_\theta)(x-\tau y,t)\right| |y|^{[\ell]+1}\,d\tau \\ &\le C|y|^\ell \int_0^1 |x|^{[\ell]+1}\, t^{-\frac{N}{\theta}-\frac{[\ell]+j+1}{\theta}-m}\left(1+t^{-\frac{1}{\theta}}|x-\tau y|\right)^{-(N+\theta m'+[\ell]+j+1)}\,d\tau \\ &\le C|y|^\ell (t^{-\frac{1}{\theta}}|x|)^{[\ell]+1}\, t^{-\frac{N}{\theta}-m-\frac{j}{\theta}}\left(1+t^{-\frac{1}{\theta}}\frac{|x|}{2}\right)^{-(N+\theta m'+[\ell]+j+1)} \\ &\le C|y|^\ell t^{-\frac{N}{\theta}-m-\frac{j}{\theta}}\left(1+t^{-\frac{1}{\theta}}\frac{|x|}{2}\right)^{-(N+\theta m'+j)}\end{aligned}$$

if $|x| \geq 2|y|$. Similarly, by (1.10) we have

$$
\begin{aligned}
|x|^\ell |\nabla^j \mathcal{S}_\ell^m(x,y,t)| &\leq |x|^\ell |(\partial_t^m \nabla^j G_\theta)(x-y,t)| + C \sum_{|\alpha|\leq \ell} |x|^\ell |(\partial_t^m \partial_x^\alpha \nabla^j G_\theta)(x,t)||y|^{|\alpha|} \\
&\leq (2|y|)^\ell |(\partial_t^m \nabla^j G_\theta)(x-y,t)| + C \sum_{|\alpha|\leq \ell} |y|^\ell |x|^{|\alpha|} |(\partial_t^m \partial_x^\alpha \nabla^j G_\theta)(x,t)|
\end{aligned}
$$

if $|x| < 2|y|$. These together with (2.2) imply that

$$
|||\nabla^j \mathcal{S}_\ell^m(\cdot,y,t)|||_{q,\ell} \leq C t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-m-\frac{j}{\theta}} |y|^\ell.
$$

Thus assertion (a) follows.

We prove assertions (b) and (c). Let $\varphi \in L_K^1$, $0 \leq \ell \leq K$, and $R > 0$. It follows from (1.10) that

$$
\begin{aligned}
&\left|\left|\left| \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |\nabla^j \mathcal{S}_K^m(\cdot,y,t)||\varphi(y)|\,dy \right|\right|\right|_{q,\ell} \\
&\leq \left|\left|\left| \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |\nabla^j \mathcal{S}_\ell^m(\cdot,y,t)||\varphi(y)|\,dy \right|\right|\right|_{q,\ell} \\
&\qquad + C \sum_{\ell<|\alpha|\leq K} \left|\left|\left| \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |(\partial_t^m \partial_x^\alpha \nabla^j G_\theta)(\cdot,t)||y|^{|\alpha|}|\varphi(y)|\,dy \right|\right|\right|_{q,\ell} \\
&\leq \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |||\nabla^j \mathcal{S}_\ell^m(\cdot,y,t)|||_{q,\ell} |\varphi(y)|\,dy \\
&\qquad + C \sum_{\ell<|\alpha|\leq K} \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |||(\partial_t^m \partial_x^\alpha \nabla^j G_\theta)(\cdot,t)|||_{q,\ell} |y|^{|\alpha|}|\varphi(y)|\,dy.
\end{aligned}
$$

This together with (2.2) and assertion (a) implies that

$$
\begin{aligned}
&t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+m+\frac{j}{\theta}} \left|\left|\left| \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |\nabla^j \mathcal{S}_K^m(\cdot,y,t)||\varphi(y)|\,dy \right|\right|\right|_{q,\ell} \\
&\leq C\left(\int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |y|^\ell |\varphi(y)|\,dy + \sum_{\ell<|\alpha|\leq K} t^{-\frac{|\alpha|-\ell}{\theta}} \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |y|^{|\alpha|}|\varphi(y)|\,dy \right) \\
&\leq C\left(\int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |y|^\ell \left(\frac{|y|}{R^{\frac{1}{\theta}}}\right)^{K-\ell} |\varphi(y)|\,dy \right. \\
&\qquad\qquad \left. + \sum_{\ell<|\alpha|\leq K} t^{-\frac{|\alpha|-\ell}{\theta}} \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} \left(\frac{|y|}{R^{\frac{1}{\theta}}}\right)^{K-|\alpha|} |y|^{|\alpha|}|\varphi(y)|\,dy \right) \\
&= C t^{-\frac{K-\ell}{\theta}} \left((R^{-1}t)^{\frac{K-\ell}{\theta}} + \sum_{\ell<|\alpha|\leq K} (R^{-1}t)^{\frac{K-|\alpha|}{\theta}} \right) \int_{\{|y|\geq R^{\frac{1}{\theta}}\}} |y|^K |\varphi(y)|\,dy.
\end{aligned}
\tag{3.1}
$$

Similarly, by (1.10) we have

$$
\begin{aligned}
&\left|\left|\left| \int_{\{|y|<R^{\frac{1}{\theta}}\}} |\nabla^j \mathcal{S}_K^m(\cdot,y,t)||\varphi(y)|\,dy \right|\right|\right|_{q,\ell} \\
&\le C \left|\left|\left| \int_{\{|y|<R^{\frac{1}{\theta}}\}} \int_0^1 |(\partial_t^m \nabla^{[K]+j+1} G_\theta)(\cdot-\tau y,t)||y|^{[K]+1}|\varphi(y)|\,d\tau\,dy \right|\right|\right|_{q,\ell} \\
&\le C \int_{\{|y|<R^{\frac{1}{\theta}}\}} \int_0^1 |||(\partial_t^m \nabla^{[K]+j+1} G_\theta)(\cdot-\tau y,t)|||_{q,\ell} |y|^{[K]+1}|\varphi(y)|\,d\tau\,dy.
\end{aligned} \tag{3.2}
$$

