

Refined asymptotics for constant scalar curvature metrics with isolated singularities

Nick Korevaar*
University of Utah
korevaar@math.utah.edu

Rafe Mazzeo†
Stanford University
mazzeo@math.stanford.edu

Frank Pacard
Université Paris XII
Frank.Pacard@univ-paris12.fr

Richard Schoen‡
Stanford University
schoen@math.stanford.edu

Abstract

We consider the asymptotic behaviour of positive solutions u of the conformal scalar curvature equation, $\Delta u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0$, in the neighbourhood of isolated singularities in the standard Euclidean ball. Although asymptotic radial symmetry for such solutions was proved some time ago, [2], we present a much simpler and more geometric derivation of this fact. We also discuss a refinement, showing that any such solution is asymptotic to one of the deformed radial singular solutions. Finally we give some applications of these refined asymptotics, first to computing the global Pohožaev invariants of solutions on the sphere with isolated singularities, and then to the regularity of the moduli space of all such solutions.

1 Introduction

The problem we consider in this paper is to derive asymptotics for positive solutions of the conformally invariant semilinear elliptic equation

$$\Delta u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0 \tag{1}$$

which are defined in the punctured ball $B^n \setminus \{0\}$, and which are singular at the origin. It is well-known that a solution u of this equation corresponds to a conformally flat metric

$$g = u^{\frac{4}{n-2}} \delta$$

which has constant scalar curvature $R_g = n(n-1)$. Equation (1) is a special case of the more general equation relating the scalar curvatures of any two conformally related metrics. More specifically, if g and $g' = u^{\frac{4}{n-2}}g$ are any two such metrics, with corresponding scalar curvature functions R_g and $R_{g'}$, respectively, then

$$\Delta_g u - \frac{n-2}{4(n-1)}R_g u + \frac{n-2}{4(n-1)}R_{g'} u^{\frac{n+2}{n-2}} = 0. \tag{2}$$

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The linear part of the operator on the left

$$L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g \quad (3)$$

is known as the *conformal Laplacian* associated with the metric g . It satisfies the conformal equivariance property that

$$L_g(u\phi) = u^{\frac{n+2}{n-2}} L_{g'}\phi \quad (4)$$

for any $\phi \in C^\infty$ and any two metrics g and g' related as above.

Asymptotics of solutions of equation (1) were proved by L.A. Caffarelli, B. Gidas and J. Spruck [2] using a rather complicated version of Alexandrov reflection. Their result states that an arbitrary positive solution u of (1) satisfies

$$u(x) = |x|^{\frac{2-n}{2}} v_\varepsilon(-\log|x| + T)(1 + o(1)), \quad (5)$$

where $u_\varepsilon(x) = |x|^{\frac{2-n}{2}} v_\varepsilon(-\log|x|)$ is one of a one-parameter family of radial solutions of (1). We call the v_ε and the u_ε *Fowler solutions* and discuss them at greater length in the next section (*Fowler solutions* are also called *Delaunay type solutions* in many works which we will refer to). In recent work [5], C.C. Chen and C.S. Lin have generalized this result and slightly simplified their argument, although their proof is still quite technical. This asymptotics theorem is related to a simpler result due to P. Aviles [1] concerning the subcritical equation, where the exponent $\frac{n+2}{n-2}$ is replaced by a smaller number p in the range $[\frac{n}{n-2}, \frac{n+2}{n-2})$. P. Aviles shows that an arbitrary positive solution of the subcritical equation with an isolated singularity at the origin converges, in the sense of (5), to a radial solution of the same equation.

The present work is a combination of two separate projects which are sufficiently closely related that it has seemed natural to publish them together. The first, undertaken several years ago by the first and fourth authors (K & S), is to supply an alternate argument, more geometric and much simpler than that of [2], to establish (5). This also gives a slightly stronger estimate, improving the $o(1)$ remainder term to $O(|x|^\alpha)$ for some $\alpha > 0$. The second project, quite recently undertaken by the second and third authors (M & P) and inspired by their recent construction [9], concerns a refinement of these asymptotics, in some sense obtaining a second term in the expansion, along with various applications for these refined asymptotics. More precisely, it is possible to improve (5) using, instead of v_ε , a family of deformations of the Fowler solutions, parametrized by vectors $a \in \mathbb{R}^n$, which we denote $v_{\varepsilon,a}$ and which are defined by

$$v_{\varepsilon,a}(t, \theta) = |\theta - ae^{-t}|^{\frac{2-n}{2}} v_\varepsilon(t + \log|\theta - ae^{-t}|),$$

where $\theta = x/|x|$. For all $a \in \mathbb{R}^n$ and $T \in \mathbb{R}$ we define

$$u_{\varepsilon,a,T}(x) = |x|^{\frac{2-n}{2}} v_{\varepsilon,a}(-\log|x| + T, x/|x|).$$

The main result is

Theorem 1 *Let u be an arbitrary weak solution of the equation (1) which is positive on the ball $B^n(0) \subset \mathbb{R}^n$ and in $C^\infty(B^n(0) \setminus \{0\})$. Then either $u \in C^\infty(B^n(0))$, or else there exists some choice of parameters $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 = ((n-2)/n)^{(n-2)/4}$), $a \in \mathbb{R}^n$ and some constants $T \in \mathbb{R}$ and $\alpha > 1$ such that*

$$u(x) = |x|^{\frac{2-n}{2}} (v_{\varepsilon,a}(-\log|x| + T, x/|x|) + O(|x|^\alpha)), \quad \text{as } |x| \rightarrow 0.$$

Alternatively,

$$u(x) = u_{\varepsilon,a,T}(x) + O(|x|^\beta) \quad \text{as } |x| \rightarrow 0$$

for some constant $\beta > \frac{4-n}{2}$. In particular, there are upper and lower bounds for u of the form $C_1|x|^{\frac{2-n}{2}} \leq u \leq C_2|x|^{\frac{2-n}{2}}$, with $0 < C_1 \leq C_2$, and the corresponding metric $g = u^{\frac{n+2}{n-2}}\delta$ is complete at 0.

Solutions of equation (2) on the sphere S^n , singular along a closed set Λ , giving conformal factors relating the standard metric to a new metric of scalar curvature $n(n-1)$ complete on $S^n \setminus \Lambda$, were first considered in the work [20] of the fourth author and S.T. Yau. For solutions to exist it is necessary that the Hausdorff dimension of Λ be less than or equal to $(n-2)/2$. Under mild geometric hypotheses, these solutions are shown there to extend to global weak solutions on S^n . Around the same time, solutions with isolated singularities (and with some more complicated singular sets arising essentially as limit sets of Kleinian groups) on S^n were constructed by one of us [17]. Since then, solutions singular along an arbitrary disjoint collection of submanifolds, and giving metrics of *positive* scalar curvature, have been constructed in a succession of works, [13], [15], [8], [9]. (We do not discuss the by now extensive set of results on the analogous problem for metrics of negative scalar curvature.)

Emerging from [8] and [9] (as well as [17]) is the lesson that solutions with isolated singularities are far more rigid objects than solutions with higher dimensional singular sets. This is apparent if one fixes the singular set Λ and studies the moduli space \mathcal{M}_Λ of all solutions singular along Λ . We shall henceforth restrict attention to the case where Λ is a finite disjoint collection of submanifolds, of (possibly varying) dimensions less than or equal to $(n-2)/2$. When at least one component of Λ is of positive dimension, \mathcal{M}_Λ is infinite dimensional and is modelled locally on a Banach manifold. Closely related to Theorem 1 is a result on refined asymptotics, analogous to the one here but for solutions singular along higher dimensional submanifolds, given in [7]. One sees from these asymptotics that solution metrics need not have bounded curvature, and that the correct way to parametrize the moduli space \mathcal{M}_Λ then is via an asymptotic Dirichlet problem. On the other hand, when Λ is a set of k points on the sphere, then \mathcal{M}_Λ is finite dimensional. It is proved in [11] that in this case, \mathcal{M}_Λ is even a locally real analytic set with virtual dimension k . In [9] it is shown that for generic configurations Λ , \mathcal{M}_Λ attains this dimension. The sharper asymptotics proved here have applications to the study of \mathcal{M}_Λ when Λ is finite. In particular, corresponding to any conformal Killing vector field on the sphere one can define a Pohožaev invariant of the (global) solution, depending on the parameters ε and a associated to each singular point. Using the refined asymptotics one can evaluate these invariants explicitly; one obtains a finite number of real analytic equations satisfied by the set of parameters associated to each solution. While these equations are not enough to determine the moduli space completely, they do lead to new information about it. This is done in §6. There are two other applications of the refined asymptotics considered here. Then, in §7, we discuss the nondegeneracy of the moduli space \mathcal{M}_Λ near its ends.

For any $k > 0$, Theorem 1 gives the existence of a ‘parameter map’ Φ on the space of solutions defined on all of \mathbb{R}^n , with exactly k singularities; Φ associates to a solution its set of singular points p_1, \dots, p_k and the associated parameters T_j, a_j and ε_j at each p_j . Φ is defined more carefully in (21) below. This map should be quite useful in the global study of the moduli spaces \mathcal{M}_Λ when Λ is finite. It seems quite likely that Φ is injective, but the proof of this fact may be subtle. We leave this as an interesting open problem.

We conclude this introduction by remarking on the relationship of the results here with what is known for noncompact constant (nonzero) mean curvature surfaces $\Sigma \subset \mathbb{R}^3$ which are (Alexandrov) embedded and have k ends. Although seemingly quite different, this extrinsic geometric problem bears many similarities to the intrinsic conformal scalar curvature equation. Indeed, all the results known in one setting seem to have direct analogues in the other setting, although the proofs are sometimes different. Any end of such a surface Σ is asymptotic to one of the Delaunay surfaces (these are a family of rotationally symmetric CMC surfaces in \mathbb{R}^3); this follows from work of the first author, Kusner and Solomon, [4], and Meeks, [14]. The deformations of the radial solutions used here, and the corresponding refined asymptotics, corresponds in the CMC setting to translations and rotations of the asymptotic axes. More analytically, the difference is that the Jacobi fields corresponding to these deformations, in the CMC context, are at most linearly growing, while in the scalar curvature setting they are exponentially growing or decaying. Amongst the various applications of the refined asymptotics we develop here, the global balanc-

ing formula of §6 corresponds to the computationally more simple balancing formula in the CMC context discussed, for example, in [4]. The nondegeneracy argument of §7 has a direct analogue in the CMC setting, and to our knowledge, is new there as well.

2 Preliminaries

In this section we define and discuss in some detail the $(n + 1)$ -parameter family of functions $v_{\varepsilon, a}$ which arise in the statement of Theorem 1. We also study the linearization of (1) about any one of the the Fowler solutions v_ε , since this will be a crucial tool in the later analysis. Parts of this material can also be found in [17] and [11].

In the ensuing discussion it will be convenient to use the conformal equivariance of the problem to express (1) relative to different background metrics in the flat conformal class. The three most natural choices are the standard Euclidean metric, the cylindrical metric $dt^2 + d\theta^2$ on $\mathbb{R} \times S^{n-1}$, and the standard spherical metric on S^n . Equation (1) is already expressed relative to the Euclidean metric. Relative to the cylindrical metric this equation becomes

$$\frac{d^2v}{dt^2} + \Delta_\theta v - \frac{(n-2)^2}{4}v + \frac{n(n-2)}{4}v^{\frac{n+2}{n-2}} = 0. \quad (6)$$

The transformation between (1) and (6) can also be seen directly by setting

$$u(x) = |x|^{\frac{2-n}{2}}v(-\log|x|, x/|x|)$$

and choosing a new independent variable $t = -\log|x|$. We shall not need the explicit form of this equation relative to the spherical metric.

At each stage below we shall use whichever of these ambient metrics is most convenient for the computations at hand.

2.1 Fowler solutions

We now define the special family of solutions mentioned in the introduction and develop their relevant properties. We start by considering the radial solutions of (1) on \mathbb{R}^n which are singular at the origin. For this problem it is simplest to use the cylindrical metric and equation (6).

The first result characterizes the global solutions of (6).

Proposition 1 *Let v be any positive solution of (6) defined on the whole cylinder $\mathbb{R} \times S^{n-1}$. Then v is independent of the spherical variable θ , hence depends only on the variable t . Moreover we always have $0 < v(t) \leq 1$ for all $t \in \mathbb{R}$.*

The proof of this uses an Alexandrov reflection argument, and is contained in [2] and also in [18].

