REFINED BOUNDS ON THE NUMBER OF CONNECTED COMPONENTS OF SIGN CONDITIONS ON A VARIETY

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Abstract. Let R be a real closed field, $\mathcal{P}, \mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$ finite subsets of polynomials, with the degrees of the polynomials in $\mathcal{P}$ (resp. $\mathcal{Q}$) bounded by d (resp. d_0). Let $V \subset \mathrm{R}^k$ be the real algebraic variety defined by the polynomials in $\mathcal{Q}$ and suppose that the real dimension of V is bounded by k'. We prove that the number of semi-algebraically connected components of the realizations of all realizable sign conditions of the family $\mathcal{P}$ on V is bounded by

$$\sum_{j=0}^{k'} 4^j \binom{s+1}{j} F_{d,d_0,k,k'}(j),$$

where $s = \mathrm{card}\, \mathcal{P}$, and

$$F_{d,d_0,k,k'}(j) = \binom{k+1}{k-k'+j+1} (2d_0)^{k-k'} d^j \ \max\{2d_0, d\}^{k'-j} + 2(k-j+1).$$

In case $2d_0 \leq d$, the above bound can be written simply as

$$\sum_{j=0}^{k'} \binom{s+1}{j} d^{k'} d_0^{k-k'} O(1)^k = (sd)^{k'} d_0^{k-k'} O(1)^k$$

(in this form the bound was suggested by J. Matousek [?]). Our result improves in certain cases (when $d_0 \ll d$) the best known bound of

$$\sum_{1 \leq j \leq k'} \binom{s}{j} 4^j d(2d-1)^{k-1}$$

on the same number proved earlier by Basu, Pollack and Roy (2009), in the case $d = d_0$.

The distinction between the bound d_0 on the degrees of the polynomials defining the variety V and the bound d on the degrees of the polynomials in $\mathcal{P}$ that appears in the new bound is motivated by several applications in discrete geometry.

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