On the other hand, it follows that

$$
\begin{aligned}
|x|^\ell |(\partial_t^m \nabla^{[K]+j+1} G_\theta)(x-\tau y,t)| &= |z+\tau y|^\ell |(\partial_t^m \nabla^{[K]+j+1} G_\theta)(z,t)| \\
&\le C(|z|^\ell + |y|^\ell)|(\partial_t^m \nabla^{[K]+j+1} G_\theta)(z,t)|
\end{aligned}
$$

for x, $y \in \mathbb{R}^N$, $t>0$, and $\tau \in (0,1)$, where $z := x-\tau y$. This together with (2.2) implies that

$$
\begin{aligned}
&|||(\partial_t^m \nabla^{[K]+j+1} G_\theta)(\cdot-\tau y,t)|||_{q,\ell} \\
&\le |||(\partial_t^m \nabla^{[K]+j+1} G_\theta)(t)|||_{q,\ell} + |y|^\ell |||(\partial_t^m \nabla^{[K]+j+1} G_\theta)(t)|||_q \\
&\le Ct^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-m-\frac{[K]+j+1-\ell}{\theta}} + Ct^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-m-\frac{[K]+j+1}{\theta}}|y|^\ell
\end{aligned} \tag{3.3}
$$

for $y \in \mathbb{R}^N$, $t>0$, and $\tau \in (0,1)$. By (3.2) and (3.3) we obtain

$$
\begin{aligned}
&t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+m+\frac{j}{\theta}} \left|\left|\left| \int_{\{|y|<R^{\frac{1}{\theta}}\}} |\nabla^j \mathcal{S}_K^m(\cdot,y,t)||\varphi(y)|\,dy \right|\right|\right|_{q,\ell} \\
&\le C \int_{\{|y|<R^{\frac{1}{\theta}}\}} (t^{-\frac{[K]+1-\ell}{\theta}} + t^{-\frac{[K]+1}{\theta}}|y|^\ell)|y|^{[K]+1}|\varphi(y)|\,dy \\
&\le C\left(t^{-\frac{[K]+1-\ell}{\theta}} R^{\frac{[K]+1-K}{\theta}} + t^{-\frac{[K]+1}{\theta}} R^{\frac{[K]+\ell+1-K}{\theta}}\right) \int_{\{|y|<R^{\frac{1}{\theta}}\}} |y|^K |\varphi(y)|\,dy.
\end{aligned} \tag{3.4}
$$

Combining (3.1) and (3.4) and setting $R=t$, we obtain

$$
t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+m+\frac{j}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{S}_K^m(\cdot,y,t)\varphi(y)\,dy \right|\right|\right|_{q,\ell} \le Ct^{-\frac{K-\ell}{\theta}} |||\varphi|||_{1,K}, \quad t>0,
$$

which implies assertion (b). Similarly, setting $R=\varepsilon t$ with $0<\varepsilon\le 1$, we have

$$
\begin{aligned}
&t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+m+\frac{j}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{S}_K^m(\cdot,y,t)\varphi(y)\,dy \right|\right|\right|_{q,\ell} \\
&\le Ct^{-\frac{K-\ell}{\theta}} \left((\varepsilon^{-1})^{\frac{K-\ell}{\theta}} + \sum_{\ell<|\alpha|\le K} (\varepsilon^{-1})^{\frac{K-|\alpha|}{\theta}} \right) \int_{\{|y|\ge(\varepsilon t)^{\frac{1}{\theta}}\}} |y|^K|\varphi(y)|\,dy \\
&\qquad + Ct^{-\frac{K-\ell}{\theta}} \left(\varepsilon^{\frac{[K]+1-K}{\theta}} + \varepsilon^{\frac{[K]+\ell+1-K}{\theta}} \right) |||\varphi|||_{1,K}.
\end{aligned}
$$

This together with $\varphi \in L^1_K$ implies that

$$\limsup_{t\to\infty} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+m+\frac{K+j-\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{S}_K^m(\cdot,y,t)\varphi(y)\,dy \right|\right|\right|_{q,\ell} \le C\left(\varepsilon^{\frac{[K]+1-K}{\theta}} + \varepsilon^{\frac{[K]+\ell+1-K}{\theta}}\right) |||\varphi|||_{1,K}.$$

Since ε is arbitrary, we obtain assertion (c). Thus Lemma 3.2 follows. $\Box$

By Lemmata 3.1 and 3.2 we have:

Lemma 3.3 *Let f be a measurable function in $\mathbb{R}^N \times (0,\infty)$. Assume (1.5) for some $K \ge 0$ and $1 \le q \le \infty$. Let $0 \le \ell \le K$, $m \in \mathbb{N}_0$, and $j = 0,1$. Then there exists $C > 0$ such that*

$$\begin{aligned} &(t+1)^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{K+j-\ell}{\theta}} \left|\left|\left| \int_T^t \int_{\mathbb{R}^N} (s+1)^m \nabla^j \mathcal{S}_K^m(\cdot,y,t+1) f(y,s)\,dy\,ds \right|\right|\right|_{q,\ell} \\ &\le C \int_T^t E_{K,q}[f](s)\,ds \end{aligned} \tag{3.5}$$

for $t > T \ge 0$. Furthermore,

$$\lim_{t\to\infty} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{K+j-\ell}{\theta}} \left|\left|\left| \int_0^T \int_{\mathbb{R}^N} (s+1)^m \nabla^j \mathcal{S}_K^m(\cdot,y,t+1) f(y,s)\,dy\,ds \right|\right|\right|_{q,\ell} = 0$$

for $T > 0$.

Proof. It follows from Lemma 3.1 that

$$|||f(s)|||_{1,K} \le C E_{K,q}[f](s), \quad s > 0. \tag{3.6}$$

This together with Lemma 3.2 (c) implies that, for any $T > 0$,

$$\lim_{t\to\infty} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{K+j-\ell}{\theta}} (s+1)^m \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{S}_K^m(\cdot,y,t+1) f(y,s)\,dy \right|\right|\right|_{q,\ell} = 0 \tag{3.7}$$

for $0 < s < T$. Furthermore, by (3.6) with Lemma 3.2 (b) we see that

$$\begin{aligned} &(t+1)^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{K+j-\ell}{\theta}} (s+1)^m \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{S}_K^m(\cdot,y,t+1) f(y,s)\,dy \right|\right|\right|_{q,\ell} \\ &\le C(t+1)^{-m}(s+1)^m \|f(s)\|_{1,K} \le C E_{K,q}[f](s) \end{aligned} \tag{3.8}$$

for $0 < s < t$. Inequality (3.8) implies (3.5). Furthermore, by (3.7) and (3.8) we apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{t\to\infty} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{K+j-\ell}{\theta}} \left|\left|\left| \int_0^T \int_{\mathbb{R}^N} (s+1)^m \nabla^j \mathcal{S}_K^m(\cdot,y,t+1) f(y,s)\,dy\,ds \right|\right|\right|_{q,\ell} = 0.$$

Thus Lemma 3.3 follows. $\Box$

Next we prove the following lemma on the integral kernel $\mathcal{T}(x,y,t,s)$.

Lemma 3.4 *Let $1 \le q \le \infty$, and $0 \le \ell \le K$.*

(a) *Let $\varphi \in L^1_K$ with $K \ge 0$ and $j = 0, 1$. Then there exists $C_1 > 0$ such that*

$$t^{\frac{N}{q}\left(1-\frac{1}{q}\right)+\frac{j}{\theta}}(t+1)^{\frac{K-\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, 1)\varphi(y)\, dy \right|\right|\right|_{q,\ell} \le C_1 \|\varphi\|_{L^1_K} \tag{3.9}$$

for $t > 0$. Furthermore,

$$\lim_{t\to\infty} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{K+j-\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, 1)\varphi(y)\, dy \right|\right|\right|_{q,\ell} = 0. \tag{3.10}$$

(b) *Let f be a measurable function in $\mathbb{R}^N \times (0,\infty)$ and satisfy (1.5). Let $j = 0$ if $0 < \theta \le 1$ and $j \in \{0, 1\}$ if $1 \le \theta < 2$. Then there exists $C_2 > 0$ such that*

$$\begin{aligned} &t^{\frac{N}{q}\left(1-\frac{1}{q}\right)}(t+1)^{\frac{K-\ell}{\theta}} \left|\left|\left| \int_T^t \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy\, ds \right|\right|\right|_{q,\ell} \\ &\le C_2 \int_T^t (t-s)^{-\frac{j}{\theta}} E_{K,q}[f](s)\, ds \end{aligned} \tag{3.11}$$

for $t > T \ge 0$. Furthermore,

$$\lim_{t\to\infty} t^{\frac{N}{q}\left(1-\frac{1}{q}\right)+\frac{K-\ell}{\theta}} \left|\left|\left| \int_0^T \int_{\mathbb{R}^N} \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy\, ds \right|\right|\right|_{q,\ell} = 0 \tag{3.12}$$

for $T > 0$.