Hence if v is a global solution, then it satisfies the autonomous ODE obtained by dropping the θ differentiation in (6). We analyze this by converting it to a system of first order equations on the phase plane. Namely, setting $w = \partial_t v$, the equation becomes

$$\frac{d}{dt}v = w, \quad \frac{d}{dt}w = \frac{(n-2)^2}{4}v - \frac{n(n-2)}{4}v^{\frac{n+2}{n-2}}.$$

This system is Hamiltonian, with corresponding energy function

$$H(v, w) = w^2 - \frac{(n-2)^2}{4}v^2 + \frac{(n-2)^2}{4}v^{\frac{2n}{n-2}}. \quad (7)$$

In particular, if $v(t)$ is a solution of (6) then the path parametrized by $(v(t), v'(t))$ is contained within a level set of H , hence $H(v, v')$ does not depend on t . There is a homoclinic trajectory lying in the level set $\{H = 0\} \cap \{v > 0\}$, and a one-parameter family of closed level sets contained

in the bounded set $\mathcal{O} \equiv \{H < 0\} \cap \{v > 0\}$. All other trajectories and level sets pass into the region where $v \leq 0$, so we do not consider them here. The compact level sets in \mathcal{O} correspond to a family of periodic solutions of (6) which we shall denote $v_\varepsilon(t)$. The parameter ε denotes the minimum value attained by the solution, and we will often refer to it as the *necksize* of the solution. It varies in the interval $(0, \varepsilon_0]$, where $\varepsilon_0^{\frac{4}{n-2}} = \frac{n-2}{n}$. The level set of H corresponding to v_ε is

$$H \equiv H(\varepsilon) = \frac{(n-2)^2}{4} \left(\varepsilon^{\frac{2n}{n-2}} - \varepsilon^2 \right). \quad (8)$$

Before continuing with the rest of our treatment of this special family of solutions we digress slightly to discuss the energy of arbitrary (nonnegative) solutions of (6). Thus let v be any such solution. Multiply this equation by $\partial_t v$; a small calculation yields

$$\frac{1}{2} \partial_t (\partial_t v)^2 + \operatorname{div}_\theta (\partial_t v \nabla_\theta v) - \frac{1}{2} \partial_t |\nabla_\theta v|^2 - \frac{(n-2)^2}{8} \partial_t (v^2) + \frac{(n-2)^2}{8} \partial_t (v^{\frac{2n}{n-2}}) = 0.$$

We can rephrase this by noting that it is equivalent to the vanishing of the divergence of the vector field

$$W = \left(\frac{1}{2} (\partial_t v)^2 - \frac{1}{2} |\nabla_\theta v|^2 - \frac{(n-2)^2}{8} v^2 + \frac{(n-2)^2}{8} v^{\frac{2n}{n-2}} \right) \partial_t + (\partial_t v) \nabla_\theta v \nabla_\theta.$$

Hence the integral of $\langle W, \partial_t \rangle$ on the sphere $t = T$ (with respect to the volume form for the metric g) gives a number which, by the divergence theorem, does not depend on T . Thus we obtain an invariant of the solution v , which we shall call the ‘radial Pohožaev invariant’ of v and denote $\mathcal{P}_{\text{ra}}(v)$. It is a special case of the more general Pohožaev invariants we shall discuss later. Note that the radial Pohožaev invariant of the Fowler solution v_ε is simply $\frac{1}{2} \omega_{n-1} H(\varepsilon)$, where ω_{n-1} is the volume of S^{n-1} and $H(\varepsilon)$ is the Hamiltonian energy (7).

We now return to the discussion of the special family of solutions. There are actually two parameters for these periodic solutions. The first is this necksize parameter ε , while the second corresponds to the value of the solution at $t = 0$ and we will denote by T_ε the period of v_ε . We normalize the functions v_ε by assuming that $v_\varepsilon(0) = \min v_\varepsilon$. Hence the complete family of positive radial solutions is $v_\varepsilon(t + T)$ for $T \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0]$. The two extremes of this family are when v is the constant solution $v \equiv \varepsilon_0$ (geometrically, this corresponds to the correct magnification of the cylinder which has scalar curvature $n(n-1)$), and the limits $v_\varepsilon(t) \rightarrow 0$ and $v_\varepsilon(t + \frac{1}{2}T_\varepsilon) \rightarrow (\cosh t)^{\frac{2-n}{2}}$ as $\varepsilon \rightarrow 0$. This last explicit solution corresponds to the conformal factor $(\cosh t)^{-2}$ transforming the cylinder to the punctured sphere $S^n \setminus \{p, -p\}$.

It is known that T_ε , the period of v_ε , is monotone in ε , converging to $2\pi/\sqrt{n-2}$ as $\varepsilon \rightarrow \varepsilon_0$ and increasing to ∞ as $\varepsilon \rightarrow 0$. Geometrically, however, the metrics $g_\varepsilon \equiv v_\varepsilon^{\frac{n+2}{n-2}} (dt^2 + d\theta^2)$ converge to a bead of spheres arranged along a fixed axis. This family of radial solutions interpolates between the cylinder and this singular limit.

We may also transform these back to solutions of (1), to obtain the solutions

$$e^{\frac{2-n}{2}T} u_\varepsilon(e^{-T}x) = |x|^{\frac{2-n}{2}} v_\varepsilon(-\log|x| + T)$$

on $\mathbb{R}^n \setminus \{0\}$.

As indicated earlier, we shall consider, in addition, an n parameter family of deformations of these radial solutions. Geometrically these may be understood as follows. Using the Euclidean background metric, and starting with a radial solution u , we first take the Kelvin transform $\tilde{u}(x) = |x|^{2-n} u(x/|x|^2)$. It is not difficult to check that \tilde{u} is also a solution of (1). Then translate \tilde{u} by some vector $a \in \mathbb{R}^n$, and finally take the Kelvin transform once again to obtain $u_{\varepsilon,a}$. Note that this is the pullback by the composition of three conformal transformations, first inversion in the unit sphere $\{|x| = 1\}$, then Euclidean translation, which is a parabolic transformation fixing infinity, and then inversion once again. Each of these elementary transformations takes the

space of solutions of (1) to itself. Using instead the spherical metric as the background, if u is transformed to a solution on S^n which is singular at two antipodal points $\{p, -p\}$, then these steps correspond to first reflecting across the equator determined by p and $-p$, then applying a parabolic transformation F which fixes the point p and carries another point q to p , and finally reflecting across the equator again. Finally, note that relative to the cylindrical background metric, the final and initial transformations correspond even more simply to the reflection $t \rightarrow -t$. The parabolic translation is more complicated to describe on the cylinder.

Now let us perform these steps explicitly on $u_\varepsilon(x) = |x|^{\frac{2-n}{2}} v_\varepsilon(-\log|x|)$. After the first inversion, we obtain $|x|^{\frac{2-n}{2}} v_\varepsilon(\log|x|)$. Translation carries this to $|x-a|^{\frac{2-n}{2}} v_\varepsilon(\log|x-a|)$. Performing the final inversion yields at last

$$u_{\varepsilon,a}(x) = \left| \frac{x}{|x|^2} - a \right|^{\frac{2-n}{2}} v_\varepsilon\left(\log \left| \frac{x}{|x|^2} - a \right|\right) = |x|^{\frac{2-n}{2}} |\theta - a|x|^{\frac{2-n}{2}} v_\varepsilon(-\log|x| + \log|\theta - a|x|),$$

where $\theta = x/|x|$. We can also define the associated functions

$$v_{\varepsilon,a}(t, \theta) = |\theta - ae^{-t}|^{\frac{2-n}{2}} v_\varepsilon(t + \log|\theta - ae^{-t}|)$$

on the cylinder; these are solutions of (6). Notice that these functions are regular except at $t = \log|a|$, $\theta = a/|a|$, and in particular are smooth for t sufficiently large.

We conclude this section by calculating the expansion of $u_{\varepsilon,a}$ near $|x| = 0$. To do this we first observe that

$$\left| \frac{x}{|x|} - a|x| \right|^{\frac{2-n}{2}} = 1 + \frac{n-2}{2} a \cdot x + O(|x|^2).$$

Similarly we calculate

$$\log \left| \frac{x}{|x|} - a|x| \right| = -a \cdot x + O(|x|^2),$$

and so

$$v_\varepsilon(-\log|x| - a \cdot x + O(|x|^2)) = v_\varepsilon(-\log|x|) - v'_\varepsilon(-\log|x|)(a \cdot x) + O(|x|^2).$$

Putting these altogether we finally obtain

$$\begin{aligned} u_{\varepsilon,a}(x) &= |x|^{\frac{2-n}{2}} \left(v_\varepsilon(-\log|x|) + a \cdot x \left(-v'_\varepsilon(-\log|x|) + \frac{n-2}{2} v_\varepsilon(-\log|x|) \right) + O(|x|^2) \right) \\ &= u_\varepsilon(x) + |x|^{\frac{2-n}{2}} (a \cdot x) \left(-v'_\varepsilon + \frac{n-2}{2} v_\varepsilon \right) + O(|x|^{\frac{6-n}{2}}). \end{aligned} \quad (9)$$

In particular, $|v_{\varepsilon,a}(t, \theta) - v_\varepsilon(t)| \leq Ce^{-t}$.

We have now obtained the families of solutions $u_{\varepsilon,a}(x)$ and $v_{\varepsilon,a}(t, \theta)$ for the problems (1) and (6), respectively. For all $T \in \mathbb{R}$, changing $v_\varepsilon(t)$ into $v_\varepsilon(t+T)$ in the construction above leads to families of solutions $u_{\varepsilon,a,T}(x)$ and $v_{\varepsilon,a,T}(t, \theta)$ for the problems (1) and (6), respectively. We note again that the parameters $a \in \mathbb{R}^n$ and $T \in \mathbb{R}$ correspond to explicit geometric (conformal) motions; only the parameter ε does not arise from an extrinsic motion. For simplicity, when $a = 0$, we shall denote these functions simply by u_ε and v_ε .

2.2 The linearized equation

We shall now consider the linearization

$$\mathcal{L}_\varepsilon = \frac{\partial^2}{\partial t^2} + \Delta_\theta - \frac{(n-2)^2}{4} + \frac{n(n+2)}{4} v_\varepsilon^{\frac{4}{n-2}} \quad (10)$$

of the operator in (6) around one of the *radial* Fowler solutions v_ε . Our primary interest is in the mapping properties of \mathcal{L}_ε , which we shall review from [11] and [9].

This operator has periodic coefficients, hence may be studied by classical Floquet - or Bloch wave - theoretic methods, as in [11], but also by separation of variables and elementary ODE methods, as in [9]. We shall need to refer to results derived from both of these methods.

Let $\{\lambda_j, \chi_j(\theta)\}$ be the eigendata of $\Delta_{S^{n-1}}$. We use the convention that the eigenvalues are listed with multiplicity, so that $\lambda_0 = 0, \lambda_1 = \dots = \lambda_n = n-1, \lambda_{n+1} = 2n$, etc. Then \mathcal{L}_ε decouples into infinitely many ordinary differential operators

$$\mathcal{L}_{\varepsilon,j} = \frac{d^2}{dt^2} + \left(\frac{n(n+2)}{4} v_\varepsilon^{\frac{4}{n-2}} - \frac{(n-2)^2}{4} - \lambda_j \right).$$

When $j > n+1$, the term of order zero in each of the $\mathcal{L}_{\varepsilon,j}$ is negative because $\lambda_j \geq 2n$ and $v_\varepsilon < 1$; hence for these values of j , $\mathcal{L}_{\varepsilon,j}$ satisfies the maximum principle. The same conclusion holds for $\mathcal{L}_{\varepsilon,j}$, $j = 1, \dots, n$, because conjugating by an appropriate power of v_ε yields an operator also with negative term of order zero. It follows from these facts that the L^2 spectrum of $-\mathcal{L}_{\varepsilon,j}$ is contained in $(0, \infty)$ when $j > 0$. On the other hand, 0 is contained in the *essential* spectrum of $-\mathcal{L}_{\varepsilon,0}$. In particular, \mathcal{L} does not have closed range on $L^2(\mathbb{R} \times S^{n-1}; dt d\theta)$, but the difficulty is localized to the ground eigenspace of the Laplacian on the cross-section.

The failure of \mathcal{L}_ε to have closed range is caused by its Jacobi fields, i.e. by the solutions of $\mathcal{L}_\varepsilon \psi = 0$. To study these, it suffices to consider the solutions of the induced problems $\mathcal{L}_{\varepsilon,j} \psi_j = 0$. It can be proved, cf. [11], that for each $j > 0$ there are two normalized linearly independent solutions $\psi_{\varepsilon,j}^\pm$ and an associated constant $\gamma_{\varepsilon,j}$ such that $|\psi_{\varepsilon,j}^\pm(t)| \leq e^{\mp \gamma_{\varepsilon,j} t}$ for all t . The solutions $\psi_{\varepsilon,0}^+(t), \psi_{\varepsilon,0}^-(t)$ for $\mathcal{L}_{\varepsilon,0}$ are bounded and linearly growing, respectively, and so it is natural to define $\gamma_{\varepsilon,0} = 0$. We immediately deduce the following

Proposition 2 *Suppose that $\mathcal{L}_\varepsilon \psi = 0$ on $\mathbb{R}^+ \times S^{n-1}$ and $\psi = O(e^{-\gamma t})$ as $t \rightarrow \infty$, for some $\gamma > 0$. Then $\psi(t, \theta) = \sum_{j=j_0}^{\infty} \psi_{\varepsilon,j}^+(t) \chi_j(\theta)$, where j_0 is the first integer such that $\gamma_{j_0} > \gamma$.*

Somewhat remarkably, we may deduce the explicit forms of these solutions when $j = 0, \dots, n$. This rests on the fact that whenever we have a one-parameter family of solutions of the nonlinear equation (6), then the differential of this family gives a solution of the linearized equation. In the previous subsection we found several different one-parameter families of solutions. The simplest are the families

$$T \longrightarrow v_\varepsilon(t+T), \quad \text{and} \quad \varepsilon \longrightarrow v_\varepsilon(t).$$

Differentiation with respect to either of these two parameters yields the solutions corresponding to the cross-sectional eigenvalue $\lambda_0 = 0$.

$$\psi_{\varepsilon,0}^+(t) \equiv \frac{d}{dT} v_\varepsilon(t+T) = v'_\varepsilon(t), \quad \psi_{\varepsilon,0}^-(t) \equiv \frac{d}{d\varepsilon} v_\varepsilon(t) = \dot{v}_\varepsilon(t). \quad (11)$$

We may also differentiate $v_{\varepsilon,a}$ with respect to a_j , using the expansion (9), to get

$$\psi_{\varepsilon,j}^+(t) \chi_j(\theta) \equiv \frac{d}{da_j} v_{\varepsilon,a}(t, \theta) = e^{-t} \left(-v'_\varepsilon(t) + \frac{n-2}{2} v_\varepsilon(t) \right) \chi_j(\theta); \quad (12)$$

these give the exponentially decreasing solutions of $\mathcal{L}_{\varepsilon,j} \psi = 0$, $j = 1, \dots, n$. Since these functions are all equal to one another, it is often simpler to denote any one of them by just $\psi_{\varepsilon,1}^+$. In any case, using them we may restate (9) as

$$\begin{aligned} u_{\varepsilon,a}(x) &= |x|^{\frac{2-n}{2}} \left(v_\varepsilon(-\log|x|) + |x| \left(\sum_{j=1}^n a_j \chi_j(\theta) \psi_{\varepsilon,j}^+(-\log|x|) \right) + O(|x|^2) \right) \\ &= |x|^{\frac{2-n}{2}} \left(v_\varepsilon(-\log|x|) + (a \cdot x) \psi_{\varepsilon,1}^+(-\log|x|) + O(|x|^2) \right). \end{aligned} \quad (13)$$

Notice that $\psi_{\varepsilon,1}^+ = \dots = \psi_{\varepsilon,n}^+$. Although they are not required later, we also note that the exponentially increasing solutions $\psi_{\varepsilon,j}^-(t)$ of $\mathcal{L}_{\varepsilon,j}$, $j = 1, \dots, n$, are obtained from $\psi_{\varepsilon,j}^+(t)$ simply by replacing t by $-t$.

