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REFINEMENTS, EXTENSIONS AND GENERALIZATIONS OF THE SECOND KERSHAW'S DOUBLE INEQUALITY

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Abstract. In the paper, the second Kershaw's double inequality concerning the ratio of two gamma functions is refined, extended and generalized elegantly.

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Key words and phrases: Kershaw's double inequality, refinement, extension, generalization, generalized logarithmic mean, ratio of two gamma functions, psi function, polygamma function, inequality, monotonicity.

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