Proof. Let $0 \le \ell \le K$ and $j = 0, 1$. We find $\ell' > 0$ such that

$$\ell \le \ell', \qquad \theta K_\theta < \ell' < \theta(K_\theta + 1). \tag{3.13}$$

Let x, $y \in \mathbb{R}^N$ and $t > 0$. It follows that

$$t^{-\frac{\ell}{\theta}}|x|^\ell \le t^{-\frac{\ell}{\theta}}(|x-y|^\ell + |y|^\ell) \le C\left(1 + t^{-\frac{\ell'}{\theta}}|x-y|^{\ell'} + t^{-\frac{K}{\theta}}|y|^K\right). \tag{3.14}$$

This together with (1.10) implies that

$$\begin{aligned} &t^{-\frac{\ell}{\theta}}|x|^\ell |\nabla^j \mathcal{T}(x,y,t,s)| \\ &\le C t^{-\frac{\ell'}{\theta}} s^{K_\theta+1} \int_0^1 |x-y|^{\ell'} |(\partial_t^{K_\theta+1}\nabla^j G_\theta)(x-y, t-\tau s)|\, d\tau \\ &\quad + C\left(1 + t^{-\frac{K}{\theta}}|y|^K\right)\left[|(\nabla^j G_\theta)(x-y,t-s)| + \sum_{m=0}^{K_\theta} s^m |(\partial_t^m \nabla^j G_\theta)(x-y,t)|\right] \end{aligned} \tag{3.15}$$

for $0 < s < t$. Let $\psi \in L^{r_1}_K \cap L^{r_2}_K$ with $1 \le r_1, r_2 \le q$. Let $1 \le r_i' \le \infty$ $(i = 1, 2)$ be such that

$$\frac{1}{q} = \frac{1}{r_i} + \frac{1}{r_i'} - 1.$$

Then we observe from the Young inequality, (2.2) and (3.15) that

$$
\begin{aligned}
&t^{-\frac{\ell}{\theta}}\left|\!\left|\!\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot,y,t,s)\psi(y)\,dy \right|\!\right|\!\right|_{q,\ell} \\
&\le Ct^{-\frac{\ell'}{\theta}}s^{K_\theta+1}\int_0^1 |||\partial_t^{K_\theta+1}\nabla^j G_\theta(t-\tau s)|||_{r_1',\ell'}\|\psi\|_{r_1}\,d\tau \\
&\qquad + C\|\nabla^j G_\theta(t-s)\|_{r_1'}\|\varphi\|_{r_1} + Ct^{-\frac{K}{\theta}}\|\nabla^j G_\theta(t-s)\|_{r_1'}|||\varphi|||_{r_1,K} \\
&\qquad\quad + C\sum_{m=0}^{K_\theta} s^m\left[\|\partial_t^m\nabla^j G_\theta(t)\|_{r_2'}\|\varphi\|_{r_2} + t^{-\frac{K}{\theta}}\|\partial_t^m\nabla^j G_\theta(t)\|_{r_2'}|||\varphi|||_{r_2,K}\right] \\
&\le Ct^{-\frac{\ell'}{\theta}}s^{K_\theta+1}\|\psi\|_{r_1}\int_0^1 (t-\tau s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-(K_\theta+1)+\frac{\ell'-j}{\theta}}\,d\tau \\
&\qquad + C(t-s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-\frac{j}{\theta}}(\|\psi\|_{r_1}+t^{-\frac{K}{\theta}}|||\psi|||_{r_1,K}) \\
&\qquad\quad + Ct^{-\frac{N}{\theta}\left(\frac{1}{r_2}-\frac{1}{q}\right)-\frac{j}{\theta}}\left(1+\sum_{m=0}^{K_\theta}s^m t^{-m}\right)(\|\psi\|_{r_2}+t^{-\frac{K}{\theta}}|||\psi|||_{r_2,K})
\end{aligned}
\tag{3.16}
$$

for $t/2 < s < t$. On the other hand, it follows from (3.13) that

$$
\begin{aligned}
\int_0^1 (t-\tau s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-(K_\theta+1)+\frac{\ell'-j}{\theta}}\,d\tau &\le (t-s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)}\int_0^1 (t-\tau s)^{-(K_\theta+1)+\frac{\ell'-j}{\theta}}\,d\tau \\
&\le C(t-s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-\frac{j}{\theta}}s^{-1}t^{-K_\theta+\frac{\ell'}{\theta}}
\end{aligned}
$$

for $0 < s < t$. This together with (3.16) implies that

$$
\begin{aligned}
&t^{-\frac{\ell}{\theta}}\left|\!\left|\!\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot,y,t,s)\psi(y)\,dy \right|\!\right|\!\right|_{q,\ell} \\
&\le C(t-s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-\frac{j}{\theta}}(\|\psi\|_{r_1}+t^{-\frac{K}{\theta}}|||\psi|||_{r_1,K}) \\
&\qquad + Ct^{-\frac{N}{\theta}\left(\frac{1}{r_2}-\frac{1}{q}\right)}(t-s)^{-\frac{j}{\theta}}(\|\psi\|_{r_2}+t^{-\frac{K}{\theta}}|||\psi|||_{r_2,K})
\end{aligned}
\tag{3.17}
$$

for $t/2 < s < t$. Similarly, by (1.10) and (3.14) we have

$$
\begin{aligned}
t^{-\frac{\ell}{\theta}}|x|^\ell|\nabla^j\mathcal{T}(x,y,t,s)| &\le Cs^{K_\theta+1}\left(1+t^{-\frac{\ell'}{\theta}}|x-y|^{\ell'}+t^{-\frac{K}{\theta}}|y|^K\right) \\
&\qquad \times\int_0^1 |(\partial_t^{K_\theta+1}\nabla^j G_\theta)(x-y,t-\tau s)|\,d\tau
\end{aligned}
\tag{3.18}
$$