It follows from (12) that the ‘indicial root’ $\gamma_{\varepsilon,j} \equiv 1$, $j = 1, \dots, n$. It is somewhat unexpected that for $j \leq n$, $\gamma_{\varepsilon,j}$ is independent of ε ; one can show however that all other $\gamma_{\varepsilon,j}$ depend nontrivially on ε , and in fact tend to $(\frac{(n-2)^2}{4} + \lambda_j)^{1/2}$ as $\varepsilon \rightarrow 0$ [9].

The numbers $\{\pm\gamma_{\varepsilon,j}\}$ are analogous to the indicial roots of a Fuchsian operator, and they influence the mapping properties in exactly the same way. To state these mapping properties most suitably for our purposes we define weighted Hölder spaces on the (half-) cylinder

$$\forall t_0 \in \mathbb{R}, \quad \mathcal{C}_\gamma^{k,\alpha}([t_0, +\infty) \times S^{n-1}) = \{w = e^{t\gamma\bar{w}} : \bar{w} \in \mathcal{C}^{k,\alpha}([t_0, +\infty) \times S^{n-1})\}.$$

If \mathcal{F} is any space of functions defined on the half-cylinder $[t_0, +\infty) \times S^{n-1}$, we let $\mathcal{F}_{\mathcal{D}}$ denote the subspace of functions of \mathcal{F} vanishing at $t = t_0$ for some fixed t_0 . We cannot choose t_0 arbitrarily because it is necessary for the following result that $\psi_j^\pm(t_0) \neq 0$ for $j \leq n$; this condition is always fulfilled for $j = 1, \dots, n$ but fails for $j = 0$ when $t_0 \in \{(\ell/2)T_\varepsilon : \ell \in \mathbb{Z}\}$, T_ε being the period of v_ε . We also define

$$E_{\varepsilon,0} = \text{Span}\{\psi_{\varepsilon,0}^+(t)\} \subset \mathcal{C}_0^{2,\alpha}([t_0, +\infty) \times S^{n-1})$$

and

$$E_{\varepsilon,1} = \text{Span}\{\psi_{\varepsilon,j}^+(t)\chi_j(\theta) : j = 1, \dots, n\} \subset \mathcal{C}_{-1}^{2,\alpha}([t_0, +\infty) \times S^{n-1}).$$

Using this notation, the result we need is

Proposition 3 *Assume that $t_0 \neq (\ell/2)T_\varepsilon$, $\ell \in \mathbb{Z}$. Recalling that $\gamma_{\varepsilon,0} = 0$, $\gamma_{\varepsilon,1} = \dots = \gamma_{\varepsilon,n} = 1$ and $\gamma_{\varepsilon,n+1} > 1$, if $0 < \gamma < 1$, then*

$$\mathcal{L}_\varepsilon : [\mathcal{C}_{-\gamma}^{2,\alpha}([t_0, \infty) \times S^{n-1}) \oplus E_{\varepsilon,0}]_{\mathcal{D}} \longrightarrow \mathcal{C}_{-\gamma}^{0,\alpha}([t_0, \infty) \times S^{n-1})$$

is a surjective Fredholm mapping. In particular, there exists a bounded right inverse

$$\mathcal{G}_{\varepsilon,0} : \mathcal{C}_{-\gamma}^{0,\alpha}([t_0, \infty) \times S^{n-1}) \longrightarrow [\mathcal{C}_{-\gamma}^{2,\alpha}([t_0, \infty) \times S^{n-1}) \oplus E_{\varepsilon,0}]_{\mathcal{D}}$$

so that $\mathcal{L}_\varepsilon \mathcal{G}_{\varepsilon,0} = I$. If $1 < \gamma < \gamma_{\varepsilon,n+1}$, then

$$\mathcal{L}_\varepsilon : [\mathcal{C}_{-\gamma}^{2,\alpha}([t_0, \infty) \times S^{n-1}) \oplus E_{\varepsilon,0} \oplus E_{\varepsilon,1}]_{\mathcal{D}} \longrightarrow \mathcal{C}_{-\gamma}^{0,\alpha}([t_0, \infty) \times S^{n-1})$$

is also a surjective Fredholm mapping, and there exists a bounded right inverse

$$\mathcal{G}_{\varepsilon,1} : \mathcal{C}_{-\gamma}^{0,\alpha}([t_0, \infty) \times S^{n-1}) \longrightarrow [\mathcal{C}_{-\gamma}^{2,\alpha}([t_0, \infty) \times S^{n-1}) \oplus E_{\varepsilon,0} \oplus E_{\varepsilon,1}]_{\mathcal{D}}$$

with $\mathcal{L}_\varepsilon \mathcal{G}_{\varepsilon,1} = I$.

One may find solutions of $\mathcal{L}_\varepsilon w = f$ for $f \in \mathcal{C}_{-\gamma}^{0,\alpha}([t_0, +\infty) \times S^{n-1})$ by considering the induced differential operators $\mathcal{L}_{\varepsilon,j}$ on each eigenspace of Δ_θ separately. When $j > n$, solutions are easily obtained using the fact that $\mathcal{L}_{\varepsilon,j}$ satisfies the maximum principle. This is done explicitly in [9]. To obtain solutions when $j \leq n$ it is easiest to use the Fourier-Laplace methods of [11]. Although these methods are L^2 -based, it is not hard to check that $w \in \mathcal{C}_{-\gamma}^{2,\alpha}([t_0, +\infty) \times S^{n-1})$ if $f \in \mathcal{C}_{-\gamma}^{0,\alpha}([t_0, +\infty) \times S^{n-1})$. The necessity of adding $E_{\varepsilon,0}$ and $E_{\varepsilon,1}$ to the domains to obtain surjectivity for the homogeneous Dirichlet problem is a simple form of the linear regularity theorem of [11], but also follows from easy ODE arguments.

It is important in this result that the weight factor $-\gamma$ does not equal one of the $\pm\gamma_{\varepsilon,j}$. In fact, on any one of the spaces $\mathcal{C}_{\pm\gamma_{\varepsilon,j}}^{2,\alpha}([t_0, +\infty) \times S^{n-1})$ the operator \mathcal{L}_ε does not have closed range.

This proposition may be used not only to find solutions of the equation $\mathcal{L}_\varepsilon u = f$, but also to obtain decay properties of solutions which are already given.

Corollary 1 *Suppose that μ and γ are weight parameters and $\mathcal{L}_\varepsilon u = f$ where $u \in \mathcal{C}_{-\mu}^{2,\alpha}(\mathbb{R}^+ \times S^{n-1})$ and $f \in \mathcal{C}_{-\gamma}^{0,\alpha}(\mathbb{R}^+ \times S^{n-1})$. If $0 < \mu < \gamma < 1$, then $u \in \mathcal{C}_{-\gamma}^{2,\alpha}(\mathbb{R}^+ \times S^{n-1})$. If $0 < \mu < 1 < \gamma < \gamma_{\varepsilon,n+1}$, then $u \in \mathcal{C}_{-\gamma}^{2,\alpha}(\mathbb{R}^+ \times S^{n-1}) \oplus E_{\varepsilon,1}$.*

The proof of this corollary is straightforward. For example, if $0 < \mu < \gamma < 1$, fix some $t_0 > 0$ satisfying the assumptions of Proposition 3, then $v + c\psi_{\varepsilon,0}^+ = \mathcal{G}_{\varepsilon,0}f \in [\mathcal{C}_{-\gamma}^{2,\alpha}([t_0, +\infty) \times S^{n-1}) \oplus E_{\varepsilon,0}]_{\mathcal{D}}$ is also a solution of $\mathcal{L}_{\varepsilon}v = f$. Hence $h = u - v$ is a homogeneous solution, $\mathcal{L}_{\varepsilon}h = 0$, which is also exponentially decaying as $t \rightarrow +\infty$. This means that $h \in \mathcal{C}_{-1}^{2,\alpha}([t_0, +\infty) \times S^{n-1})$, and so we conclude that u has the stated decay. The other cases are proved similarly.

3 Upper and lower bounds for solutions

In this section we derive upper and lower bounds on solutions defined in the punctured unit ball. We further show that any such solution has a nonzero radial Pohožaev invariant \mathcal{P}_{ra} . Specifically we prove the following.

Theorem 2 *Assume that u is a nonnegative smooth solution of (1) defined in the punctured unit ball $B^n \setminus \{0\}$. Either u extends as a smooth solution to the ball, or there exist positive constants C_1, C_2 such that*

$$C_1|x|^{(2-n)/2} \leq u(x) \leq C_2|x|^{(2-n)/2}.$$

Furthermore, the radial Pohožaev invariant of u is nonzero.

As stated, the constants C_1 and C_2 depend on the solution u . We will see in this section that the upper bound C_2 follows from a more precise universal upper bound, while the discussion of the previous section, and in particular the fact that the infimum of v_{ε} tends to zero with ε , shows that the lower bound depends in a more delicate way on the solution. It seems to be unknown how one should express the lower bound in a universal form. A universal upper bound for global weak solutions was given in [19], and was later recorded and used in [16]. A local version for smooth solutions was obtained by the fourth author, and presented in his course at the Courant Institute in 1989-90. This result was written up and applied by Y.-Y. Li [6].

The main subtlety in obtaining an upper bound is the existence of spherical solutions on \mathbb{R}^n .

Definition 1 *For any $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, the spherical solution of dilation λ and center x_0 is given by*

$$u_{\lambda,x_0}(x) = |x - x_0|^{\frac{2-n}{2}} (\cosh(-\log|x - x_0| + \log \lambda))^{\frac{2-n}{2}} = \left(\frac{2\lambda}{1 + \lambda^2|x - x_0|^2} \right)^{\frac{n-2}{2}}.$$

Any one of these functions gives a conformal factor transforming the flat metric on \mathbb{R}^n to the pullback of the standard metric on the sphere by a conformal transformation. The danger is that a general solution may be well approximated by one of these, in particular by a strongly dilated one, in some neighbourhood. This corresponds to the phenomenon of bubbling.

We present here a local version of the upper bound for solutions defined in a region which extend to be weak supersolutions across the complement. A positive function u in $L_{\text{loc}}^{\frac{n+2}{n-2}}(B^n)$ is a weak supersolution of (1) if it satisfies

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} \leq 0$$

in the distributional sense on B^n . We will be especially interested in supersolutions which are smooth solutions on an open subset Ω of B^n . The complement $\Lambda = B^n \setminus \Omega$ is then relatively closed in B^n . Note that such a function u is superharmonic, and thus can be redefined on a set of measure zero so as to be upper semicontinuous. As such, the restriction of u to any compact subset of B^n achieves its infimum.

We remark that in the main application, u will be a weak solution. The reason for considering supersolutions is that it was shown in ([20], Theorem 5.1) that any solution u which defines a complete metric on a region Ω in the sphere extends as a weak supersolution to the full sphere.

Moreover, any solution u on Ω which lies in $L^{\frac{n+2}{n-2}}_{\text{loc}}(B^n)$ and tends locally uniformly to infinity on approach to Λ extends as a weak supersolution on B^n . This can be seen by observing that for any large constant L , the function $u_L = \min\{u, L\}$ is a weak solution of the inequality

$$\Delta u_L \leq -\frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} \chi_{\{u \leq L\}}$$

where χ_E denotes the characteristic function of a set E . A simple application of the dominated convergence theorem then shows that u defines a weak supersolution.

The proof of the upper bound estimate for a solution u in B^n which is singular only at the origin will require that u extends to a weak solution of (1) on all of B^n . This follows from a more general result. In fact, any solution of (1) defined outside a ‘sufficiently thin’ set automatically extends as a weak solution. To make this precise, we let $\Lambda \subseteq B^n$ be a relatively closed set of Lebesgue measure zero. We will call Λ **thin** if there is a sequence ϕ_i of smooth functions on B^n with values in $[0, 1]$ such that $\phi_i \equiv 0$ in a neighborhood of Λ , $\lim \phi_i = 1$ on $B^n \setminus \Lambda$, and

$$\lim_{i \rightarrow \infty} \int_{B(0,r)} (|\Delta \phi_i|^{(n+2)/4} + |\nabla \phi_i|^{(n+2)/2}) dx = 0$$

for any $r < 1$. One sees that a smooth submanifold of dimension less than $\frac{n-2}{2}$ is thin by choosing ϕ_i to be a function of distance to the submanifold; in particular, a point is thin in any dimension.

Lemma 1 *Suppose u is a solution of (1) defined on $B^n \setminus \Lambda$ where Λ is a relatively closed thin set. Then u lies locally in $L^{(n+2)/(n-2)}(B^n)$, and defines a weak solution of (1) on B^n .*

To prove this we first show that u is locally in $L^{\frac{n+2}{n-2}}(B^n)$. Let ζ be a smooth nonnegative function with compact support in $B^n \setminus \Lambda$, and multiply (1) by $\zeta^{(n+2)/2}$ and integrate by parts to obtain

$$\frac{n(n-2)}{4} \int \zeta^{(n+2)/2} u^{(n+2)/(n-2)} dx = - \int u \Delta (\zeta^{(n+2)/2}) dx.$$

This implies the bound

$$\int (\zeta^{(n-2)/2} u)^{(n+2)/(n-2)} dx \leq c \int \zeta^{(n-2)/2} u (\zeta |\Delta \zeta| + |\nabla \zeta|^2) dx.$$

An application of Hölder’s inequality then implies

$$\int (\zeta^{(n-2)/2} u)^{(n+2)/(n-2)} dx \leq c \int (\zeta |\Delta \zeta| + |\nabla \zeta|^2)^{(n+2)/4} dx.$$

Now we choose $\zeta = \psi \phi_i$ where ψ is a function which is equal to one on $B(0, r)$ for some $r < 1$, and equal to zero outside $B(0, r_1)$ for some $r_1 \in (r, 1)$. We then have by an easy estimate

$$(\zeta |\Delta \zeta| + |\nabla \zeta|^2)^{(n+2)/4} \leq c(1 + |\Delta \phi_i|^{(n+2)/4} + |\nabla \phi_i|^{(n+2)/2})$$

with a constant c depending on r, r_1 . Letting i tend to infinity we then conclude that u is in $L^{\frac{n+2}{n-2}}(B(0, r))$ for any $r < 1$.

To complete the proof of the lemma, we now let ζ be any chosen smooth compactly supported function in B^n , and we multiply (1) by $\zeta \phi_i$ and integrate by parts to obtain

$$\int_{B^n} (u \Delta (\zeta \phi_i) + \frac{n(n-2)}{2} \zeta \phi_i u^{(n+2)/(n-2)}) dx = 0.$$

We expand

$$\Delta (\zeta \phi_i) = \zeta \Delta (\phi_i) + 2 \langle \nabla \zeta, \nabla \phi_i \rangle + \phi_i \Delta \zeta.$$

By Hölder's inequality

$$\left| \int u \zeta \Delta(\phi_i) \right| \leq \|u \zeta\|_{L^{(n+2)/(n-2)}} \|\Delta(\phi_i)\|_{L^{(n+2)/4}(\text{spt}\zeta)},$$

and this term tends to zero as i tends to infinity. Similarly,

$$\left| \int u \langle \nabla \zeta, \nabla \phi_i \rangle \right| \leq \|u |\nabla \zeta|\|_{L^{(n+2)/(n-2)}} \|\nabla(\phi_i)\|_{L^{(n+2)/4}(\text{spt}\zeta)}$$

which also goes to zero. Therefore we may apply the bounded convergence theorem to let i tend to infinity and conclude

$$\int_{B^n} (u \Delta(\zeta) + \frac{n(n-2)}{2} \zeta u^{(n+2)/(n-2)}) dx = 0.$$

This completes the proof of Lemma 1.

In the following proposition we let $d(x, \Lambda)$ denote the distance from x to Λ for $x \in B^n$.

Proposition 4 *Let u be a positive supersolution of (1) which is a smooth solution on an open set $\Omega \subset B^n$. Then there exists a constant c depending only on n such that*

$$u(x) \leq c d(x, \Lambda)^{\frac{2-n}{2}} \left(\inf_{\partial B(0, 3/4)} u \right)^{-1}$$

for all $x \in \Omega \cap B(0, 1/2)$.

Let Λ_ϵ denote the neighborhood of radius ϵ about Λ . For $x \in B(0, 5/8)$, set $U_\epsilon(x) = d_\epsilon(x)^{\frac{n-2}{2}} u(x)$ where $d_\epsilon(x) = \min\{d(x, \Lambda_\epsilon), \frac{5}{8} - |x|\}$. In order to prove the proposition it suffices to show that

$$\sup_{B(0, 5/8)} U_\epsilon \leq c \left(\inf_{\partial B(0, 3/4)} u \right)^{-1} \quad (14)$$

for some constant c not depending on ϵ .

The first step in proving this is to show that given any constants R and $\delta \in (0, 1/2]$, there is a constant c_1 depending only on n , R , and δ such that if $M_\epsilon \equiv \sup_{B(0, 5/8)} U_\epsilon \geq c_1$, then the solution u “differs by at most δ ” from a spherical piece “of size R ” which is concentrated near any chosen maximum point $x_0 \in B(0, 5/8) \setminus \Lambda_\epsilon$ of U_ϵ . Precisely we mean that, if x_0 is a maximum point of U_ϵ , and we set $\lambda = u(x_0)^{2/(2-n)}$, then the rescaled function $w_\lambda(x) \equiv \lambda^{(n-2)/2} u(x_0 + \lambda x)$ satisfies

$$\|u_{\mu, y_0} - w_\lambda\|_{C^2(B(0, R))} < \delta \quad (15)$$

where u_{μ, y_0} is the standard spherical solution defined above with $|y_0| \leq c_2$ and $1/c_2 \leq \mu \leq c_2$ for a constant c_2 depending only on n . To prove this assertion, we use an indirect blow-up argument. Assume that there is a sequence u_i with corresponding values $M_{i, \epsilon}$ tending to infinity. Let x_i be a maximum point of $U_{i, \epsilon}$, and let $w_i(x) = \lambda_i^{(n-2)/2} u_i(x_i + \lambda_i x)$ where $\lambda_i = u_i(x_i)^{2/(2-n)}$. Since x_i is a maximum point of $U_{i, \epsilon}$, we have $U_{i, \epsilon}(x) \leq U_{i, \epsilon}(x_i)$ for $x \in B(x_i, r_i)$ where $r_i = \frac{1}{2} d_{i, \epsilon}(x_i)$. Since $d_{i, \epsilon}(x) \geq \frac{1}{2} d_{i, \epsilon}(x_i)$ for $x \in B(x_i, r_i)$ we have $u_i(x) \leq 2^{(n-2)/2} u_i(x_i)$ for $x \in B(x_i, r_i)$. We then see that w_i is a solution of (1) which is regular and bounded by $2^{(n-2)/2}$ in $B(0, R_i)$ where $R_i = r_i / \lambda_i = \frac{1}{2} M_i^{2/(n-2)}$. Since R_i tends to infinity, we see in a standard way that a subsequence of the w_i converges in C^2 norm on compact subsets to a global positive solution w of (1). By the theorem of [3] this must be a standard spherical solution.

We now argue by contradiction. If the result were not true, there would exist a sequence u_i with corresponding values $M_{i, \epsilon}$ tending to infinity such that (15) does not hold for any $\mu > 0$ and any $y_0 \in \mathbb{R}^n$. This clearly contradicts the previous analysis.

Once (15) is proved, the bounds on y_0 and μ follow from the conditions $w_\lambda \leq 2^{(n-2)/2}$ in $B(0, R')$, where $R' = \frac{1}{2}d_\varepsilon(x_0)/\lambda \geq \frac{1}{2}c_1^{2/(n-2)}$ and $w_\lambda(0) = 1$. Indeed, increasing c_1 if necessary, we may always assume that $c_1 \geq 2^{n-2}$ so that $R' \geq \frac{1}{2}c_1^{2/(n-2)} \geq 2$. It follows from $w_\lambda \leq 2^{(n-2)/2}$ and $\delta \leq 1/2$ that $u_{\mu, y_0}(y) \leq 2^{(n-2)/2} + \delta \leq 2^{n/2}$ for all $y \in B(0, 2) \subset B(0, R')$ and it follows from $w_\lambda(0) = 1$ that $u_{\mu, y_0}(0) \geq 1 - \delta \geq 1/2$. Now assume that $y_0 \in B(0, 2)$, we may then take $y = y_0$ in the inequality $u_{\mu, y_0}(y) \leq 2^{n/2}$ to obtain $2\mu \leq 2^{n/(n-2)}$ which gives a bound on μ . Finally, if $y_0 \notin B(0, 2)$, there exists $y_1 \in B(0, 2)$ such that $|y_1 - y_0|^2 = |y_0|^2 - 1$ and it then follows from the inequality $u_{\mu, y_0}(y_1) \leq 2^{n/2}$ that

$$2^{-2/(n-2)}\mu \leq 1 + \mu^2|y_0|^2 - \mu^2$$

and from the inequality $u_{\mu, y_0}(0) \geq 1/2$ that

$$1 + \mu^2|y_0|^2 \leq 2^{n/(n-2)}\mu.$$

These inequalities imply first that $2^{-n/(n-2)} \leq \mu \leq 2^{n/(n-2)}$ and then $|y_0| \leq 2^{n/(n-2)}$.

Notice that the conclusion (15) can be replaced by the simpler inequality

$$\|u_{1,0} - w_\lambda\|_{C^2(B(0,R))} < \delta \tag{16}$$

if we allow ourselves to shift the center point x_0 to a new point x_1 which is within distance $2c_2\lambda$ where $c_2 = c_2(n)$ is the constant above, and which is a local maximum point of u . Thus we now are taking $w_\lambda(x) \equiv \lambda^{(n-2)/2}u(x_1 + \lambda x)$ with $\lambda = u(x_1)^{2/(2-n)}$. To see this we observe from the blow-up argument above that w_i will have a absolute maximum point near y_0 for i sufficiently large, since $|y_0|$ is bounded and since μ is bounded from below and from above. Shifting center to this point, and rescaling so that the maximum value is one then produces $u_{1,0}$ as the limit of the sequence. Note that the distance $2c_2\lambda_i$ is an arbitrarily small constant times r_i since $\lambda_i = 2M_i^{2/(2-n)}r_i$, and thus x_1 can be used in place of x_0 above and we can assume, increasing c_1 if necessary, that $2c_2\lambda_i < 1/16$.

To complete the proof of (14), we first note that $\inf_{\partial B(0,3/4)} u$ is bounded above since one can see easily from the fact that u is a supersolution (as in the proof of Lemma 1) that u is locally bounded in $L^{(n+2)/(n-2)}(B^n)$. Thus if the left hand side is bounded above by the constant c_1 (for a fixed chosen R and δ) of the previous paragraph, then we are done. Therefore we may assume that this is not the case, and that u satisfies the above bounds with δ chosen as small as desired, and R chosen as large as we like. In this case, we are going to prove that (14) still holds. We now introduce the function $v(t, \theta)$ on the cylinder $\mathbb{R} \times S^{n-1}$ which corresponds to the function $w = w_\lambda$ with $\lambda = u(x_1)^{2/(2-n)}$. Thus we define $v(t, \theta) = |x|^{(n-2)/2}w(x)$, where $t = -\log|x|$ and $\theta = x/|x|$. It is important to note in the following argument that because the point x_1 lies in the ball of radius $5/8 + 1/16 = 11/16$ about the origin, its distance to $\partial B(0, 3/4)$ is at least $1/16$; upon rescaling, we see that the ball $B(0, \frac{1}{16}\lambda^{-1})$ lies in the domain of w , and corresponds to a portion of $B(0, 3/4)$ in the domain of u . From (16), we see that v is close in C^2 norm on $[-\log R, \infty) \times S^{n-1}$ to the function $v_0(t, \theta) = (\cosh t)^{(2-n)/2}$. We now fix $R = e^2$ so that $-\log R = -2$. It follows that $\partial_t v(-1, \theta) > 0$ for all $\theta \in S^{n-1}$. Then apply the Alexandrov technique to v on the region $[-\log(\frac{1}{16}\lambda^{-1}), \infty) \times S^{n-1}$, reflecting across the spheres $\{t_1\} \times S^{n-1}$, starting with t_1 very positive, and continuing as far as possible. Because $\partial_t v(-1, \theta) > 0$, this procedure must end before t_1 reaches -1 . We will show below that because v is a supersolution, and the reflected function v^* is a solution, there can be no interior contact point with $v^* \leq v$. Since v is a regular solution near $t = t_1$ for $t_1 \geq -1$, the Hopf boundary point lemma implies that $\partial_t v(t_1, \theta) < 0$. Thus the reflection argument must end because a contact between v^* and v occurs on the boundary. In order for this to happen we must have

$$\inf\{v(-\log(\frac{1}{16}\lambda^{-1}), \theta) : \theta \in S^{n-1}\} < \sup\{v(t, \theta) : t > \log(\frac{1}{16}\lambda^{-1}) - 2, \theta \in S^{n-1}\}.$$

Recalling the definitions of v and λ , we see that

$$\inf\{v(-\log(\frac{1}{16}\lambda^{-1}), \theta) : \theta \in S^{n-1}\} = 16^{(2-n)/2} \inf_{\partial B(x_1, 1/16)} u \geq 16^{(2-n)/2} \inf_{\partial B(0, 3/4)} u$$

where the inequality holds because u is superharmonic. Finally, for δ small and *fixed*, we have $v(t, \theta) \leq 2v_0(t)$ for $t \geq 0$, and therefore we have

$$\inf_{\partial B(0, 3/4)} u \leq c \sup_{[\log(\frac{1}{16}\lambda^{-1})-2, \infty)} v_0 \leq c\lambda^{(n-2)/2} = cu(x_1)^{-1}$$

where we have used the definition of v_0 and λ . This implies (14) since U_ϵ is bounded above by a constant times u .

In this argument we needed to know that if $v^* \leq v$ in a connected open set, then either $v^* < v$ or $v^* \equiv v$. To see this, observe that $L(v - v^*) \leq (v^*)^{(n+2)/(n-2)} - v^{(n+2)/(n-2)} \leq 0$ weakly. The standard mean value inequality then gives us the desired conclusion. This completes the proof of the proposition.

For the proof of the lower bound, we require the following result.

Lemma 2 *If u is a positive solution of (1) defined in $B^n \setminus \{0\}$ with $\lim_{x \rightarrow 0} |x|^{(n-2)/2} u(x) = 0$, then u extends as a smooth solution to all of B^n .*

To prove this, first note that by Lemma 1 u extends as a weak solution to B^n . If we could show that $u \in L^p$ for some $p > 2n/(n-2)$, then standard results would imply first that u is bounded near the origin, and then by linear elliptic theory that it extends smoothly across 0. Let $v(t, \theta)$ be the corresponding cylindrical solution, and observe that the hypothesis implies that $v(t, \theta)$ tends to 0 uniformly as t tends to ∞ . Therefore (1) implies that $\Delta v \geq \beta v$ for some $\beta > 0$ and $t \geq t_0$ for sufficiently large t_0 . Now consider the function $w = ce^{-\sqrt{\beta}t} + \epsilon e^{\sqrt{\beta}t}$ where c is chosen large enough that $ce^{-\sqrt{\beta}t_0} > v(t_0, \theta)$ for all $\theta \in S^{n-1}$. Note that w is a solution of $\Delta w = \beta w$. We will use the maximum principle on $[t_0, T] \times S^{n-1}$ for any sufficiently large T to obtain the necessary decay estimate on v . If T is chosen sufficiently positive, then we have $v(T, \theta) < w(T, \theta)$ since v tends to zero as t tends to ∞ while w is growing at an exponential rate. By the maximum principle it follows that for all $t_0 \leq t \leq T$, all θ , and any chosen $\epsilon > 0$ we have

$$v(t, \theta) \leq ce^{-\sqrt{\beta}t} + \epsilon e^{\sqrt{\beta}t}.$$

Since T is arbitrarily large, this inequality holds for all $t \geq t_0$, and since c is independent of ϵ , we may let ϵ go to zero. We have shown that $e^{\sqrt{\beta}t} v(t, \theta) \leq c$ for $t \geq t_0$. Writing this in terms of u we have shown that $u(x) \leq c|x|^q$ for $q = (2-n)/2 + \sqrt{\beta}$. This implies that $u \in L^p$ for some $p > 2n/(n-2)$ as required. This completes the proof of Lemma 2.