for $0 < s < t$. It follows from the Young inequality, (2.2), and (3.18) that

$$
\begin{aligned}
& t^{-\frac{\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t, s) \psi(y)\, dy \right|\right|\right|_{q,\ell} \\
& \le C s^{K_\theta+1} \int_0^1 \|(\partial_t^{K_\theta+1} \nabla^j G_\theta)(t-\tau s)\|_{r_1'} \|\psi\|_{r_1}\, d\tau \\
& \qquad + C s^{K_\theta+1} t^{-\frac{\ell'}{\theta}} \int_0^1 |||(\partial_t^{K_\theta+1} \nabla^j G_\theta)(t-\tau s)|||_{r_1',\ell'} \|\psi\|_{r_1}\, d\tau \\
& \qquad\quad + C s^{K_\theta+1} t^{-\frac{K}{\theta}} \int_0^1 \|(\partial_t^{K_\theta+1} \nabla^j G_\theta)(t-\tau s)\|_{r_1'} |||\psi|||_{r_1,K}\, d\tau \\
& \le C s^{K_\theta+1} \|\psi\|_{r_1} \int_0^1 (t-\tau s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-(K_\theta+1)-\frac{j}{\theta}}\, d\tau \\
& \qquad + C s^{K_\theta+1} t^{-\frac{\ell'}{\theta}} \|\psi\|_{r_1} \int_0^1 (t-\tau s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-(K_\theta+1)+\frac{\ell'-j}{\theta}}\, d\tau \\
& \qquad\quad + C s^{K_\theta+1} t^{-\frac{K}{\theta}} |||\psi|||_{r_1,K} \int_0^1 (t-\tau s)^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-(K_\theta+1)-\frac{j}{\theta}}\, d\tau \\
& \le C s^{K_\theta+1} t^{-\frac{N}{\theta}\left(\frac{1}{r_1}-\frac{1}{q}\right)-(K_\theta+1)} (t-s)^{-\frac{j}{\theta}} \left\{ \|\psi\|_{r_1} + t^{-\frac{K}{\theta}} |||\psi|||_{r_1,K} \right\}
\end{aligned}
\tag{3.19}
$$

for $0 < s \le t/2$.

We prove assertion (a). Since $(t+1)/2 \le 1$ for $0 < t \le 1$, by (3.17) with $\psi = \varphi$ and $r_1 = r_2 = 1$ we have

$$
\begin{aligned}
& (t+1)^{-\frac{\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, 1) \varphi(y)\, dy \right|\right|\right|_{q,\ell} \\
& \le C t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{j}{\theta}} (\|\varphi\|_1 + (t+1)^{-\frac{K}{\theta}} |||\varphi|||_{1,K}) \\
& \le C t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{j}{\theta}} \|\varphi\|_{L^1_K}
\end{aligned}
\tag{3.20}
$$

for $0 < t \le 1$. On the other hand, since $(t+1)/2 > 1$ for $t > 1$ by (3.19) with $\psi = \varphi$ and $r_1 = 1$ we see that

$$
\begin{aligned}
& (t+1)^{-\frac{\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, 1) \varphi(y)\, dy \right|\right|\right|_{q,\ell} \\
& \le C (t+1)^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-(K_\theta+1)} t^{-\frac{j}{\theta}} \left\{ \|\varphi\|_1 + (t+1)^{-\frac{K}{\theta}} |||\varphi|||_{1,K} \right\} \\
& \le C t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{j}{\theta}} (t+1)^{-(K_\theta+1)} \|\varphi\|_{L^1_K}
\end{aligned}
$$

for $t > 1$. This together with $\theta(K_\theta + 1) > K$ implies (3.10) and

$$
t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{j}{\theta}} (t+1)^{\frac{K-\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \mathcal{T}(\cdot, y, t+1, 1) \varphi(y)\, dy \right|\right|\right|_{q,\ell} \le C \|\varphi\|_{L^1_K}
\tag{3.21}
$$

for $t > 1$. Combining (3.20) and (3.21), we obtain (3.9) and (3.10). Thus assertion (a) follows.

We prove assertion (b). Since $(t+1)/2 < s+1$ for $t/2 < s < t$, by (3.13) and (3.17) with $\psi = f(s)$ and $(r_1, r_2) = (q, 1)$ we have

$$
\begin{aligned}
&(t+1)^{-\frac{\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy \right|\right|\right|_{q,\ell} \\
&\le C(t-s)^{-\frac{j}{\theta}} (\|f(s)\|_q + (t+1)^{-\frac{K}{\theta}} |||f(s)|||_{q,K}) \\
&\qquad + Ct^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} (t-s)^{-\frac{j}{\theta}} (\|f(s)\|_1 + (t+1)^{-\frac{K}{\theta}} |||f(s)|||_{1,K})
\end{aligned}
$$

for $t/2 < s < t$. This together with Lemma 3.1 implies that

$$
\begin{aligned}
&(t+1)^{-\frac{\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy \right|\right|\right|_{q,\ell} \\
&\le C \left(s^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} + t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} \right) \left((s+1)^{-\frac{K}{\theta}} + (t+1)^{-\frac{K}{\theta}} \right) (t-s)^{-\frac{j}{\theta}} E_{K,q}[f](s) \\
&\le Ct^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} (t+1)^{-\frac{K}{\theta}} (t-s)^{-\frac{j}{\theta}} E_{K,q}[f](s)
\end{aligned}
\tag{3.22}
$$

for $t/2 < s < t$. On the other hand, by (3.13) and (3.17) with $\psi = f(s)$ and $r_1 = r_2 = 1$ we have

$$
\begin{aligned}
&(t+1)^{-\frac{\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy \right|\right|\right|_{q,\ell} \\
&\le Ct^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} (t-s)^{-\frac{j}{\theta}} (\|f(s)\|_1 + (t+1)^{-\frac{K}{\theta}} |||f(s)|||_{1,K})
\end{aligned}
$$

for $0 < s \le t/2$ with $(t+1)/2 < s+1$. This together with Lemma 3.1 implies that

$$
\begin{aligned}
&(t+1)^{-\frac{\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy \right|\right|\right|_{q,\ell} \\
&\le Ct^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} \left((s+1)^{-\frac{K}{\theta}} + (t+1)^{-\frac{K}{\theta}} \right) (t-s)^{-\frac{j}{\theta}} E_{K,q}[f](s) \\
&\le Ct^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} (t+1)^{-\frac{K}{\theta}} (t-s)^{-\frac{j}{\theta}} E_{K,q}[f](s)
\end{aligned}
\tag{3.23}
$$

for $0 < s \le t/2$ with $(t+1)/2 < s+1$. Furthermore, by (3.19) with $\psi = f(s)$ and $r_1 = 1$ we have

$$
\begin{aligned}
&(t+1)^{-\frac{\ell}{\theta}} \left|\left|\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy \right|\right|\right|_{q,\ell} \\
&\le C(s+1)^{K_\theta+1} (t+1)^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)-(K_\theta+1)} (t-s)^{-\frac{j}{\theta}} \times \\
&\qquad\qquad \times \left\{ \|f(s)\|_1 + (t+1)^{-\frac{K}{\theta}} |||f(s)|||_{1,K} \right\}
\end{aligned}
$$

for $0 < s \le t/2$ with $(t+1)/2 \ge s+1$. This together with Lemma 3.1 again implies that