We now are in a position to prove Theorem 2. The upper bound follows from Lemma 1 and Proposition 4, as already noted. To prove the lower bound, it is most convenient to use the cylindrical background, and so consider the function $v(t, \theta)$. We first observe that the upper bound $v \leq C_2$ implies a Harnack inequality of the form

$$\sup\{v(t, \theta) : T \leq t \leq T + T_0\} \leq c \inf\{v(t, \theta) : T \leq t \leq T + T_0\} \quad (17)$$

for all $T > 2$, and where the constant c depends on T_0 and C_2 , but not on T . Define $\bar{v}(t) = \int_{S^{n-1}} v(t, \theta) d\theta$. From the Harnack inequality we see that v and \bar{v} are bounded in ratio. Suppose then that the singularity at 0 is not removable. Applying Lemma 2 we see that either the lower bound holds or else there is a sequence t_i tending to ∞ such that $\bar{v}(t_i)$ tends to 0.

Using (17), the Harnack inequality, standard elliptic estimates as well as the fact that v is a solution of (1), we can estimate the radial Pohožaev invariant $\mathcal{P}_{\text{ra}}(v)$, as defined in §2 (cf. also §6), at any one of the t_i . Using the fact that it is independent of i , we see that the invariant must vanish. Now set $w_i(t, \theta) = v(t + t_i, \theta)/\bar{v}(t_i)$. Some subsequence of the w_i converges in C^2

on compact subsets of $\mathbb{R} \times S^{n-1}$ to a positive solution of $Lw = 0$. The associated function $h(x)$ defined on $\mathbb{R}^n \setminus \{0\}$ is then a positive harmonic function, and can therefore be written $a|x|^{n-2} + b$ for some $a, b \geq 0$. This is the same as the condition that $w(t, \theta) = ae^{(n-2)/2t} + be^{-(n-2)/2t}$. Since \bar{w} has a critical point at $t = 0$, it follows that $a = b$, and both are positive. Then, a direct calculation shows that

$$\begin{aligned} & \lim_{i \rightarrow +\infty} \int_{S^{n-1}} \left(\frac{1}{2} (\partial_t w_i)^2 - \frac{1}{2} |\nabla_{\theta} w_i|^2 - \frac{(n-2)^2}{8} w_i^2 + \bar{v}(t_i)^{\frac{4}{n-2}} \frac{(n-2)^2}{8} w_i^{\frac{2n}{n-2}} \right) d\theta \\ &= \int_{S^{n-1}} \frac{1}{2} (\partial_t w)^2 - \frac{(n-2)^2}{8} w^2 d\theta = -\omega_n \frac{(n-2)^2}{2} ab \neq 0 \end{aligned}$$

But this is a contradiction, for this is also a limit of rescalings of the Pohožaev invariant for v , and hence must be zero. Indeed

$$0 = \bar{v}(t_i)^{-2} \mathcal{P}_{\text{ra}}(v) = \int_{S^{n-1}} \left(\frac{1}{2} (\partial_t w_i)^2 - \frac{1}{2} |\nabla_{\theta} w_i|^2 - \frac{(n-2)^2}{8} w_i^2 + \bar{v}(t_i)^{\frac{4}{n-2}} \frac{(n-2)^2}{8} w_i^{\frac{2n}{n-2}} \right) d\theta \neq 0$$

for i sufficiently large. This establishes the lower bound.

To show that $\mathcal{P}_{\text{ra}}(v) \neq 0$, we observe that by the upper and lower bounds, we can choose a sequence t'_i tending to ∞ so that the corresponding translated solutions $t \rightarrow v(t + t'_i, \theta)$ converge in C^2 norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to a solution satisfying the same bounds and defined on all of $\mathbb{R} \times S^{n-1}$. By Proposition 1 such a solution must be independent of θ , and hence must have nonzero radial Pohožaev invariant. Since this invariant must be the same as $\mathcal{P}_{\text{ra}}(v)$ because of the C^2 convergence, we have shown $\mathcal{P}_{\text{ra}}(v) \neq 0$ as desired. The proof of Theorem 2 is complete.

4 Simple convergence to a radial solution

Our aim in this section is to prove

Proposition 5 *Assume that u is a nonnegative smooth solution of (1) defined in the punctured unit ball $B^n \setminus \{0\}$ with a nonremovable singularity at the origin. For $t > 0$ and $\theta \in S^{n-1}$, let us define $v(t, \theta)$ so that $u(x) = |x|^{\frac{2-n}{2}} v(-\log|x|, x/|x|)$. Then, there exists a Fowler parameter $\varepsilon \in (0, \varepsilon_0]$, constants $T \in [0, T_\varepsilon)$ and $C > 0$, and some exponent $\alpha > 0$ such that*

$$|v(t, \theta) - v_\varepsilon(t + T)| \leq Ce^{-\alpha t}, \quad \text{for } t \geq 0.$$

We first show that any sequence of translates of v has a subsequence converging to one of the v_ε , then that any angular derivative of v converges to zero. After this, a somewhat delicate rescaling argument due originally to Leon Simon, in a different context, gives the final convergence.

By the results of the last section we know that $0 < C_1 \leq v(t, \theta) \leq C_2$ for all $t \geq 0$. By standard elliptic estimates, we also get the uniform boundedness of any derivative $|\partial_t^j \partial_\theta^\gamma v(t, \theta)| \leq C_{j,\gamma}$ for all $t \geq 0$.

Let $\{\tau_j\}$ be any sequence of numbers converging to ∞ , and define $v_j(t, \theta) = v(t + \tau_j, \theta)$. Then v_j is defined on $[-\tau_j, \infty) \times S^{n-1}$ and satisfies (6) there. Using the uniform bounds on any derivative of v_j , we may choose a subsequence of the v_j converging in the C^∞ topology on any compact subset of $\mathbb{R} \times S^{n-1}$. The limit function, v_∞ , still satisfies (6), is nonvanishing because of the lower bound for v and is defined on the whole cylinder. By Proposition 1, the only functions with these properties are the translated Fowler solutions; hence we deduce that for some ε and T , $v_\infty(t, \theta) = v_\varepsilon(t + T)$.

The radial Pohožaev invariant of v equals that of any one of the v_j , hence also equals that of $v_\varepsilon(t + T)$. Hence the limiting necksize ε is independent of the original sequence of numbers τ_j and of the subsequence.

The fact that any sequence of translates of v has a subsequence converging to a θ -independent solution is very strong. It implies immediately that any angular derivative $\partial_\theta v$ tends to zero uniformly. For if this were false, then there would be a sequence of points (τ_j, θ_j) with $\tau_j \rightarrow \infty$ for which $|\partial_\theta v(\tau_j, \theta_j)| \geq C > 0$. But then, translating back by τ_j and rotating θ_j to some fixed point $\theta_0 \in S^{n-1}$ to get a new sequence of solutions v_j , we can again extract some subsequence converging to a radial function. But this contradicts the positive lower bound on $|\partial_\theta v_j(0, \theta_0)|$.

Next, let X be any infinitesimal rotation on S^{n-1} , which we denote for simplicity by ∂_θ ; applying it to (6) and using that it commutes with Δ_θ , we see that $\partial_\theta v$ is a Jacobi field for the Jacobi operator at v , i.e. the linearization of (6) at v . By the discussion above, we know that $\partial_\theta v = o(1)$, but since we know little about this Jacobi operator, we cannot deduce better decay directly. However, for some $\tau_j \rightarrow \infty$ consider the corresponding sequence of translates $\partial_\theta v_j$ and let $A_j = \sup_{t \geq 0} |\partial_\theta v_j|$. If this supremum is attained at some point (s_j, θ_j) then s_j must stay bounded. Otherwise we could translate back further by s_j to obtain, after passing to a subsequence, a Jacobi field ϕ for the Jacobi operator at $v_\varepsilon(t+T)$, for some T , defined on all of \mathbb{R} . By construction, ϕ is bounded and not identically zero, but on the other hand, since it clearly has no zero eigenvector relative to Δ_θ , the results above show that it must increase exponentially either as $t \rightarrow +\infty$ or as $t \rightarrow -\infty$. This contradiction shows that s_j stays bounded. Now fix a positive integer N such that $NT_\varepsilon > \sup s_j$ and define $I_N \equiv \{0 \leq t \leq NT_\varepsilon\}$; here, of course, T_ε is the period of v_ε . Then $A_j = \sup_{I_N} |\partial_\theta v_j|$, and we obtain again, after passing to a subsequence, a Jacobi field ϕ for the Jacobi operator at some $v_\varepsilon(t+T)$ which is bounded for $t \geq 0$ and attains its supremum of 1 in I_N .

By the results on this Jacobi operator stated earlier, ϕ must decay at least like e^{-t} . Actually, something slightly stronger is true. Although the limit ϕ is not necessarily unique, there exists a constant c , independent of all other choices, such that $|\phi(t)| \leq ce^{-t}$ for $t \geq 0$. To see this, write $\phi = \sum_{j=1}^n c_j \psi_{\varepsilon,j}^+ + \phi'' = \phi' + \phi''$, where $\phi'' \in \text{Span}\{h_j(t)\chi_j(\theta) : j \geq n+1\} \equiv E''$. Since $|\phi| \leq 1$ for $t \geq 0$, the c_j , $j = 1, \dots, n$, are absolutely bounded because they are given by the (normalized) inner products of ϕ with $\chi_j(\theta)$. Hence $|\sum_{j=1}^n c_j \psi_{\varepsilon,j}^+| \leq ce^{-t}$ with $c = c(n)$. Next, we claim that $|\phi''| \leq ce^{-t}$ for some c independent of all other choices, and that this is a consequence of the fact that ϕ'' is absolutely bounded for $t \geq 0$ and $\phi'' \in E''$. We argue by contradiction and assume that there exists a sequence of Jacobi fields $\phi_i'' \in E''$ such that $A_i'' = \sup_{t \geq 0} e^t |\phi_i''(t, \theta)|$ tends to infinity. If this supremum is achieved at a point (t_i, θ_i) , then define $\tilde{\phi}_i(t, \theta) = e^{t_i} A_i''^{-1} \phi_i''(t + t_i, \theta)$; these are Jacobi fields on $[-t_i, \infty)$ which are bounded by e^{-t} , and which take the value 1 at $(0, \theta_i)$. The t_i clearly cannot be bounded, because the Jacobi equation is uniformly elliptic on any finite piece of the cylinder. Passing to a subsequence, if necessary, we obtain a Jacobi field $\tilde{\phi}$ defined and satisfying $|\tilde{\phi}| \leq e^{-t}$ on the whole cylinder. Since we also know that $\tilde{\phi} \in E''$, it must grow faster than e^{-t} either at $+\infty$ or $-\infty$, which is a contradiction. This proves the claim.

At this stage, it is now possible to show that $\partial_\theta v$ decays exponentially. Although we do not, strictly speaking, need this result, we sketch the proof anyway as a warm-up to the arguments later. Let $J_N \equiv \{NT_\varepsilon \leq t \leq 2NT_\varepsilon\}$, and denote by $v_\tau(t, \theta)$ the translate $v(t + \tau, \theta)$. We claim that if N is chosen large enough then

$$\forall \tau \geq 0, \quad \sup_{J_N} |\partial_\theta v_\tau| \leq \frac{1}{2} \sup_{I_N} |\partial_\theta v_\tau|.$$

As usual, this is proved by contradiction. If it were to fail, there would exist some sequence $\tau_j \rightarrow \infty$ such that $B_j > \frac{1}{2} A_j$, where

$$A_j \equiv \sup_{I_N} |\partial_\theta v_j|, \quad B_j \equiv \sup_{J_N} |\partial_\theta v_j|.$$

However, since $A_j^{-1} \partial_\theta v_j$ converges to the Jacobi field ϕ uniformly on compact sets, we see that $\partial_\theta v_j = A_j \phi + o(A_j)$ on $I_N \cup J_N$. Since $|\phi| \leq Ce^{-t}$, we then get that $B_j \leq CA_j e^{-NT_\varepsilon} + \frac{1}{4} A_j$ for j sufficiently large. However, as we have seen, the constant C is bounded independently of other

choices since ϕ is normalized to be less than 1 on I_N , so if N is sufficiently large, $Ce^{-NT_\varepsilon} \leq \frac{1}{4}$, and we obtain a contradiction. This proves the assertion. It is now trivial to deduce that $\partial_\theta v$ decays at some exponential rate.