$$\begin{aligned}
&(t+1)^{-\frac{\ell}{\theta}} \left|\!\left|\!\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy \right|\!\right|\!\right|_{q,\ell} \\
&\le C t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} (s+1)^{K_\theta+1} (t+1)^{-(K_\theta+1)} \left((s+1)^{-\frac{K}{\theta}} + (t+1)^{-\frac{K}{\theta}} \right) \times \\
&\qquad\qquad \times (t-s)^{-\frac{j}{\theta}} E_{K,q}[f](s) \\
&\le C t^{-\frac{N}{\theta}\left(1-\frac{1}{q}\right)} (s+1)^{K_\theta+1-\frac{K}{\theta}} (t+1)^{-(K_\theta+1)} (t-s)^{-\frac{j}{\theta}} E_{K,q}[f](s)
\end{aligned} \tag{3.24}$$

for $0 < s \le t/2$ with $(t+1)/2 \ge s+1$. Then, by (3.24), for any $T > 0$, we observe from $\theta(K_\theta + 1) > K$ that

$$\lim_{t\to\infty} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)+\frac{j}{\theta}} (t+1)^{\frac{K-\ell}{\theta}} \left|\!\left|\!\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy \right|\!\right|\!\right|_{q,\ell} = 0 \tag{3.25}$$

for $0 < s < T$. Furthermore, by (3.22), (3.23), and (3.24) we see that

$$\begin{aligned}
&t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)} (t+1)^{\frac{K-\ell}{\theta}} \left|\!\left|\!\left| \int_{\mathbb{R}^N} \nabla^j \mathcal{T}(\cdot, y, t+1, s+1) f(y,s)\, dy \right|\!\right|\!\right|_{q,\ell} \\
&\le C (t-s)^{-\frac{j}{\theta}} E_{K,q}[f](s)
\end{aligned} \tag{3.26}$$

for $0 < s < t$. This implies (3.11). Furthermore, by (3.25) and (3.26) we apply the Lebesgue dominated convergence theorem to obtain (3.12). Thus assertion (b) follows. The proof of Lemma 3.4 is complete. □

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let u and w be as in Theorem 1.1. Then, by (1.11) and (1.12) we have

$$\begin{aligned}
&u(x,t) - w(x,t) \\
&= \int_{\mathbb{R}^N} \mathcal{T}(x,y,t+1,1)\varphi(y)\, dy + \int_0^t \int_{\mathbb{R}^N} \mathcal{T}(x,y,t+1,s+1) f(y,s)\, dy\, ds \\
&\quad + \sum_{m=0}^{K_\theta} \frac{(-1)^m}{m!} \int_{\mathbb{R}^N} \mathcal{S}_K^m(x,y,t+1)\varphi(y)\, dy \\
&\qquad + \sum_{m=0}^{K_\theta} \frac{(-1)^m}{m!} \int_0^t \int_{\mathbb{R}^N} (s+1)^m \mathcal{S}_K^m(x,y,t+1) f(y,s)\, dy\, ds.
\end{aligned}$$

We apply Lemmata 3.2, 3.3, and 3.4 to obtain (1.6) and (1.8). Then we easily see that (1.9) holds. Thus Theorem 1.1 follows. □

Proof of Corollary 1.1. By property (G)-(i) we see that $g_{\alpha,m}(0) \in L^q_K$ is equivalent to $g_{\alpha,m}(t) \in L^q_K$ for $t \ge 0$. Then Corollary 1.1 follows from Theorem 1.1. □

Remark 3.1 (i) *The arguments of* [11,12,15] *are in the frameworks of* L^q *and* L^1_K*. On the other hand, the arguments in the proof of Theorem* 1.1 *are in the framework of* L^q_K*. This improvement enables us to obtain HOAE of solutions to the Cauchy problem for nonlinear fractional diffusion equations such as* (1.2)*. See Section* 5*.*

(ii) *Let* $0 \le K < \theta$ *and* $\ell = 0$*. By similar arguments to those in the proof of Theorem* 1.1 *we see that Theorem* 1.1 *holds with* $E_{K,q}[f]$ *replaced by*

$$E'_{K,q}[f](t) := (t+1)^{\frac{K}{\theta}} \left[t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)} \|f(t)\|_q + \|f(t)\|_1 \right] + |||f(t)|||_{1,K}.$$

See also [14, Theorem 1.2].

4 Fractional convection-diffusion equation

In this section we consider the Cauchy problem for a convection type inhomogeneous fractional diffusion equation

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = \mathrm{div} f(x,t) \quad \text{in} \quad \mathbb{R}^N \times (0,\infty), \quad u(x,0) = \varphi(x) \quad \text{in} \quad \mathbb{R}^N, \tag{4.1}$$

where $1 < \theta < 2$, $\varphi \in L^1_K$ with $K \ge 0$, and $f = (f_1, \dots, f_N)$ is a vector-valued function in $\mathbb{R}^N \times (0,\infty)$. Similarly to Theorem 1.1, we have:

Theorem 4.1 *Let* $N \ge 1$*,* $1 < \theta < 2$*,* $K \ge 0$*, and* $1 \le q \le \infty$*. Let* $f = (f_1, \dots, f_N)$ *be a vector-valued measurable function in* $\mathbb{R}^N \times (0,\infty)$ *satisfying* (1.5)*. Let* $u \in C(\mathbb{R}^N \times (0,\infty))$ *be a solution to problem* (4.1)*, that is,* u *satisfies*

$$u(x,t) = \int_{\mathbb{R}^N} G_\theta(x-y,t)\varphi(y)\,dy + \int_0^t \int_{\mathbb{R}^N} \nabla G_\theta(x-y,t-s) \cdot f(y,s)\,dy\,ds$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$*, where* $\varphi \in L^1_K$*. Let* $0 \le \ell \le K$*. Then*

$$\sup_{0<t<\tau} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)} |||u(t) - z(t)|||_{q,\ell} < \infty \quad \textit{for} \quad \tau > 0, \tag{4.2}$$

where

$$\begin{aligned} z(x,t) := &\sum_{m=0}^{K_\theta} \sum_{|\alpha| \le K} M_\alpha(\varphi)\, g_{\alpha,m}(x,t) \\ &+ \sum_{m=0}^{K_\theta} \sum_{|\alpha| \le K} \sum_{j=1}^{N} \left(\int_0^t (s+1)^m M_\alpha(f_j(s))\,ds \right) \partial_{x_j} g_{\alpha,m}(x,t). \end{aligned}$$

Furthermore, there exists $C > 0$ *such that, for any* $\varepsilon > 0$ *and* $T > 0$*,*

$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t) - z(t)|||_{q,\ell} \le \varepsilon t^{-\frac{K}{\theta}} + C t^{-\frac{K}{\theta}} \int_T^t (t-s)^{-\frac{1}{\theta}} E_{K,q}[f](s)\,ds \tag{4.3}$$

holds for large enough $t > 0$*.*

Proof of Theorem 4.1. Let u and z be as in Theorem 4.1. Then, similarly to (1.11) and (1.12), we have

$$
\begin{aligned}
&u(x,t) - z(x,t) \\
&= \int_{\mathbb{R}^N} \mathcal{T}(x,y,t+1,1)\varphi(y)\,dy + \int_0^t \int_{\mathbb{R}^N} \nabla \mathcal{T}(x,y,t+1,s+1)\cdot f(y,s)\,dy\,ds \\
&\quad + \sum_{m=0}^{K_\theta} \frac{(-1)^m}{m!} \int_{\mathbb{R}^N} \mathcal{S}_K^m(x,y,t+1)\varphi(y)\,dy \\
&\quad + \sum_{m=0}^{K_\theta} \frac{(-1)^m}{m!} \int_0^t \int_{\mathbb{R}^N} (s+1)^m \nabla \mathcal{S}_K^m(x,y,t+1)\cdot f(y,s)\,dy\,ds.
\end{aligned}
$$

Similarly to the proof of Theorem 1.1, we apply Lemmata 3.2, 3.3, and 3.4 to obtain (4.2) and (4.3). Thus Theorem 4.1 follows. □

5 Nonlinear fractional diffusion equation

Let $N \geq 1$, $0 < \theta < 2$, and $F \in C(\mathbb{R}^N \times [0,\infty) \times \mathbb{R})$. Consider the Cauchy problem for a nonlinear fractional diffusion equation

$$
\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = F(x,t,u) \quad \text{in} \quad \mathbb{R}^N \times (0,\infty), \quad u(x,0) = \varphi(x) \quad \text{in} \quad \mathbb{R}^N, \tag{P}
$$

where $\varphi \in L^1_K \cap L^\infty$ for some $K \geq 0$ under the following condition (F):

(F) there exists $p > 1 + \theta/N$ such that

$$
|F(x,t,v) - F(x,t,w)| \leq C(|v| + |w|)^{p-1}|v - w|
$$

for $(x,t,v,w) \in \mathbb{R}^N \times [0,\infty) \times \mathbb{R}^2$.

Let $u \in C(\mathbb{R}^N \times (0,\infty))$ be a solution to problem (P) that is, u satisfies

$$
u(x,t) = \left[e^{-t(-\Delta)^{\theta/2}}\varphi\right](x) + \int_0^t \left[e^{-(t-s)(-\Delta)^{\theta/2}} F(\cdot,s,u(\cdot,s))\right](x)\,ds
$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$. In this section, under condition (1.13), we obtain HOAE of the solution u. Theorem 5.1 is an application of Theorem 1.1.

Theorem 5.1 *Let $N \geq 1$, $0 < \theta < 2$, and $\varphi \in L^1_K$ with $K \geq 0$. Assume condition* (F). *Let $u \in C(\mathbb{R}^N \times (0,\infty))$ be a solution to problem* (P). *Set*

$$
F(x,t) := F(x,t,u(x,t)), \quad (x,t) \in \mathbb{R}^N \times (0,\infty).
$$

(a) *Assume that $\varphi \in L^\infty_k$ with $k = \min\{N+\theta, K\}$. Let u satisfy* (1.13). *Then*

$$
\sup_{t>0}\,(t+1)^{\frac{N}{q}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t)|||_{q,\ell} < \infty \tag{5.1}
$$

for $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$ with $\ell < \theta + N(1-1/q)$.

(b) *Let u satisfy* (5.1). *If $p(N+\theta) > K+N$, then*

$$E_{K,q}[F](t) \le C(t+1)^{\frac{K}{\theta}-A_p}, \quad t>0, \tag{5.2}$$

for $1 \le q \le \infty$, where $A_p := N(p-1)/\theta > 1$.

(c) *Assume that* (5.2) *holds. Set*

$$U_0(x,t) := \sum_{m=0}^{K_\theta} \sum_{|\alpha|\le K} \left(M_\alpha(\varphi) + \int_0^t (s+1)^m M_\alpha(F(s))\,ds \right) g_{\alpha,m}(x,t). \tag{5.3}$$

Then

$$\sup_{t>0} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)} (t+1)^{-\frac{\ell}{\theta}} |||u(t)-U_0(t)|||_{q,\ell} < \infty \tag{5.4}$$

and

$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t)-U_0(t)|||_{q,\ell} = \begin{cases} o(t^{-\frac{K}{\theta}}) + O(t^{-A_p+1}) & \text{if} \quad A_p - 1 \ne K/\theta, \\ O(t^{-\frac{K}{\theta}} \log t) & \text{if} \quad A_p - 1 = K/\theta, \end{cases} \tag{5.5}$$

as $t \to \infty$, for $1 \le q \le \infty$ and $0 \le \ell \le K$.

Remark 5.1 *Let $N \ge 1$, $0 < \theta < 2$, and $\varphi \in L^\infty$. Assume condition* (F).

(i) *There exists $\delta > 0$ such that, if $\|\varphi\|_{L^{N(p-1)/\theta}} < \delta$, then problem* (P) *possesses a solution $u \in C(\mathbb{R}^N \times (0,\infty))$ satisfying* (1.13). *See* [13,16,20].

(ii) *Let $F(x,t,u) := \lambda |u|^{p-1}u$ with $\lambda \le 0$. Then the comparison principle implies that*

$$|u(x,t)| \le \left[e^{-t(-\Delta)^{\theta/2}} |\varphi| \right](x), \quad (x,t) \in \mathbb{R}^N \times (0,\infty).$$

This together with Lemma 2.1 *implies* (1.13).

We prepare the following lemma for the proof of Theorem 5.1.

Lemma 5.1 *Assume condition* (F). *Let $K \ge 0$. Let v_1 and v_2 be measurable functions in $\mathbb{R}^N \times (0,\infty)$ and h in $(0,\infty)$ such that*

$$\begin{aligned} &(t+1)^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||v_i(t)|||_{q,\ell} < \infty, \quad i = 1,2, \\ &(t+1)^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||v_1(t) - v_2(t)|||_{q,\ell} \le h(t), \end{aligned} \tag{5.6}$$

for $t > 0$, $1 \le q \le \infty$, and $0 \le \ell \le K$ with $\ell < \theta + N(1-1/q)$. Assume that $p(N+\theta) > K+N$. Then

$$E_{K,q}[F(v_1) - F(v_2)](t) \le C(t+1)^{-A_p + \frac{K}{\theta}} h(t), \quad t > 0.$$

Proof. Let $0 \le \ell \le K$ and $1 \le q \le \infty$. Since $p(N+\theta) > K+N$, we find $\ell_1,\, \ell_2 \ge 0$ such that

$$0 \le \ell_1 < \theta + N, \quad 0 \le \ell_2 < \theta + N\left(1 - \frac{1}{q}\right), \quad \ell = (p-1)\ell_1 + \ell_2.$$