We may use the sequence of translates v_j to obtain a Jacobi field in another way. Suppose that we have chosen a subsequence converging to some $v_\varepsilon(t+T)$. Define

$$w_j(t, \theta) = v_j(t, \theta) - v_\varepsilon(t+T), \quad \alpha_j = \sup_{I_N} |w_j|, \quad \text{and} \quad \phi_j = \alpha_j^{-1} w_j.$$

Then it is easy to check that ϕ_j converges to a solution ϕ of the Jacobi operator at $v_\varepsilon(t+T)$. This Jacobi field is bounded on $t \geq 0$. To see this we examine its eigencomponents with respect to Δ_θ . First consider the sum of eigencomponents over all nonzero eigenvalues, which we call $\tilde{\phi}$. To show that $\tilde{\phi}$ is bounded, and hence exponentially decaying, it is sufficient to show that $\partial_\theta \phi$ is also bounded for $t \geq 0$. We may assume that $\tilde{\phi}$ is not identically zero, for otherwise this case is trivial. This function arises as the limit of a subsequence of the sequence $\alpha_j^{-1} \partial_\theta v_j$. If we knew that α_j were commensurate with A_j , the supremum of $\partial_\theta v_j$ on I_N , i.e. if $C_1 A_j \leq \alpha_j \leq C_2 A_j$ for some constants $C_1, C_2 > 0$, then we could appeal to our earlier argument. However, this must be the case, for otherwise $\alpha_j^{-1} \partial_\theta v_j$ would either blow up or tend to zero on I_N , whereas we know that it converges to the nontrivial function $\partial_\theta \phi$. As for the remaining eigencomponent, with eigenvalue zero, we know that

$$\phi(t, \theta) = a^+ \psi_{\varepsilon,0}^+ + a^- \psi_{\varepsilon,0}^- + \tilde{\phi},$$

where $\tilde{\phi}$ decreases exponentially. We claim that $a^- = 0$. Intuitively this is true because this coefficient corresponds to an infinitesimal change of Fowler parameter. Rigorously, simply write $v_j = v_\varepsilon + \alpha_j \phi + o(\alpha_j)$ on I_N , say; if a^- were not to vanish, then the radial Pohožaev invariants of the two sides would differ, which is impossible. Thus we have shown that ϕ must be bounded for $t \geq 0$.

We are finally in a position to show that the difference between $v(t, \theta)$ and some $v_\varepsilon(t+T)$ converges to zero uniformly. The subtlety here arises because the displacement T cannot be detected by any of the Pohožaev integrals. We use again the same sort of argument as above, which is due to L. Simon. Define $v_\tau(t, \theta) = v(t+\tau, \theta)$ and set $w_\tau(t, \theta) = v_\tau(t, \theta) - v_\varepsilon(t)$. (Since we do not know the correct translation parameter T beforehand, we choose one arbitrarily, say $T=0$.) The idea is to establish an improvement of approximation in the following sense. Use the interval I_N as before and fix a constant $B > 0$, and let $\eta(\tau) = \sup_{I_N} |w_\tau|$. Then we claim :

If τ is sufficiently large and $\eta(\tau)$ sufficiently small, then there exists an s with $|s| \leq B\eta(\tau)$ such that $\eta(\tau + NT_\varepsilon + s) \leq \frac{1}{2}\eta(\tau)$.

To prove this, suppose that, for fixed values of N and B , it fails. Then there exists some sequence $\tau_j \rightarrow \infty$ such that $\eta_j \equiv \eta(\tau_j) \rightarrow 0$, but such that for any s with $|s| \leq B\eta_j$ we have $\eta(\tau_j + NT_\varepsilon + s) > \frac{1}{2}\eta_j$. Let $\phi_j = \eta_j^{-1} w_{\tau_j}$. We have shown already that ϕ_j converges in C^∞ on compact sets to a Jacobi field ϕ which is bounded for $t \geq 0$. By construction, $|\phi| \geq 1/2$ on $NT_\varepsilon \leq t \leq 2NT_\varepsilon$, so $\phi \not\equiv 0$. Expanding ϕ into eigenfunctions again, we get $\phi = a^+ \psi_{\varepsilon,0}^+ + \tilde{\phi}$, where $\tilde{\phi}$ is exponentially decreasing. Note that a^+ , which we relabel simply a , is uniformly bounded, independently of the sequence, because $|\phi| \leq 1$ on $0 \leq t \leq NT_\varepsilon$. We assume that the constant B has been chosen larger than this upper bound. Because $\psi_{\varepsilon,0}^+$ corresponds to infinitesimal translation, this argument indicates that we should adjust v_ε by some translation, and to first order, this adjustment should just be $s_j = -\eta_j a$; by our choice of B , this is less than $B\eta_j$. Thus we estimate

$$w_{\tau_j+s_j}(t, \theta) = v(t+\tau_j-\eta_j a, \theta) - v_\varepsilon(t) = w_{\tau_j}(t, \theta) - a\eta_j \psi_{\varepsilon,0}^+(t) + o(\eta_j).$$

In particular,

$$w_{\tau_j+s_j} = \eta_j \tilde{\phi} + o(\eta_j).$$

But the function $\tilde{\phi}$ decays at a fixed exponential rate and its supremum on $0 \leq t \leq NT_\varepsilon$ is bounded by one, so that if N is chosen large enough (but again, independent of our original

sequence τ_j , etc.) then

$$\sup_{0 \leq t \leq NT_\varepsilon} |w_{\tau_j + s_j + NT_\varepsilon}| = \sup_{NT_\varepsilon \leq t \leq 2NT_\varepsilon} |w_{\tau_j + s_j}| = \eta_j \sup_{NT_\varepsilon \leq t \leq 2NT_\varepsilon} |\tilde{\phi}| + o(\eta_j) \leq \frac{\eta_j}{4}.$$

But this contradicts the assumption that $\eta(\tau_j + NT_\varepsilon + s) > \frac{1}{2}\eta(\tau_j)$, and thus we have proved our claim.

It remains only to show how this approximation improvement property yields the correct simple asymptotics. In fact, we must prove that $w_\sigma \rightarrow 0$ for some fixed translation σ . We may assume, by starting at some sufficiently large t , that $\eta(0)$ is sufficiently small and that $B\eta(0) \leq \frac{1}{2}NT_\varepsilon$. Let $\tau_0 = 0$ and s_0 be determined by the first step of this argument. Define, for any integer $j > 0$,

$$\sigma_j = \sum_{i=0}^{j-1} s_i, \quad \tau_j = \tau_{j-1} + s_{j-1} + NT_\varepsilon.$$

Then by iteration, $\eta(\tau_j) \leq 2^{-j}\eta(0)$, $|s_j| \leq 2^{-j-1}NT_\varepsilon$, and hence in particular $\sigma = \lim \sigma_j$ is well defined and less than NT_ε . To show that σ is the correct translation parameter, for any $t > 0$, let $[t]$ denote the reduction mod NT_ε , so that $t = jNT_\varepsilon + [t]$ for some $j \geq 0$. Then

$$\begin{aligned} w_\sigma(t, \theta) &= v(t + \sigma, \theta) - v_\varepsilon(t) = (v(t + \sigma_j, \theta) - v_\varepsilon(t)) + (v(t + \sigma, \theta) - v(t + \sigma_j, \theta)) \\ &= w_{\tau_j}([t], \theta) + O(2^{-j}), \end{aligned}$$

where we have used Taylor's theorem and the uniform boundedness of $\partial_t v$ and $\partial_t v_\varepsilon$. But using the bound on $\eta(\tau_j)$, we finally conclude that

$$|w_\sigma(t, \theta)| \leq C2^{-j}, \quad \text{or equivalently} \quad |w_\sigma(t, \theta)| \leq C'e^{-\frac{\log 2}{NT_\varepsilon}t}.$$

This completes our proof of Proposition 5.

5 Refined asymptotics

In this last step of the proof of Theorem 1, we improve the asymptotics one step further. We show that for some $a \in \mathbb{R}^n$, u has an expansion of the form

$$u(x) = |x|^{\frac{2-n}{2}} \left(v_\varepsilon(-\log|x| + T) + (a \cdot x)\psi_{\varepsilon,1}^+(-\log|x| + T) + O(|x|^\beta) \right), \quad (18)$$

$$\text{where} \quad \beta = \min\{2, \gamma_2^+\}.$$

On the other hand, we also know that the deformed Fowler solution $u_{\varepsilon,a}$ has an expansion of this exact form, (9); comparing these two expansions completes the proof of the main theorem.

Using the simple asymptotics already established, write

$$u(x) = |x|^{\frac{2-n}{2}} v(-\log|x|) = |x|^{\frac{2-n}{2}} \left(v_\varepsilon(-\log|x| + T) + w(-\log|x|) \right)$$

where $w(t) \in \mathcal{C}_{-\gamma}^{2,\alpha}(\mathbb{R}^+ \times S^{n-1})$ for some $\gamma > 0$. The function $v(t)$ satisfies (6), which we write as $N(v) = 0$. Expanding this in a Taylor series about $v_\varepsilon(t + T)$ gives

$$\mathcal{L}_\varepsilon w = \frac{n(n-2)}{4} \left((v_\varepsilon + w)^{\frac{n+2}{n-2}} - v_\varepsilon^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} v_\varepsilon^{\frac{4}{n-2}} w \right) \equiv Q(w). \quad (19)$$

It is straightforward to check that if $w \in \mathcal{C}_{-\gamma}^{0,\alpha}(\mathbb{R}^+ \times S^{n-1})$ and $v_\varepsilon + w > 0$, then $Q(w) \in \mathcal{C}_{-2\gamma}^{0,\alpha}(\mathbb{R}^+ \times S^{n-1})$.

First assume that $0 < \gamma < \frac{1}{2}$. Then Corollary 1 gives that $w \in \mathcal{C}_{-2\gamma}^{2,\alpha}(\mathbb{R}^+ \times S^{n-1})$, and so $Q(w) \in \mathcal{C}_{-4\gamma}^{2,\alpha}(\mathbb{R}^+ \times S^{n-1})$. Continuing on, we deduce after finitely many steps that $w \in \mathcal{C}_{-\gamma'}^{2,\alpha}(\mathbb{R}^+ \times S^{n-1})$ for some $\gamma' \in (1/2, 1)$. Applying Corollary 1 again gives $w \in \mathcal{C}_{-\beta}^{2,\alpha}(\mathbb{R}^+ \times S^{n-1}) \oplus E_{\varepsilon,1}$, where $\beta = \min\{2\gamma', \gamma_2\}$.

The optimal β we could expect is $\min\{2, \gamma_2\}$. We have proved the expansion (18).

6 The global balancing formula

In this section we shall present an application of the refined asymptotics theorem. We consider solutions of the singular Yamabe problem with discrete singular set Λ in the standard conformal class $[g_0]$ on S^n . More specifically, let $\Lambda = \{p_1, \dots, p_k\} \subset S^n$ be arbitrary, $k \geq 2$. The solutions we consider are functions $u > 0$ on $S^n \setminus \Lambda$ such that $u^{\frac{4}{n-2}}g_0$ is a complete metric of constant positive scalar curvature $n(n-1)$ on $S^n \setminus \Lambda$. More analytically, these functions satisfy the special case of (2),

$$\Delta_{S^n} u - \frac{(n-2)^2}{4}u + \frac{(n-2)^2}{4}u^{\frac{n+2}{n-2}} = 0, \quad (20)$$

with $u > 0$ on $S^n \setminus \Lambda$ and u singular at the points of Λ . The ‘unmarked moduli space’ \mathcal{M}_k consists of all such solutions with k singular points, but with Λ allowed to vary. (One may also consider the ‘marked moduli space’ \mathcal{M}_Λ , consisting of all such solutions with fixed singular set Λ .) The most basic question about these moduli spaces, whether they are nonempty, was answered originally by the fourth author [17] some time ago; an alternate construction was recently given in [9].

The refined asymptotics theorem implies that to any $u \in \mathcal{M}_k$ one can associate a set of parameters: the singular points p_j , and then at each p_j the Fowler and translation parameters $\varepsilon_j \in (0, \varepsilon_0]$ and $a_j \in \mathbb{R}^n$. This yields a map

$$\Phi : \mathcal{M}_k \longrightarrow \mathcal{X}_k \equiv \left(\left(\prod_{j=1}^k S^n \right) \setminus \cup_{i,j} \Delta_{ij} \times (0, \varepsilon_0]^k \times (\mathbb{R}^n)^k \right) / \Sigma_k, \quad (21)$$

where Δ_{ij} is the diagonal of the product of the i^{th} factor and the j^{th} factor and Σ_k is the permutation group on k letters, which acts in the obvious way. It is shown in [10] (following the proof of the analogous fact for \mathcal{M}_Λ in [11]) that \mathcal{M}_k is a real analytic set of dimension $k(n+1)$; on the other hand, $\dim \mathcal{X}_k = k(2n+1)$. There are actually another k parameters, arising from the translations (what we called T earlier) at each p_j . Thus the correct parameter space is $2k(n+1)$ -dimensional. In current work, D. Pollack shows that this extended version of Φ is a Lagrangian immersion which is equivariant under the obvious action of the conformal group on either side.

Our purpose here is to use the refined asymptotics theorem to show that the image of \mathcal{M}_k in \mathcal{X}_k lies in the zero set of a collection of real analytic equations obtained by computing the global Pohožaev integral for each solution u with respect to any conformal Killing vector field X on S^n . Although there are not enough equations here in general to fully determine the moduli space \mathcal{M}_k , the relations we can obtain here are still interesting in their own right. Before embarking on this computation, though, we first define these integral invariants.

6.1 Pohožaev invariants

We now turn to a discussion of the existence and specific form of a family of homological integral invariants of solutions of equation (1). These ‘Pohožaev invariants’ were discovered in their simplest form by Pohožaev, and applied by him to prove the nonexistence of nonnegative smooth solutions of (1) satisfying Dirichlet conditions on smooth, star-shaped domains. The identity he discovered was later put into a natural and general Riemannian setting in [17]. These extended invariants associate to any metric g of constant scalar curvature on a manifold Ω , any closed hypersurface $\Sigma \subset \Omega$ and any conformal Killing field X , a real number $\mathcal{P}(X, \Sigma)$ which depends only on the the homology class of Σ in $H^{n-1}(\Omega)$. The invariant associated to X , g and Σ is

$$\mathcal{P}(X, g, \Sigma) \equiv \frac{n-2}{2(n-1)} \int_{\Sigma} T(X, \nu) d\sigma, \quad (22)$$

where $T(\cdot, \cdot)$ is the trace-free Ricci tensor for the metric g , ν is the unit normal to Σ and $d\sigma$ is the volume form induced by g on Σ .