Then, by condition (F) and (5.6) we see that

$$\begin{aligned}
&(t+1)^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||F(v_1(s)) - F(v_2(s))|||_{q,\ell} \\
&\le C(t+1)^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} (|||v_1(t)|||^{p-1}_{\infty,\ell_1} + |||v_2(t)|||^{p-1}_{\infty,\ell_1}) |||v_1(t) - v_2(t)|||_{q,\ell_2} \\
&\le C(t+1)^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}-\frac{N(p-1)}{\theta}+\frac{(p-1)\ell_1}{\theta}} |||v_1(t) - v_2(t)|||_{q,\ell_2} \\
&\le C(t+1)^{-\frac{\ell}{\theta}-\frac{N(p-1)}{\theta}+\frac{(p-1)\ell_1}{\theta}+\frac{\ell_2}{\theta}} h(t) = C(t+1)^{-A_p} h(t), \quad t > 0.
\end{aligned}$$

Thus Lemma 5.1 follows. □

Proof of Theorem 5.1. We prove assertion (a). Since $A_p = N(p-1)/\theta > 1$, the comparison principle together with condition (F) and (1.13) implies that

$$|u(x,t)| \le \exp\left(C \int_0^t (s+1)^{-A_p}\, ds\right) \left[e^{-t(-\Delta)^{\theta/2}} |\varphi|\right](x) \le C \left[e^{-t(-\Delta)^{\theta/2}} |\varphi|\right](x)$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$. This together with Lemma 2.1 implies assertion (a). Furthermore, assertion (b) follows from Lemma 5.1 with $v_1 = u$ and $v_2 = 0$. On the other hand, by Theorem 1.1 with (5.2) we obtain (5.4). Furthermore, for any $\varepsilon > 0$ and $T > 0$, we have

$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t) - U_0(t)|||_{q,\ell} \le \varepsilon t^{-\frac{K}{\theta}} + C t^{-\frac{K}{\theta}} \int_T^t (s+1)^{\frac{K}{\theta}-A_p}\, ds$$

for large enough $t > 0$. This implies (5.5). Thus assertion (c) follows. The proof of Theorem 5.1 is complete. □

As a corollary of Theorem 5.1, we have:

Corollary 5.1 *Let $N \ge 1$, $0 < \theta < 2$, and $\varphi \in L^1_K$ with $K \ge 0$. Assume condition* (F) *and*

$$p > 1 + \frac{2K + \theta}{N}. \tag{5.7}$$

Let $u \in C(\mathbb{R}^N \times (0,\infty))$ be a solution to problem (P) *and satisfy* (5.1)*. Then there exists a set $\{M_{\alpha,m}\} \subset \mathbb{R}$, where $m \in \{0,\dots,K_\theta\}$ and $\alpha \in \mathbb{M}$ with $|\alpha| \le K$, such that*

$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t) - U_*(t)|||_{q,\ell} = o\left(t^{-\frac{K}{\theta}}\right) \quad \text{as} \quad t \to \infty \tag{5.8}$$

for $1 \le q \le \infty$ and $0 \le \ell \le K$, where

$$U_*(x,t) := \sum_{m=0}^{K_\theta} \sum_{|\alpha| \le K} M_{\alpha,m}\, g_{\alpha,m}(x,t). \tag{5.9}$$

Proof. It follows from (5.7) that

$$p(N+\theta) > K+N, \qquad A_p - 1 = \frac{N}{\theta}(p-1) - 1 > \frac{2K}{\theta}. \tag{5.10}$$

By Theorem 5.1 we have

$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t) - U_0(t)|||_{q,\ell} = o\left(t^{-\frac{K}{\theta}}\right) \quad \text{as} \quad t \to \infty \tag{5.11}$$

for $1 \le q \le \infty$ and $1 \le \ell \le K$. Here U_0 is as in Theorem 5.1.

Let $m \in \{0, \dots, K_\theta\}$ and $\alpha \in \mathbb{M}$ with $|\alpha| \le K$. Assertion (b) of Theorem 5.1 implies that

$$|M_\alpha(F(t))| \le C(t+1)^{-A_p + \frac{|\alpha|}{\theta}}, \quad t > 0. \tag{5.12}$$

Then, by (5.10) we find $M_{\alpha,m} \in \mathbb{R}$ such that

$$M_{\alpha,0} = M_\alpha(\varphi) + \int_0^\infty M_\alpha(F(s))\, ds, \quad M_{\alpha,m} = \int_0^\infty (s+1)^m M_\alpha(F(s))\, ds \quad (m \ge 1).$$

Furthermore, by (2.2), (5.3), (5.9), (5.10), and (5.12) we have

$$\begin{aligned}
&t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||U_0(t) - U_*(t)||_{q,\ell} \\
&\le t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} \sum_{m=0}^{K_\theta} \sum_{|\alpha| \le K} \left(\int_t^\infty (s+1)^m |M_\alpha(F(s))|\, ds \right) |||g_{\alpha,m}(t)|||_{q,\ell} \\
&\le C \sum_{m=0}^{K_\theta} \sum_{|\alpha| \le K} (t+1)^{-m-\frac{|\alpha|}{\theta}} \int_t^\infty (s+1)^m (s+1)^{-A_p + \frac{|\alpha|}{\theta}}\, ds \le C t^{-A_p+1}, \quad t \ge 1.
\end{aligned}$$

This together with (5.10) and (5.11) implies (5.8). Thus Corollary 5.1 follows. □

Combining Theorems 1.1 and 5.1, we obtain a refined asymptotic expansion of the solution to problem (P).

Theorem 5.2 *Assume the same conditions as in Theorem* 5.1. *Let u satisfy* (5.1). *For $n = 1, 2, \dots,$ define a function $U_n = U_n(x,t)$ in $\mathbb{R}^N \times (0,\infty)$ inductively by*

$$\begin{aligned}
U_n(x,t) := U_0(x,t) &+ \int_0^t \left[e^{-(t-s)(-\Delta)^{\theta/2}} F_{n-1}(s) \right](x)\, ds \\
&- \sum_{m=0}^{K_\theta} \sum_{|\alpha| \le K} \left(\int_0^t (s+1)^m M_\alpha(F_{n-1}(s))\, ds \right) g_{\alpha,m}(x,t),
\end{aligned}$$

where U_0 is as in Theorem 5.1 *and $F_{n-1}(x,t) := F(x,t,U_{n-1}(x,t))$. Then*

$$\sup_{t>0} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}} |||u(t) - U_n(t)|||_{q,\ell} < \infty \tag{5.13}$$

and

$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}}|||u(t)-U_n(t)|||_{q,\ell} = \begin{cases} o(t^{-\frac{K}{\theta}})+O(t^{-(n+1)(A_p-1)}) & \text{if} \quad (n+1)(A_p-1)\neq K/\theta, \\ O(t^{-\frac{K}{\theta}}\log t) & \text{if} \quad (n+1)(A_p-1)=K/\theta, \end{cases} \tag{5.14}$$

as $t\to\infty$, *for* $1\le q\le\infty$ *and* $0\le\ell\le K$.

Proof. Let $K\ge 0$. By Theorem 5.1 we have (5.13) and (5.14) with $n=0$. Assume that (5.13) and (5.14) hold for some $n=k\in\{0,1,\dots\}$. Then, by (5.1) and (5.13) with $n=k$ we have

$$\begin{aligned} &\sup_{t>0} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}}|||U_k(t)|||_{q,\ell} \\ &\le \sup_{t>0} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}}|||u(t)-U_k(t)|||_{q,\ell}+\sup_{t>0} t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}}|||u(t)|||_{q,\ell}<\infty \end{aligned} \tag{5.15}$$

for $1\le q\le\infty$ and $0\le\ell\le K$ with $\ell<\theta+N(1-1/q)$. On the other hand, it follows that

$$\begin{aligned} &u(x,t)-\int_0^t\left[e^{-(t-s)(-\Delta)^{\theta/2}}F_k(s)\right](x)\,ds \\ &=\left[e^{-t(-\Delta)^{\theta/2}}\varphi\right](x)+\int_0^t e^{-(t-s)(-\Delta)^{\theta/2}}[F(s)-F_k(s)]\,ds \end{aligned} \tag{5.16}$$

for $(x,t)\in\mathbb{R}^N\times(0,\infty)$. By (5.2) and (5.15) we apply Lemma 5.1 to obtain

$$E_{K,q}[F-F_k]\in L^\infty(0,\tau) \quad \text{for} \quad \tau>0$$

and

$$E_{K,q}[F-F_k](t)=\begin{cases} o\left(t^{-A_p}\right) & \text{if} \quad (k+1)(A_p-1)<K/\theta, \\ O\left(t^{-A_p}\log t\right) & \text{if} \quad (k+1)(A_p-1)=K/\theta, \\ O\left(t^{-A_p+\frac{K}{\theta}-(k+1)(A_p-1)}\right) & \text{if} \quad (k+1)(A_p-1)>K/\theta, \end{cases}$$

as $t\to\infty$. Then we apply Theorem 1.1 to (5.16), namely $f(x,t)=F(x,t)-F_k(x,t)$, and obtain (5.13) and (5.14) with $n=k+1$. Therefore, by induction we obtain (5.13) and (5.14) for $n=0,1,2,\dots$. Thus Theorem 5.2 follows. □

Similarly to the proof of Theorem 5.2 for the case $n=1$, we prove Theorem 1.2.

Proof of Theorem 1.2. Let $1\le q\le\infty$ and $0\le\ell\le K$ with $\ell<\theta+N(1-1/q)$. Assume $p>1+\theta/N$ and put $F(u(x,t)):=\lambda|u(x,t)|^{p-1}u(x,t)$. Assertion (a) follows from the similar argument to that of the proof of Corollary 5.1. Furthermore, for any $\sigma > 0$, by (2.2) and (5.12) we have

$$
\begin{aligned}
&t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}}|||U_0(t) - M_* g(t)|||_{q,\ell} \\
&\le C\int_t^\infty |M_0(F(s))|\,ds + C\sum_{1\le|\alpha|\le K} t^{-\frac{|\alpha|}{\theta}}|M_\alpha(\varphi)| + C\sum_{m=1}^{K_\theta}\sum_{|\alpha|\le K} t^{-m-\frac{|\alpha|}{\theta}}|M_\alpha(\varphi)| \\
&\qquad\quad + C\sum_{m=1}^{K_\theta}\sum_{|\alpha|\le K} t^{-m-\frac{|\alpha|}{\theta}}\int_0^t (s+1)^m |M_\alpha(F(s))|\,ds \\
&\qquad\qquad\qquad + C\sum_{1\le|\alpha|\le K} t^{-\frac{|\alpha|}{\theta}}\int_0^t |M_\alpha(F(s))|\,ds \\
&= O\left(t^{-(A_p-1)}\right) + O(t^{-\frac{1}{\theta}}) + O(t^{-1}) + O\left(t^{-1}\int_1^t s^{1-A_p}\,ds\right) + O\left(t^{-\frac{1}{\theta}}\int_1^t s^{\frac{1}{\theta}-A_p}\,ds\right) \\
&= O\left(t^{-(A_p-1)+\sigma}\right) + O(t^{-1}) + O(t^{-\frac{1}{\theta}}) = O(h_\sigma(t))
\end{aligned}
$$

as $t \to \infty$. This together with (5.5) implies that

$$
t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}}|||u(t) - M_* g(t)|||_{q,\ell} = o\left(t^{-\frac{K}{\theta}}\right) + O(h_\sigma(t)) \quad \text{as} \quad t \to \infty. \tag{5.17}
$$

Furthermore, combining Lemma 5.1 and (5.17), we have

$$
E_{K,q}[F(u) - F_\infty](t) = o\left(t^{-A_p-\frac{K}{\theta}}\right) + O\left(t^{-A_p}h_\sigma(t)\right) \tag{5.18}
$$

as $t \to \infty$. On the other hand, it follows that

$$
\begin{aligned}
w(x,t) &:= u(x,t) - \int_0^t e^{-(t-s)(-\Delta)^{\frac{\theta}{2}}} F_\infty(s)\,ds \\
&= \left[e^{-t(-\Delta)^{\frac{\theta}{2}}}\varphi\right](x) + \int_0^t e^{-(t-s)(-\Delta)^{\frac{\theta}{2}}}[F(u(s)) - F_\infty(s)]\,ds.
\end{aligned}
$$

Applying Theorem 1.1, for any $\varepsilon > 0$ and $T > 0$, we obtain

$$
\begin{aligned}
&t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}}|||w(t) - w_*(t)|||_{q,\ell} \\
&\le \varepsilon t^{-\frac{K}{\theta}} + C_* t^{-\frac{K}{\theta}}\int_T^t s^{\frac{K}{\theta}} E_{K,q}[F(u) - F_\infty](s)\,ds
\end{aligned} \tag{5.19}
$$

as $t \to \infty$, where C_* is a positive constant independent of ε and T and

$$
w_*(x,t) = \sum_{m=0}^{K_\theta}\sum_{|\alpha|\le K}\left(M_\alpha(\varphi) + \int_0^t (s+1)^m M_\alpha(F(u(s)) - F_\infty(s))\,ds\right) g_{\alpha,m}(x,t).
$$

Since ε is arbitrary, by (5.18) and (5.19) we see that

$$t^{\frac{N}{\theta}\left(1-\frac{1}{q}\right)-\frac{\ell}{\theta}}|||w(t)-w_*(t)|||_{q,\ell} = o\left(t^{-\frac{K}{\theta}}\right) + O\left(t^{-\frac{K}{\theta}}\int_T^t s^{\frac{K}{\theta}-A_p}h_\sigma(s)\,ds\right)$$

as $t \to \infty$. This implies assertion (b). Thus Theorem 1.2 follows. □

Remark 5.2 *Let u be a solution to the Cauchy problem for a nonlinear fractional diffusion equation and possess the mass conservation law, that is, $\int_{\mathbb{R}^N} u(x,t)\,dx$ is independent of t. The mass conservation law has often played an important role in the study of HOAE of solutions to various nonlinear problems, see e.g.* [5, 6, 9, 18, 21–26]*. Then the arguments in the proof of Theorem* 4.1 *are valid for the Cauchy problem.*

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