In this subsection we shall compute these invariants in our particular case of interest, where $g = u^{\frac{4}{n-2}}\delta$, u is a solution of (1) in the ball $B(0,1)$ with an isolated singularity at 0 and $\Sigma = \Sigma_{|\eta|}$ is the sphere $|x| = \eta < 1$. To do this, we need two ingredients. First, the trace-free Ricci tensor T transforms fairly simply under conformal changes: if $g = \phi^{-2}g_0$ are any two conformally related metrics, with trace-free Ricci tensors T and T_0 , then

$$T = T_0 + (n-2)\phi^{-1} \left(Dd\phi - \frac{\Delta_0\phi}{n}g_0 \right).$$

Next, the conformal Killing vector fields on the sphere constitute a vector space of dimension $(n+1)(n+2)/2$. A spanning set is given by the following four basic types of conformal Killing fields:

- $X^{(b)} = \sum b_i \partial_{x_i}$, the generators of parabolic motions fixing infinity,
- $\mathcal{D} = \sum x_i \partial_{x_i}$, the generator of dilation,
- $R^{(b,c)} = \sum ((b \cdot x)c_i - (c \cdot x)b_i) \partial_{x_i}$, the generators of rotations, and
- $Y^{(b)} = \sum ((b \cdot x)x_i - \frac{1}{2}|x|^2 b_i) \partial_{x_i}$, the generators of parabolic motions fixing zero.

We shall compute relative to the Euclidean metric δ . The transformation rule for the trace-free Ricci tensor yields

$$T = \frac{2n}{n-2}u^{-2}du \cdot du - 2u^{-1}Ddu - \left(\frac{2}{n-2}u^{-2}|\nabla u|^2 + \frac{n-2}{2}u^{\frac{4}{n-2}} \right) \delta. \quad (23)$$

Next, the hypersurfaces Σ_η have unit normal and volume form

$$\nu = u^{\frac{-2}{n-2}} \frac{x}{|x|} = u^{\frac{-2}{n-2}} |x| \mathcal{D}, \quad \text{and} \quad d\sigma = u^{\frac{2(n-1)}{n-2}} |x|^{n-1} d\theta.$$

Thus we need to evaluate

$$\begin{aligned} \frac{n-2}{2(n-1)} \int_{|x|=\eta} T(X, \nu) d\sigma &= \int_{|x|=\eta} |x|^{n-2} \left\{ \frac{n}{n-1} (X \cdot \nabla u)(\mathcal{D} \cdot \nabla u) - \frac{n-2}{n-1} u (Ddu)(X, \mathcal{D}) \right. \\ &\quad \left. - \left(\frac{1}{n-1} |\nabla u|^2 + \frac{(n-2)^2}{4(n-1)} u^{\frac{2n}{n-2}} \right) (X \cdot \mathcal{D}) \right\} d\theta. \end{aligned} \quad (24)$$

Denote by S the symmetric 2-tensor appearing on the right here (and including the factor $|x|^{n-2}$).

The main considerations in the computations below involve homogeneity. Let us say that a term is of order j if it is a sum of products of terms homogeneous of order j and functions periodic in $\log|x|$. Thus, from the refined asymptotics of u we see that

$$S = S_{(-2)} + S_{(-1)} + S',$$

where $S_{(j)}$, $j = -2, -1$, is a symmetric bilinear form, the coefficients of which (in a standard Euclidean basis) are of order j . All coefficients in the remainder term S' are bounded. Each of the conformal fields X listed above is homogeneous, and so the corresponding term $S_{(j)}(X, \mathcal{D})$ is of order $j + \ell + 1$ if the coefficients of X are homogeneous of order ℓ . Since the integral itself is independent of the radius η , the invariant in each case, coincides with the integral of the term of order zero.

The easiest case is when $X = Y^{(b)}$, since it has coefficients homogeneous of order 2, and so $S(X, \mathcal{D}) = O(|x|)$. We conclude that

$$\mathcal{P}(Y^{(b)}, g) = 0. \quad (25)$$

The other cases require more work. First we make some general calculations. Using the refined asymptotics of u always, we first see that

$$\mathcal{D} \cdot \nabla u = |x|^{\frac{2-n}{2}} \left\{ -(v'_\varepsilon + \frac{n-2}{2}v_\varepsilon) + (a \cdot x) \left[-v'_\varepsilon + \frac{n-2}{2}v_\varepsilon - \frac{n(n-2)}{4}v_\varepsilon^{\frac{n+2}{n-2}} \right] + \dots \right\}. \quad (26)$$

Next, by definition of the Hessian of a function,

$$Ddu(X, \mathcal{D}) = X(\mathcal{D}u) - (\nabla_X \mathcal{D})u.$$

Since $\nabla_X \mathcal{D} = X$, this reduces to $X(\mathcal{D} \cdot \nabla u - u)$. Thus we will need that

$$\mathcal{D} \cdot \nabla u - u = |x|^{\frac{2-n}{2}} \left\{ -(v'_\varepsilon + \frac{n-2}{2}v_\varepsilon) - \frac{n(n-2)}{4}(a \cdot x)v_\varepsilon^{\frac{n+2}{n-2}} + \dots \right\}. \quad (27)$$

Finally,

$$u^{\frac{2n}{n-2}} = |x|^{-n} \left(v_\varepsilon^{\frac{2n}{n-2}} + \frac{2n}{n-2}(a \cdot x)(-v'_\varepsilon + \frac{n-2}{2}v_\varepsilon) + \dots \right). \quad (28)$$

First consider $X = \mathcal{D}$. Since $(\mathcal{D} \cdot \nabla u)^2 = |x|^2 |\nabla u|^2$, the first and third terms combine to give

$$\begin{aligned} (\mathcal{D} \cdot \nabla u)^2 = |x|^{2-n} \left((v'_\varepsilon + \frac{n-2}{2}v_\varepsilon)^2 + (a \cdot x) \left((v'_\varepsilon)^2 - \frac{(n-2)^2}{4} + \right. \right. \\ \left. \left. \frac{n(n-2)}{2}v_\varepsilon^{\frac{n+2}{n-2}}(v'_\varepsilon + \frac{n-2}{2}v_\varepsilon) \right) + \dots \right). \end{aligned} \quad (29)$$

Combining this with the appropriate multiples of \mathcal{D} applied to (27) and $|x|^2$ multiplied by (28), we find that

$$S(\mathcal{D}, \mathcal{D}) = H(\varepsilon)(1 + n(a \cdot x) + \dots).$$

Integrating on S^{n-1} , and taking the limit as $\eta \rightarrow 0$, we get

$$\mathcal{P}(\mathcal{D}, g) = \omega_{n-1} H(\varepsilon). \quad (30)$$

Next, consider the case $X = R^{(b,c)}$ (or simply R , for short). The third and fourth terms of the integrand involve the inner product of R with \mathcal{D} , and hence vanish. Also, R annihilates any function of $|x|$. Thus, the components of order zero in each of the remaining terms $(R \cdot \nabla u)(\mathcal{D} \cdot \nabla u)$ and $R(\mathcal{D} \cdot \nabla u - u)$ are of the form $F(|x|)(e \cdot x)$ for some vectors e . These integrate to zero, and so

$$\mathcal{P}(R^{(b,c)}, g) = 0. \quad (31)$$

For the final case, when $X = X^{(b)}$, we decompose X into the orthogonal sum of two vectors $X^{(1)} + X^{(2)}$, where

$$X^{(1)} = \frac{b \cdot x}{|x|^2} \mathcal{D}$$

is the radial component. Then

$$S(X^{(1)}, \mathcal{D}) = \frac{b \cdot x}{|x|^2} S(\mathcal{D}, \mathcal{D}).$$

Using the expansion above, this becomes

$$\frac{b \cdot x}{|x|^2} H(\varepsilon) + n \frac{(a \cdot x)(b \cdot x)}{|x|^2} H(\varepsilon) + \dots$$

The first term here integrates to zero, while for the second we use the identity

$$\int_{|x|=1} (a \cdot x)(b \cdot x) d\theta = \frac{1}{n} \omega_{n-1}(a \cdot b),$$

to conclude that its integral equals

$$\omega_{n-1}(a \cdot b)H(\varepsilon).$$

For the final term, $S(X^{(2)}, \mathcal{D})$, we continue with the same methods, noting that since $X^{(2)}$ is orthogonal to the radial direction, the third and fourth terms once again vanish. A straightforward calculation, using the same sorts of parity considerations, shows that the integral of the remaining two terms is exactly the same as for the first component. Putting these together, we get, at last, that

$$\mathcal{P}(X^{(b)}, g) = \frac{2n}{n-2} \omega_{n-1} H(\varepsilon)(a \cdot b). \quad (32)$$

Notice that one consequence of these calculations is that for any solution u of (1) with an isolated singularity at the origin, we can recover the values of the parameters ε and a in its refined expansion from the Pohožaev invariants.

6.2 The balancing formulæ

We now put the information from the last subsection together as follows. Let u correspond to any element of \mathcal{M}_k . We regard u as a function on \mathbb{R}^n rather than on the sphere S^n for simplicity. Assume that u is singular at the points $\{p_1, \dots, p_k\}$, and has Fowler and translation parameters at p_j given by ε_j , T_j and a_j , respectively. Let X be any one of the conformal Killing vector fields listed earlier. We compute the invariants for the homologically trivial hypersurfaces

$$\Sigma_\eta = \cup_{j=1}^k B(p_j, \eta) \cup B(0, 1/\eta)$$

by letting η tend to zero. To do this, first note that $u = O(|x|^{2-n})$ as $|x|$ tends to infinity (since ∞ corresponds to a regular point of u on the sphere), and thus the integral around the sphere of radius $1/\eta$ tends to zero with η . As for the integrals around the other components of Σ_η , we decompose X at each p_j as a sum of these four types of basis vector fields translated by p_j . Thus

- $X^{(b)} = \sum b_i \partial_{x_i}$
- $D = \sum (x_i - p_{j,i}) \partial_{x_i} + \sum p_{j,i} \partial_{x_i}$
- $R^{(b,c)} = \sum ((b \cdot p_j) c_i - (c \cdot p_j) b_i) \partial_{x_i} + \sum ((b \cdot (x - p_j)) c_i - (c \cdot (x - p_j)) b_i) \partial_{x_i}$
- $Y^{(b)} = \sum ((b \cdot p_j) p_{j,i} - \frac{1}{2} |p_j|^2 b_i) + \sum ((b \cdot (x - p_j)) p_{j,i} - (p_j \cdot (x - p_j)) b_i) \partial_{x_i} + \sum (b \cdot p_j) (x_i - p_{j,i}) \partial_{x_i} + O(|x - p_j|^2)$.

Here $p_{j,i}$ is the i^{th} coordinate of the point p_j , and the last term in the last item is some vector field with coefficients vanishing quadratically at p_j .

Now apply the computations of the last subsection to evaluate the limit as η tends to zero of the integral around the sphere $|x - p_j| = \eta$. For $X = X^{(b)}$ we obtain $\sum H(\varepsilon_j)(b \cdot a_j) = 0$ and since this holds for any vector $b \in \mathbb{R}^n$, we conclude that

$$\sum_{j=1}^k H(\varepsilon_j) a_j = 0. \quad (33)$$

Next, with $X = \mathcal{D}$ we get

$$\sum_{j=1}^k H(\varepsilon_j) ((p_j \cdot a_j) + \frac{1}{2}) = 0. \quad (34)$$

Next, with $X = R^{(b,c)}$ we get

$$\sum_{j=1}^k H(\varepsilon_j) ((b \cdot p_j)(c \cdot a_j) - (c \cdot p_j)(b \cdot a_j)) = 0 \quad (35)$$

for any vectors $b, c \in \mathbb{R}^n$. Finally, with $X = Y^{(b)}$ we get

$$\sum_{j=1}^k H(\varepsilon_j) \left((b \cdot p_j) \left(a_j \cdot p_j + \frac{1}{2} \right) - \frac{1}{2} |p_j|^2 (b \cdot a_j) \right) = 0 \quad (36)$$

for any $b \in \mathbb{R}^n$. These are the global balancing formulæ which hold for the parameters associated to any solution $u \in \mathcal{M}_k$.

This set of analytic equations does not determine all the parameters of the solution u , and in particular, gives absolutely no information about the ‘translation parameter’ T_j appearing in the asymptotics formula at each p_j . Still, particularly when k is small relative to n , these equations do shed more light on the global nature of the moduli space. For example, when $k = 3$ this set of equations can be solved and leads to an explicit formula for the parameters a_j in terms of the ε_j and p_j . This is not possible in general, but nevertheless we still have the following result:

Proposition 6 *There exists a constant $C > 0$, depending only on $\Lambda = \{p_1, \dots, p_k\}$, such that for any $u \in \mathcal{M}_\Lambda$, the corresponding translation parameters and necksizes satisfy $|H(\varepsilon_j)a_j| \leq C$, $j = 1, \dots, k$.*

We have already seen that $\mathcal{P}(X^{(b)}, g, \partial B(p_j, \eta))$ determines $\omega_{n-1}(a_j \cdot b)H(\varepsilon_j)$. On the other hand, using the universal upper bound of Theorem 2 and elliptic estimates, we also get an *a priori* bound for this invariant. Since this is true for all $b \in \mathbb{R}^n$, the result follows.

7 Nondegeneracy of the moduli spaces near their ends

In this final section we give another application of the asymptotics theorem and address the issue of the nondegeneracy of the unmarked moduli space \mathcal{M}_k . Unlike in the previous sections, we use only the simpler asymptotics result, not the more refined one. We have included this here because the arguments are soft, and not too different in spirit from some of the ones used above. This result is an adaptation of one in [9], and our desire here is to show its validity beyond the more limited setting of that paper.

It is unknown whether degenerate solutions ever exist. If they do they are quite unstable: it is shown in [11] that under an arbitrarily small generic change in the conformal class $[g_0]$, the moduli space becomes smooth. On the other hand, it is also nontrivial to show that a given solution is nondegenerate. This should be easier with explicitly constructed solutions, but unfortunately this was still impossible to do with the first-known solutions from [17]. One construction of nondegenerate solutions, in a somewhat limited setting, was given in [12], and another much more general one was given in [9]. In this last paper it was shown that given any singular set Λ , there is a nondegenerate solution singular at the points of Λ ; these solutions have very small necksizes (i.e. Fowler parameters). (It was also shown that for generic configurations Λ , these solutions are also nondegenerate in the marked moduli space \mathcal{M}_Λ .)

We show here that an argument from [9] may be adapted to prove something slightly weaker, although probably optimal. Before we state it, we discuss briefly the compactification theory of these moduli spaces. It was shown in [16] that if u_ℓ is any sequence of elements in \mathcal{M}_k such that the singular points $p_j^{(\ell)}$ are bounded away from one another and all Fowler parameters $\varepsilon_j^{(\ell)}$ are bounded away from zero, then some subsequence of u_ℓ converges to an element $u_\infty \in \mathcal{M}_k$. This result limits the ways in which noncompactness in \mathcal{M}_k can occur. This was discussed further in [11] and [10], where it was shown that if u_ℓ is a sequence in \mathcal{M}_k with one or both of these

restrictions not satisfied, then it is possible find conformal transformations F_ℓ such that some subsequence of $F_\ell^* u_\ell$ converges to an element u'_∞ in some $\mathcal{M}_{k'}$ with $k' < k$, or else converges to zero uniformly on compact sets. (To make things consistent here, we let \mathcal{M}_0 denote the set of pullbacks of the standard (smooth) metric on S^n by conformal transformations.) This result states, then, that \mathcal{M}_k may be compactified by adding to its ends certain subsets of moduli spaces $\mathcal{M}_{k'}$ of solutions with fewer or no singular points.

Finally then we can state our result.

Proposition 7 *Let $u_\ell \in \mathcal{M}_k$ be any sequence of elements such that the singular points $p_j^{(\ell)}$ stay bounded away from one another. Suppose that this sequence converges to $u_\infty \in \mathcal{M}_{k'}$ with $k' \leq k$. Then either u_∞ is a degenerate solution in $\mathcal{M}_{k'}$ or else u_ℓ is nondegenerate for sufficiently large ℓ .*

For simplicity in the notations, we will assume that the singularities $\{p_1^{(\ell)}, \dots, p_k^{(\ell)}\}$ do not depend on ℓ .

The proof is by contradiction. First let us apply the refined asymptotics theorem as follows. Choose small balls $B(p_j, \rho)$ which are disjoint from one another and such that we may write each u_ℓ as a sum of two functions

$$u_\ell = u_\ell^{\bar{\varepsilon}, \bar{a}, \bar{T}} + w_\ell.$$

Here $\bar{\varepsilon} = \{\varepsilon_1^{(\ell)}, \dots, \varepsilon_k^{(\ell)}\}$, $\bar{T} = \{T_1^{(\ell)}, \dots, T_k^{(\ell)}\}$ and $\bar{a} = \{a_1^{(\ell)}, \dots, a_k^{(\ell)}\}$ are the Fowler and translation parameters at each p_j for u_ℓ and $u_\ell^{\bar{\varepsilon}, \bar{a}, \bar{T}}$ is a function agreeing with u_ℓ outside the balls $B(p_j, \rho)$ and equalling the model deformed Fowler solution (relative to the background spherical metric) $u_{\varepsilon_j, a_j, T_j}$ in $B(p_j, \frac{1}{2}\rho)$; finally, $w_\ell \in \mathcal{C}_{\tilde{\gamma}}^{2, \alpha}$ for some fixed $\tilde{\gamma} > (4 - n)/2$ close to $(4 - n)/2$. Assume that at least some of the $\varepsilon_j^{(\ell)}$ tend to zero; otherwise the theorem is trivial. Relabel the points so that, after passing to a subsequence, $\varepsilon_j^{(\ell)}$ tends to zero for $k' + 1 \leq j \leq k$ while $\varepsilon_j^{(\ell)}$ converges to some nonzero values ε_j for $j \leq k'$. Finally, assume that there exists, for each ℓ , a function $\phi_\ell \in \mathcal{C}_{\tilde{\gamma}}^{2, \alpha}$ such that $\mathbb{L}_\ell \phi_\ell = 0$, where \mathbb{L}_ℓ is the Jacobi operator at u_ℓ .

Although it is not literally true, we shall assume that the singular points p_j do not vary with ℓ . Since we are assuming that they stay a bounded distance away from one another, we could transform to this case where Λ is fixed by a convergent set of diffeomorphisms of the sphere, but the only effect this would have would be to complicate notation.

Normalize ϕ_ℓ , multiplying it by a suitable constant, so that $\sup d(y)^{-\tilde{\gamma}} |\phi_\ell(y)| = 1$, where $d(y)$ is the distance of the point y from the singular set Λ in the spherical metric. Choose a point $y_\ell \in S^n \setminus \Lambda$ realizing this supremum, i.e. such that $d(y_\ell)^{-\tilde{\gamma}} |\phi_\ell(y_\ell)| = 1$.

If some subsequence of the y_ℓ converges to a point $y_0 \in S^n \setminus \Lambda$, then we may extract a subsequence of the v_ℓ converging to an element $v' \in S^n \setminus \Lambda'$, where $\Lambda' = \{p_1, \dots, p_{k'}\}$ and also so that ϕ_ℓ converges to a nontrivial function ϕ on $S^n \setminus \Lambda'$. Clearly, $|\phi| \leq d(y)^{\tilde{\gamma}}$, and also $\mathbb{L}' \phi = 0$ weakly on all of S^n , where \mathbb{L}' is the Jacobi operator at v' . Since $\tilde{\gamma} > (4 - N)/2$ and v' is smooth at the points p_j , $j > k'$, it follows from a standard removable singularities theorem that ϕ is actually smooth across these points. Thus $\phi \in \mathcal{C}_{\tilde{\gamma}}^{2, \alpha}(S^n \setminus \Lambda')$, and so we have shown that the limiting solution $v' \in \mathcal{M}_{k'}$, is degenerate.

If, on the other hand, some subsequence of the y_ℓ converges to one of the points p_j , then it is more convenient to transform the problem, using its conformal equivariance, to one on a cylinder before proceeding further. First, choose a function A on $S^n \setminus \{p_j, q\}$ (for any second point q) such that $A^{-\frac{4}{n-2}} g_0$ is the product metric $g_C = \frac{n-2}{n} (dt^2 + d\theta^2)$ on the cylinder $C = \mathbb{R} \times S^{n-1}$, with $t = \infty$ corresponding to p_j and $t = -\infty$ corresponding to q . On C , the function A is simply a multiple of $(\cosh t)^{\frac{n-2}{2}}$, and on S^n is of the order $\text{dist}(y, \{p_j, q\})^{\frac{n-2}{2}}$. The solutions v_ℓ on S^n correspond to solutions Av_ℓ on C . Since the metrics g_C and g_0 both have scalar curvature $n(n-1)$, a straightforward calculation (cf. [12]) shows that the linearized scalar curvature operators \mathbb{L}_ℓ on S^n at v_ℓ and $\mathbb{L}_{C, \ell}$ on C at Av_ℓ satisfy the same conformal equivariance property as the conformal

Laplacians for these two metrics, namely

$$\mathbb{L}_{C,\ell}(A\phi) = A^{\frac{N+2}{N-2}} \mathbb{L}_\ell \phi, \quad (37)$$

for any function ϕ . (This depends strongly on the fact that both metrics have the same scalar curvature.)

Using this transformation, we now have a sequence Av_ℓ of solutions of the nonlinear equation as well as a sequence $A\phi_\ell$ of solutions of the Jacobi operator at Av_ℓ on C . For simplicity we relabel these functions and the operator by v_ℓ , ϕ_ℓ and \mathbb{L}_ℓ again. Let $y = (t, \theta)$ denote the variable on the cylinder and define $\gamma = \tilde{\gamma} + (n-2)/2$, so that $\gamma > 1$, close to 1. Then

$$\sup d(y)^{-\gamma} |\phi_\ell| = 1, \quad (38)$$

where $d(y)$ is once again a smoothed distance function to the singular points in $\Lambda \setminus \{p_j\}$, transplanted to C , in some large compact set and equal to $\text{sech } t$ outside this neighbourhood. By (38), ϕ_ℓ decays at both ends of the cylinder.

As before, let the supremum in (38) be attained at the point $y_\ell = (t_\ell, \theta_\ell)$. By assumption, $t_\ell \rightarrow \infty$. Translating back by t_ℓ and renormalizing the solution, we find yet another sequence of solutions, which we again call w_ℓ , attaining their maximum at $t = 0$, and which solve the translated equation, which we again write as $\mathbb{L}_\ell \phi_\ell = 0$. Here \mathbb{L}_ℓ is the Jacobi operator at $v_\ell(t + t_\ell, \theta)$. As usual, some subsequence of the ϕ_ℓ converge to a nontrivial solution ϕ of the limiting equation $L\phi = 0$, and ϕ is bounded by $e^{-\gamma t}$ for all t .

There are two cases to consider. In the first, p_j is one of the singular points for which $\varepsilon_j^{(\ell)}$ tends to zero. Here there are two subcases, depending on whether $v_\ell(t_\ell, \theta_\ell)$ tends to zero or not. If it does tend to zero, then ϕ satisfies the equation

$$\frac{n}{n-2} (\partial_t^2 + \Delta_\theta) \phi - \frac{n(n-2)}{4} \phi = \frac{n}{n-2} \left(\partial_t^2 + \Delta_\theta - \frac{(n-2)^2}{4} \right) \phi = 0.$$

Decomposing ϕ into its Δ_θ eigenvectors, we see that any eigenvector ϕ_j is a sum of exponentials, $\phi_j = a_j^+ e^{\mu_j t} + a_j^- e^{-\mu_j t}$. Since ϕ decays as $t \rightarrow +\infty$, $a_j^+ = 0$. But then it is clear that no function of the form $e^{-\mu_j t}$ can be bounded for all t by $e^{-\gamma t}$ unless $\gamma = \mu_j$, which is not the case, so we arrive at a contradiction. In the other subcase, $v_\ell(t_\ell, \theta_\ell)$ does not tend to zero. Translating by a fixed finite amount, we may assume that v_ℓ tends to the function $(\cosh t)^{(2-N)/2}$, and hence, after pulling out the superfluous constants, that the limiting function ϕ satisfies

$$\left(\partial_t^2 + \Delta_\theta - \frac{(n-2)^2}{4} + \frac{n^2-4}{4} \text{sech}^2 t \right) \phi = 0.$$

Again separate ϕ into its eigenvectors ϕ_j . Then

$$\partial_t^2 \phi_j - \left(\frac{(n-2)^2}{4} + \lambda_j \right) \phi_j + \frac{n^2-4}{4} \text{sech}^2 t \phi_j = 0.$$

For $j = 0$ the indicial roots of this equation at both $\pm\infty$ are $\pm(n-2)/2$, for $j = 1$ they are $\pm n/2$, and for $j > 1$ the indicial roots are all $\geq (n+2)/2$.

The components ϕ_j with $j > 1$ are easy to eliminate. In fact, these ϕ_j must decay faster than $e^{\pm(n+2)|t|/2}$ at $\pm\infty$, so we may multiply the equation satisfied by ϕ_j and integrate by parts to obtain

$$\int_{-\infty}^{\infty} (\partial_t w_j)^2 + \left(\lambda_j + \frac{(n-2)^2}{4} \right) \phi_j^2 - \frac{n^2-4}{4} \text{sech}^2 t \phi_j^2 dt = 0.$$

Since $\lambda_j \geq 2n$, the integrand is nonnegative, hence $\phi_j = 0$.

For the remaining cases, when $j = 0, 1$, the indicial roots at $\pm\infty$ are less than $(n+2)/2$ in absolute value. On the other hand, to check unmarked nondegeneracy it suffices to use any

$\gamma_{\varepsilon_j, n+1} > \gamma > 1$. But because ε_j tends to zero, this upper limit tends to $(n+2)/2$. Thus if we choose γ in the range $(n/2, (n+2)/2)$ then both ϕ_0 and ϕ_1 decay less quickly than $e^{-\gamma t}$ as $t \rightarrow +\infty$, which implies that these ϕ_j too must vanish. This is a contradiction.

The final case to consider is when $\varepsilon_j^{(\ell)}$ does not converge to zero, so that v_ℓ is converging to some Fowler solution v_ε . In this case ϕ_ℓ converges to a solution of $\mathcal{L}_\varepsilon \phi = 0$ which satisfies $|\phi| \leq C e^{-\gamma t}$. But we have shown that all solutions of this equation are sums of terms for each eigenspace of Δ_θ which satisfy bounds $|\psi_{\varepsilon, j}^\pm| e^{\mp \gamma_{\varepsilon, j} t} \leq C$. Clearly this is incompatible with the previous bound, so we arrive here too at a contradiction.

We have shown, finally, that unless it is converging to a degenerate solution in some $\mathcal{M}_{k'}$, v_ℓ is nondegenerate in \mathcal{M}_k .

It may well seem disappointing that we can not exclude degeneracy near any end, but as remarked earlier, this form of the result is probably optimal. However, in certain cases we can deduce nondegeneracy without restriction; this is due to the fact that only the first of the several cases treated in the proof did not necessarily lead to a contradiction. However, for example, if the solutions v_ℓ are converging to zero uniformly on compact sets of $S^n \setminus \Lambda$, then they must be nondegenerate for sufficiently large ℓ . This is because the equation satisfied by the limiting Jacobi field ϕ is $(\Delta_{S^n} - (n-2)^2/4)\phi = 0$ which has no nontrivial solution. This is the case studied in [9]. Another case where we can deduce nondegeneracy for sufficiently large ℓ is when all but two of the singular points p_j disappear in the limit, i.e. when $k' = 2$. This is because any Fowler solution is nondegenerate. On the other hand, it is not possible, using this argument, to deduce nondegeneracy for the solutions constructed in [17], for there v_ℓ converges to the constant function 1 on S^n , and the sphere is degenerate.

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